CHAPTER I

PRELIMINARIES

In this chapter we shall give some notations, definitions and theorems used in this thesis. Our notations are :

$\mathbb{Z}$ is the set of all integers,

$\mathbb{Z}^+$ is the set of all positive integers,

$\mathbb{Z}_0^+ = \mathbb{Z}^+ \cup \{0\}$,

$\mathbb{Q}^+$ is the set of all positive rational numbers,

$\mathbb{Q}_0^+ = \mathbb{Q}^+ \cup \{0\}$,

$\mathbb{R}^+$ is the set of all positive real numbers and

$\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$.

In this thesis, if we do not give the definitions of a binary operations or order on any subset of $\mathbb{R}_0^+$ then we shall mean the usual binary operations and order on it.

As usual one may write $y \geq x$ for $x \leq y$ and $x < y$ or $y > x$ to mean that $x \leq y$ and $x \neq y$. If neither $x \leq y$ nor $y \leq x$ then x and y are said to be <u>incomparable</u> and this is denoted by $x \,\|\, y$.

<u>Definition 1.1.</u> Let $(P, \leq)$ be a partially ordered set. P is said to be <u>complete</u> iff every subset of P which has a lower bound has an infimum.

In [5], p. 5 it was shown that a partially ordered set is complete iff every subset of P which has an upper bound has a supremum.

<u>Definition 1.2.</u> Let $(P, \leq)$ be a partially ordered set, P is a <u>lower</u> [upper] <u>semilattice</u> iff $\inf\{x, y\}$ $[\sup\{x, y\}]$ exists for all $x, y \in P$ and denoted by $x \wedge y$ $[x \vee y]$. P is said to be a <u>lattice</u> iff P is both a lower and upper semilattice.

<u>Definition 1.3.</u> Let $(P, \leq)$ be a partially ordered set. A nonempty subset S of P is called <u>dense</u> in P iff for every $x, y \in P$, $x < y$ implies that there exists $z \in S$ such that $x < z < y$.

<u>Definition 1.4.</u> Let $(S, +)$ be a semigroup. S is said to be a <u>band</u> iff for every $x \in S$, $x + x = x$.

Let $(L, \leq)$ be an upper [lower] semilattice. Define a binary operation $+_{\leq}$ on L by $x +_{\leq} y = x \vee y$ $[x \wedge y]$ for all $x, y \in L$. Then we have that $(L, +_{\leq})$ is a commutative band.

Let $(L, +)$ be a commutative band. Define a binary operation $\leq_+$ on L by $x \leq_+ y$ iff $x + y = y$ for all $x, y \in L$. Then we have that $(L, \leq_+)$ is an upper semilattice such that $x \vee y = x + y$ for all $x, y \in L$.

<u>Proposition 1.5.</u> Let L be a nonempty set. Let $\mathcal{S}$ be the set of all semilattice structure on L and $\mathcal{C}$ the set of all commutative band structures on L. Then there exists a bijection between $\mathcal{S}$ and $\mathcal{C}$.

<u>Proof</u> Define $\varphi : \mathcal{C} \to \mathcal{S}$ by $\varphi((L, +)) = (L, \leq_+)$ for all $(L, +) \in \mathcal{C}$ and define $\Psi : \mathcal{S} \to \mathcal{C}$ by $\Psi((L, \leq)) = (L, +_{\leq})$ for all $(L, \leq) \in \mathcal{S}$. To show that $\Psi \circ \varphi = Id_{\mathcal{C}}$ and $\varphi \circ \Psi = Id_{\mathcal{S}}$, let $(L, +) \in \mathcal{C}$. Then $\Psi \circ \varphi((L, +)) = \Psi(\varphi((L, +))) = \Psi((L, \leq_+)) = (L, +_{\leq_+})$. Let $x, y \in L$.

Therefore $x +_{\leq_+} y = x \vee y = x + y$. So $+_{\leq_+} = +$, hence $\Psi \circ \varphi = Id_{\mathscr{S}}$. Next, let $(L, \leq) \in \mathscr{P}$. Then $\varphi \circ \Psi((L, \leq)) = \varphi(\Psi(L, \leq)) = \varphi((L, +_{\leq})) = (L, \leq_{+_\leq})$. Let $x, y \in L$ be such that $x \leq_{+_\leq} y$. Then $x +_{\leq} y = y$. Since $x +_{\leq} y = x \vee y$, $x \vee y = y$. So $x \leq y$, hence $\leq_{+_\leq} \subseteq \leq$. Similarly, $\leq \subseteq \leq_{+_\leq}$, hence $\leq = \leq_{+_\leq}$. Thus $\varphi \circ \Psi = Id_{\mathscr{P}}$. Therefore Ψ is a bijection. #

<u>Definition 1.6.</u> Let L be a nonempty set and $\wedge, \vee$ be binary operations on L such that

i) $(L, \wedge)$ and $(L, \vee)$ are commutative bands and

ii) for every $x, y \in L$, $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$

Then $(L, \wedge, \vee)$ is called a <u>lattice algebra.</u>

Let $(L, \wedge, \vee)$ be a lattice algebra. Define $\leq_{\wedge\vee}$ on L by $x \leq_{\wedge\vee} y$ iff $x \wedge y = x$ for all $x, y \in L$. Then we have that $(L, \leq_{\wedge\vee})$ is a partially ordered set.

Note that for every $x, y \in L$, we define $x \leq_{\wedge\vee} y$ iff $x \wedge y = x$ is equivalent to $x \vee y = y$.

Next to show that $(L, \leq_{\wedge\vee})$ is lattice, claim that $\inf\{x, y\} = x \wedge y$ and $\sup\{x, y\} = x \vee y$ for all $x, y \in L$. Let $x, y \in L$, clear that $x \wedge y \leq_{\wedge\vee} x$ and $x \wedge y \leq_{\wedge\vee} y$. Thus $x \wedge y$ is a lower bound of x and y. Let $w \in L$ be such that $w \leq_{\wedge\vee} x$ and $w \leq_{\wedge\vee} y$. Then $w \wedge x = w$ and $w \wedge y = w$. Therefore

$w \wedge (x \wedge y) = (w \wedge x) \wedge y = w \wedge y = w$, so $w \leq_{\wedge\vee} x \wedge y$. Thus $\inf\{x, y\} = x \wedge y$.

Similarly, $\sup\{x, y\} = x \vee y$ for all $x, y \in L$.

Let $(L, \leq)$ be a lattice. Then we have that $(L, \wedge_{\leq}, \vee_{\leq})$ is a lattice algebra where $x \wedge_{\leq} y = \inf\{x, y\}$ and $x \vee_{\leq} y = \sup\{x, y\}$ for all $x, y \in L$.

Proposition 1.7. Let L be a nonempty set. Let $\mathscr{S}$ be the set of all lattice algebra structures on L and $\mathscr{L}$ the set of all lattice structures on L. Then there exists a bijection between $\mathscr{S}$ and $\mathscr{L}$.

Proof Define $\varphi : \mathscr{L} \to \mathscr{S}$ by $\varphi((L, \leq)) = (L, \wedge_{\leq}, \vee_{\leq})$ for all $(L, \leq) \in \mathscr{L}$ and define $\Psi : \mathscr{S} \to \mathscr{L}$ by $\Psi((L, \wedge, \vee)) = (L, \leq_{\wedge\vee})$ for all $(L, \wedge, \vee) \in \mathscr{S}$. To show that $\Psi \circ \varphi = Id_{\mathscr{L}}$ and $\varphi \circ \Psi = Id_{\mathscr{S}}$ let $(L, \leq) \in \mathscr{L}$. Then $\Psi \circ \varphi((L, \leq)) = \Psi(\varphi((L, \leq))) = \Psi((L, \wedge_{\leq}, \vee_{\leq})) = (L, \leq_{\wedge_{\leq}\vee_{\leq}})$.

To show that $\leq = \leq_{\wedge_{\leq}\vee_{\leq}}$, let $x, y \in L$. Assume that $x \leq_{\wedge_{\leq}\vee_{\leq}} y$ iff $x \wedge_{\leq} y = x$, so $x \wedge y = x$. Then $x \leq y$, so $\leq_{\wedge_{\leq}\vee_{\leq}} \subseteq \leq$. Similarly, $\leq \subseteq \leq_{\wedge_{\leq}\vee_{\leq}}$, hence $\Psi \circ \varphi = Id_{\mathscr{L}}$. Next, let $(L, \wedge, \vee) \in \mathscr{S}$. Then $\varphi \circ \Psi((L, \wedge, \vee)) = \varphi(\Psi((L, \wedge, \vee)) = \varphi((L, \leq_{\wedge\vee})) = (L, \wedge_{\leq_{\wedge\vee}}, \vee_{\leq_{\wedge\vee}})$. To show that $\vee = \vee_{\leq_{\wedge\vee}}$ and $\wedge = \wedge_{\leq_{\wedge\vee}}$, let $x, y \in L$. Then $x \wedge_{\leq_{\wedge\vee}} y = \inf\{x, y\} = x \wedge y$ and $x \vee_{\leq_{\wedge\vee}} y = \sup\{x, y\} = x \vee y$. Hence $\varphi \circ \Psi = Id_{\mathscr{S}}$. Therefore Ψ is a bijection. #

Definition 1.8. Let L be a lattice algebra. L is said to be a distributive lattice algebra iff for every $x, y, z \in L$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Remark 1.9. Let L be a lattice algebra. Then L is a distributive lattice algebra iff for every $x, y, z \in L$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Proof Assume that L is lattice distributive algebra. Let $x, y, z \in L$. Then

$$\begin{aligned}(x \wedge y) \vee (x \wedge z) &= [(x \wedge y) \vee x] \wedge [(x \wedge y) \vee z] \\ &= x \wedge [(x \vee z) \wedge (y \vee z)] \\ &= [x \wedge (x \vee z)] \wedge (y \vee z) \\ &= x \wedge (y \vee z).\end{aligned}$$

The proof of the converse is similar. #

Definition 1.10. Let L be a lattice. L is said to be a distributive lattice iff for every $x, y, z \in L$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Corollary1.11. Let L be a nonempty set. Let $\mathscr{A}$ be the set of all distributive lattice algebra L and $\mathscr{B}$ be the set of all distributive lattice structures on L. Then there exists a bijection between $\mathscr{A}$ and $\mathscr{B}$.

Proof Similar to the proof of Proposition 1.7.. #

Definition 1.12. Let $(P, \leq)$ be a partially ordered set. P is a totally ordered set if for every $x, y \in P$, $x \leq y$ or $y \leq x$.

Definition 1.13. Let $(P, \leq)$ and $(P', \leq')$ be partially ordered sets. A function $f : P \to P'$ is said to be isotone iff $x \leq y$ implies $f(x) \leq' f(y)$ for all $x, y \in P$, f is said to be an order isomorphism iff f is bijection and both f and f^{-1} are isotone. In this case, P and P' are called order isomorphic.

Definition 1.14. Let P and P' be lattices and $f : P \to P'$ is said to be a lattice homomorphism iff for every $x, y \in P$, $f(x \vee y) = f(x) \vee f(y)$ and $f(x \wedge y) = f(x) \wedge f(y)$.

Remark 1.15. Let P and P' be lattices and $f : P \to P'$. Then the following statements clearly hold :

(1) if f is a lattice homomorphism then f is isotone

(2) if f is an order isomorphism then f is a lattice homomorphism.

<u>Definition 1.16.</u> A subset C of P is said to be <u>ordered convex subset</u> iff for every $x, y \in C$ and $z \in P$ the inequalities $x \leq z \leq y$ imply $z \in C$.

From now on we shall call an ordered convex subset an <u>o-convex</u> subset.

<u>Example 1.17.</u> (1) Let P be a partially ordered set, $x \in P$. $\{x\}$ is an o-convex subset of P.

(2) Every interval of $\mathbb{R}$ is an o-convex subset of $\mathbb{R}$.

(3) In $\mathbb{R} \times \mathbb{R}$, $\{(x, y) \mid x^2 + y^2 \leq 4\}$ is an o-convex subset of $\mathbb{R} \times \mathbb{R}$ where $(x, y) \leq (z, w)$ iff $x \leq z$ and $y \leq w$ for all $x, y, z, w \in \mathbb{R}$.

<u>Remark 1.18.</u> (1) The intersection of a family of o-convex subsets of a partially oredered set is o-convex. Also the union of an increasing chain of o-convex subsets is o-convex.

(2) If $f : P \to P'$ is an isotone map and C' an o-convex subset of P'. Then $f^{-1}(C')$ is an o-convex subset of P.

<u>Proof</u> 1) Clearly.

2) Let C' be an o-convex subset of P'. Let $x, y \in f^{-1}(C')$ and $z \in P$ be such that $x \leq z \leq y$. Since f is isotone, $f(x) \leq f(z) \leq f(y)$. Since $f(x), f(y) \in C'$ and C' is o-convex, $f(z) \in C'$. Therefore $z \in f^{-1}(C')$. #

<u>Definition 1.19.</u> A triple $(R, +, .)$ is a <u>semiring</u> iff

(1) $(R, .)$ is a semigroup,

(2) $(R, +)$ is a commutative semigroup and

(3) for every $x, y, z \in R$, $x(y + z) = xy + xz$ and $(y + z)x = yx + zx$.

Definition 1.20. Let $(R, +, .)$ be a commutative semiring with multiplicative zero 0. R is said to be a 0-semifield iff $(R^*, .)$ is a group and, $x + 0 = x$ for all $x \in K$. A subset H of 0-semifield K is called a subsemifield of K iff H is an 0-semifield under the same operations. And a subset S of K^* is said to be conic iff $S \cap S^{-1} = \{1\}$.

Remark 1.21. The intersection of a family subsemifields of a 0-semifield is a subsemifield. Hence the intersection of all subsemifields is the smallest subsemifield of a semifield and will be called the prime semifield.

Proposition 1.22. ([4]) Let K be a 0-semifield. If there is $x \in K^*$ such that x has an additive inverse. Then every element in K has an additive inverse and hence K is a field.

Proposition 1.23. ([4]) If K is a 0-semifield then the prime semifield of K is either isomorphic to Q_0^+ or Z_p where p is a prime number or the semifield $\{0, 1\}$ with $1 + 1 = 1$.

In our thesis, we shall study only 0-semifields which are not fields. So from now on we shall use the word semifield for 0-semifields.

Example 1.24. (1) Q_0^+, $\mathbb{R}_0^+$ are semifields.

(2) Let G be a lattice abelian group with zero 0. Then we can define a binary operation + on G by $x + y = x \vee y$ and $x + 0 = 0 + x = x$ for all $x, y \in G$. Then $G \cup \{0\}$ is a semifield.

(3) Let $K = \{2^n \mid n \in Z\} \cup \{0\}$. then K is a semifield with usual multiplication and $2^n + 2^m = 2^{\max\{n, m\}}$ for all $m, n \in Z$.

Definition 1.25. Let S be a semiring with multiplicative zero 0. Then S is said to be multiplicatively cancellative (M.C.) iff for every $x, y, z \in S$, $xy = xz$ implies that $x = 0$ or $y = z$. And S is said to be additively cancellative (A.C.) iff for every $x, y, z \in S$, $x + y = x + z$ implies that $y = z$.

Theorem 1.26. ([4]) Let S be a commutative semiring with multiplicative zero 0. Then S can be embedded into a semifield iff S is multiplicatively cancellative.

We shall now give the construction of semifield of quotients of semiring S which appears in [4], pp. 27-28.

Assume that S is having M.C. the property. Define a relation ~ on $S \times (S - \{0\})$ by

$(x, y) \sim (z, w)$ iff $xw = zy$ for all $(x , y), (z , w) \in S \times (S - \{0\})$. It is easy to show that ~ is an equivalence relation.

Let $\alpha, \beta \in S \times (S - \{0\})/_{\sim}$. Define + and . on $S \times (S - \{0\})/_{\sim}$ as follow : Choose $(x , y) \in \alpha$, $(z , w) \in \beta$ define

$$\alpha + \beta = [(xw + yz , yw)] \text{ and}$$

$$\alpha\beta = [(xz , yw)].$$

In [4], it was shown that $(S \times (S - \{0\})/_{\sim} , + , .)$ is the semifield of quotients of S.

Proposition 1.27. ([4]) Let S a commutative semiring with multiplicative zero 0 having M.C. of order > 1. Then $S \times (S - \{0\})/_{\sim}$ is the smallest semifield containing S up to isomorphism.

Examples 1.28. (1) Z_0^+ is a commutative semiring with multiplicative zero which is M.C..

(2) Let $S = Z^+ \times Z^+ \cup \{(0, 0)\}$. Then S with the usual addition and multiplication are a commutative semiring with multiplicative zero which is M.C..

(3) Let $Z_0^+[\sqrt{2}] = \{ a + b\sqrt{2} \mid a, b \in Z_0^+\}$. Then $Z_0^+[\sqrt{2}]$ is a commutative semiring with multiplicative zero which is M.C..

(4) Let S be a commutative semiring with multiplicative zero which is A.C. and M.C.. Let $S[x] = \{(a_n)_{n \in Z^+} \mid a_n \in S$ for all $n \in Z^+$ and $a_n \neq 0$ for only finitely many n$\}$. Define + and . on S[x] as follows : for $(a_n)_{n \in Z^+}, (b_n)_{n \in Z^+} \in S[x]$ define

$$(a_n)_{n \in Z^+} + (b_n)_{n \in Z^+} = (a_n + b_n)_{n \in Z^+} \text{ and}$$

$$(a_n)_{n \in Z^+} . (b_n)_{n \in Z^+} = (\sum_{i+j=n} a_i b_j)_{n \in Z^+}$$

Then we have that S[x] is a commutative semiring with multiplicative zero which is M.C..

Definition1.29. Let K be a semifield. A subset C of K is called an algebraically convex subset of K iff for every $x, y \in C$ and $a, b \in K$ such that $a + b = 1$, $ax + by \in C$.

From now on we shall call algebraically convex subsets a-convex subsets.

Remark1.30. (1) The intersection of a family of a-convex subsets of semifield is a-convex. And union of the increasing chain of a-convex sets is an a-convex.

(2) Let C be an a-convex subset of a semifield K. For every $n \in Z^+$, $a_1, a_2, \ldots, a_n \in K$, $x_1, x_2, \ldots, x_n \in C$ and $\sum_{i=1}^{n} a_i = 1$, $\sum_{i=1}^{n} a_i x_i \in C$.

(3) Let C be a subset of a semifield K, the smallest an a-convex subset of K containing C is $\{\sum_{i=1}^{n} a_i x_i \mid n \in Z^+, a_i \in K, x_i \in C$ and $\sum_{i=1}^{n} a_i = 1$ for all $i \in \{1, 2, \ldots, n\}\}$.

Proof To prove (3), let $B = \{\sum_{i=1}^{n} a_i x_i \mid n \in Z^+, a_i \in K, x_i \in C$ and $\sum_{i=1}^{n} a_i = 1$ for all $i \in \{1, 2, \ldots, n\}\}$. To show that B is an a-convex subset of K containing C, let $\sum_{i=1}^{n} a_i x_i$, $\sum_{i=1}^{n} b_i y_i \in B$ and $a, b \in K$ be such that $a + b = 1$. So $a(\sum_{i=1}^{n} a_i x_i) + b(\sum_{i=1}^{n} b_i y_i) = \sum_{i=1}^{n} (aa_i x_i + bb_i y_i) \in B$ since $\sum_{i=1}^{n} (aa_i + bb_i) = 1$. Clearly C is contained in B, so B is an a-convex subset of K which containing C.

Let D be an a-convex subset of K containing C. To show that $B \subseteq D$, let $\sum_{i=1}^{n} a_i x_i \in B$. Assume that $a_1, a_2, \ldots, a_n$ are nonzero. Since $x_1, x_2, \ldots, x_n \in C \subseteq D$, $x_1, x_2, \ldots, x_n \in D$. By the a-convexity of D, $\sum_{i=1}^{n} a_i x_i \in D$. #

Definition1.31. A subset C of a semifield K is called an a-convex subgroup of K iff C is a mutiplicative subgroup of K^* and it is an a-convex subset of K.

Remark 1.32. Let K be a semifield.

(1) $\{1\}$ and K^* are trivial a-convex subgroups of K.

(2) The intersection of a family of a-convex subgroups of K is an a-convex subgroup of K. Also the union of an increasing chain of a-convex subgroups is an a-convex subgroup of K.

Proposition 1.33. Let K be a semifield and C a multiplicative subgroup of K. Then the following statements are equivalent :

(1) C is a-convex.

(2) for every $x, y \in C$ and $a \in K$, $(x + a)(y + a)^{-1} \in C$.

(3) for every $x \in C$ and $a, b \in K$ such that $a + b = 1$, $ax + b \in C$.

(4) for every $x, y \in C$ and $a, b \in K$ such that $a + b \in C$, $ax + by \in C$.

(5) for every $x \in C$ and $a, b \in K$ such that $a + b \in C$, $ax + b \in C$.

Proof (1) $\rightarrow$ (2) Let $x, y \in C$ and $a \in K$. Then $xy^{-1} \in C$. By (1), $(x + a)(y + a)^{-1} = y(y + a)^{-1}(xy^{-1}) + a(y + a)^{-1} \in C$.

(2) $\rightarrow$ (1) Let $x, y \in C$ and $a, b \in K$ be such that $a + b = 1$. If $a = 0$ then $b = 1$. So $ax + by = y \in C$. Suppose that $a \neq 0$. Thus $(ax + by)y^{-1}$ $= (ax + by)(ay + by)^{-1} = [x + (by)a^{-1}][y + (by)a^{-1}]^{-1} \in C$. Therefore $ax + by \in C$.

(3) $\rightarrow$ (1) Let $x, y \in C$ and $a, b \in K$ be such that $a + b = 1$. Since $x, y \in C$, $xy^{-1} \in C$. By (3), $(ax + by)y^{-1} = a(xy^{-1}) + b \in C$. Therefore $ax + by \in C$.

(1) $\rightarrow$ (4) Let $x, y \in C$ and $a, b \in K$ be such that $a + b \in C$. By (1), $(ax + by)(a + b)^{-1} = [a(a + b)^{-1}]x + b(a + b)^{-1}y \in C$. Since $a + b \in C$, $ax + by \in C$.

(5) $\rightarrow$ (4) Let $x, y \in C$ and $a, b \in K$ be such that $a + b \in C$. Then $xy^{-1} \in C$. By (5), $(ax + by)y^{-1} = (ax)y^{-1} + b \in C$. Hence $ax + by \in C$.

The remaining cases are clearly seen to be true. #

Proposition 1.34. Let A and B be a-convex subgroups of a semifield K. Then AB is an a-convex subgroup of K.

Proof Clearly, AB is a multiplicative subgroup of K. Let $x \in A$, $y \in B$ and $a, b \in K$ be such that $a + b = 1$. Since A is a-convex, $ax + b \in A$. Since B is a-convex, $(axy + b)(ax + b)^{-1} = [ax(ax + b)^{-1}]y + [b(ax + b)^{-1}] \in B$. Thus $a(xy) + b = (ax + b)[(axy + b)(ax + b)^{-1}] \in AB$. Hence AB is an a-convex subgroup of K. #

Notation Let K be a semifield and $S \subseteq K^*$. Let (S) be the multiplicative subsemigroup of K generated by S and $< S >_a$ is the a-convex subgroup of K generated by S. Therefore $< S >_a = \{(\sum_{i=1}^{m} a_i x_i)(\sum_{j=1}^{n} b_j y_j)^{-1} \mid m, n \in Z^+, a_i, b_j \in K, x_i, y_j \in (S) \text{ and } \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = 1 \text{ for all } i \in \{1, 2, ..., m\} \text{ and } j \in \{1, 2, ..., n\}\}$

To prove this, let $B = \{(\sum_{i=1}^{m} a_i x_i)(\sum_{j=1}^{n} b_j y_j)^{-1} \mid m, n \in Z^+, a_i, b_j \in K, x_i, y_j \in (S) \text{ and } \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = 1 \text{ for all } i \in \{1, 2, ..., m\} \text{ and } j \in \{1, 2, ...,n\}\}$.

Let $(\sum_{i=1}^{n} a_i x_i)(\sum_{i=1}^{n} b_i y_i)^{-1}$, $(\sum_{j=1}^{m} c_j z_j)(\sum_{j=1}^{m} d_j w_j)^{-1} \in B$. Then

$$(\sum_{i=1}^{n} a_i x_i)(\sum_{i=1}^{n} b_i y_i)^{-1}][(\sum_{j=1}^{m} c_j z_j)(\sum_{j=1}^{m} d_j w_j)^{-1}]^{-1} = (\sum_{i=1}^{n}\sum_{j=1}^{m} a_i d_j x_i w_j)(\sum_{i=1}^{n}\sum_{j=1}^{m} b_i c_j y_i z_j)^{-1} \in B.$$

Let $a, b \in K$ be such that $a + b = 1$.

Then $a[(\sum_{i=1}^{n} a_i x_i)(\sum_{i=1}^{n} b_i y_i)^{-1}] + b[(\sum_{j=1}^{m} c_j z_j)(\sum_{j=1}^{m} d_j w_j)^{-1}]$

$$= [a(\sum_{i=1}^{n} a_i x_i)(\sum_{j=1}^{m} d_j w_j) + b(\sum_{j=1}^{m} c_j z_j)(\sum_{i=1}^{n} b_i y_i)](\sum_{i=1}^{n}\sum_{j=1}^{m} b_i d_j y_i w_j)^{-1}$$

$$= [\sum_{i=1}^{n}\sum_{j=1}^{m} aa_i d_j x_i w_j + \sum_{i=1}^{n}\sum_{j=1}^{m} bb_i c_j y_i z_j](\sum_{i=1}^{n}\sum_{j=1}^{m} b_i d_j y_i w_j)^{-1} \in B.$$ Clearly, $S \subseteq B$.

Therefore B is an a-convex subgroup of K containing S. Let D be an a-convex subgroup containing S. Let $(\sum_{i=1}^{n} a_i x_i)(\sum_{i=1}^{n} b_i y_i)^{-1} \in B$. Then $x_i, y_i \in (S)$ for all $i \in \{1, 2, \dots, n\}$. So $x_i, y_i \in D$ for all i. Since D is a-convex, $\sum_{i=1}^{n} a_i x_i, \sum_{i=1}^{n} b_i y_i \in D$. Hence $(\sum_{i=1}^{n} a_i x_i)(\sum_{i=1}^{n} b_i y_i)^{-1} \in D$ since D is a subgroup. So $B \subseteq D$, hence $B = \langle S \rangle_a$. #

<u>Definition 1.35.</u> Let K and M be semifields. A function $f : K \to M$ is called a <u>homomorphism</u> of K into M iff for every $x, y \in K$

(1) $f(x) = 0$ if and only if $x = 0$,

(2) $f(x + y) = f(x) + f(x)$ and

(3) $f(xy) = f(x)f(y)$.

And the <u>kernel of f</u> is the set $\{x \in K \mid f(x) = 1\}$, which is denoted by ker f.

A homomorphism $f : K \to M$ is called a <u>monomorphism</u> iff f is injective, an <u>epimorphism</u> if f is onto and an <u>isomorphism</u> if f is bijection. K and M are said to be <u>isomorphic</u> if there exists an isomorphism of K onto M and we denote this by $K \cong M$. Note that if $f : K \to M$ is an isomorphism then f^{-1} is also an isomorphism.

<u>Remark 1.36.</u> Let $f : K \to M$ be a homomorphism of semifields. Then the following statements hold :

(1) $f(x^{-1}) = (f(x))^{-1}$ for all $x \in K^*$.

(2) $\ker f$ is an a-convex subgroup of K.

(3) If C' is an a-convex subgroup of M then $f^{-1}(C')$ is an a-convex subgroup of K containing $\ker f$.

(4) If f is onto and C an a-convex subgroup of K then f(C) is an a-convex subgroup of M.

Proof (1) Obviously.

(2) Clear that $\ker f$ is a multiplicative subgroup of K. Let $x \in \ker f$ and $a, b \in K$ be such that $a + b \in \ker f$. Then $f(x) = 1$ and $f(a) + f(b) = f(a + b) = 1$, so $f(ax + b) = f(a)f(x) + f(b) = f(a) + f(b) = 1$. Hence $ax + b \in \ker f$ Therefore $\ker f$ is an a-convex subgroup of K.

(3) Clearly, $f^{-1}(C')$ is a multiplicative subgroup of K containing $\ker f$. Let $x \in f^{-1}(C')$ and $a, b \in K$ be such that $a + b \in f^{-1}(C')$. Then $f(x) \in C'$ and $f(a) + f(b) = f(a + b) \in C'$. By the a-convexity of C', $f(ax + b) = f(a)f(x) + f(b) \in C'$. So $ax + b \in f^{-1}(C')$, hence $f^{-1}(C')$ is an a-convex subgroup of K.

(4) Let C be an a-convex subgroup of K. Clearly f(C) is a multiplicative subgroup of M. Let $x \in f(C)$ and $a', b' \in K$ be such that $a' + b' = 1$. Then there is $c \in C$ such that $f(c) = x$. Since f is onto, $f(a) = a'$ and $f(b) = b'$ for some $a, b \in K$. Then $f(a + b) = f(a) + f(b) = a' + b' = 1$. Since C is a-convex, $(ac + b)(a + b)^{-1} = [a(a + b)^{-1}]c + b(a + b)^{-1} \in C$. Hence $f[(ac + b)(a + b)^{-1}] = [f(ac + b)][f(a + b)]^{-1} = f(a)f(c) + f(b) = a'x + b' \in f(C)$. Therefore f(C) is an a-convex subgroup of M. #

We shall give some examples of an a-convex subgroups of semifield.

Example1.37. Let $\overset{+}{\mathbb{R}} \times \overset{+}{\mathbb{R}} \cup \{(0,0)\}$ be a semifield. Define $f : \overset{+}{\mathbb{R}} \times \overset{+}{\mathbb{R}} \cup \{(0,0)\} \to \overset{+}{\mathbb{R}}_0$ by $f((x,y)) = x$ for all $(x,y) \in \overset{+}{\mathbb{R}} \times \overset{+}{\mathbb{R}} \cup \{(0,0)\}$. It is easy to show that f is a homomorphism and $\ker f = \{(1,x) \mid x \in \overset{+}{\mathbb{R}}\}$.

By Remark 1.36. (2), $\{(1,x) \mid x \in \overset{+}{\mathbb{R}}\}$ is an a-convex subgroup of $\overset{+}{\mathbb{R}} \times \overset{+}{\mathbb{R}} \cup \{(0,0)\}$.

Proposition 1.38. Let $f : K \to M$ be an epimorphism of semifields. Let $\mathscr{C}$ be the set of all a-convex subgroups of K containing ker f and $\mathscr{C}'$ the set of all a-convex subgroups of M. Then there exists an order isomorphism from $\mathscr{C}$ onto $\mathscr{C}'$.

Proof Define $\varphi : \mathscr{C}' \to \mathscr{C}$ by $\varphi(C') = f^{-1}(C')$ for all $C' \in \mathscr{C}'$ and define $\Psi(C) = f(C)$ for all $C \in \mathscr{C}$. To show that $\varphi \circ \Psi = Id_{\mathscr{C}}$ and $\Psi \circ \varphi = Id_{\mathscr{C}'}$, let $C \in \mathscr{C}$. Then $\varphi \circ \Psi(C) = \varphi(\Psi(C)) = \varphi(f(C)) = f^{-1}(f(C))$. Clearly, $C \subseteq f^{-1}(f(C))$. Let $x \in f^{-1}(f(C))$. Then $f(x) \in f(C)$, so $f(x) = f(c)$ for some $c \in C$. Then $f(xc^{-1}) = f(x)f(c)^{-1} = 1$. So $xc^{-1} \in \ker f$. Since $\ker f \subseteq C$, $xc^{-1} \in C$. Hence $x \in C$. Thus $f^{-1}(f(C)) \subseteq C$, so $f^{-1}(f(C)) = C$. Therefore $\varphi \circ \Psi = Id_{\mathscr{C}}$

Next, let $C' \in \mathscr{C}'$. Then $\Psi \circ \varphi(C') = \Psi(\varphi(C')) = \Psi(f^{-1}(C')) = f(f^{-1}(C'))$. Since f is onto, $f(f^{-1}(C')) = C'$. Thus $\Psi \circ \varphi = Id_{\mathscr{C}'}$. Hence φ is a bijection. Clearly φ and Ψ are isotone. Therefore φ is an order isomorphism. #

Definition 1.39. Let K be a semifield and ρ an equivalence relation on K. ρ is called a congruence on K if for any $x, y, z \in K$,

(1) $x \rho 0$ if and only if $x = 0$,

(2) $x \rho y$ implies that $(xz) \rho (yz)$ and

(3) $x \rho y$ implies that $(x + z) \rho (y + z)$.

Remark 1.40. (1) The intersection of a family of congruences on semifield K is a congruence on K.

(2) $x \rho y$ implies $x^{-1} \rho y^{-1}$ for all $x, y \in K^*$.

Let K be a semifield and ρ a congruence on K. Let K/ρ be the set of all equivalence classes of K with respect to ρ.

We shall show that $[1]_\rho = \{x \in K \mid x \rho 1\}$ is an a-convex subgroup of K.

Let $x, y \in [1]_\rho$. Then $x \rho 1$ and $y \rho 1$. Thus $(xy) \rho y$, so $(xy) \rho 1$. Hence $xy \in [1]_\rho$. Since $x \rho 1$, $x \neq 0$. Then $xx^{-1} \rho x^{-1}$, so $x^{-1} \in [1]_\rho$. Hence $[1]_\rho$ is a multiplicative subgroup. Next, let $a, b \in K$ be such that $a + b = 1$. Since $x \rho 1$ and $y \rho 1$, $(ax) \rho a$ and $(by) \rho b$, so $(ax + by) \rho (a + by)$ and $(by + a) \rho (a + b)$. Hence $(ax + by) \rho (a + b)$, so $(ax + by) \rho = 1$. Therefore $[1]_\rho$ is an a-convex subgroup of K.

Let C be an a-convex subgroup of a semifield K. Define a relation ρ_C on K by $x \rho_C y$ iff $xy^{-1} \in C$ or $x = y = 0$ for all $x, y \in K$. Then we have that ρ_C is a congruence on K and denoted $[x]_{\rho_C} = xC$.

Let K/ρ_C be the set of all equivalence classes of K with repect to ρ_C, we shall use the notation K/C instead of K/ρ_C.

Define + and . on K/C as follows : for $xC, yC \in K/C$

$$xC + yC = (x + y)C \text{ and } xC \,.\, yC = (xy)C.$$

To prove that + and . are well-defined, let $xC, yC \in K/C$. Choose $a \in xC$ and $b \in yC$. Then $xa^{-1} \in C$ or $x = a = 0$ and $yb^{-1} \in C$ or $y = b = 0$. If $x = a = 0$ then $ab = xy = 0$. Thus $(ab)C = (xy)C$ and $(a + b)C = bC = yC = (x + y)C$. This prove is similar for $y = b = 0$ then done. Suppose that $a, b \neq 0$. Then $xa^{-1}, yb^{-1} \in C$. Then $(xy)(ab)^{-1} \in C$, so $(xy)C = (ab)C$. So . is well-defined.

Since C is a-convex, $(x + y)(a + b)^{-1} = [a(a + b)^{-1}](xa^{-1}) + [b(a + b)^{-1}](yb^{-1}) \in C$, so $(x + y)C = (a + b)C$. Therefore $+$ is well-defined.

Then we have that $(K/_C, +, .)$ is a semifield.

<u>Proposition 1.41.</u> Let K be a semifield. Let $\mathcal{A}$ be the set of all congruence structures on K and $\mathcal{B}$ the set of all a-convex subgroups of K. Then there exists an order isomorphism form $\mathcal{A}$ onto $\mathcal{B}$.

<u>Proof</u> Define $\varphi : \mathcal{A} \to \mathcal{B}$ by $\varphi(\rho) = [1]_\rho$ for all $\rho \in \mathcal{A}$ and define $\Psi : \mathcal{B} \to \mathcal{A}$ by $\Psi(C) = \rho_C$ for all $C \in \mathcal{B}$. To show that $\varphi \circ \Psi = Id_{\mathcal{B}}$ and $\Psi \circ \varphi = Id_{\mathcal{A}}$, let $\rho^* \in \mathcal{A}$. Then $\Psi \circ \varphi(\rho^*) = \Psi(\varphi(\rho^*)) = \Psi([1]_{\rho^*}) = \rho_{[1]_{\rho^*}}$. Let $x, y \in K$ be such that $x \, \rho_{[1]_{\rho^*}} \, y$. Then $xy^{-1} \in [1]_{\rho^*}$ or $x = y = 0$. If $x = y = 0$ then $x \, \rho^* \, y$. If $xy^{-1} \in [1]_{\rho^*}$ then $xy^{-1} \, \rho^* \, 1$. Hence $x \, \rho^* \, y$. Therefore $\rho_{[1]_{\rho^*}} \subseteq \rho^*$. Similarly, $\rho^* \subseteq \rho_{[1]_{\rho^*}}$. Thus $\rho_{[1]_{\rho^*}} = \rho^*$.
Let $C \in \mathcal{B}$. Then $\varphi \circ \Psi(C) = \varphi(\Psi(C)) = \varphi(\rho_C) = [1]_{\rho_C}$. Let $x \in [1]_{\rho_C}$. Then $x \, \rho_C \, 1$, so $x \in C$. Hence $[1]_{\rho_C} \subseteq C$. If $x \in C$ then $x \, \rho_C \, 1$. So $x \in [1]_{\rho_C}$. Thus $C \subseteq [1]_{\rho_C}$, so $C = [1]_{\rho_C}$. Therefore φ is a bijection. Clearly φ and Ψ are isotone, hence φ is an order isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. #

<u>Corollary 1.42.</u> Let K be a semifield and C an a-convex subgroup of K. Let $\mathcal{A}$ be the set of all a-convex subgroups of $K/_C$ except $\{C\}$ and $\mathcal{B}$ the set of all a-convex subgroups of K that strictly contain C. Then there exists an order isomorphism form $\mathcal{A}$ onto $\mathcal{B}$.

<u>Proof</u> Note that for every $\mathcal{D} \in \mathcal{A}$, $\bigcup_{\alpha \in \mathcal{D}} \alpha$ is an a-convex subgroup of K which is strictly containing C. To prove this, let $x, y \in \bigcup_{\alpha \in \mathcal{D}} \alpha$. Then there

exist $\alpha, \beta \in \mathscr{D}$ such that $x \in \alpha$ and $y \in \beta$. Then $xy \in \alpha\beta \in \mathscr{D}$.

Since $\alpha \in \mathscr{D}$, there is an $\alpha^{-1} \in \mathscr{D}$ such that $\alpha\alpha^{-1} = C$. Since $1 \in C$, $1 = uv$ for some $u \in \alpha$ and $v \in \alpha^{-1}$. By the definition of α, $ux^{-1} \in C$. Therefore $1 = uv = (uvx^{-1})x$, and $(ux^{-1})v \in vC = \alpha^{-1} \subseteq \bigcup_{\alpha\in\mathscr{D}}\alpha$. Thus $x^{-1} \in \bigcup_{\alpha\in\mathscr{D}}\alpha$. Let $a, b \in K$ be such that $a + b = 1$. Then $aC + bC = (a + b)C = C$. Hence $(ax + by)C = (aC)(xC) + (bC)(yC) = (aC)\alpha + (bC)\beta \in \mathscr{D}$, so $ax + by \in \bigcup_{\alpha\in\mathscr{D}}\alpha$. Since $\mathscr{D} \neq \{C\}$, there exists an $\alpha \in \mathscr{D} - \{C\}$. Choose $x \in \alpha$. Hence $x \notin C$. Thus $\bigcup_{\alpha\in\mathscr{D}}\alpha$ is an a-convex subgroup of K which is a strictly containing C.

Define $\varphi : \mathscr{A} \to \mathscr{B}$ by $\varphi(\mathscr{D}) = \bigcup_{\alpha\in\mathscr{D}}\alpha$ for all $\mathscr{D} \in \mathscr{A}$ and define $\Psi : \mathscr{B} \to \mathscr{A}$ by $\Psi(D) = \Pi(D)$ for all $D \in \mathscr{B}$ where Π is the projection map of K onto $K/_C$. To show that $\varphi \circ \Psi = Id_{\mathscr{B}}$ and $\Psi \circ \varphi = Id_{\mathscr{A}}$, let $\mathscr{D} \in \mathscr{A}$. Then $\Psi \circ \varphi(\mathscr{D}) = \Psi(\varphi(\mathscr{D})) = \Psi(\bigcup_{\alpha\in\mathscr{D}}\alpha) = \Pi(\bigcup_{\alpha\in\mathscr{D}}\alpha)$. Let $\alpha \in \mathscr{D}$. Choose $x \in \alpha$. $\Pi(x) = xC = \alpha \in \Pi(\bigcup_{\alpha\in\mathscr{D}}\alpha)$. Let $\beta \in \Pi(\bigcup_{\alpha\in\mathscr{D}}\alpha)$. There exists $x \in \bigcup_{\alpha\in\mathscr{D}}\alpha$ such that $\Pi(x) = \beta$. Since $x \in \bigcup_{\alpha\in\mathscr{D}}\alpha$, $x \in \alpha_0$ for some $\alpha_0 \in \mathscr{D}$. Then $\alpha_0 = xC = \Pi(x) = \beta \in \mathscr{D}$.

Next, let $D \in \mathscr{B}$. Then $\varphi \circ \Psi(D) = \varphi(\Psi(D)) = \varphi(\Pi(D)) = \bigcup_{x\in D}\Pi(x)$. Let $d \in D$. Since $d \in dC = \Pi(d)$, $d \in \bigcup_{x\in D}\Pi(x)$. Let $x \in \bigcup_{x\in D}\Pi(x)$. There exists $d_0 \in D$ such that $x \in \Pi(d_0) = d_0C$. So $x \in d_0C$, $xd_0^{-1} \in C \subseteq D$. Since $d_0 \in D$, $x \in D$. Therefore φ is bijection. Clearly φ and Ψ are isotone, hence φ is an order isomorphism from $\mathscr{A}$ onto $\mathscr{B}$. #

Proposition 1.43. Let $f : K \to M$ be a homomorphism of semifields. Define ρ_f on K by $x \,\rho_f\, y$ if and only if $f(x) = f(y)$ for all $x, y \in K$. Then there exists a unique monomorphism $f^* : K/_{\rho_f} \to M$ such that $f^* \circ \Pi = f$ and moreover if f is onto then f^* is an isomorphism, that is $K/_{\rho_f} \cong M$ where Π is the projection map of K onto $K/_{\rho_f}$.

Proof Clearly, ρ_f is a congruence on K. Define $f^* : K/_{\rho_f} \to M$ by $f^*([x]_{\rho_f}) = f(x)$ for all $x \in K$. Then we have that f^* is a monomorphism.

Let $x \in K$. Then $f^* \circ \Pi(x) = f^*(\Pi(x)) = f^*([x]_{\rho_f}) = f(x)$. Hence $f^* \circ \Pi = f$. Suppose that $g : K/_{\rho_f} \to M$ is a monomorphism such that $g \circ \Pi = f$. Let $[x]_{\rho_f} \in K/_{\rho_f}$. Then $g([x]_{\rho_f}) = g \circ \Pi(x) = f(x) = f^* \circ \Pi(x) = f^*([x]_{\rho_f})$. Therefore $g = f^*$. #

Propositonn 1.44. Let K be a semifield and $C \subseteq K^*$. Then C is an a-convex subgroup of K iff C is a kernel of some epimorphism.

Proof Assume that C is an a-convex subgroup of K. Define $\Pi : K \to K/_C$ by $\Pi(x) = xC$ for all $x \in K$. Then Π is an epimorphism and $\ker \Pi = C$.

The converse follows from Remark 1.36. (2). #

From Proposition 1.44, a map $\Pi : K \to K/_C$ is called the canonical projection of K onto $K/_C$.

Theorem 1.45. (First Isomorphism Theorem)

Let $f : K \to M$ be a homomorphism of semifields. Then $K/_{\ker f} \cong \operatorname{Im} f$. Hence if f is onto then $K/_{\ker f} \cong M$.

Proof Clearly, by Remark 1.36. (2) $\ker f$ is an a-convex subgroup of K. Define $\varphi : K/_{\ker f} \to \operatorname{Im} f$ as follows : let $\alpha \in K/_{\ker f}$ choose $x \in \alpha$ define $\varphi(\alpha) = f(x)$. To show that φ is well-defined, let $x, y \in K$ be such that xker f = ykerf. Then $xy^{-1} \in \ker f$, so $f(xy^{-1}) = 1$. Thus $\varphi(x\ker f) = f(x) = f(y) = \varphi(y\ker f)$. Hence φ is well-defined. And $\varphi(0) = f(0) = 0$.

Clearly, φ is a homomorphism and bijection.

Hence $K/_{\ker f} \cong \operatorname{Im} f$. #

Lemma 1.46. Let H be a subsemifield of a semifield K and C an a-convex subgroup of K. Then $H \cap C$ is an a-convex subgroup of H. And HC is a subsemifield of K.

Proof Clearly, $H \cap C$ is a multiplicative subgroup of H. Let $x \in H \cap C$ and $a, b \in H$ be such that $a + b \in H$. Since H is of K subsemifield and $x, a, b \in H$, $ax + b \in H$. Since C is a-convex subgroup, $ax + b \in C$. Hence $ax + b \in H \cap C$. Therefore $H \cap C$ is an a-convex subgroup of H.

To show that HC is a subsemifield of K. $HC \neq \varnothing$ since $0 = 01$ for $0 \in H$ and $1 \in C$. Let $x, y \in (HC)^*$. Then $x = h_1c_1$ and $y = h_2c_2$ for some $h_1, h_2 \in H^*$ and $c_1, c_2 \in C$. Thus $xy^{-1} = (h_1c_1)(h_2c_2)^{-1} = (h_1h_2^{-1})(c_1c_2^{-1}) \in HC^*$. Since C is a-convex, $[h_1(h_1 + h_2)^{-1}]c_1 + [h_2(h_1 + h_2)^{-1}]c_2 \in C$. Thus $x + y = h_1c_1 + h_2c_2 = (h_1 + h_2)[(h_1c_1 + h_2c_2)(h_1 + h_2)^{-1}] \in HC$.

Therefore HC is a subsemifield of K. #

Theorem 1.47. (Second Isomorphism Theorem)

Let H be a subsemifield of a semifield K and C an a-convex subgroup of K. Then $H/_{H \cap C} \cong HC/_C$.

Proof Define $\varphi : H \to HC/_C$ by $\varphi(x) = xC$ for all $x \in H$. Then φ is an epimorphism. For each $x \in \ker \varphi$, $xC = \varphi(x) = C$, so $x \in C$.

Then $x \in H \cap C$, so $\ker \varphi \subseteq H \cap C$. Clearly, $H \cap C \subseteq \ker \varphi$, hence $\ker \varphi = H \cap C$. By Theorem 1.45., $H/_{H \cap C} \cong HC/_{C}$. #

Lemma 1.48. Let D and H be a-convex subgroups of a semifield K such that $H \subseteq D$. Then $D/_{H}$ is an a-convex subgroup of $K/_{H}$.

Proof Clearly, $D/_{H}$ is a multiplicative subgroup of $K/_{H}$. To show that $D/_{H}$ is a-convex, let $xH \in D/_{H}$ and $aH, bH \in K/_{H}$ be such that $aH + bH = H$. Choose $h_1, h_2 \in H$ such that $ah_1 + bh_2 = 1$.
Since D is an a-convex subgroup of K, $(ah_1)x + bh_2 \in D$. So $(aH)(xH) + (bH) = (ah_1)H(xH) + (bh_2)H = (ah_1x + bh_2)H \in D/_{H}$. Therefore $D/_{H}$ is an a-convex subgroup of $K/_{H}$. #

Theorem 1.49. (Third Isomorphism Theorem)

Let K be a semifield, D and H a-convex subgroups of K such that $H \subseteq D$. Then $(K/_{H})/_{(D/_{H})} \cong K/_{D}$.

Proof Define $\varphi : K/_{H} \to K/_{D}$ by $\varphi(xH) = xD$ for all $x \in K$. We have that φ is well-defined and epimorphism. Let $xH \in \ker \varphi$. Then $xD = \varphi(xH) = D$, so $x \in D$. Hence $xH \in D/_{H}$. Next, let $xH \in D/_{H}$. Thus $x \in D$ and $\varphi(xH) = xD = D$, so $xH \in \ker \varphi$. Therefore $\ker \varphi = D/_{H}$.
By Theorem 1.45., $(K/_{H})/_{(D/_{H})} \cong K/_{D}$. #

Proposition 1.50. Let $f : K \to M$ be an epimorphism of semifields. If C' is an a-convex subgroup of M then $K/_{f^{-1}(C')} \cong M/_{C'}$.

Proof By Remark 1.36. (3), $f^{-1}(C')$ is an a-convex subgroup of K. Define $\varphi : K \to M/C'$ by $\varphi(x) = f(x)C'$ for all $x \in K$. Then φ is an epimorphism. Let $x \in \operatorname{Ker} \varphi$. Then $f(x)C' = \varphi(x) = C'$, so $f(x) \in C'$. Hence $x \in f^{-1}(C')$, so $\ker \varphi \subseteq f^{-1}(C')$. Similarly, $f^{-1}(C') \subseteq \ker \varphi$. Therefore $f^{-1}(C') = \ker \varphi$. By Theorem 1.45., $K/f^{-1}(C') \cong M/C'$. #

Definition 1.51. Let $\{K_i \mid i \in I\}$ be a family of semifields. The direct product of the family $\{K_i \mid i \in I\}$, denoted by $\prod_{i\in I} K_i$, is the set of all elements $(x_i)_{i \in I}$ in the cartesian product of the family $\{K_i^* \mid i \in I\}$ and 0 where $0 = (0_i)_{i \in I}$ together with operations + and · defined as usual, that is for any $(x_i)_{i \in I}, (y_i)_{i \in I} \in \prod_{i\in I} K_i$,

$$(x_i)_{i \in I} + (y_i)_{i \in I} = (x_i + y_i)_{i \in I} \text{ and}$$

$$(x_i)_{i \in I} \cdot (y_i)_{i \in I} = (x_i y_i)_{i \in I}.$$

Then we have that $\prod_{i\in I} K_i$ is a semifield.

Proposition 1.52. Let $\{K_i \mid i \in I\}$ be a family of semifields. Then the following statements hold :

(1) for each $k \in I$, the canonical projection $\Pi_k : \prod_{i\in I} K_i \to K_k$ given by $\Pi_k((x_i)_{i \in I}) = x_k$ is an epimorphism of semifields,

(2) if $1_i + 1_i = 1_i$ for all $i \in I$ then for each $k \in I$ the canonical injection $\iota_k : K_k \to \prod_{i\in I} K_i$ given by $\iota_k(x_k) = (x_i)_{i \in I}$ where $x_i = 1_i$ for $i \neq k$ for all $x_k \in K_k^*$ and $\iota_k(0) = 0$, is a monomorphism of semifields.

Proof Obvious. #

Note that for every $(x_i)_{i\in I}$, $(y_i)_{i\in I} \in \prod_{i\in I} K_i$, suppose that $\Pi_k((x_i)_{i\in I}) = \Pi_k((y_i)_{i\in I})$ for all $k \in I$. Then $x_i = y_i$ for all $i \in I$, hence $(x_i)_{i\in I} = (y_i)_{i\in I}$.

Proposition 1.53. Let $\{K_i \mid i \in I\}$ be a family of semifields and L a semifield. For each $k \in I$, let $\varphi_k: L \to K_k$ be a homomorphism of semifields. Then there is a unique homomorphism $\varphi : L \to \prod_{i\in I} K_i$ such that $\Pi_k \circ \varphi = \varphi_k$ for all $k \in I$.

Proof Define $\varphi : L \to \prod_{i\in I} K_i$ by $\varphi(x) = ((\varphi_i(x))_{i\in I})$ for all $x \in L$. We shall show that φ is a homomorphism, $\varphi(0) = ((\varphi_i(0))_{i\in I}) = (0_i)_{i\in I} = 0$. Let $x, y \in L$. Then $\varphi(x + y) = ((\varphi_i(x + y))_{i\in I}) = ((\varphi_i(x) + \varphi_i(y))_{i\in I}) = (\varphi_i(x)_{i\in I}) + (\varphi_i(y)_{i\in I}) = \varphi(x) + \varphi(y)$, $\varphi(xy) = ((\varphi_i(xy))_{i\in I}) = (\varphi_i(x)\varphi_i(y)_{i\in I}) = ((\varphi_i(x))_{i\in I})((\varphi_i(y)_{i\in I}) = \varphi(x)\varphi(y)$. Therefore φ is a homomorphism.

For each $k \in I$, $\Pi_k \circ \varphi(x) = \Pi_k(\varphi(x)) = \Pi_k((\varphi_i(x))_{i\in I}) = \varphi_k(x)$ for all $x \in L$. Hence $\Pi_k \circ \varphi = \varphi_k$ for all $k \in I$. To prove uniqueness, suppose that there is a homomorphism $\Psi : L \to \prod_{i\in I} K_i$ such that $\Pi_k \circ \Psi = \varphi_k$ for all $k \in I$. Let $x \in L$ and $k \in I$. So $\Pi_k(\Psi(x)) = \Pi_k \circ \Psi(x) = \varphi_k(x) = \Pi_k \circ \varphi(x) = \Pi_k(\varphi(x))$ which implies that $\Psi(x) = \varphi(x)$. Therefore $\Psi = \varphi$. #

Proposition 1.54. Let $\{K_i \mid i \in I\}$ be a family of semifields and let C_i be an a-convex subgroup of K_i for all $i \in I$. Then $\prod_{i\in I} C_i$ is an a-convex subgroup of $\prod_{i\in I} K_i$ and $\prod_{i\in I} K_i / \prod_{i\in I} C_i \cong \prod_{i\in I} (K_i/C_i)$.

Proof Define $\varphi : \prod_{i\in I} K_i \to \prod_{i\in I} (K_i/C_i)$ by $\varphi((x_i)_{i\in I}) = ((x_iC_i)_{i\in I})$ for all $(x_i)_{i\in I} \in \prod_{i\in I} K_i$. Then φ is an epimorphism.

To prove that $\ker \varphi = \prod_{i\in I} C_i$, let $(x_i)_{i \in I} \in \ker \varphi$.

Then $(x_iC_i)_{i \in I} = \varphi((x_i)_{i \in I}) = (C_i)_{i \in I}$, so $x_iC_i = C_i$ for all $i \in I$. Thus $x_i \in C_i$ for all $i \in I$, so $(x_i)_{i \in I} \in \prod_{i\in I} C_i$. Therefore $\ker \varphi \subseteq \prod_{i\in I} C_i$. It is clear that $\prod_{i\in I} C_i \subseteq \ker \varphi$. Hence $\prod_{i\in I} C_i = \ker \varphi$. By Remark 1.36 (2), we get that $\prod_{i\in I} C_i$ is an a-convex subgroup of $\prod_{i\in I} K_i$. By Theorem 1.45., $\prod_{i\in I} K_i / \prod_{i\in I} C_i \cong \prod_{i\in I} (K_i/C_i)$. #

<u>Definition 1.55.</u> Let L be a subsemifield of a direct product of family of semifields $\{K_i \mid i \in I\}$. L is said to be a <u>subdirect product</u> of $\{K_i \mid i \in I\}$ iff for every $k \in I$, $\Pi_k(L) = K_k$ where Π_k is the projection map.

<u>Example 1.56.</u> Let K be a semifield. Let $L = \{(x , x) \mid x \in K\}$. Then L is a subdirect product of $K^* \times K^* \cup \{(0 , 0)\}$

<u>Definition 1.57.</u> Let $\{K_i \mid i \in I\}$ be a family of semifields and L a semifield. Let $g : L \to \prod_{i\in I} K_i$ be a homomorphism g is said to be a <u>representation</u> of L as a subdirect product of $\{K_i \mid i \in I\}$ iff Im g is a subdirect product of $\{K_i \mid i \in I\}$.

<u>Definition 1.58.</u> Let K be a semifield. K is said to be a <u>subdirectly irreducible</u> iff for every family $\{K_i \mid i \in I\}$ of semifields and for every monomorphic representation $f : K \to \prod_{i\in I} K_i$ there exists $k \in I$ such that $\Pi_k \circ f$ is an isomorphism.

A semifield K is not subdirectly irreducible, we shall call K that a <u>subdirectly reducible</u>.

<u>Theorem 1.59.</u> Let $g : L \to \prod_{i\in I} K_i$ be a representation of L as a subdirect product of $\{K_i \mid i \in I\}$. Then $\operatorname{Im} g \cong L/_{\cap \ker \Pi_k \circ g}$.

Proof Define $\varphi : L \to \text{Im } g$ by $\varphi(x) = g(x)$ for all $x \in L$. Then φ is an epimorphism. To show that $\ker \varphi = \cap \ker \Pi_k \circ g$, let $x \in L$ be such that $\varphi(x) = (1_i)_{i \in I}$. So $g(x) = (1_i)_{i \in I}$. For each $k \in I$, $\Pi_k \circ g(x) = 1_k$, then $x \in \ker \Pi_k \circ g$ Hence $x \in \cap \ker \Pi_k \circ g$. Thus $\ker \varphi \subseteq \cap \ker \Pi_k \circ g$.

Next, let $x \in \cap \ker \Pi_k \circ g$. Then $\Pi_k \circ g(x) = 1_k$ for all $k \in I$ which implies that $g(x) = (1_i)_{i \in I}$. Since $\varphi(x) = g(x) = 1$, $x \in \ker \varphi$.
Hence $\cap \ker \Pi_k \circ g \subseteq \ker \varphi$. By Theorem 1.45, $\text{Im } g \cong L/_{\cap \ker \Pi_k \circ g}$. #

Corollary 1.60. Let $g : L \to \prod_{i \in I} K_i$ be a monomorphic representation of L as a subdirect product of $\{K_i \mid i \in I\}$. Then $\cap \ker \Pi_k \circ g = \{1\}$, hence $\text{Im } g \cong L$.

Proof To show that $\cap \ker \Pi_k \circ g = \{1\}$, let $x \in \cap \ker \Pi_k \circ g$. Then $\Pi_k \circ g(x) = 1_k$ for all $k \in I$. This implies that $g(x) = (1_i)_{i \in I}$. Since g is monomorphism, $x = 1$. Therefore $\cap \ker \Pi_k \circ g = \{1\}$. #

Proposition 1.61. Let L be a semifield and $\mathscr{C} = \{C_i \mid C_i$ is an a-convex subgroup of L, for all $i \in I\}$. Define $f_{\mathscr{C}} : L \to \prod_{i \in I}(L/_{C_i})$ by $f_{\mathscr{C}}(x) = (xC_i)_{i \in I}$ for all $x \in L$. Then $f_{\mathscr{C}}$ is a representation of L as a subdirect product of $\{L/_{C_i} \mid i \in I\}$. Furthermore, if $\bigcap_{i \in I} C_i = \{1\}$ then $f_{\mathscr{C}}$ is a monomorphic representation of L.

Proof Clearly, $f_{\mathscr{C}}$ is a homomorphism of L. To show that $\text{Im } f_{\mathscr{C}}$ is subdirect product, let $k \in I$ and $x \in L$. $\Pi_k \circ f_{\mathscr{C}}(x) = \Pi_k(f_{\mathscr{C}}(x)) = \Pi_k((xC_i)_{i \in I})$ $= xC_k \in L/_{C_k}$. Thus $\Pi_k(\text{Im } f_{\mathscr{C}}) \subseteq L/_{C_k}$. Let $x \in L$. Then $xC_k \in L/_{C_k}$. Then $f_{\mathscr{C}}(x) \in \prod_{i \in I}(L/_{C_i})$ and $\Pi_k(f_{\mathscr{C}}(x)) = xC_k \in \Pi_k(\text{Im } f_{\mathscr{C}})$. Hence $L/_{C_k} \subseteq \Pi_k(\text{Im } f_{\mathscr{C}})$.

Therefore $\Pi_k \circ f_{\mathscr{C}}(L) = \Pi_k(\operatorname{Im} f_{\mathscr{C}}) = L/C_k$. Hence $f_{\mathscr{C}}$ is a representation of L as a subdirect product of $\{L/C_i \mid i \in I\}$.

Next, assume that $\bigcap_{i\in I} C_i = \{1\}$. To prove that $f_{\mathscr{C}}$ is 1-1, let $x \in L$ be such that $f_{\mathscr{C}}(x) = (C_i)_{i\in I}$. Then $(xC_i)_{i\in I} = (C_i)_{i\in I}$, so $x \in C_i$ for all $i \in I$. By assumption, $x = 1$. Hence $f_{\mathscr{C}}$ is 1-1. #

<u>Proposition 1.62.</u> Let K be a semifield and $\mathscr{C}$ the set of all a-convex subgroups of K except $\{1\}$. Then K is subdirectly irreducible iff $\mathscr{C}$ has a minimum element.

<u>Proof</u> Assume that K is a subdirectly irreducible. Suppose that $\mathscr{C}$ has no minimun element. Then $\cap \mathscr{C} = \{1\}$. By Proposition 1.61., we have that $f_{\mathscr{C}}: K \to \prod_{C\in\mathscr{C}}(K/C)$ defined by $f_{\mathscr{C}}(x) = (xC)_{C\in\mathscr{C}}$ which is a monomorphic representation of L as a subdirect product of $\{K/C \mid C \in \mathscr{C}\}$. By assumption, there exists a $C_0 \in \mathscr{C}$ such that $\Pi_{C_0} \circ f_{\mathscr{C}}$ is an isomorphism of L. Claim that $C_0 = \{1\}$. Let $x \in C_0$. Then $\Pi_{C_0} \circ f_{\mathscr{C}}(x) = \Pi_{C_0}(f_{\mathscr{C}}(x)) = \Pi_{C_0}((xC)_{C\in\mathscr{C}}) = xC_0$. Since $x \in C_0$, $x \in \ker \Pi_{C_0} \circ f_{\mathscr{C}}$. Since $\Pi_{C_0} \circ f_{\mathscr{C}}$ is an isomorphism, $x = 1$. So we have the claim. It is a contradiction, since $\{1\} = C_0 \in \mathscr{C}$. Therefore $\mathscr{C}$ has a minimum element.

Conversely, assume that $\mathscr{C}$ has a minimum element say C_m. Let $\{K_i \mid i \in I\}$ be a family of semifields and $f : K \to \prod_{i\in I} K_i$ a monomorphic representation of K as a subdirect product of $\{K_i \mid i \in I\}$. Then by Remark 1.36 (2), we have that $\{\ker \Pi_i \circ f \mid i \in I\}$ is a set of a-convex subgroups of K. Since f is a monomorphism, $\bigcap_{i\in I} \ker \Pi_i \circ f = \{1\}$. Suppose that for $i \in I$, $\ker \Pi_i \circ f \neq \{1\}$. Then $\{\ker \Pi_i \circ f \mid i \in I\} \subseteq \mathscr{C}$. Therefore $C_m \subseteq \bigcap_{i\in I} \ker \Pi_i \circ f =$

$\{1\}$. So $C_m = \{1\}$, a contradiction. Therefore there exists $k \in I$ such that $\ker \Pi_k o f = \{1\}$. Claim that $\Pi_k o f$ is an isomorphism. Let $x, y \in K^*$ be such that $\Pi_k o f(x) = \Pi_k o f(y)$. Hence $xy^{-1} \in \ker \Pi_k o f = \{1\}$, so $x = y$. So we have the claim. Therefore K is a subdirectly irreducible. #

Next, we want to show that every semifield is a subdirect product of subdirectly irreducible semifields. First we need the lemmas.

Lemma 1.63. Let K be a semifield and $x, y \in K^*$ such that $x \neq y$. Then there is a maximal a-convex subgroup M of K such that $xy^{-1} \notin M$.

Proof Let $x, y \in K^*$ be such that $x \neq y$. Let $\mathscr{C} = \{C \mid C$ is an a-convex subgroup of K and $xy^{-1} \notin C\}$. $\mathscr{C} \neq \varnothing$ since $\{1\} \in \mathscr{C}$.
By Zorn's Lemma, $\mathscr{C}$ has a maximal element. #

Lemma 1.64. By assumption of Lemma 1.63., let $\mathscr{A} = \{C \mid C$ is an a-convex subgroup of K such that $M \subset C\}$. Then $\mathscr{A}$ has a minimum element.

Proof $\mathscr{A} \neq \varnothing$ since $K^* \in \mathscr{A}$. If there is $C \in \mathscr{A}$ and $xy^{-1} \notin C$ then this contradicts the minimality of M. Therefore for every $C \in \mathscr{A}$, $xy^{-1} \in C$. Then we have that $\cap \mathscr{A}$ is an a-convex subgroup of K which is the minimum element and $xy^{-1} \in \cap \mathscr{A}$. Hence $\cap \mathscr{A} \neq M$. #

Lemma 1.65. By assumption of Lemma 1.63., $K/_M$ is a subdirectly irreducible semifields.

Proof Let $\mathscr{C}$ be the set of all a-convex subgroups of $K/_M$ except $\{M\}$. By Corollary 1.42., we have that $\mathscr{C}$ is isomorphic to the set of all a-convex subgroups of K strictly containing M. By Lemma 1.64., $\mathscr{C}$ has a minimum element. By Proposition 1.62., $K_{/M}$ is a subdirectly irreducible semifields. #

Theorem 1.66. Let K be a semifield. Then K is a subdirect product of subdirectly irreducible semifields.

Proof By Lemma 1.63., for $x, y \in K^*$ and $x \neq y$, we have that C_{xy} is an a-convex subgroup of K such that $xy^{-1} \notin C_{xy}$. By Lemma 1.65., $K_{/C_{xy}}$ is a subdirectly irredudible for all $x, y \in K^*$ and $x \neq y$.

Let $\mathscr{C} = \{ C_{xy} \mid x, y \in K^* \text{ and } x \neq y\}$. Let $x \in \cap \mathscr{C}$. Suppose that $x \neq 1$. Then $x \notin C_{x1}$, a contradiction since $x \in \cap \mathscr{C}$. So $\cap \mathscr{C} = \{1\}$. Thus by Proposition 1.61., we have $f_{\mathscr{C}}: K \rightarrow \prod_{c \in \mathscr{C}} K_{/C}$ is a monomorphic representation of K as a subdirect product of $\{K_{/C} \mid C \in \mathscr{C}\}$. Therefore $f_{\mathscr{C}}(K)$ is a subdirect product of $\{K_{/C} \mid C \in \mathscr{C}\}$. Since $K \cong f_{\mathscr{C}}(K)$, K is a subdirect product of a subdirectly irreducible semifields. #