วิธีเชิงตัวเลขสำหรับกระบวนการกลับสู่ค่าเฉลี่ยรากที่สองแบบกระโดด

นายเรวัต ถนัดกิจหิรัญ

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2552

NUMERICAL METHODS FOR MEAN-REVERTING SQUARE ROOT PROCESSES WITH JUMPS

Mr. Raywat Tanadkithirun

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Mathematics
Department of Mathematics
Faculty of Science
Chulalongkorn University
Academic Year 2009

Thesis Title	NUMERICAL METHODS FOR MEAN-REVERTING SQUARE ROOT PROCESSES WITH JUMPS
By	Mr. Raywat Tanadkithirun
Field of Study	Mathematics
Thesis Advisor	Kittipat Wong, Ph.D.
Thesis Co-Advisor	Sirod Sirisup, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

S. Hannongbua ..... Dean of the Faculty of Science

(Professor Supot Hannongbua, Dr.rer.nat.)

THESIS COMMITTEE

Kritsana Neammanee ..... Chairman

(Professor Kritsana Neammanee, Ph.D.)

K. W ..... Thesis Advisor

(Kittipat Wong, Ph.D.)

Sirod Sirisup ..... Thesis Co-Advisor

(Sirod Sirisup, Ph.D.)

..... Examiner

(Khamron Mekchay, Ph.D.)

Sanae Rujivan ..... External Examiner

(Sanae Rujivan, Dr.rer.nat.)

เรวัต ถนัดกิจหิรัญ : วิธีเชิงตัวเลขสำหรับกระบวนการกลับสู่ค่าเฉลี่ยรากที่สองแบบกระโดด. (NUMERICAL METHODS FOR MEAN-REVERTING SQUARE ROOT PROCESSES WITH JUMPS) อ. ที่ปรึกษาวิทยานิพนธ์หลัก : อ.ดร. กิตติพัฒน์ วอง, อ. ที่ปรึกษาวิทยานิพนธ์ร่วม : ดร. ศิโรจน์ ศิริทรัพย์, 38 หน้า.

ในวิทยานิพนธ์นี้ เราศึกษาวิธีเชิงตัวเลข 3 วิธี ได้แก่ วิธี Euler-Maruyama, วิธี compensated split-step backward Euler และวิธี jump-adapted Euler โดยการตรวจสอบพฤติกรรมของแต่ละวิธีเชิงตัวเลขรวมถึงความถูกต้องแบบอ่อนในการแก้กระบวนการกลับสู่ค่าเฉลี่ยรากที่สองแบบกระโดด นอกจากนี้เรายังให้สูตรของขอบเขตความคลาดเคลื่อนแบบอ่อนสำหรับวิธี Euler-Maruyama และวิธี compensated split-step backward Euler

ภาควิชา.....คณิตศาสตร์..... ลายมือชื่อนิสิต.....เรวัต ถนัดกิจหิรัญ.....

สาขาวิชา.....คณิตศาสตร์..... ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก.....

ปีการศึกษา.....2552..... ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์ร่วม.....

##5172422523 : MAJOR MATHEMATICS
KEYWORDS : STOCHASTIC DIFFERENTIAL EQUATIONS WITH JUMPS / MEAN-REVERTING SQUARE ROOT PROCESSES WITH JUMPS / NUMERICAL METHODS FOR STOCHASTIC DIFFERENTIAL EQUATIONS

RAYWAT TANADKITHIRUN : NUMERICAL METHODS FOR MEAN-REVERTING SQUARE ROOT PROCESSES WITH JUMPS. THESIS ADVISOR : KITTIPAT WONG, Ph.D., THESIS CO-ADVISOR : SIROD SIRISUP, Ph.D., 38 pp.

We study three numerical methods: Euler-Maruyama method, compensated split-step backward Euler method, and jump-adapted Euler method by numerically investigating on their performance as well as accuracy in solving the mean-reverting square root process with jumps in weak sense. Rigorous error bounds in weak sense for Euler-Maruyama and compensated split-step backward Euler methods will also be provided.

Department	: ....Mathematics....	Student's Signature	: Raywat Tanadkithirun
Field of Study	: ....Mathematics....	Advisor's Signature	: K. W.
Academic Year	: .....2009.....	Co-Advisor's Signature	: Sirod Sirisup

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to Dr. Kittipat Wong, my thesis advisor, and Dr. Sirod Sirisup, my thesis co-advisor, for their kind supervision and consistent encouragement while doing this thesis. Without their constructive suggestions and knowledgeable guidance in this study, this research would never have successfully been completed.

Sincere thanks and deep appreciation are also extended to Professor Dr. Kritsana Neammanee, Dr. Khamron Mekchay and Dr. Sanae Rujivan, my thesis committee, for their comments and suggestions. Besides, I would like to thank all teachers who have taught me all along.

I am also grateful to the Chulalongkorn University Graduate Scholarship to Commemorate the 72^{nd} Anniversary of His Majesty King Bhumibol Adulyadej for granting me financial support throughout my study in Master of Science in Mathematics at Chulalongkorn University.

Last but not least, I would like to thank my family and dear friends for their cheer, love and encouragement during my study.

CONTENTS

CHAPTER I

INTRODUCTION

Stochastic differential equation (SDE) models find applications in various areas such as in economics and finance, civil and mechanical engineering, environmental science, signal processing and filtering, chemistry and physics, population dynamics and psychology, pharmacology and medicine. The mean-reverting square root process [3] is an SDE which has found considerable use in mathematical finance as an alternative to geometric Brownian motion. It is used as a model for volatility, interest rate, and other financial quantities, and forms the stochastic volatility component of Heston's asset price model [4]. Moreover, it can be used for pricing bonds and barrier options [6].

However, introducing a jump process into such process makes the model become more realistic. The mean-reverting square root process with jumps on which we focus in this work has the form

$$dS(t) = \alpha(\mu - S(t^-))dt + \sigma\sqrt{S(t^-)}dW(t) + \delta S(t^-)d\widetilde{N}(t) \tag{1.1}$$

where $t \in [0, T]$, $S(t^-)$ denotes $\lim_{r \to t^-} S(r)$, W is a Wiener process and $\widetilde{N}$ is a compensated Poisson process. $S(t)$ represents the spot price at time t. The parameter μ is the long run equilibrium price or mean reversion level, α is the mean reversion rate, σ is the degree of volatility around it caused by noise from the Wiener process, and δ is the degree of jumps.

If we have the strong solution in explicit form of an SDE with jumps, we can determine its expectation, variance and covariance functions or even higher-order moments. Unfortunately, this SDE with jumps has no strong solution in explicit form. Thus, we would like to find its numerical approximation. Note

that even though all coefficient functions satisfy the linear growth condition, we cannot directly apply the standard convergence theory for numerical simulations to this model due to the non-Lipschitz diffusion coefficient which is the square root function.

In this work, we study three different numerical methods: Euler-Maruyama (EM) method, compensated split-step backward Euler (CSSBE) method, and jump-adapted Euler (JAE) method by numerically investigating on their performance as well as accuracy in solving this particular model in weak sense. Computable error bounds in weak sense for EM and CSSBE methods will also be provided.

CHAPTER II

PRELIMINARIES

In this chapter, we give some basic concepts in probability and summarize important definitions and theorems relating to SDEs with jumps. The proof will be omitted but can be found in [2, 7, 8, 9]. We assume that readers have knowledge in measure theory at graduate level.

2.1 Basic Probability Theory

Let $(\Omega, \mathscr{F}, P)$ be a probability space and (Y, Σ) a measurable space. A **random variable** is a measurable function from Ω to Y. Typically, the measurable space (Y, Σ) is the measurable space over the real numbers $(\mathbb{R}, \mathscr{B})$, where $\mathscr{B}$ is the Borel σ-algebra. Let X be a random variable. It can be shown that a function $\mu : \mathscr{B} \to [0, 1]$ defined by $\mu(B) := P(X \in B)$ is a measure on $(\mathbb{R}, \mathscr{B})$ and is called a **distribution** of X. A function $F : \mathbb{R} \to [0, 1]$ defined by $F(x) := P(X \leq x) = \mu((-\infty, x])$ is called a **distribution function** of X. A **discrete** random variable is a random variable whose distribution function is discrete. Similarly, a **continuous** random variable is a random variable whose distribution function is continuous. If the distribution μ of a discrete random variable X is absolutely continuous with respect to the counting measure N, then a Radon-Nikodym derivative $\frac{d\mu}{dN}$ is called a **probability mass function** of X. If the distribution μ of a continuous random variable X is absolutely continuous with respect to the Lebesgue measure λ, then a Radon-Nikodym derivative $\frac{d\mu}{d\lambda}$ is called a **probability density function** of X.

The **expected value** or mean of X, denoted by $E[X]$ or just EX, is defined

as $E[X] := \int_\Omega X dP$. If X is a discrete random variable with probability mass function $p(x)$, then the expected value becomes $E[X] = \sum_{x_i} x_i p(x_i)$. If the distribution of X admits a probability density function $f(x)$, then the expected value can be computed as $E[X] = \int_{-\infty}^{\infty} x f(x) dx$. If a random variable X has mean μ, then the **variance** of X, denoted by $Var(X)$,is given by $Var(X) := E[(X-\mu)^2]$.

Two random variables X and Y are **equal in distribution**, denoted by $X \stackrel{d}{=} Y$, if they have the same distributions, i.e. $P(X \in B) = P(Y \in B)$ for all $B \in \mathscr{B}$, and are **independent** if $P(X \in B \wedge Y \in C) = P(X \in B)P(Y \in C)$ for all $B, C \in \mathscr{B}$. If X and Y are independent, then $E[XY] = E[X]E[Y]$.

2.2 Stochastic Processes

Given a probability space $(\Omega, \mathscr{F}, P)$, a **stochastic process** with state space Y is a collection of Y-valued random variables indexed by a set I, i.e. a stochastic process X is a collection $\{X_t : t \in I\}$ where each X_t is a Y-valued random variable. For a fixed $\omega \in \Omega$, a function $X(\omega) : I \to Y$, $X(\omega)(t) := X_t(\omega)$, is called a realization, a trajectory, or a **sample path** of the process X. Usually, the state space Y is $\mathbb{R}$ which comes with the Borel σ-algebra $\mathscr{B}$, and the index set I is an interval $[0, T]$ or $[0, \infty)$ on $\mathbb{R}$. Two stochastic processes $U = \{U_t : t \in I\}$ and $V = \{V_t : t \in I\}$ on the same probability space are **independent** if U_s and V_t are independent for all $s, t \in I$.

For a random variable Y, the **σ-algebra generated by** Y, denoted by $\sigma(Y)$, is the smallest σ-algebra which makes Y measurable. For a stochastic process $X = \{X_t : t \in I\}$, the **σ-algebra generated by** X, denoted by $\sigma(X)$, is the smallest σ-algebra which makes X_t measurable for all $t \in I$. A collection $\{\mathscr{F}_t : t \in I\}$ of σ-algebras on Ω is called a **filtration** if $\mathscr{F}_s \subseteq \mathscr{F}_t$ for all $s \leq t$. A stochastic process $X = \{X_t : t \in I\}$ is said to be **adapted to the filtration** $\{\mathscr{F}_t : t \in I\}$ if $\sigma(X_t) \subseteq \mathscr{F}_t$ for all $t \in I$ and we will call X an **adapted process** $\{X_t, \mathscr{F}_t : t \in I\}$. Every stochastic process $X = \{X_t : t \in I\}$ is always adapted to the **natural filtration**

generated by X: $\{\mathscr{F}_t = \sigma(\{X_s : s \leq t\}) : t \in I\}$. If a stochastic process U is adapted to the natural filtration generated by a stochastic process V, we say that U is **adapted to the stochastic process** V. A filtration$\{\mathscr{F}_t\}_{t\in I}$ is said to satisfy the **usual conditions** if it is right-continuous, i.e. $\bigcap_{\epsilon>0} \mathscr{F}_{t+\epsilon} = \mathscr{F}_t$ for all $t \in I$, and $\mathscr{F}_0$ contains all the P-null sets in $\mathscr{F}$.

A stochastic process $X = \{X_t : t \in I\}$ is said to have **stationary increments** if $X_t - X_s \overset{d}{=} X_{t+h} - X_{s+h}$ for all $t, s \in I$ and h with $t + h, s + h \in I$, and **independent increments** if for every choice of $t_i \in I$ with $t_1 < \ldots < t_n$, $X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent random variables.

A **normal distribution** $N(\mu, \sigma^2)$ with parameter $\mu \in \mathbb{R}$ and $\sigma^2 > 0$ is a continuous distribution whose probability density function is of the form $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. A normal distribution $N(\mu, \sigma^2)$ has mean μ and variance σ^2.

An **exponential distribution** $Exp(\lambda)$ with parameter $\lambda > 0$ is a continuous distribution whose probability density function is of the form $f(x) = \lambda e^{-\lambda x} 1_{x\geq 0}$. An exponential distribution $Exp(\lambda)$ has mean $\frac{1}{\lambda}$ and variance $\frac{1}{\lambda^2}$.

A **Poisson distribution** $Poi(\lambda)$ with parameter $\lambda > 0$ is a discrete distribution whose probability mass function is of the form $p(k) = e^{-\lambda}\dfrac{\lambda^k}{k!}, k \in \mathbb{N} \cup \{0\}$. A Poisson distribution $Poi(\lambda)$ has mean λ and variance λ.

Let $(\Omega, \mathscr{F}, P)$ be a probability space and $\{\mathscr{F}_t\}_{t\geq 0}$ a filtration. An adapted stochastic process $W = \{W_t, \mathscr{F}_t : t \geq 0\}$ is called Brownian motion or a **Wiener process** if the following conditions are satisfied:

- $W_0 = 0$ almost surely.
- It has stationary and independent increments.
- For every $t > 0$, W_t has a normal $N(0, t)$ distribution.
- It has almost surely continuous sample paths.

Let $\{\tau_i\}$ be a sequence of independent exponential random variables with parameter λ and $T_n = \sum_{i=1}^{n} \tau_i$. An adapted stochastic process $N = \{N_t, \mathscr{F}_t : t \geq 0\}$ defined by $N_t := \sum_{n\in\mathbb{N}} 1_{t\geq T_n} = \#\{n \in \mathbb{N} : T_n \geq t\}$ is called a **Poisson process** with

intensity λ. Moreover, a Poisson process N with intensity λ satisfies the following properties which show nicely what kind of sample path a Poisson process has:

- $N_0 = 0$ almost surely.
- It has stationary and independent increments.
- For every $t > 0$, N_t has a Poisson $Poi(\lambda t)$ distribution.
- It has right continuous and piecewise constant sample paths which increase by jumps of size 1.

An adapted stochastic process $\widetilde{N} = \{\widetilde{N}_t, \mathscr{F}_t : t \geq 0\}$ defined by $\widetilde{N}_t = N_t - \lambda t$ is called a **compensated Poisson process**.

An adapted stochastic process $X = \{X_t, \mathscr{F}_t : t \in I\}$ such that $E|X_t| < \infty$ for all $t \in I$ is said to be a **martingale** if for every $s, t \in I$ such that $s < t$, $E(X_t \,|\, \mathscr{F}_s) = X_s$ almost surely. A Wiener process W and a compensated Poisson process $\widetilde{N}$ are martingales. Therefore, $E[W_t] = 0$ and $E[\widetilde{N}_t] = 0$ for all $t \geq 0$.

2.3 Stochastic Integrals

In what follows, we will consider all processes on a fixed interval $[0, T]$. First, we introduce an appropriate class of Itô integrable processes. Let $\{\mathscr{F}_t\}_{t\in[0,T]}$ be the natural filtration of a Wiener process on $[0, T]$. Then, a stochastic process $C = \{C_t : t \in [0, T]\}$ is said to be **simple** if there exists a partition Π: $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = T$ and a sequence $\{Z_i : i = 1, \ldots, n\}$ of random variables such that the sequence $\{Z_i\}$ is adapted to $\{\mathscr{F}_{t_{i-1}} : i = 1, \ldots, n\}$, $E[Z_i^2] < \infty$ for all i, and $C_t = \sum_{i=1}^{n} Z_i 1_{[t_{i-1}, t_i)} + Z_n 1_{\{T\}}$.

The **Itô stochastic integral of a simple process** C on $[0, T]$ is given by

$$\int_0^T C_s dW_s := \sum_{i=1}^{n} C_{t_{i-1}}(W_{t_i} - W_{t_{i-1}}) = \sum_{i=1}^{n} Z_i(W_{t_i} - W_{t_{i-1}}).$$

Also, for each t such that $t_{k-1} \leq t < t_k$, we define

$$\int_0^t C_s dW_s := \int_0^T C_s 1_{[0,t]} dW_s = \sum_{i=1}^{k-1} Z_i(W_{t_i} - W_{t_{i-1}}) + Z_k(W_t - W_{t_{k-1}}),$$

where $\sum_{i=1}^{0} Z_i(W_{t_i} - W_{t_{i-1}}) := 0$.

Now, we will introduce the definition of Itô stochastic integral of any general process. Let $C = \{C_t : t \in [0,T]\}$ be a stochastic process which satisfies the following conditions:

- C is adapted to a Wiener process on $[0,T]$.
- The integral $\int_0^T E[C_s^2]ds$ is finite.

Then, there exists a sequence $\{C^{(n)}\}$ of simple processes such that

$$\int_0^T E[C_s - C_s^{(n)}]^2 ds \to 0.$$

For each $n \in \mathbb{N}$, we can consider stochastic process

$$I(C)^{(n)} = \left\{ I_t(C^{(n)}) = \int_0^t C_s^{(n)} dW_s : t \in [0,T] \right\}.$$

It can be shown that there exists a stochastic process $I(C) = \{I_t(C) : t \in [0,T]\}$ to which the sequence $\left\{I(C^{(n)})\right\}$ of stochastic processes converges in mean square:

$$E\left[\sup_{0\le t\le T} \left[I_t(C) - I_t(C^{(n)})\right]^2\right] \to 0.$$

The mean square limit $I(C)$ is called the **Itô stochastic integral of the stochastic process** C, and for each $t \in [0,T]$, $I_t(C)$ is denoted by $\int_0^t C_s dW_s$. It can be shown that this definition of Itô stochastic integrals is well-defined, see [8, 9].

Next, we will define stochastic integrals with respect to compensated Poisson random measure. The key of this definition is to construct a Poisson random measure.

Let $G \subseteq \mathbb{R}^d$ and $\mathscr{G}$ the collection of Borel sets on G. A **Radon measure** on $(G, \mathscr{G})$ is a measure μ such that $\mu(B) < \infty$ for every compact Borel set $B \in \mathscr{G}$. Let $(\Omega, \mathscr{F}, P)$ be a probability space, $G \subseteq \mathbb{R}^d$ and μ a Radon measure on $(G, \mathscr{G})$. A **Poisson random measure** on G with intensity measure μ is an integer-valued random measure $M : \Omega \times \mathscr{G} \to \mathbb{N}$ such that

- For almost all $\omega \in \Omega$, $M(\omega, \cdot)$ is an integer-valued Radon measure on G.

- For each $B \in \mathscr{G}$, $M(B) := M(\cdot, B)$ is a Poisson random variable with parameter $\mu(B)$.
- For disjoint sets $B_1, \ldots, B_n \in \mathscr{G}$, the random variables $M(B_1), \ldots, M(B_n)$ are independent.

It can be shown that for any Radon measure μ on $G \subseteq \mathbb{R}^d$, there exists a Poisson random measure M on G with intensity μ. We define the **compensated Poisson random measure** $\widetilde{M}$ by subtracting from M its intensity measure:

$$\widetilde{M}(B) = M(B) - \mu(B).$$

Now, consider a Poisson random measure M with intensity Lebesgue measure on $G = [0,T] \times \mathbb{R}$. We can define the **stochastic integral with respect to compensated Poisson random measure** $\widetilde{M}$ of a square integrable process X which is adapted to a Poisson process N, denoted by $\left\{ \int_0^t X_s d\widetilde{N}_s, t \in [0,T] \right\}$, in analogous way of defining the Itô stochastic integral, see [2].

Both Itô stochastic integral and stochastic integral with respect to compensated Poisson random measure are martingales; hence, they have expectation zero at any time $t \in [0,T]$.

2.4 SDEs with Jumps

We interpret a stochastic differential equation with jumps

$$dX_t = f(t, X_t)dt + g(t, X_t)dW_t + h(t, X_t)d\widetilde{N}_t, \quad X_0(\omega) = Y(\omega) \tag{2.1}$$

where $t \in I = [0,T]$ or $[0,\infty)$ as the stochastic integral equation

$$X_t = X_0 + \int_0^t f(s, X_s)ds + \int_0^t g(s, X_s)dW_s + \int_0^t h(s, X_s)d\widetilde{N}_s \tag{2.2}$$

where the first integral on the right-hand side is a Lebesgue integral (or a Riemann integral since the set of discontinuous points of the process has Lebesgue measure zero), the second one is an Itô stochastic integral, and the last one is a stochastic integral with respect to compensated Poisson random measure.

There are two kinds of solutions of an SDE with jumps called strong and weak solutions. A **strong solution** of the SDE with jumps (2.1), on the given probability space $(\Omega, \mathscr{F}, P)$ and with respect to the fixed Wiener process W and the compensated Poisson process $\widetilde{N}$ and initial condition Y, is a process $X = \{X_t : t \in I\}$ with the following properties:

- X is adapted to the augmented filtration generated by the Wiener process W, the compensated Poisson process $\widetilde{N}$ and the initial condition Y.
- $X_0 = Y$ almost surely.
- The integrals occurring in (2.1) are well-defined as Lebesgue or Riemann integral, Itô stochastic integral and stochastic integral with respect to compensated Poisson random measure, respectively, and (2.2) holds almost surely.
- X is a function of the underlying Wiener and compensated Poisson sample paths and of the coefficient functions $f(t,x), g(t,x)$ and $h(t,x)$.

Thus, a strong solution of (2.1) is based on the paths of the underlying Wiener and compensated Poisson processes. If we changed the Wiener and compensated Poisson processes by other Wiener and compensated Poisson processes, we would get another strong solution which would be given by the same functional relationship, but with the new Wiener and compensated Poisson processes in it. For a weak solution, the path behavior is not essential. We are only interested in the distribution of X. The initial condition Y and the coefficient functions $f(t,x), g(t,x)$ and $h(t,x)$ are given, and we have to find one Wiener process and one compensated Poisson process such that (2.2) holds almost surely.

In this thesis, we only consider strong solutions of SDEs with jumps. We now give sufficient conditions for the existence and uniqueness of such solutions. From (2.1), suppose that the coefficient functions $f(t,x), g(t,x)$ and $h(t,x)$ satisfy the Lipschitz and linear growth conditions:

$$|f(t,x) - f(t,y)| + |g(t,x) - g(t,y)| + |h(t,x) - h(t,y)| \leq K\,|x-y|,$$

$$|f(t,x)|^2 + |g(t,x)|^2 + |h(t,x)|^2 \leq K^2\left(1 + |x|^2\right),$$

for every $t \in I, x \in \mathbb{R}, y \in \mathbb{R}$ where K is a positive constant. In addition, the random variable Y is independent of the Wiener and compensated Poisson processes and has finite second moment:

$$E|Y|^2 < \infty.$$

Then, there exists a unique strong solution of the SDE with jumps (2.1), see [7].

CHAPTER III

NUMERICAL SCHEMES

In this chapter, we present the three numerical schemes investigated in this thesis and give the concept of strong and weak convergence of numerical solutions. From now on, for a stochastic process X, we write $X(t)$ in place of X_t and use the subscript when we refer to a numerical solution.

First of all, we will talk about our assumptions for the mean-reverting square root process with jumps. Throughout this thesis, let $(\Omega, \mathscr{F}, P)$ be a complete probability space with a filtration $\{\mathscr{F}_t : t \geq 0\}$ satisfying the usual conditions. Let W be a Wiener process and N a Poisson process with intensity λ such that $\widetilde{N}(t) = N(t) - \lambda t$ is the corresponding compensated process. Assume that W and N are independent, and all of these processes are defined on this probability space. This thesis considers (1.1) in which α, λ and σ are positive, μ is nonnegative, $\alpha + \lambda\delta > 0$, and $S(0) > 0$ almost surely.

For any given initial value $S(0) = S_0 > 0$ almost surely, the condition $\alpha + \lambda\delta > 0$ will force (1.1) to have a unique strong solution which will never become negative with probability one, see [13]. The following theorem yields the expectation of the solution of (1.1) for any time t.

Theorem 3.1. [13] *For the equation (1.1),*

$$ES(t) - \mu = e^{-\alpha t}(ES_0 - \mu)$$

so that $\lim_{t\to\infty} ES(t) = \mu$.

Now, we will focus on our three numerical schemes. We write s_n to denote the numerical approximation of (1.1) and simulate the model on a fixed finite interval $[0, T]$.

For the first method, we divide the interval $[0, T]$ into N parts with a fixed time step $\Delta = \frac{T}{N}$. Then, we acquire an equidistant time discretization $\{t_0, t_1, \ldots, t_N\}$ with $t_n = n\Delta$. Then, we define the EM approximation to (1.1) by setting $s_0 = ES_0$ and forming

$$\begin{aligned} s_{n+1} &= s_n + \alpha(\mu - s_n)\Delta + \sigma\sqrt{|s_n|}\Delta W_n + \delta s_n \Delta \widetilde{N}_n \\ &= (1 - \alpha\Delta)s_n + \alpha\mu\Delta + \sigma\sqrt{|s_n|}\Delta W_n + \delta s_n \Delta \widetilde{N}_n, \end{aligned} \tag{3.1}$$

where $\Delta W_n = W(t_{n+1}) - W(t_n)$, which is normally distributed with mean zero and variance Δ, is a Wiener process increment and $\Delta\widetilde{N}_n = \widetilde{N}(t_{n+1}) - \widetilde{N}(t_n)$, which has the distribution $Poi(\lambda\Delta) - \lambda\Delta$, is a compensated Poisson process increment. Note that a numerical method that is directly applied to (1.1) may break down due to negative values being supplied to the square root function. However, we have known that the solution $S(t)$ will never become negative almost surely. Thus, the SDE with jumps (1.1) is equivalent to

$$dS(t) = \alpha(\mu - S(t^-))dt + \sigma\sqrt{|S(t^-)|}dW(t) + \delta S(t^-)d\widetilde{N}(t)$$

which is a computationally safer problem. For this reason, we use $|s_n|$ instead of s_n under the square root function.

Next, with the same equidistant time discretization $\{t_0, t_1, \ldots, t_N\}$, the CSSBE scheme for (1.1) introduced in [5] is defined by letting $s_0 = ES_0$ and forming

$$\begin{aligned} s^*_{n+1} &= s_n + \alpha(\mu - s^*_{n+1})\Delta, \\ s_{n+1} &= s^*_{n+1} + \sigma\sqrt{|s^*_{n+1}|}\Delta W_n + \delta s^*_{n+1}\Delta\widetilde{N}_n \\ &= s_n + \alpha(\mu - s^*_{n+1})\Delta + \sigma\sqrt{|s^*_{n+1}|}\Delta W_n + \delta s^*_{n+1}\Delta\widetilde{N}_n. \end{aligned} \tag{3.2}$$

We note here that $s^*_{n+1} = \dfrac{s_n + \alpha\mu\Delta}{1 + \alpha\Delta}$. For each step from s_n to s_{n+1}, this method has two substeps. The first substep which is to find s^*_{n+1} concerns only with the deterministic component, and the second substep which is to substitute s^*_{n+1} from the first substep into the formula to obtain s_{n+1} deals with the random parts from the Wiener process and the compensated Poisson process.

Our last method based on time discretizations that include all jump times is originally introduced in [11]. We adapt this method from [1] where a compound Poisson process, which is a pure jump process, is used to be the jump process in the model. Note that in this thesis, our jump process in the SDE with jumps is a compensated Poisson process $\widetilde{N}$ which is not a pure jump process but can be separated into a pure jump part $N(t)$ and a nonjump part $-\lambda t$. Recall that waiting time between two consecutive jump times of a Poisson process with intensity λ is exponentially distributed with parameter λ, which has mean $\frac{1}{\lambda}$. We construct a jump-adapted time discretization by merging the old equidistant time discretization with step size Δ and the jump times $\{\tau_1, \tau_2, \ldots, \tau_M\}$ generated by the Poisson process N, and then orderly rename all points in this new jump-adapted time discretization, namely $\{t_0, t_1, \ldots, t_L\}$. Note that the jump times $\{\tau_1, \tau_2, \ldots, \tau_M\}$ and the number of jump times are random, so M and L is not a fixed number. Moreover, the jump-adapted time discretization $\{t_0, t_1, \ldots, t_L\}$ may have different step sizes, so we define $\Delta_n = t_{n+1} - t_n$ and $\Delta W_n = W(t_{n+1}) - W(t_n)$, which is normally distributed with mean zero and variance $t_{n+1} - t_n$. The JAE scheme for (1.1) is then given by setting $s_0 = ES_0$ and forming

$$\begin{aligned} s_{n+1^-} &= s_n + \alpha(\mu - s_n)\Delta_n + \sigma\sqrt{|s_n|}\Delta W_n - \lambda\delta s_n \Delta_n, \\ s_{n+1} &= \begin{cases} s_{n+1^-}, & \text{if } t_{n+1} \text{ is not a jump time;} \\ s_{n+1^-} + \delta s_{n+1^-}, & \text{if } t_{n+1} \text{ is a jump time.} \end{cases} \end{aligned} \tag{3.3}$$

For each step from s_n to s_{n+1}, this method has two substeps. The first substep which is to find s_{n+1^-} deals with nonjump component, and the second substep concerns only with pure jump part. Here, we break $\Delta\widetilde{N}_n$ into pure jump part ΔN_n and nonjump component $-\lambda\Delta_n$. The nonjump part will be assigned to the first substep, and for the second substep we add a jump of size $\delta s_{n+1^-} * 1$ if it is a jump time and do nothing if it is not a jump time because we have already known which step is a jump time of the Poisson process N whose jumps have size 1.

A numerical solution s_n with the grid size Δ is said to **converge strongly**

with order γ to S at time T if there exists a constant C, independent of Δ, such that

$$E|S(T) - s_{n,T}| \leq C\Delta^{\gamma}$$

for any Δ sufficiently small, where $s_{n,T}$ denote the numerical solution at time T. As one can notice from the definition of the strong error, strong schemes provide pathwise approximations. Therefore, these methods are suitable for problems such as filtering, scenario analysis and hedge simulation.

A numerical solution s_n with the grid size Δ is said to **converge weakly with order** γ to S at time T if for each $f \in C_P^{2(\gamma+1)}$ there exists a constant C, independent of Δ, such that

$$|Ef(S(T)) - Ef(s_{n,T})| \leq C\Delta^{\gamma}$$

for any Δ sufficiently small, where $s_{n,T}$ denote the numerical solution at time T and $C_P^{2(\gamma+1)}$ denote the space of $2(\gamma+1)$ continuously differentiable functions which have polynomial growth. It is customary to choose the function f to be the identity function when we simulate a model and measure the error in weak sense. Weak schemes provide approximations of the probability measure and are appropriate for problems such as derivative pricing and the evaluation of moments, risk measures and expected utilities.

CHAPTER IV

ERROR BOUNDS

This chapter provides rigorous error bounds in weak sense for Euler-Maruyama and compensated split-step backward Euler methods. The key ingredients of our proof is the Fubini's theorem and the Gronwall's inequality. Let's state these two theorems in the versions that we will use in our proof.

Theorem 4.1. [12] *If A and B are σ-finite measure spaces, and either*

$$\int_A \int_B |f(x,y)|\ dydx < \infty \text{ or } \int_B \int_A |f(x,y)|\ dxdy < \infty,$$

then $\int_A \int_B f(x,y)\ dydx = \int_B \int_A f(x,y)\ dxdy.$

Theorem 4.2. [10] *Let u, f, g and h be nonnegative continuous functions defined on $J = [a,b]$, and $u(t) \leq f(t) + g(t) \int_a^t h(s)u(s)ds$ for all $t \in J$. Then, for any $t \in J$,*

$$u(t) \leq f(t) + g(t) \int_a^t h(s)f(s)e^{\int_s^t h(r)g(r)dr} ds.$$

Recall here that $E[\Delta W_n] = E[\Delta \widetilde{N}_n] = 0$. Moreover, every Itô stochastic integral and stochastic integral with respect to compensated Poisson random measure have expectation zero. These properties will be repeatedly used throughout this chapter.

4.1 Euler-Maruyama Method

We will first deal with the EM method. Throughout this section, s_n will denote the EM numerical solution of (1.1). Let us define the continuous-time EM approximation

$$s(t) = s_0 + \alpha \int_0^t (\mu - \bar{s}(r))dr + \sigma \int_0^t \sqrt{|\bar{s}(r)|}dW(r) + \delta \int_0^t \bar{s}(r)d\widetilde{N}(r) \qquad (4.1)$$

where $t \in [0,T]$, and $\bar{s}(t)$ is the step function $\bar{s}(t) := s_n$ for $t \in [t_n, t_{n+1})$. From (3.1) and (4.1), we see that at each grid point $t = t_n$, $\bar{s}(t) = s_n = s(t_n)$. This yields that an error bound for $s(t)$ will automatically imply an error bound for our numerical solution s_n. Hence, we will aim at the error bound for $s(t)$.

Theorem 4.3. $Es_n - \mu = (1-\alpha\Delta)^n(Es_0 - \mu)$.

Proof. First, notice that after we fix Δ, $E|s_n|$ is bounded due to the linear growth condition of all coefficients and the finite number of time steps, and s_n is independent of ΔW_n and $\Delta\widetilde{N}_n$ thanks to the fixed time step Δ. Note that if the time step is not fixed, s_n may not be independent of $\Delta\widetilde{N}_n$. Taking expectation in (3.1) yields that $Es_{n+1} = (1-\alpha\Delta)Es_n + \alpha\mu\Delta$. Therefore, for each n, $Es_{n+1} - \mu = (1-\alpha\Delta)(Es_n - \mu)$; hence, this can lead to the desired result. □

This theorem yields that $\lim_{n\to\infty} Es_n = \mu$ for $\Delta < \frac{2}{\alpha}$. Thus, the discrete approximation of $S(t)$ still keeps mean reversion when Δ is sufficiently small. Note that we usually choose Δ so small that $\Delta < \frac{2}{\alpha}$. From Theorem 3.1, we also immediately obtain the following corollary.

Corollary 4.4. $|ES(n\Delta) - Es_n| = \left| e^{-\alpha n\Delta} - (1-\alpha\Delta)^n \right| |ES_0 - \mu|$.

It is worth remarking here that from the above corollary we can show that when $|1-\alpha\Delta| < 1$ or equivalently $\Delta < \frac{2}{\alpha}$, $|ES(T) - Es_N| \le C\Delta$ for some constant C, where N is the number of time steps. Therefore, the order of weak convergence for the EM numerical approximation when Δ is sufficiently small is 1.0. To see this claim, we start at

$$\begin{aligned} |ES(T) - Es_N| &= \left| (e^{-\alpha\Delta})^N - (1-\alpha\Delta)^N \right| |ES_0 - \mu| \\ &= \left| \left(\sum_{k=0}^{\infty} \frac{(-\alpha\Delta)^k}{k!} \right)^N - (1-\alpha\Delta)^N \right| |ES_0 - \mu|. \end{aligned}$$

We will show that $\left(\sum_{k=0}^{\infty} \frac{(-\alpha\Delta)^k}{k!} \right)^N - (1-\alpha\Delta)^N = A\Delta$ for some constant A which does not depend on N and Δ, and then set $C = |A||ES_0 - \mu|$ in order to obtain our desired result. Observe that

$$\left(\sum_{k=0}^{\infty}\frac{(-\alpha\Delta)^k}{k!}\right)^N = \left(1-\alpha\Delta+\frac{(-\alpha\Delta)^2}{2!}+\frac{(-\alpha\Delta)^3}{3!}+\dots\right)^N$$
$$= 1-N\alpha\Delta+\left[\binom{N}{2}(-\alpha\Delta)^2+\binom{N}{1}\frac{(-\alpha\Delta)^2}{2!}\right]+$$
$$\left[\binom{N}{3}(-\alpha\Delta)^3+\binom{N}{1}\binom{N-1}{1}\frac{(-\alpha\Delta)^3}{2!}+\binom{N}{1}\frac{(-\alpha\Delta)^3}{3!}\right]$$
$$+\dots$$

and

$$(1-\alpha\Delta)^N = 1-N\alpha\Delta+\binom{N}{2}(-\alpha\Delta)^2+\binom{N}{3}(-\alpha\Delta)^3+\dots+\binom{N}{N}(-\alpha\Delta)^N.$$

Then, $\left(\sum_{k=0}^{\infty}\frac{(-\alpha\Delta)^k}{k!}\right)^N-(1-\alpha\Delta)^N$ has the form $\alpha\Delta\sum_{k=1}^{\infty}f_k(N)(\alpha\Delta)^k$ where $f_k(N)$ is a polynomial over N of degree k. For each k, $f_k(N)(\alpha\Delta)^k$ does not depend on N and Δ but T and α, since we can eliminate N^l where $l\le k$ by multiplying it by Δ^l so that $N^l\Delta^l=T^l$ and also annihilate the remainder Δ^{k-l} by matching it with α^{k-l} available from α^k so that $\Delta^{k-l}\alpha^{k-l}<2^{k-l}$. Whatever N is, $\sum_{k=1}^{\infty}f_k(N)(\alpha\Delta)^k$ is still convergent. Hence, we have our claim. However, it is hard to explicitly calculate such constant A. As we will see later, Theorem 4.7, which is the main result of this section, will provides rigorous error bound of the EM method for the mean-reverting square root process with jumps.

Lemma 4.5. *$E|\bar{s}(t)|$ is bounded on $[0,T]$.*

Proof. Observe that $E|s_n|$ is bounded, and for any $t\in[t_n,t_{n+1}),\bar{s}(t)=s_n$. Hence, $E|\bar{s}(t)|=E|s_n|$ is also bounded on $[0,T]$. □

Lemma 4.6. *$|E[s(t)-\bar{s}(t)]|\le D_1(\alpha,\Delta,S_0,\mu,N)$ for any $t\in[0,T]$ where*

$$D_1(\alpha,\Delta,S_0,\mu,N) := \begin{cases}\alpha\Delta|ES_0-\mu|, & \text{if } |1-\alpha\Delta|<1,\\ \alpha\Delta|ES_0-\mu||1-\alpha\Delta|^N, & \text{if } |1-\alpha\Delta|\ge 1\end{cases}$$

and N is the number of time steps.

Proof. Let $t \in [0,T]$ and $n = \left\lfloor \frac{t}{\Delta} \right\rfloor$, the integer part of $\frac{t}{\Delta}$. Since $\bar{s}(t) = s_n = s(t_n)$ at each grid point $t = t_n$, we have

$$\begin{aligned} s(t) - \bar{s}(t) = \alpha(\mu - s_n)(t - n\Delta) + \sigma\sqrt{|s_n|}\,(W(t) - W(n\Delta)) \\ + \delta s_n \left(\widetilde{N}(t) - \widetilde{N}(n\Delta)\right). \end{aligned}$$

Taking expectation through this equation and applying Theorem 4.3, we acquire

$$\begin{aligned} E[s(t) - \bar{s}(t)] &= \alpha E(\mu - s_n)(t - n\Delta) \\ &= \alpha(1 - \alpha\Delta)^n(\mu - Es_0)(t - n\Delta). \end{aligned}$$

Taking absolution through this equation and noting that $t - n\Delta \leq \Delta$, we obtain that

$$|E[s(t) - \bar{s}(t)]| \leq \alpha|1 - \alpha\Delta|^n|ES_0 - \mu|\Delta.$$

Note that n can be any integer varied from 0 to N. If $|1 - \alpha\Delta| < 1$, then we have that $|1 - \alpha\Delta|^n < 1$; thus, we get the desired result. If $|1 - \alpha\Delta| \geq 1$, then $|1 - \alpha\Delta|^n < |1 - \alpha\Delta|^N$; hence, we acquire the desired result. □

Remark that Δ is usually so small that $|1 - \alpha\Delta| < 1$; therefore, in this case, we choose $D_1 = \alpha\Delta|ES_0 - \mu|$ whose formula does not depend on the number of time steps N.

Theorem 4.7. *For any $t \in [0,T]$,*

$$|E[S(t) - s(t)]| \leq D_1(\alpha, \Delta, S_0, \mu, N)(e^{\alpha T} - 1),$$

where $D_1(\alpha, \Delta, S_0, \mu, N)$ is defined as in Lemma 4.6.

Proof. Let $t \in [0,T]$. From (1.1) and (4.1), we have

$$\begin{aligned} S(t) - s(t) = &- \alpha \int_0^t \left(S(r^-) - \bar{s}(r)\right) dr \\ &+ \sigma \int_0^t \left(\sqrt{S(r^-)} - \sqrt{|\bar{s}(r)|}\right) dW(r) + \delta \int_0^t \left(S(r^-) - \bar{s}(r)\right) d\widetilde{N}(r). \end{aligned}$$

Taking expectation through this equation and replacing r^- by r will not have any effect on the Lebesgue integrals yield

$$E[S(t) - s(t)] = -\alpha E \int_0^t (S(r) - \bar{s}(r))\, dr.$$

Next, for the right hand side of the above equation, we will apply Theorem 4.1 in order to interchange the order between the expectation and the integral. Note that the expectation is the integral with respect to the probability measure P over the whole space Ω, and both Ω and $[0, t]$ are σ-finite measure spaces. We will verify that

$$\int_0^t E|S(r) - \bar{s}(r)|dr < \infty.$$

Since the exact solution $S(t)$ will never become negative with probability one, $|S(t)| = S(t)$ almost surely. Thus, $E|S(r)| = ES(r) = \mu + e^{-\alpha r}(ES_0 - \mu)$ which is bounded on $[0, t]$. By Lemma 4.5, $E|S(r) - \bar{s}(r)| \leq E|S(r)| + E|\bar{s}(r)|$ is also bounded on $[0, t]$. Therefore, the integral $\int_0^t E|S(r) - \bar{s}(r)|dr$ is bounded. Then, we can interchange the order between the expectation and the integral as desired. After that taking the absolution yields

$$\begin{aligned}
|E[S(t) - s(t)]| &= \alpha \left| \int_0^t E\left(S(r) - \bar{s}(r)\right) dr \right| \\
&\leq \alpha \int_0^t \left| E\left(S(r) - \bar{s}(r)\right) \right| dr \\
&\leq \alpha \int_0^t \left| E\left(S(r) - s(r)\right) \right| dr + \alpha \int_0^t \left| E\left(s(r) - \bar{s}(r)\right) \right| dr \\
&\leq \alpha \int_0^t \left| E\left(S(r) - s(r)\right) \right| dr + \alpha \int_0^t D_1 dr \qquad (*) \\
&= \alpha D_1 t + \alpha \int_0^t \left| E\left(S(r) - s(r)\right) \right| dr.
\end{aligned}$$

Note that we obtain the inequality in line (*) from Lemma 4.6. Now, we will apply Theorem 4.2 to this inequality. Observe that we have to check only that $|E[S(t) - s(t)]|$ is continuous. Also, notice that $ES(t) = \mu + e^{-\alpha t}(ES_0 - \mu)$ and the absolution are continuous. We claim that $Es(t)$ is also continuous; hence, we will acquire the desired condition for applying Theorem 4.2. To show our claim,

we first note that for every measurable function f, $\int_0^t f(r)dr$ is continuous in t. From (4.1),

$$Es(t) = Es_0 + \alpha E\int_0^t (\mu - \bar{s}(r))dr = Es_0 + \alpha\mu t - E\int_0^t \bar{s}(r)dr.$$

Since $E|\bar{s}(t)|$ is bounded by Lemma 4.5, again, we can interchange the order between the expectation and the integral and finally have

$$Es(t) = Es_0 + \alpha\mu t - \int_0^t E\bar{s}(r)dr$$

which is continuous. Therefore, we can now apply Theorem 4.2 to the inequality $|E[S(t) - s(t)]| \leq \alpha D_1 t + \alpha\int_0^t |E(S(r) - s(r))|\, dr$ which yields

$$\begin{aligned}
|E[S(t) - s(t)]| &\leq \alpha D_1 t + \alpha\int_0^t \alpha D_1 r e^{\int_r^t \alpha dv} dr \\
&= \alpha D_1 t + \alpha^2 D_1 \int_0^t r e^{\alpha t - \alpha r} dr \\
&= \alpha D_1 t + \alpha^2 D_1 \left[\frac{e^{\alpha t}}{\alpha^2} - \frac{t}{\alpha} - \frac{1}{\alpha^2}\right] \qquad (**) \\
&= D_1(e^{\alpha t} - 1) \\
&\leq D_1(e^{\alpha T} - 1).
\end{aligned}$$

Note that the equation in line (**) is obtained by integration by parts. □

Note that there are other versions of Gronwall's inequality which can yield the error bound $\alpha T D_1 e^{\alpha T}$ being greater than $D_1(e^{\alpha T} - 1)$. The Gronwall's inequality in Theorem 4.2 is a version giving the best result for Theorem 4.7 as we have found from a lot of textbooks.

By Theorem 4.7, we see that the order of weak convergence for the EM numerical solution when Δ is sufficiently small is 1.0.

Recall that s_n and $s(t)$ agree on every grid point and Corollary 4.4 gives a weak error at each grid point t_n. Because the number of time steps is finite, $\max_n |e^{-\alpha n\Delta} - (1 - \alpha\Delta)^n||ES_0 - \mu|$ is an error bound for our numerical solution s_n for every grid point. This differs from the error bound in Theorem 4.7 which

provides the error for the continuous-time EM approximation $s(t)$ for the whole interval $[0, T]$. Thus, the error bound from Theorem 4.7 can be used instead of $\max_n |e^{-\alpha n\Delta} - (1-\alpha\Delta)^n||ES_0 - \mu|$ for Corollary 4.4. Another good aspect of the error bound in Theorem 4.7 is that it has a simple form which is easy to be calculated.

4.2 Compensated Split-Step Backward Euler Method

Now, we work on the CSSBE method. Throughout this section, s_n will denote the CSSBE numerical solution of (1.1) which is obtained from (3.2). We now define the continuous-time CSSBE approximation by

$$s(t) = s_0 + \alpha \int_0^t (\mu - \bar{s}(r))dr + \sigma \int_0^t \sqrt{|\bar{s}(r)|}dW(r) + \delta \int_0^t \bar{s}(r)d\widetilde{N}(r), \qquad (4.2)$$

where $t \in [0, T]$, and $\bar{s}(t)$ is the step function $\bar{s}(t) := s^*_{n+1}$ for $t \in [t_n, t_{n+1})$. From (3.2) and (4.2), we know that $s_n = s(t_n)$ at every grid point $t = t_n$. Like the EM method, we will seek the error bound for $s(t)$ in order to obtain an error bound for our numerical solution s_n. This section is very much like the previous section so that some remarks and details of the proof will be omitted.

Theorem 4.8. $Es_n - \mu = (\frac{1}{1+\alpha\Delta})^n(Es_0 - \mu)$.

Proof. Note that $E|s^*_{n+1}|$ is bounded. Taking expectation in (3.2) yields that $Es_{n+1} = \frac{Es_n + \alpha\mu\Delta}{1+\alpha\Delta}$. Therefore, $Es_{n+1} - \mu = \frac{1}{1+\alpha\Delta}(Es_n - \mu)$. Then, we will get the desired result. □

This theorem yields that $\lim_{n\to\infty} Es_n = \mu$ for any size of Δ. Thus, the discrete approximation s_n still keeps mean reversion for any size of Δ. This makes the CSSBE method seem better than the EM method.

Corollary 4.9. $|ES(n\Delta) - Es_n| = \left| e^{-\alpha n\Delta} - (\frac{1}{1+\alpha\Delta})^n \right| |ES_0 - \mu|$.

Lemma 4.10. $E|\bar{s}(t)|$ *is bounded on* $[0, T]$.

The proof of the above lemma is similar to Lemma 4.5's.

Lemma 4.11. $|E[s(t) - \bar{s}(t)]| \leq D_2(\alpha, \Delta, S_0, \mu)$ *for any* $t \in [0, T]$ *where*

$$D_2(\alpha, \Delta, S_0, \mu) := \frac{\alpha\Delta}{1 + \alpha\Delta}|ES_0 - \mu|.$$

Proof. Let $t \in [0, T]$ and $n = \left\lfloor \frac{t}{\Delta} \right\rfloor$, the integer part of $\frac{t}{\Delta}$. From (3.2) and (4.2), we acquire that

$$\begin{aligned} s(t) - \bar{s}(t) = s_n + \alpha(\mu - s^*_{n+1})(t - n\Delta) + \sigma\sqrt{|s^*_{n+1}|}\,(W(t) - W(n\Delta)) \\ + \delta s^*_{n+1}\left(\widetilde{N}(t) - \widetilde{N}(n\Delta)\right) - \frac{s_n + \mu\alpha\Delta}{1 + \alpha\Delta}. \end{aligned}$$

Taking expectation through this equation, we have

$$\begin{aligned} E[s(t) - \bar{s}(t)] &= \frac{\alpha\Delta}{1 + \alpha\Delta}(Es_n - \mu) + \frac{\alpha(t - n\Delta)}{1 + \alpha\Delta}(\mu - Es_n) \\ &= \frac{\alpha((n+1)\Delta - t)}{1 + \alpha\Delta}(Es_n - \mu). \end{aligned}$$

Taking absolution on both sides of this equation, noting that $(n+1)\Delta - t \leq \Delta$ and applying Theorem 4.8, we obtain

$$\begin{aligned} |E[s(t) - \bar{s}(t)]| &\leq \frac{\alpha\Delta}{1 + \alpha\Delta}\left(\frac{1}{1 + \alpha\Delta}\right)^n |Es_0 - \mu| \\ &\leq \frac{\alpha\Delta}{1 + \alpha\Delta}|Es_0 - \mu| \end{aligned}$$

which completes the proof. □

Theorem 4.12. *For any* $t \in [0, T]$,

$$|E[S(t) - s(t)]| \leq D_2(\alpha, \Delta, S_0, \mu)(e^{\alpha T} - 1),$$

where $D_2(\alpha, \Delta, S_0, \mu)$ *is defined as in Lemma 4.11.*

We can imitate the proof of Theorem 4.7 to prove this theorem by applying Lemma 4.10 and Lemma 4.11 instead of Lemma 4.5 and Lemma 4.6, respectively.

From Theorem 4.12, since $\frac{\alpha\Delta}{1+\alpha\Delta} < \alpha\Delta$, we also have that

$$|E[S(t) - s(t)]| < \alpha\Delta|ES_0 - \mu|(e^{\alpha T} - 1).$$

Hence, the order of weak convergence for the CSSBE numerical solution is 1.0.

Observe that we can still imitate the package of proof from section 4.1 to find error bounds for other methods that have a constant time step size. Unfortunately, the JAE method has random time step sizes, so we cannot use the proof from section 4.1 to show an error bound for this method. It is harder to find such bound.

CHAPTER V

COMPUTATIONAL EXPERIMENTS

First, we show computable error bounds for EM and CSSBE methods acquired from Theorem 4.7 and Theorem 4.12, respectively. Specifically, we demonstrate, in Table 5.1, the bounds in the case $\alpha = 4, \mu = 0.5, \sigma = 0.3, \lambda = 8, \delta = 0.2, s_0 = 1$, and $T = 0.5$ for a range of Δ values.

	EM	CSSBE
$\Delta = 2^{-9}$	0.024957250386448	0.024763783329189
$\Delta = 2^{-8}$	0.049914500772896	0.049146585376390
$\Delta = 2^{-7}$	0.099829001545791	0.096803880286828
$\Delta = 2^{-6}$	0.199658003091583	0.187913414674431
$\Delta = 2^{-5}$	0.399316006183166	0.354947561051703

Table 5.1: Error bounds for EM and CSSBE methods when $\alpha = 4, \mu = 0.5, \sigma = 0.3, \lambda = 8, \delta = 0.2, s_0 = 1$, and $T = 0.5$.

Next, we simulate our model through MATLAB using the three methods. In Figure 5.1 - 5.6, graphs that show the order of convergence in weak sense of our three methods are presented. For each method, we generate 1,000,000 sample paths with 5 different sizes of $\Delta : 2^{-9}, 2^{-8}, 2^{-7}, 2^{-6}$ and 2^{-5}. Then, we measure the error in weak sense: $|ES(T) - Es_{n,T}|$. Here, $ES(T)$ can be calculated from Theorem 3.1, and we find $Es_{n,T}$ by averaging $s_{n,T}$ of all 1,000,000 paths. For each graph, the x-axis is the size of Δ and the y-axis represents the weak error $|ES(T) - Es_{n,T}|$. Here, we plot these graphs in log-log scale so that the slope of

each line will represent the order of convergence in weak sense. The reference line with slope of one is also plotted in dash. Note that for Figure 5.3 - 5.6, we fix $\alpha = 6, \mu = 0.5, \sigma = 0.5, \delta = 0.1, s_0 = 1$ and $T = 0.25$ and vary λ over 4, 12, 36 and 108. For each λ, the expected number of jumps of sample paths is λT so that we examine the cases when the expected numbers of jumps of sample paths are 1, 3, 9 and 27, correspondingly.

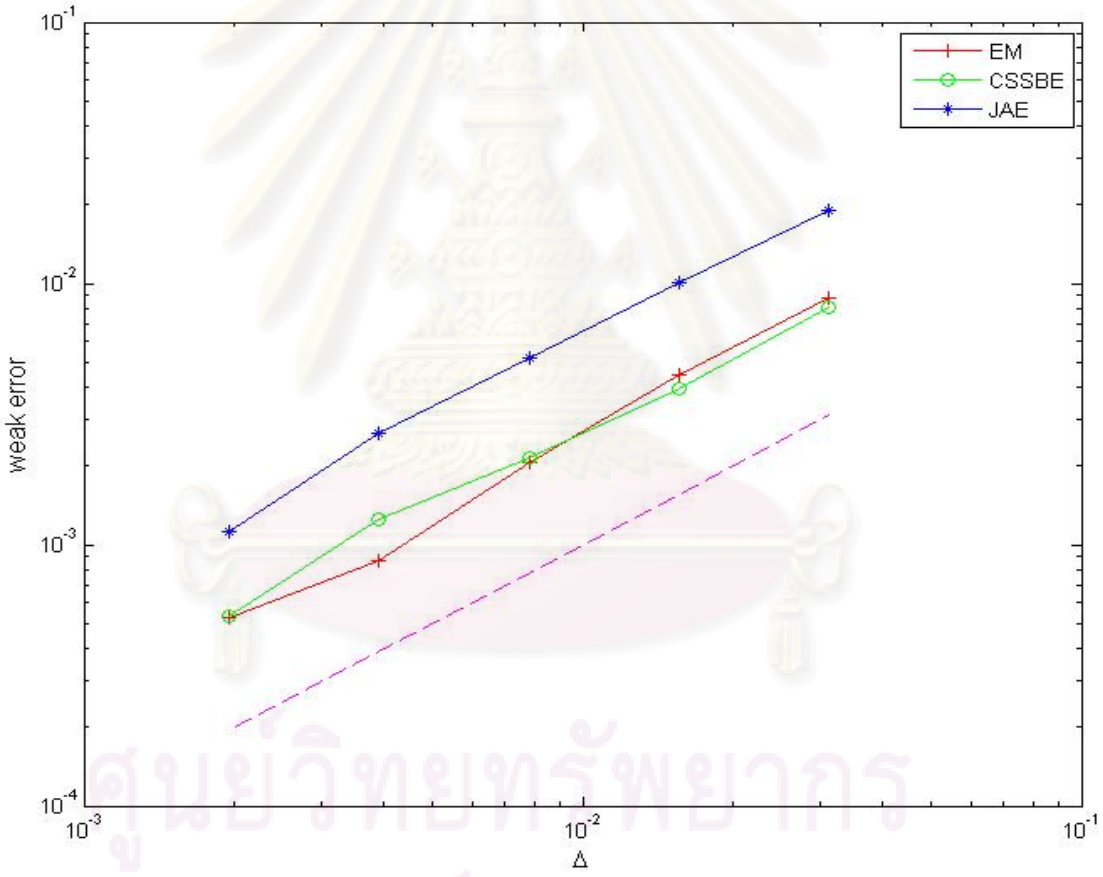

Figure 5.1: Weak error plots when $\alpha = 4, \mu = 0.5, \sigma = 0.3, \lambda = 8, \delta = 0.2, s_0 = 1$, and $T = 0.5$.

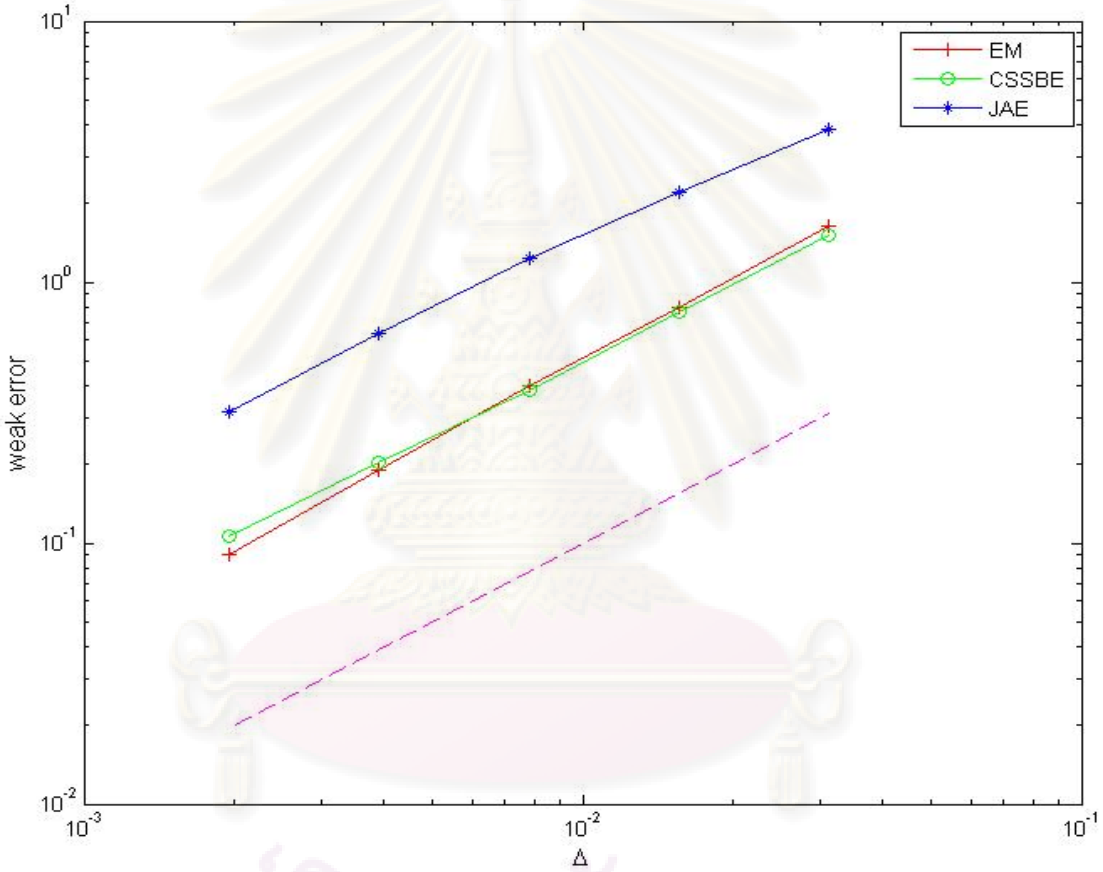

Figure 5.2: Weak error plots when $\alpha = 6, \mu = 50, \sigma = 0.5, \lambda = 36, \delta = 0.1, s_0 = 100$, and $T = 0.25$.

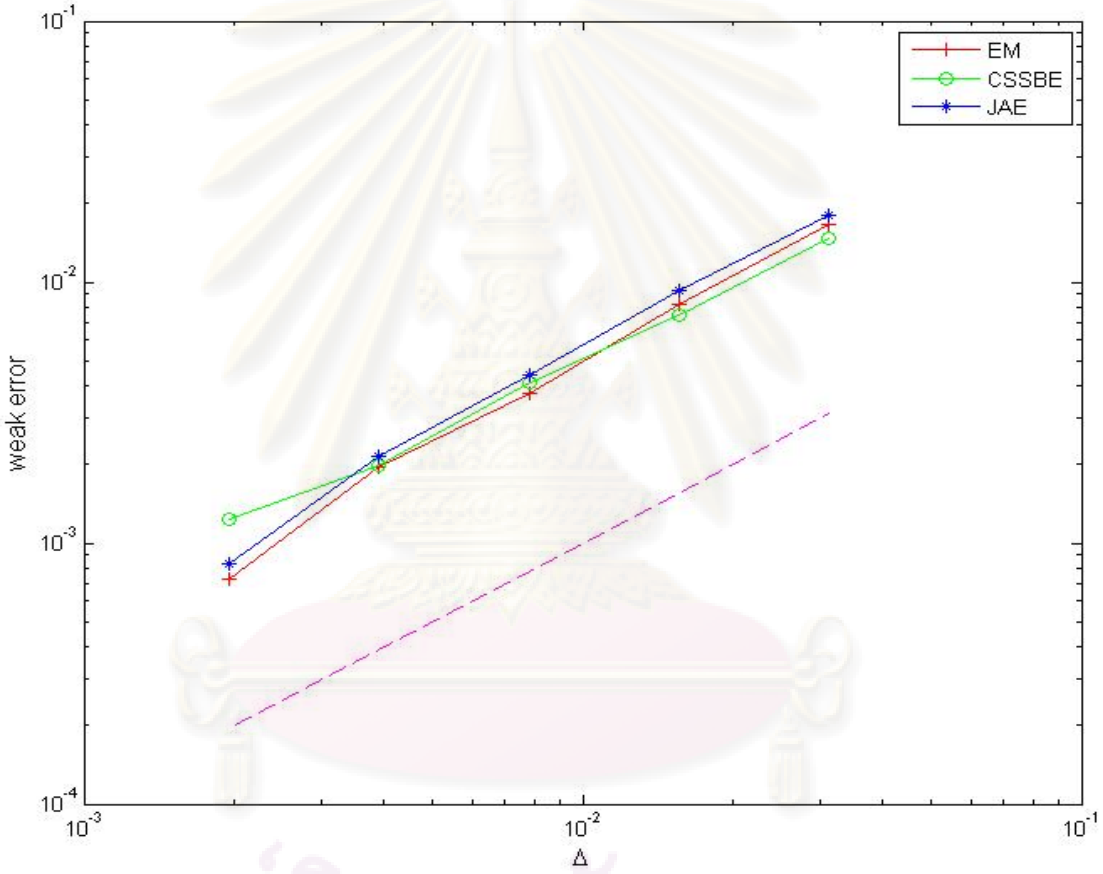

Figure 5.3: Weak error plots when $\alpha = 6, \mu = 0.5, \sigma = 0.5, \lambda = 4, \delta = 0.1, s_0 = 1$, and $T = 0.25$.

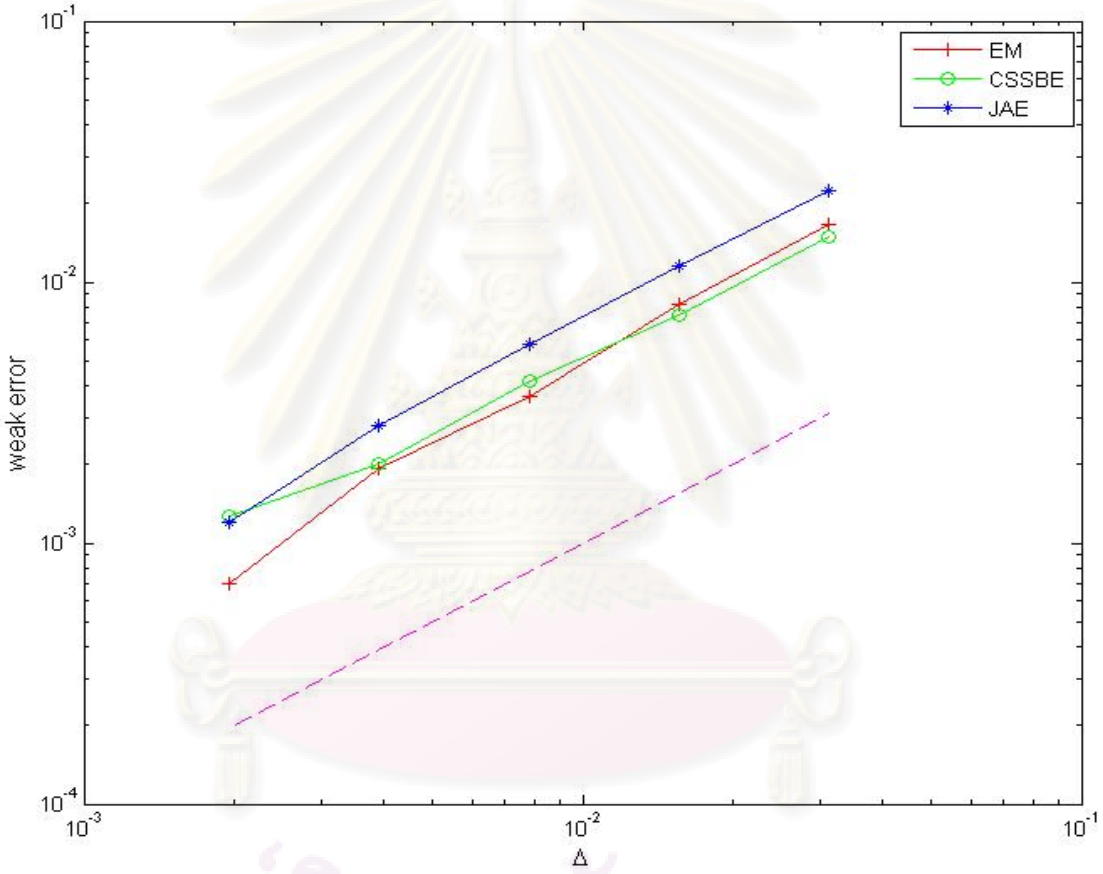

Figure 5.4: Weak error plots when $\alpha = 6, \mu = 0.5, \sigma = 0.5, \lambda = 12, \delta = 0.1, s_0 = 1$, and $T = 0.25$.

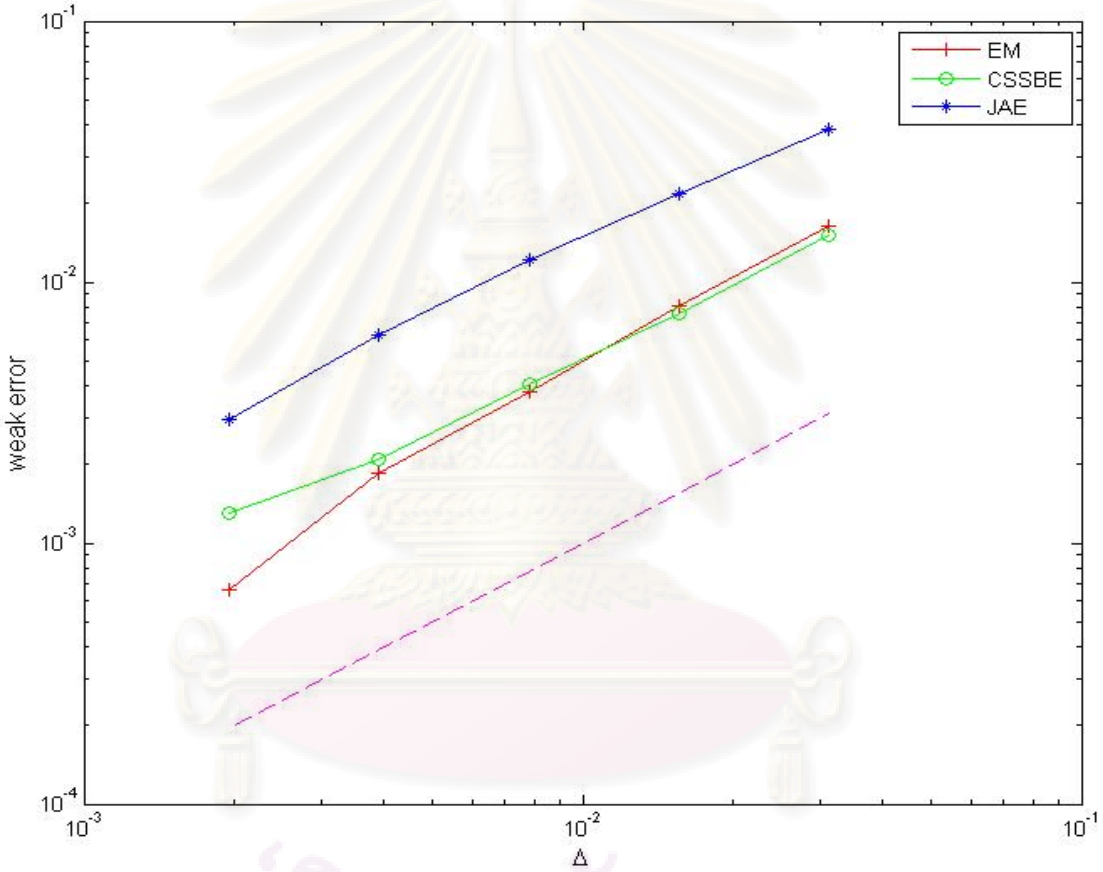

Figure 5.5: Weak error plots when $\alpha = 6, \mu = 0.5, \sigma = 0.5, \lambda = 36, \delta = 0.1, s_0 = 1,$ and $T = 0.25$.

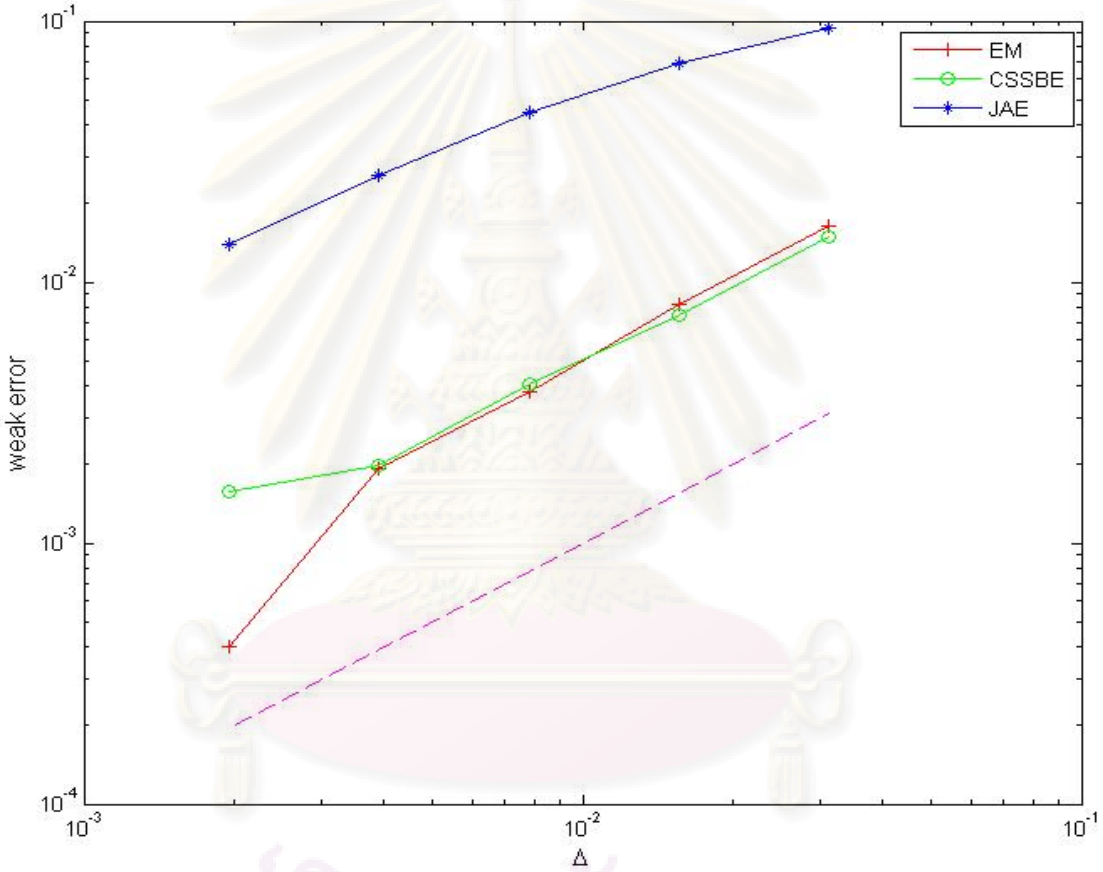

Figure 5.6: Weak error plots when $\alpha = 6, \mu = 0.5, \sigma = 0.5, \lambda = 108, \delta = 0.1, s_0 = 1$, and $T = 0.25$.

Lastly, graphs that show weak error bounds for EM and CSSBE methods coming from Theorem 4.7 and Theorem 4.12, respectively, together with the weak error plots from corresponding simulation are illustrated. Figure 5.7 - 5.8 show the relation between our theoretical error bounds and the errors from our simulation in the case $\alpha = 4, \mu = 0.5, \sigma = 0.3, \lambda = 8, \delta = 0.2, s_0 = 1$, and $T = 0.5$. Here, we use 5 different sizes of $\Delta : 2^{-9}, 2^{-8}, 2^{-7}, 2^{-6}$ and 2^{-5} so that we can obtain each point of theoretical error bounds from Table 5.1. The reference line with slope of one is also plotted in dash.

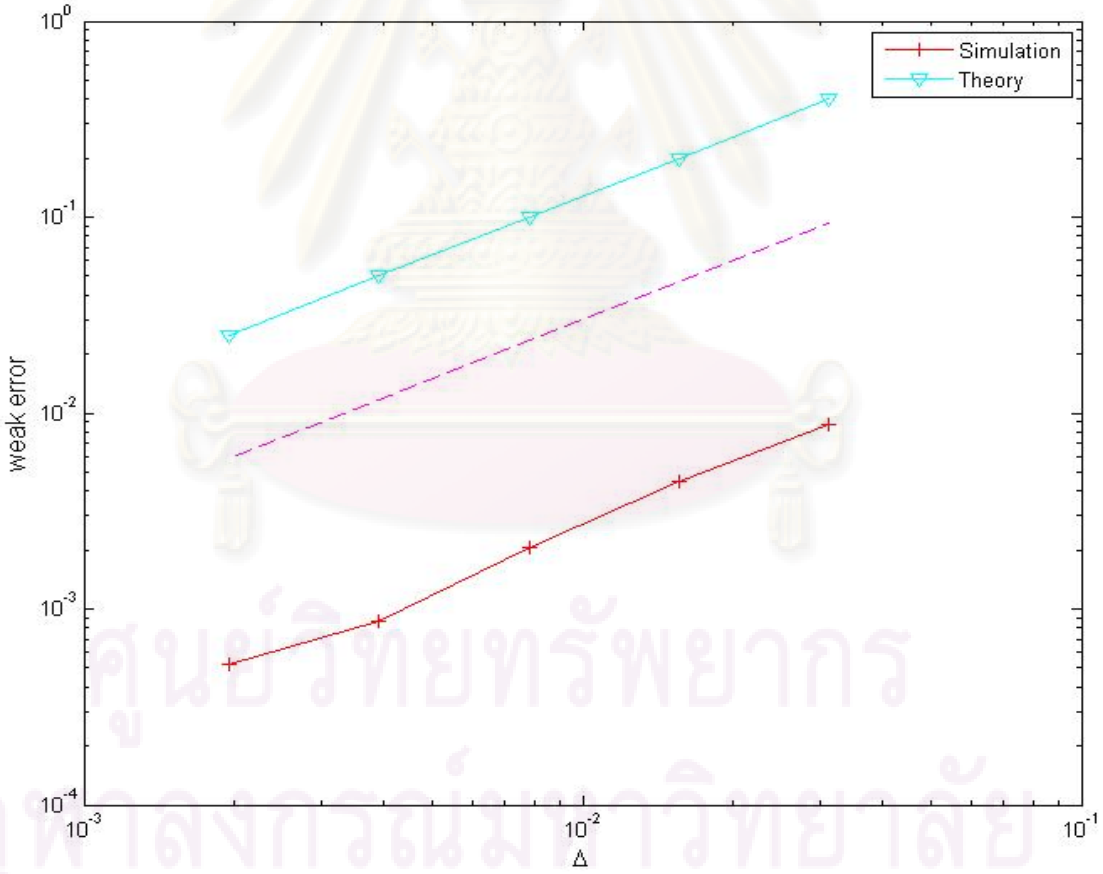

Figure 5.7: Theoretical error bound from Theorem 4.7 and weak error plot for EM method when $\alpha = 4, \mu = 0.5, \sigma = 0.3, \lambda = 8, \delta = 0.2, s_0 = 1$, and $T = 0.5$.

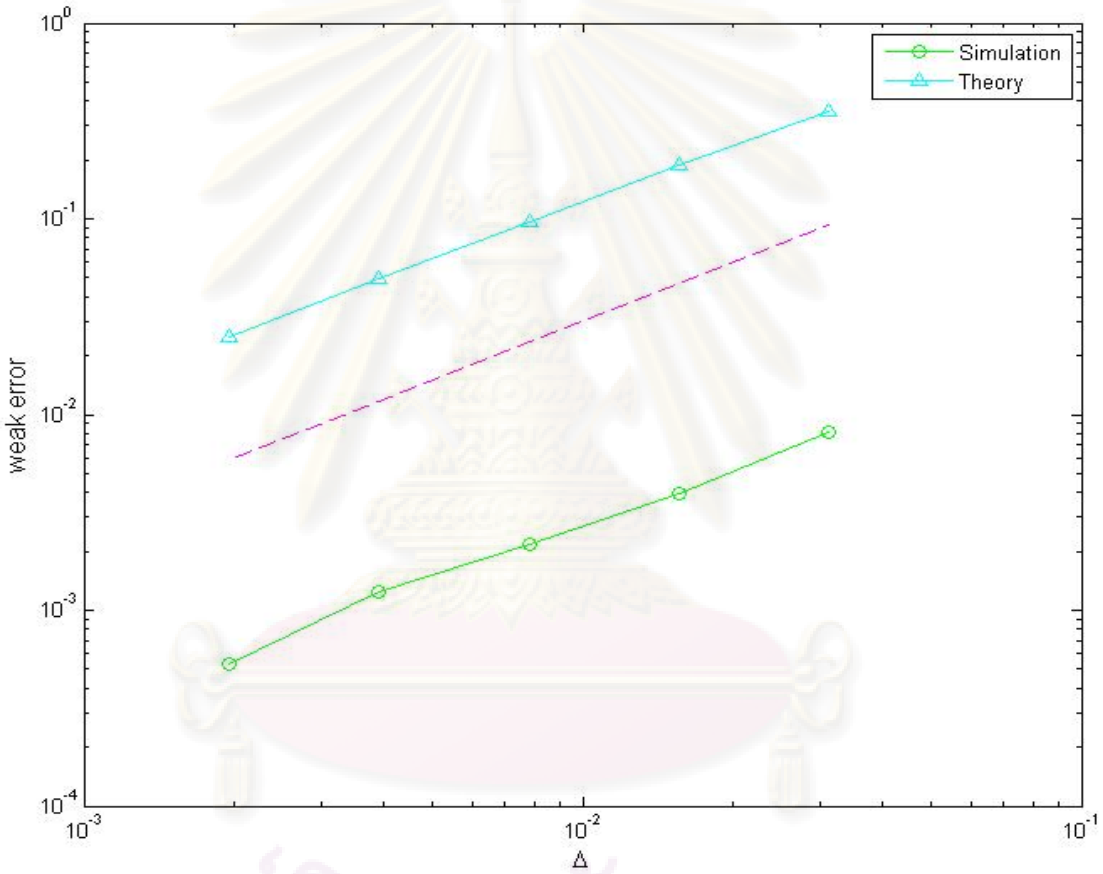

Figure 5.8: Theoretical error bound from Theorem 4.12 and weak error plot for CSSBE method when $\alpha = 4, \mu = 0.5, \sigma = 0.3, \lambda = 8, \delta = 0.2, s_0 = 1$, and $T = 0.5$.

CHAPTER VI
DISCUSSION AND CONCLUSION

In this work, we have provided rigorous numerical error bounds in weak sense for EM and CSSBE methods for the mean-reverting square root process with jumps. The numerical investigations have also been done with both methods and also with the JAE method. It is found numerically that all of three methods tend to have order of weak convergence equal to 1.0. This coincides with the general theory for SDEs with jumps with Lipschitzian coefficients. In fact, from the formulae of the error bounds for EM and CSSBE methods in chapter 4, we know that the order of weak convergence for both numerical schemes is exactly 1.0.

We notice that the formulae of error bounds for EM and CSSBE methods do not depend on parameters σ, λ, δ. This is because these parameters relate to the Wiener process and the compensated Poisson process which are martingales.

Comparing the formulae of error bounds for EM and CSSBE methods, we see that the error bound of CSSBE method is slightly better than EM method's. This agrees with our computer simulation in many cases in which both of EM and CSSBE plots are close together.

Although the JAE method should have a better numerical solution when it is compared with EM and CSSBE methods for other models, it has been observed from the experiments that the medthod gives higher errors in almost all cases of our simulation. The most suspicious factor to this phenomenon is the parameters concerning with jumps, which are λ and δ. However, δ represents the degree of jumps and should be around 0.1 or 0.2 for instance. Thus, we shift out attention to the effect of the number of jumps in sample paths to its accuracy. Figure 6.1 and 6.2 show some sample paths simulated by the JAE method with the

same parameters used in Figure 5.6. The dash line in each picture represents the expectation of the exact solution obtained from Theorem 3.1.

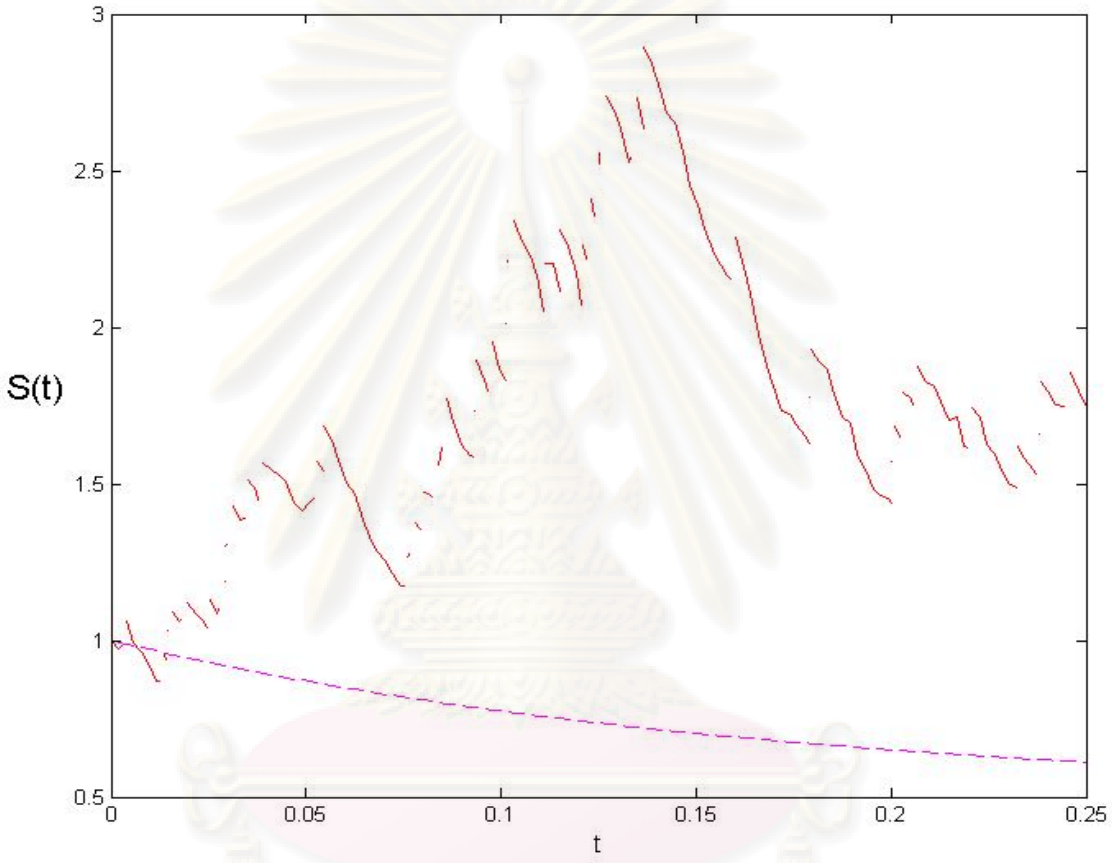

Figure 6.1: A sample path of JAE method that has a great number of jumps when $\alpha = 6, \mu = 0.5, \sigma = 0.5, \lambda = 108, \delta = 0.1, s_0 = 1, T = 0.25$ and $\Delta = 2^{-9}$.

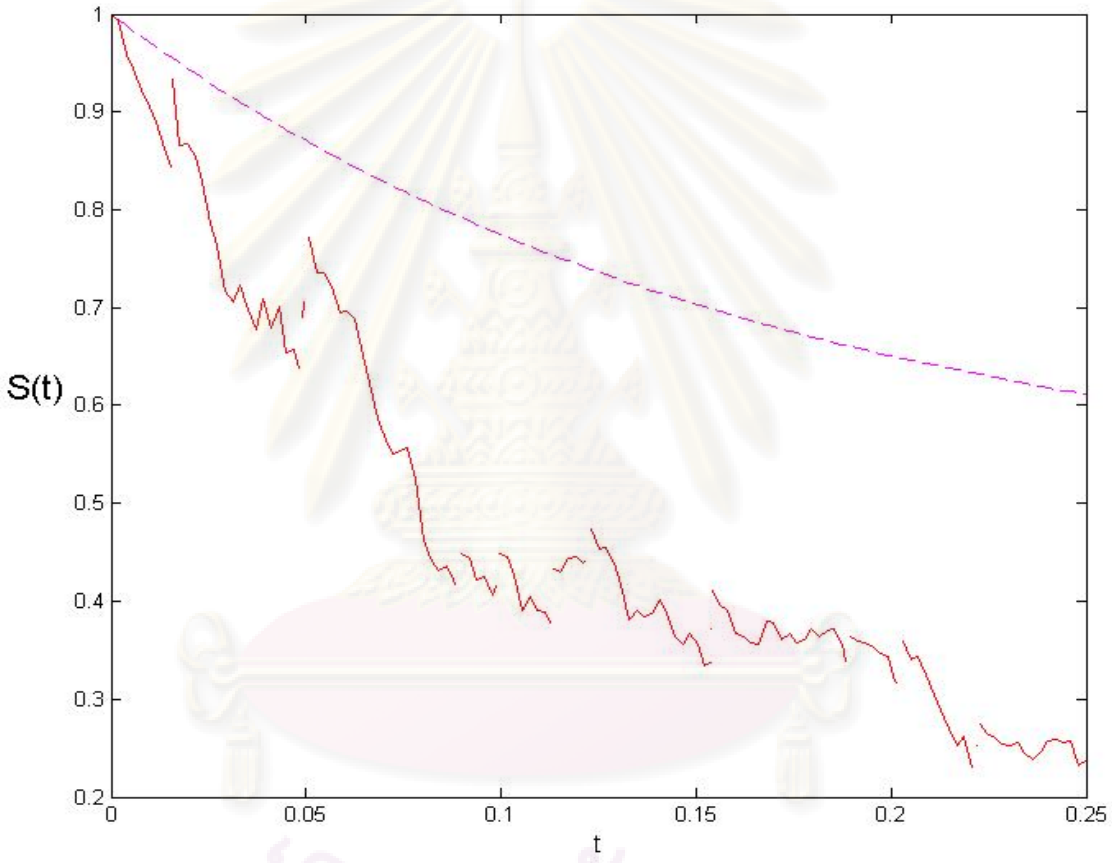

Figure 6.2: A sample path of JAE method that has a small number of jumps when $\alpha = 6, \mu = 0.5, \sigma = 0.5, \lambda = 108, \delta = 0.1, s_0 = 1, T = 0.25$ and $\Delta = 2^{-9}$.

When the number of jumps is too low, in Figure 5.3, all methods will behave indistinctively. The more the number of jumps, the more the difference of JAE method to the other two schemes. If the number of jumps is too high, the JAE method may create some paths, as in Figure 6.1, that have too many jumps or might generate some paths, as in Figure 6.2, that have a small number of jumps before they approach to the strike time T and go far off the expectation of the exact solution at time T. This means that the JAE method has high volatility when λ is too large and can affect its performance. Hence, we should use the JAE method for the scenario having the reasonable expected number of jumps. Anyway, this is just our hypothesis. The actual factors are still unknown. This might need a theoretical error bound for JAE method and deep analysis which is more complicated than the other two methods' because of the nonconstant time step size.

REFERENCES

[1] N. Bruti-Liberati, E. Platen, *Approximation of jump diffusions in finance and economics*, University of Technology Sydney, Technical report, Quantitative Finance Research Papers 176 (2006).

[2] R. Cont, P. Tankov, *Financial Modelling with Jump Processes*, CRC Press, London, 2004.

[3] J.C. Cox, J.E. Ingersoll Jr., S.A. Ross, *A theory of the term structure of interest rates*, Ecomometrica, **53** (1985), 385-407.

[4] S.I. Heston, *A closed-form solution for options with stochastic volatility with applications to bond and currency options*, Review of Financial Studies, **6** (1993), 327-343.

[5] D.J. Higham, P.E. Kloeden, *Numerical methods for nonlinear stochastic differential equations with jumps*, Numerische Mathematik, **101** (2005), 101-119.

[6] D.J. Higham, X. Mao, *Convergence of Monte Carlo simulations involving the mean-reverting square root process*, Journal of Computational Finance, **8**(3) (2005), 35-61.

[7] N. Ikeda, S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, 2nd ed., North-Holland, Amsterdam, 1989.

[8] I. Karatzas, S.E. Shreve, *Brownian Motion and Stochastic Calculus*, 2nd ed., Springer-Verlag, New York, 1999.

[9] T. Mikosch, *Elementary Stochastic Calculus–With Finance in View*, World Scientific, Singapore, 2006.

[10] B.G. Pachpatte, *Inequalities for Differential and Integral Equations*, Academic Press, London, 1998.

[11] E. Platen, *An approximation method for a class of Itô processes with jump component*, Liet. Mat. Rink., **22**(2) (1982), 124-136.

[12] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill Book Company, New York, 1987.

[13] F. Wu, X. Mao, K. Chen, *Strong convergence of Monte Carlo simulations of the mean-reverting square root process with jump*, Applied Mathematics and Computation, **206** (2008), 494-505.

VITA

Name Mr. Raywat Tanadkithirun

Date of Birth 19 August 1985

Place of Birth Bangkok, Thailand

Education B.Sc.(Mathematics, First Class Honours),
Chulalongkorn University, 2008