CHAPTER III

THE HYDROGEN ATOM IN THREE DIMENSIONS

In the preceding chapter we deal with the path integral for the Hydrogen atom in two dimensions which is rather nonrealistic since it was thought that the electron moves only about the nucleus in a plane under the influence of an attractive inverse potential. In order to complete the problems, we must consider the higher dimensions,ie,the three dimensional problem. The situation is indeed analogy with the case of the two dimensions, the path integral must be transformed into a gaussian integral. The time reparameterization and the coordinate transformation still be the key steps for the reduction procedure. But the coordinate transformation is different from that used earlier. In three dimensional problem we use the four dimensional Kustaanheimo Stiefel transformation instead of the Levi-Civita transformation in two dimensions.

3.1 The Coulomb Path Integral in Three Dimensions.

Consider the Green's function

$$G(\vec{x}'',\vec{x}';E) = \int_0^\infty K(\vec{x}'',\vec{x}';\tau)\, e^{\frac{iE\tau}{\hbar}}\, d\tau \qquad (3.1)$$

Here $K(\vec{x}'', \vec{x}'; \tau)$ is the propagator of the bound electron and is given in the form

$$K(\vec{x}'',\vec{x}';\tau) = \int_{\vec{x}'}^{\vec{x}''} \exp\left\{\frac{i}{\hbar}\int_0^\tau L(\vec{x},\dot{\vec{x}})\,dt\right\} \mathcal{D}\vec{x}(t) \qquad (3.2)$$

with the Lagrangian

$$L(\vec{x},\dot{\vec{x}}) = \frac{m}{2}\dot{\vec{x}}^2 + \frac{e^2}{r} \qquad (3.3)$$

where $\vec{x}$ is a coordinate vector in three dimensional space and

$$r = (x^2 + y^2 + z^2)^{1/2} \qquad (3.4)$$

In the short time division basis, (3.2) becomes

$$K(\vec{x}'',\vec{x}';T) = \lim_{N\to\infty} \iint\cdots\int \exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N}\left[\frac{m(\Delta\vec{x}_j)^2}{2\tau_j} + \frac{e^2}{\bar{r}_j}\tau_j\right]\right\}$$
$$\prod_{j=1}^{N}\left[\frac{m}{2\pi i\hbar\tau_j}\right]^{3/2}\prod_{j=1}^{N-1} d^3x_j \qquad (3.5)$$

where $\bar{r}_j$ indicates the radial component of the midpoint vector $\vec{x}(t)$ at time t between t_{j-1} and t_j. Notice that

$$\exp\{iET/\hbar\} = \exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N} E\tau_j\right\} \qquad (3.6)$$

and substituting (3.5) into (3.1), the Green's function becomes

$$G(\vec{x}'',\vec{x}';E) = \lim_{N\to\infty} \iiint\cdots\int \exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N}\left[\frac{m(\Delta\vec{x}_j)^2}{2\tau_j} + \frac{e^2}{\bar{r}_j}\tau_j + E\tau_j\right]\right\}$$
$$\prod_{j=1}^{N}\left[\frac{m}{2\pi i\hbar\tau_j}\right]^{3/2}\prod_{j=1}^{N-1} d^3x_j\, dT \qquad (3.7)$$

We see that (3.7) has the same form of the Green's function as in two dimensions, with the same time rescaling used in chapter II ,

$$\tau_j \rightarrow \sigma_j = \frac{\tau_j}{\bar{r}_j} \tag{3.8}$$

The application of (3.8) reduced (3.7) into the form

$$G(\vec{x}'',\vec{x}';E) = \int_0^\infty e^{\frac{ie^2\sigma}{\hbar}} Q(\vec{x}'',\vec{x}';\sigma)\,d\sigma \tag{3.9}$$

where

$$Q(\vec{x}'',\vec{x}';\sigma) = \lim_{N\to\infty} r'' \iint\cdots\int \exp\{\frac{i}{\hbar}\sum_{j=1}^{N} S(\sigma_j)\} \prod_{j=1}^{N}\left[\frac{m}{2\pi i\hbar \bar{r}_j\sigma_j}\right]^{3/2} \prod_{j=1}^{N-1} d^3x_j \tag{3.10}$$

with $$S(\sigma_j) = \frac{m(\Delta x_j)^2}{2\bar{r}_j\sigma_j} + E\bar{r}_j\sigma_j \tag{3.11}$$

Of course, (3.10) is not yet integrable. The next step is to change the integral variables in (3.10)

3.2 The Modified Kustaanheimo-Stiefel Transformation in Four Dimensions.

The path integral (3.10) with the new time parameter is not of the gaussian form. In order to carry out the integration explicitly, a further reduction of the integral is required. One may expect that the Levi-Civita transformation can be used in three dimensional problem analogy with in the two dimensions. However it is found that one can not do so because there is no three dimensional counter part of the Levi-Civita transformation. Another kind of transformation must be sought, it is the Kustaanheimo-Stiefel transformation (10), the transformation which transforms the Cartesian variables $x^a = (x, y, z)$ in R^3 into the Cartesian variables $u^b = (u^1, u^2, u^3, u^4)$ in R^4

The cartesian variables in R^3 can be transformed into the cartesian variables in R^4 by using the relation

$$x^a = \sum_{b=1}^{4} A^{ab}(u)u^b \qquad (a = 1,2,3) \tag{3.12}$$

$$0 = \sum_{b=1}^{4} A^{ab}(u)u^b \qquad (a = 4) \tag{3.13}$$

with the transformation matrix

$$A(u) = \begin{bmatrix} u^3 & u^4 & u^1 & u^2 \\ -u^4 & u^3 & u^2 & -u^1 \\ -u^1 & -u^2 & u^3 & u^4 \\ -u^2 & u^1 & -u^4 & u^3 \end{bmatrix} \tag{3.14}$$

The matrix (3.14) has the following properties

i) Each matrix element is a linear homogeneous function of u^b

ii) The matrix is orthogonal in the sense that

$$\tilde{A}(u) \cdot A(u) = rI \tag{3.15}$$

with $$r = (u^1)^2 + (u^2)^2 + (u^3)^2 + (u^4)^2 \tag{3.16}$$

The transformation (3.12) and (3.13) can be put into the matrix form

$$\begin{pmatrix} x \\ y \\ z \\ o \end{pmatrix} = \begin{pmatrix} u^3 & u^4 & u^1 & u^2 \\ -u^4 & u^3 & u^2 & -u^1 \\ -u^1 & -u^2 & u^3 & u^4 \\ -u^2 & u^1 & -u^4 & u^3 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \\ u^3 \\ u^4 \end{pmatrix} \qquad (3.17)$$

This transformation, from now on is called the Kustaanheimo-Stiefel transformation or KS transformation. It is not one to one transformation. In order to make the mapping one to one, it is necessary to impose a constraint. One way of choosing such a constraint is as follow.

First, let $(\xi^1, \xi^2, \xi^3, \xi^4)$ be the transformation of the differential du^b, namely

$$\begin{aligned} \xi^1 &= u^3du^1 + u^4du^2 + u^1du^3 + u^2du^4 \\ \xi^2 &= -u^4du^1 + u^3du^2 + u^2du^3 - u^4du^4 \\ \xi^3 &= -u^1du^1 - u^2du^2 + u^3du^3 + u^4du^4 \\ \xi^4 &= -u^2du^1 + u^1du^2 - u^4du^3 + u^3du^4 \end{aligned} \qquad (3.18)$$

or in the matrix form

$$\begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \\ \xi^4 \end{pmatrix} = \begin{pmatrix} u^3 & u^4 & u^1 & u^2 \\ -u^4 & u^3 & u^2 & -u^1 \\ -u^1 & -u^2 & u^3 & u^4 \\ -u^2 & u^1 & -u^4 & u^3 \end{pmatrix} \begin{pmatrix} du^1 \\ du^2 \\ du^3 \\ du^4 \end{pmatrix} \qquad (3.19)$$

Next, differentiate both side of (3.17) to get

$$\begin{aligned} dx &= 2(u^3du^1 + u^4du^2 + u^1du^3 + u^2du^4) \\ dy &= 2(-u^4du^4 + u^3du^2 + u^2du^3 - u^1du^4) \\ dz &= 2(-u^2du^1 - u^1du^2 + u^4du^3 + u^3du^4) \end{aligned} \qquad (3.20)$$

Comparision of (3.20) with (3.18) leads to the identities

$$\begin{aligned} dx &= 2\xi^1 \\ dy &= 2\xi^2 \\ dz &= 2\xi^3 \end{aligned} \qquad (3.21)$$

In general , ξ^4 in (3.19) does not vanish. However if $\xi^4 = 0$ is demanded then (3.18) and (3.19) can be put together as

$$\begin{pmatrix} dx \\ dy \\ dz \\ 0 \end{pmatrix} = 2 \begin{pmatrix} u^3 & u^4 & u^2 & u^1 \\ -u^4 & u^3 & u^2 & -u^1 \\ -u^1 & -u^2 & u^3 & u^4 \\ -u^2 & u^1 & -u^4 & u^3 \end{pmatrix} \begin{pmatrix} du^1 \\ du^2 \\ du^3 \\ du^4 \end{pmatrix} \qquad (3.22)$$

The constraint,

$$\xi^4 = -u^2 du^1 + u^1 du^2 - u^4 du^3 + u^3 du^4 = 0 \qquad (3.23)$$

establishes the one to one relation between the two sets of variables (x, y, z) and (u^1, u^2, u^3, u^4).

An advantage of the anihilation condition (3.23) as the constraint is that as (3.17) yields

$$(x^2 + y^2 + z^2)^{1/2} = (u^1)^2 + (u^2)^2 + (u^3)^2 + (u^4)^2 = r \qquad (3.24)$$

The constrained transformation (3.22) results in a similarly simplified expression,

$$(dx)^2 + (dy)^2 + (dz)^2 = 4r \left\{ (du^1)^2 + (du^2)^2 + (du^3)^2 + (du^4)^2 \right\} \qquad (3.25)$$

In order to use the KS transformation (3.17) in a short time integral, it is necessary to define not only the transformation of the j-th coordinate but also the transformation of the j-th interval. The former follows directly from (3.12)

$$x_j^a = \sum_{b=1}^{4} A^{ab}(u_j)u_j^b \quad (a = 1, 2, 3,) \tag{3.26}$$

The later can be obtained from (3.26) by evaluating $\Delta x_j^a = x_j^a - x_{j-1}^a$, it turns out to be

$$\Delta x_j^a = 2\sum_{b=1}^{4} A^{ab}(\bar{u}_j)\,\Delta u_j^b \tag{3.27}$$

where $\bar{u}_j^a = \frac{1}{2}(u_j^a + u_{j-1}^a)$ and $\Delta u_j^b = u_j^b - u_{j-1}^b$.

Obviously, the identity (3.13) holds for the j-th variables

$$0 = \sum_{b=1}^{4} A^{ab}(u_j)u_j^b \quad (a = 4) \tag{3.28}$$

Therefore the transformation (3.20), (3.27) and (3.28) map the three dimensional space onto a four dimensional space. To establish the one to one correspondence between the $\vec{x}$ and the $\vec{u}$ coordinates, one must either impose a constraint or introduce an additional dimension.

In analogy to (3.23)

$$\xi_j = 2\sum_{b=1}^{4} A^{4b}(\bar{u}_j)\Delta u_j^b \tag{3.29}$$

may be set to be vanish. But the constrainted transformation will induce some complicated in path integral. Here it is rather convenient to utilize ξ_j defined by (3.29) as an auxiliary variable. Combining (3.27) with (3.29) yields the one to one mapping

$$\begin{pmatrix} \Delta x_j \\ \Delta y_j \\ \Delta z_j \\ \xi_j \end{pmatrix} = 2 \begin{pmatrix} \bar{u}_j^3 & \bar{u}_j^4 & \bar{u}_j^1 & \bar{u}_j^2 \\ -\bar{u}_j^4 & \bar{u}_j^3 & \bar{u}_j^2 & -\bar{u}_j^1 \\ -\bar{u}_j^1 & -\bar{u}_j^2 & \bar{u}_j^3 & \bar{u}_j^4 \\ -\bar{u}_j^2 & \bar{u}_j^1 & -\bar{u}_j^4 & \bar{u}_j^3 \end{pmatrix} \begin{pmatrix} \Delta u_j^1 \\ \Delta u_j^2 \\ \Delta u_j^3 \\ \Delta u_j^4 \end{pmatrix} \qquad (3.30)$$

which from now on called the modified Kustaanheimo-Stiefel transformation. At this point, it is important to define the mid-point value of r_j by

$$\bar{r}_j = (\bar{u}_j^1)^2 + (\bar{u}_j^2)^2 + (\bar{u}_j^3)^2 + (\bar{u}_j^4)^2 = (\bar{u}_j)^2 \qquad (3.31)$$

With this definition, the transformation matrix $A(\bar{u}_j)$ of (3.27) satisfies the orthogonality condition,

$$\tilde{A}(\bar{u}_j) \cdot A(\bar{u}_j) = \bar{r}_j I \qquad (3.32)$$

From (3.30) and (3.31) , there follows

$$(\Delta \vec{x}_j)^2 + \xi_j^2 = 4\bar{r}_j \{ (u_j^1)^2 + (u_j^2)^2 + (u_j^3)^2 + (u_j^4)^2 \} \qquad (3.33)$$

The Jacobian of transformation can be obtained

$$\left.\frac{\partial(x,y,z,\xi)}{\partial(u^1,u^2,u^3,u^4)}\right|_j = 2^4(\bar{r}_j)^2 \qquad (3.34)$$

3.3 The Lagrangian Path Integral in Four Dimensions

Now the path integral (3.10) should be treated on the KS coordinates, the integral (3.10) is still on the three dimensional basis. In order to apply the modified KS transformation, an extra dimension has to be created. This can be done by inserting into (3.10) the unitary factor

$$\prod_{j=1}^{N}\left[\frac{m}{2\pi i\hbar \bar{r}_j\sigma_j}\right]^{1/2}\int_{-\infty}^{\infty}\exp\left\{\frac{im\,\xi_j^2}{2\hbar\bar{r}_j\sigma_j}\right\}d\xi_j = 1 \qquad (3.35)$$

After inserting (3.35) into (3.10), the path integral (3.10) becomes

$$Q(\vec{x}'',\vec{x}';\sigma) = \lim_{N\to\infty} r''\iint\cdots\int \exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N}S(\sigma_j)\right\}$$

$$\cdot\prod_{j=1}^{N}\left[\frac{m}{2\pi i\hbar\bar{r}_j\sigma_j}\right]^2\prod_{j=1}^{N-1}d^3x_j\,d\xi_j\,d\xi_N \qquad (3.36)$$

with $$S(\sigma_j) = \frac{m}{2\bar{r}_j\sigma_j}\left[(\Delta\vec{x}_j)^2+\xi_j^2\right] + E\,\bar{r}_j\sigma_j \qquad (3.37)$$

Now the path integral (3.36) is completely four dimensional coordinates. Using (3.31) and (3.33) converts the short time action (3.37) into

$$S(\sigma_j) = \frac{4m}{2\sigma_j}(\Delta\vec{u}_j)^2 + E\vec{u}_j^2\sigma_j \qquad (3.38)$$

and the measure of (3.36) into

$$\prod_{j=1}^{N-1}(\bar{r}_j)^{-2}d^3x_j\,d\xi_j = \prod_{j=1}^{N-1}2^4\,d^4u_j \qquad (3.39)$$

Utillizing of (3.38) and (3.39) completes the transformation of (3.36) into

$$Q(\vec{x}'',\vec{x}';\sigma) = 2^{-4}\int_{-\infty}^{\infty}K(\vec{u}'',\vec{u}';\sigma)\frac{1}{r''}d\xi_N \qquad (3.40)$$

where

$$K(\vec{u}'',\vec{u}';\sigma) = \lim_{N\to\infty}\iint\cdots\int \exp\Big\{\frac{i}{\hbar}\sum_{j=1}^{N}\Big[\frac{4m}{2\sigma_j}(\Delta\vec{u}_j)^2 + E\sigma_j\vec{u}_j^2\Big]\Big\}$$

$$\cdot\prod_{j=1}^{N}\left(\frac{4m}{2\pi i\hbar\sigma_j}\right)^2\prod_{j=1}^{N-1}d^4u_j \qquad (3.41)$$

The short time action (3.38) is identical with that of an isotropic oscillator of mass M = 4m and frequency ω = $(-E/2m)^{\frac{1}{2}}$ in four dimensions. Thus the path integral (3.41) represents the propagator of the four dimensional oscillator on σ - evalution. The path integral of (3.41) can be readily performed and the result is

$$K(\vec{u}'',\vec{u}';\sigma) = F^4(\sigma)\exp\{-\pi F^2(\sigma)[(\vec{u}''^2+\vec{u}'^2)\cos(\omega\sigma) - 2\vec{u}''\cdot\vec{u}']\} \qquad (3.42)$$

where

$$F(\sigma) = \left[\frac{M\omega}{2\pi i\hbar \sin(\omega\sigma)}\right]^{1/2} \qquad (3.43)$$

$$M = 4m \qquad (3.44)$$

$$\omega = (-E/2m)^{\frac{1}{2}} \qquad (3.45)$$

From (3.42), (3.40) and (3.9) we can rewrite the integral representation for the Coulomb Green's function in the form

$$G(\vec{x}'',\vec{x}';E) = 2^{-4}\iint e^{ie^2\sigma/\hbar} F^4(\sigma) \exp\left\{-\pi F^2(\sigma)\left[(\vec{u}'^2+\vec{u}''^2)\cos(\omega\sigma) - 2\vec{u}'\cdot\vec{u}''\right]\right\} \frac{d\xi}{r'} {}_N d\sigma \qquad (3.46)$$

These double integrations will be shown in cahpter IV.

3.4 The Hamiltonian Path Integral in Four Dimensions.

Now, consider the three dimensional hydrogen atom problem. As before, the three dimensional Hamiltonian path integral, can be written as the Fourier transform of the propagator

$$G(\vec{x}'',\vec{x}';E) = \int_{s'}^{\infty} r'' e^{\frac{ie^2}{\hbar}(s''-s')} Q(\vec{x}'',\vec{x}';s'',s')\, ds'' \qquad (3.47)$$

where Q($\vec{x}$", $\vec{x}$'; S", S') is the auxiliary propagator given in the phase space path integral form

$$Q(\vec{x}'',\vec{x}';s'',s') = \iint \mathcal{D}\vec{p}\, \mathcal{D}\vec{x} \frac{1}{(2\pi\hbar)^3} \exp\left\{\frac{i}{\hbar}\int_{s'}^{s''} ds\left(\vec{p}\cdot\dot{\vec{x}} - r\frac{\vec{p}^2}{2m} - rE\right)\right\} \qquad (3.48)$$

Again we shall transform the component into a gaussian form. For this purpose we need a generalization of the change of variables to square root coordinates. A transformation of this type is the same type of the preceding section for Lagrangian path integration. For phase space path integral we must transform the mementum variables too.

We now introduce a canonical change of variables from $(\vec{x}, \vec{p})$ to $(\vec{u}, \vec{p}_u)$ such that $\vec{u}^2 = r$

$$x_a = \sum_{b=1}^{4} A(u)_{ab} u_b \qquad (a = 1,2,3) \tag{3.49}$$

$$p_a = \frac{1}{2r} \sum_{b=1}^{4} A(u)_{ab} p_{u_b} \qquad (a = 1,2,3,4) \tag{3.50}$$

with a matrix

$$A(u) = \begin{pmatrix} u_3 & u_4 & u_1 & u_2 \\ -u_2 & -u_1 & u_4 & u_3 \\ -u_1 & u_2 & u_3 & -u_4 \\ u_4 & -u_3 & u_2 & -u_1 \end{pmatrix} \tag{3.51}$$

In fact, due to the four component nature of $\vec{u}$, there exist quite a natural choice also for such a fourth component for $\vec{x}$. Consider the differential change of dx as u proceeding along an arbitrary path (3.49), we find

$$\begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix} = 2 \begin{pmatrix} u_3 & u_1 & u_4 & u_2 \\ -u_2 & -u_1 & u_4 & u_3 \\ -u_1 & u_2 & u_3 & -u_4 \end{pmatrix} \begin{pmatrix} du_1 \\ du_2 \\ du_3 \\ du_4 \end{pmatrix} \tag{3.52}$$

For symmetry reasons this equation calls for a completion by means of a fourth row

$$dx_4 = 2(u_4 \;\; -u_3 \;\; u_2 \;\; -u_1) \begin{pmatrix} du_1 \\ du_2 \\ du_3 \\ du_4 \end{pmatrix} \tag{3.53}$$

This permit a unique definition of the dummy coordinate x_4 as

$$x_4(S) = 2 \int_{S'}^{S} dS(u_4\dot{u}_1 - \dot{u}_2 u_3 + u_2\dot{u}_3 - u_1\dot{u}_4) \tag{3.54}$$

where we have chosen $x_4(S') = 0$ for the initial point of path. We can verify that the transformation $\vec{x}\to\vec{u}$, $\vec{p}\to\vec{P}_u$ is really canonical. With the volume elements in momentum space being related as

$$d^4 P = \frac{1}{16} r^2 d^4 P_u \tag{3.55}$$

and the measure in configuration space

$$d^4x \quad = \quad 16r^2\, d^4u \tag{3.56}$$

so that the measure in phase space remains invariant

$$d^4x\, d^4p \quad = \quad d^4u\, d^4P_u \tag{3.57}$$

Also $\vec{p} \cdot \dot{\vec{x}} + p_4 \dot{x}_4$ go over into $\vec{P}_u \cdot \dot{\vec{u}}$ and

$$\vec{p}^{\,2} + p_4^2 \quad = \quad \frac{1}{4r}\, \vec{P}_u^{\,2} \tag{3.58}$$

Before using these transformations to (3.48) we now inserting a dummy path integral involving a new pair of canonical coordinates x_4 , P_4

$$\int_{-\infty}^{\infty} dx_4'' \iint \frac{1}{2\pi}\, \mathcal{D}x_4\, \mathcal{D}p_4 \exp\left\{ \frac{i}{\hbar} \int_{s'}^{s''} ds \left(p_4 \dot{x}_4 - \frac{r p_4^2}{2m} \right) \right\}$$

$$= \int_{-\infty}^{\infty} dp_4 \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \exp\left\{ \frac{i}{\hbar}(x_4'' - x_4') \right\} dx_4'' \exp\left\{ \frac{i}{\hbar} \frac{p_4^2}{2m} \int_{s'}^{s''} ds\, r(s) \right\}$$

$$= 1 \tag{3.59}$$

into (3.47) . This choice brings the path integral in (3.47) to the four dimensional from

$$Q(\vec{x}'', \vec{x}'; s'', s') = \int_{-\infty}^{\infty} dx_4'' \, \mathcal{D}^4 p\, \mathcal{D}^4 x \frac{1}{(2\pi\hbar)^4} \exp\left\{ \frac{i}{\hbar} \int_{s'}^{s''} ds \left(\vec{p} \cdot \dot{\vec{x}} + p_4 \dot{x}_4 - \frac{r(\vec{p}^{\,2} + p_4^2)}{2m} + Er \right) \right\} \tag{3.60}$$

We now change the variables from $(\vec{x}, \vec{P})$ to $(\vec{u}, \vec{P}_u)$ and notice that

$$\mathcal{D}^4_{\vec{P}} \mathcal{D}^4 x = \frac{2^{-4}}{(r'')^2} \mathcal{D}^4 P_u \mathcal{D}^4 \vec{u} \qquad (3.61)$$

Then we have (3.60) in the form

$$Q(\vec{u}'',\vec{u}';s'',s') = \int_{-\infty}^{\infty} \frac{1}{16(r'')^2} dx_4'' \frac{1}{(2\pi\hbar)^4} \iint \mathcal{D}^4 P_u \mathcal{D}^4 \vec{u} \exp\left\{\frac{i}{\hbar}\int_{s'}^{s''} ds\left(\vec{P}_u\cdot\dot{\vec{u}} - \frac{\vec{P}_u^2}{8m} + E\vec{u}^2\right)\right\} \qquad (3.62)$$

A part from $\int_{-\infty}^{\infty} \frac{1}{16(r'')^2} dx_4''$ this is the propagator of an harmonic oscillator in four dimensions with mass $M = 4m$ and frequency $\omega = \left(-\frac{E}{2m}\right)^{1/2}$. The result is well-known as

$$K^H(\vec{u}'',\vec{u}';s'',s) = F^4(s) \exp\left\{-\pi F^2(s)\left(\cos(\omega s)(\vec{u}''^2 + \vec{u}'^2) - \vec{u}'\cdot\vec{u}''\right)\right\} \qquad (3.63)$$

where
$$F(s) = \left(\frac{M\omega}{2\pi i\hbar \sin(\omega s)}\right)^{1/2} \qquad (3.64)$$

and
$$S = S'' - S' \qquad (3.65)$$

Then the Green's function of the hydrogen atom can be written as

$$G(\vec{x}'',\vec{x}';E) = \int_0^{\infty} ds\, e^{\frac{i e^2 s}{\hbar}} \int_{-\infty}^{\infty} \frac{1}{16(r'')^2} dx_4''\, F^4(s) \exp\left\{-\pi F^2(s)\left(\cos(\omega s)(\vec{u}''^2 + \vec{u}'^2) - 2\vec{u}'\cdot\vec{u}''\right)\right\} \qquad (3.66)$$

which equivalent to (3.46) if we set $x_4'' = \xi_N$, $S = \sigma$.

The next chapter we have tried to carry out this double integration to get the important information about the energy spectrum from the Green's function $G(\vec{x}", \vec{x}'; E)$.