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STRUCTURE OF SOME SEMIHYPERRINGS OF LINEAR TRANSFORMATIONS

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วิศรุต โพธิ์อ้น : โครงสร้างของกึ่งไฮเพอร์ริงของการแปลงเชิงเส้นบางชนิด (STRUCTURE OF SOME SEMIHYPERRINGS OF LINEAR TRANSFORMATIONS) อ. ที่ปรึกษาวิทยานิพนธ์หลัก : ผศ. ดร. สุรีย์พร ชาวแพรกน้อย, 50 หน้า

สำหรับกึ่งกรุป S นิยาม S^0 เป็น S ถ้า S มีศูนย์และ S มีสมาชิกมากกว่าหนึ่งตัว มิเช่นนั้นกำหนดให้ S^0 คือกึ่งกรุป S ที่ผนวกด้วยศูนย์ 0 เรากล่าวว่ากึ่งกรุป S *ให้โครงสร้างของกึ่งไฮเพอร์ริงที่มีศูนย์* ถ้ามีการดำเนินการไฮเพอร์ + บน S^0 ที่ทำให้ $(S^0,+,\cdot)$ เป็นกึ่งไฮเพอร์ริงที่มีศูนย์ โดยที่ $\cdot$ เป็นการดำเนินการบน S^0 เรานิยาม *กึ่งกรุปที่ให้โครงสร้างของกึ่งริงสลับที่ภายใต้การบวก[AC]ที่มีศูนย์* ในทำนองเดียวกัน

ให้ V เป็นปริภูมิเวกเตอร์บนริงการหาร R, W เป็นปริภูมิย่อยของ V, $L_R(V,W)$ เป็นกึ่งกรุปภายใต้การประกอบที่ประกอบด้วยการแปลงเชิงเส้น $\alpha : V \to W$ ทั้งหมด และ $PL_R(V,W)$ เป็น*กึ่งกรุปการแปลงเชิงเส้นบางส่วนจาก V ไปยัง W*, กึ่งกรุปภายใต้การประกอบที่ประกอบด้วยการแปลงเชิงเส้นจากปริภูมิย่อยของ V ไปยัง W ทั้งหมด ถ้ากึ่งกรุปย่อยของ $L_R(V,W)$ มีศูนย์ เราศึกษาว่าเมื่อใดกึ่งกรุปเหล่านี้ให้โครงสร้างของกึ่งไฮเพอร์ริงที่มีศูนย์ มิเช่นนั้นเราให้ลักษณะว่าเมื่อใดกึ่งกรุปเหล่านี้ให้โครงสร้างของกึ่งริง AC ที่มีศูนย์ นอกจากนั้น เราให้เงื่อนไขจำเป็นที่ทำให้ $PL_R(V,W)$ ให้โครงสร้างของกึ่งริง AC ที่มีศูนย์

ภาควิชา............คณิตศาสตร์.......... ลายมือชื่อนิสิต วิศรุต โพธิ์อ้น

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For a semigroup S, the semigroup S^0 is defined to be S if S has a zero and S contains more than one element, otherwise, let S^0 be S with a zero 0 adjoined. We say that a semigroup S *admits the structure of a semihyperring with zero* if there exists a hyperoperation $+$ on S^0 such that $(S^0, +, \cdot)$ is a semihyperring with zero where $\cdot$ is the operation on S^0. Semigroups *admitting the structure of an additively commutative* [*AC*] *semiring with zero* are defined analogously.

Let V be a vector space over a division ring R, W a subspace of V, $L_R(V,W)$ the semigroup under the composition of all linear transformations $\alpha : V \to W$, and $PL_R(V,W)$ the *partial linear transformation semigroup from* V *into* W, the semigroup under the composition of all linear transformations from a subspace of V into W. If subsemigroups of $L_R(V,W)$ contain a zero, we determine when they admit the structure of a semihyperring with zero. Otherwise, we characterize when they admit the structure of an AC semiring with zero. Moreover, necessary conditions for $PL_R(V,W)$ to admit the structure of an AC semiring with zero are given.

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CONTENTS

CHAPTER I

INTRODUCTION

The multiplicative structure of a ring is given by definition a semigroup with zero. However, ring theory is a classical subject in mathematics and had been widely studied before semigroup theory was considered. Because the multiplicative structure of a ring is a semigroup with zero, it is reasonable to ask which semigroups joining zero are isomorphic to the multiplicative structure of some ring. If they do, they are said to *admit a ring structure.* In 1970, Peinado R.E. [10] gave a brief survey of semigroups admitting ring structure. Chu D.D. and Shyr H.I. [5] proved a nice result that the multiplicative semigroup $\mathbb{N}$ of natural numbers admits a ring structure. For various studies in this area, see [12] and [13].

On the other hand, the hyperstructure theory was first known in 1934 by Marty F. He gave the definition of a hypergroup as a generalization of a group. Ten years after that, Krasner hyperrings were introduced as a nice generalization of rings by Krasner M. By the definition of Krasner hyperrings, their multiplicative structures are also semigroups with zero. Semigroups admitting hyperring structure have been defined in the same way. Besides that, semigroups admitting other algebraic structures of a semigroup have been defined and studied. Many researchers from many places have developed this area. The linear transformation semigroup is one type of semigroups that have been developed and studied whether they admit some kinds of algebraic structures. We can see in [1], [2], [3], [4], [9], [11] and [14]. The work on linear transformation semigroups inspired us to investigate some specific linear transformation semigroups. The semigroups we considered are adopted from Kemprasit Y. and Chaopraknoi S. in [1], [2], [3] and [4]. They studied linear transformation semigroups from a vector space into itself. Here, we generalize to linear transformation semigroups from a vector space into its subspace. We then seperate the generalized linear transformation semigroups into two groups. The first group is linear transformation semigroups containing a zero. We shall detemine whether or when they admit the structure of a semihyperring with zero. The other group is linear transformation semigroups without zero which always admit the structure of a semiring with zero. However, they need not admit the structure of an additively commutative(AC) semiring with zero. Our purpose for semigroups in this group is to characterize whether or when they admit the structure of an additively commutative semiring with zero.

The next chapter will give precise definitions, notations and basic knowledges which will be used throughout this thesis and also give short brief for Chapter III and Chapter IV.

CHAPTER II

PRELIMINARIES

2.1 Basic definitions and examples

For any set X, let $P(X)$ denote the power set of X, $P^*(X) = P(X) \smallsetminus \{\varnothing\}$ and the notation $|X|$ means the cardinality of X.

A *hyperoperation* on a nonempty set H is a mapping of $H \times H$ into $P^*(H)$. A *hypergroupoid* is a system $(H, \circ)$ consisting of a nonempty set H and a hyperoperation $\circ$ on H. Let $(H, \circ)$ be a hypergroupoid. For nonempty subsets A and B of H, let

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} (a \circ b),$$

$A \circ x = A \circ \{x\}$ and $x \circ A = \{x\} \circ A$ for all $x \in H$. We call $(H, \circ)$ a *commutative hypergroupoid* if and only if $x \circ y = y \circ x$ for all $x, y \in H$. An element e of H is called an *identity* of $(H, \circ)$ if $x \in (x \circ e) \cap (e \circ x)$ for all $x \in H$. An identity e of $(H, \circ)$ is called the *scalar identity* if $(x \circ e) \cap (e \circ x) = \{x\}$ for all $x \in H$. Then H has at most one scalar identity.

A *semihypergroup* is a hypergroupoid $(H, \circ)$ such that $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in H$, that is,

$$\bigcup_{t \in x \circ y} t \circ z = \bigcup_{t \in y \circ z} x \circ t \quad \text{for all } \; x, y, z \in H.$$

A *hypergroup* is a semihypergroup $(H, \circ)$ such that $H \circ x = x \circ H = H$ for all $x \in H$. For x, y in a hypergroup $(H, \circ)$, x is called an *inverse* of y if there exists an identity e of H such that $e \in (x \circ y) \cap (y \circ x)$. A hypergroup $(H, \circ)$ is called *regular* if every element of H has an inverse in H. A regular hypergroup $(H, \circ)$ is said to be *reversible* if for $x, y, z \in H, x \in y \circ z$ implies $z \in u \circ x$ and $y \in x \circ v$ for some inverse u of y and some inverse v of z.

A *canonical hypergroup* is a hypergroup $(H, \circ)$ such that

(i) $(H, \circ)$ is commutative,

(ii) $(H, \circ)$ has the scalar identity,

(iii) every element of H has a unique inverse in H and

(iv) $(H, \circ)$ is reversible.

A triple $(A, +, \cdot)$ is called a *semihyperring* [*semiring*] if

(i) $(A, +)$ is a semihypergroup [semigroup],

(ii) $(A, \cdot)$ is a semigroup and

(iii) the operation $\cdot$ is distributive over the hyperoperation [operation] $+$.

A semihyperring [semiring] $(A, +, \cdot)$ is said to be *additively commutative* if $x+y = y+x$ for all $x, y \in A$. For this case, we call $(A, +, \cdot)$ an *AC semihyperring* [*AC semiring*]. An element 0 of a semihyperring [semiring] $(A, +, \cdot)$ is called a *zero* of $(A, +, \cdot)$ if $x + 0 = 0 + x = \{x\}$ $[x]$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in A$.

A (*Krasner*) *hyperring* is a system $(A, +, \cdot)$ where

(i) $(A, +)$ is a canonical hypergroup,

(ii) $(A, \cdot)$ is a semigroup with zero 0 where 0 is the scalar identity of $(A, +)$ and

(iii) the operation $\cdot$ is distributive over the hyperoperation $+$.

We can see by the definitions that every ring is a hyperring and every hyperring and every AC semiring with zero are AC semihyperrings with zero.

For a semigroup $(S, \cdot)$, the semigroup S^0 is defined to be S if S has a zero and S contains more than one element, otherwise, let S^0 be the semigroup S with a zero 0 adjoined, that is, $S^0 = (S \cup \{0\}, \circ)$ where $0 \notin S, 0 \circ x = x \circ 0 = 0$ for all $x \in S \cup \{0\}$ and $x \circ y = x \cdot y$ for all $x, y \in S$. Note that if $|S| = 1$, then

S^0 is a semigroup of two elements and $S^0 \cong (\mathbb{Z}_2, \cdot)$. Also, if G is a group, then $G^0 = (G \cup \{0\}, \circ)$ is defined as above.

Example 2.1.1. ([6] and [11]) Let G be a group. Define a hyperoperation $+$ on G^0 by

$$
\begin{aligned}
&x + 0 = 0 + x = \{x\} && \text{for all } x \in G^0, \\
&x + x = G^0 \setminus \{x\} && \text{for all } x \in G, \\
&x + y = \{x, y\} && \text{for all distinct } x, y \in G.
\end{aligned}
$$

Then $(G^0, +, \cdot)$ is a hyperring where $\cdot$ is the operation on G^0. Note that the zero of the hyperring $(G^0, +, \cdot)$ is 0 and the inverse of $x \in G$ in $(G^0, +)$ is x itself. Also, $(G^0, +, \cdot)$ is not a ring if $|G| > 1$.

Example 2.1.2. ([6]) Let A be a set with $|A| > 2$ such that 0 is an element of A. Define a hyperoperation $+$ and an operation $\cdot$ on A by

$$
\begin{aligned}
&x + 0 = 0 + x = \{x\} && \text{for all } x \in A, \\
&x + y = A && \text{for all } x, y \in A \setminus \{0\}, \\
&x \cdot y = 0 && \text{for all } x, y \in A.
\end{aligned}
$$

Then $(A, +, \cdot)$ is an AC semihyperring with zero 0 but it is neither a hyperring nor semiring with zero.

From Example 2.1.1 and Example 2.1.2, we see that hyperrings generalize rings and semihyperrings with zero generalize both semirings with zero and hyperrings.

A semigroup S is said to *admit a ring* [*hyperring*] *structure* if $(S^0, +, \cdot)$ is a ring [hyperring] for some operation [hyperoperation] $+$ on S^0 where $\cdot$ is the operation on S^0. Semigroups *admitting the structure of an AC semihyperring* [*AC semiring*] *with zero* are defined analogously. Observe that if S is a trivial semigroup, then

$S^0 \cong (\mathbb{Z}_2, \cdot)$ where $\cdot$ is the multiplication on $\mathbb{Z}_2$, so S admits a ring structure. Also, every semigroup without zero admits the structure of a semiring with zero as shown.

Example 2.1.3. Let S be a semigroup without zero. Define an operation $+$ on S^0 by

$$x + 0 = 0 + x = x \qquad \text{if } x \in S^0,$$

$$x + y = x \qquad \text{if } x, y \in S.$$

Then $(S^0, +)$ is obviously a semigroup having 0 as its identity. Since $xy \neq 0$ for all $x, y \in S$, we deduce that the multiplication $\cdot$ of S^0 distributes over the operation $+$. Hence $(S^0, +, \cdot)$ is a semiring with zero, but it is not additively commutative if $|S| > 1$.

2.2 Basic propositions and notations

For a vector space V over a division ring R, let

$$L_R(V) = \{\alpha : V \to V \mid \alpha \text{ is a linear transformation}\},$$

$$G_R(V) = \{\alpha \in L_R(V) \mid \alpha \text{ is an isomorphism}\}.$$

Then $L_R(V)$ is a semigroup under the composition of all linear transformations and $G_R(V)$ is the unit group of $L_R(V)$. Moreover, $L_R(V)$ admits a ring structure under the usual addition of linear transformation. The image of v under $\alpha \in L_R(V)$ is written by $v\alpha$. For $\alpha \in L_R(V)$, let Ker α, Dom α and Im α denote the kernel, the domain and the image of α, respectively. If α is a function or linear transformation, the notation $-\alpha$ denotes the inverse under the usual addition and α^{-1} denotes the inverse under a composition if they exist. For $A \subseteq V$, let $\langle A \rangle$ stand for the subspace of V spanned by A. The following three propositions are simple facts of vector spaces and linear transformations which will be used. The proofs are routine and elementary and they will be omitted.

Proposition 2.2.1. *Let* B *be a basis of* V. *If* u *and* w *are distinct elements of* B, *then* $\{u+w\} \cup (B \setminus \{w\})$ *is also a basis of* V.

Proposition 2.2.2. *Let* B *be a basis of* $V, A \subseteq B$ *and* $\varphi : B \setminus A \to V$ *a one-to-one map such that* $(B \setminus A)\varphi$ *is a linearly independent subset of* V. *If* $\alpha \in L_R(V)$ *is defined by*

$$v\alpha = \begin{cases} 0 & \text{if } v \in A, \\ v\varphi & \text{if } v \in B \setminus A, \end{cases}$$

then Ker $\alpha = \langle A \rangle$ *and Im* $\alpha = \langle B \setminus A \rangle \varphi$.

Proposition 2.2.3. *Let* B *be a basis of* V *and* $A \subseteq B$. *Then*

(i) $\{v + \langle A \rangle \mid v \in B \setminus A\}$ *is a basis of the quotient space* $V/\langle A \rangle$ *and*

(ii) $\dim_R(V/\langle A \rangle) = |B \setminus A|$.

In this thesis, let V be a vector space over a division ring R, W a subspace of V and $L_R(V,W)$ the semigroup under the composition of all linear transformations $\alpha : V \to W$. We can see that $L_R(V,W) \subseteq L_R(V)$. Moreover, $L_R(V,W)$ admits a ring structure under the usual addition of linear transformations. For $\alpha \in L_R(V,W)$, the notation $\alpha_{|W}$ is a linear transformation in $L_R(W)$ such that for every $w \in W$, $\alpha_{|W}$ maps w into $w\alpha$. Moreover, we have

Proposition 2.2.4. *If* $\alpha, \beta \in L_R(V,W)$, *then* $\alpha_{|W}\beta_{|W} = (\alpha\beta)_{|W}$

Since our works relate to cardinal numbers, some facts and notations about cardinal numbers will be used. Let k be a cardinal number. We denote k' be the successor of k. If k is a finite cardinal number, then $k' = k+1$. For a set X, if $T \subseteq X$, we then have $|X| = |T| + |X \smallsetminus T|$.

Proposition 2.2.5. ([7] page 145) *For any cardinal numbers κ and λ such that at least one of them is an infinite cardinal number, $\kappa + \lambda = \max\{\kappa, \lambda\}$.*

Proposition 2.2.6. *Assume that $\dim_R V$ is infinite and $\dim_R V > \dim_R W$. Then $\dim_R(V/W) = \dim_R V$.*

Proof. Let B be a basis of W and B' a basis of V extended from B. By Proposition 2.2.3,

$$|B' \smallsetminus B| = \dim_R(\langle B' \rangle \,/\, \langle B \rangle) = \dim_R(V/W).$$

Since $\dim_R V$ is infinite and $\dim_R V = |B'| = |B| + |B' \smallsetminus B|$, at least one of $|B|$ and $|B' \smallsetminus B|$ is an infinite cardinal number. By Proposition 2.2.5, $\dim_R V = \max\{|B|, |B' \smallsetminus B|\}$. Since $\dim_R V > \dim_R W$, we have $\dim_R(V/W) = \dim_R V$. □

Since every linear transformation can be defined on its basis, for convenience, we may write $\alpha \in L_R(V)$ by using a blanket notation as follows,

$$\alpha = \begin{pmatrix} B_1 & u & w & v \\ 0 & w & u & v \end{pmatrix}_{v \in B \smallsetminus (B_1 \cup \{u,w\})}$$

means that α is a linear transformation on a vector space V having B as a basis,

$B_1 \subseteq B$, u and w are distinct elements of $B \setminus B_1$ and

$$v\alpha = \begin{cases} 0 & \text{if } v \in B_1, \\ w & \text{if } v = u, \\ u & \text{if } v = w, \\ v & \text{if } v \in B \setminus (B_1 \cup \{u, w\}), \end{cases}$$

if $B_1 = \varnothing$, then

$$v\alpha = \begin{cases} w & \text{if } v = u, \\ u & \text{if } v = w, \\ v & \text{if } v \in B \setminus \{u, w\}. \end{cases}$$

For any cardinal number k with $k \leq \dim_R V$, let

$$\begin{aligned} K_R((V,W),k) &= \{\alpha \in L_R(V,W) \mid \dim_R \text{Ker } \alpha \geq k\}, \\ CI_R((V,W),k) &= \{\alpha \in L_R(V,W) \mid \dim_R(V/\text{Im } \alpha) \geq k\}, \\ I_R((V,W),k) &= \{\alpha \in L_R(V,W) \mid \dim_R \text{Im } \alpha \leq k\} \text{ where } k \leq \dim_R W. \end{aligned}$$

Then the zero map on V or we may write V_0 belongs to all of the above three subsets of $L_R(V,W)$. Since for $\alpha, \beta \in L_R(V,W)$, Ker $\alpha\beta \supseteq$ Ker α and Im $\alpha\beta \subseteq$ Im β, we conclude that all of $K_R((V,W),k), CI_R((V,W),k)$ and $I_R((V,W),k)$ are subsemigroups of $L_R(V,W)$. Moreover, their zero element is the zero map. If $V = W$, then we denote $K_R((V,W),k), CI_R((V,W),k)$ and $I_R((V,W),k)$ by $K_R(V,k), CI_R(V,k)$ and $I_R(V,k)$, respectively. We know that if $\dim_R V$ is finite, then for $\alpha \in L_R(V)$,

$$\dim_R \text{Ker } \alpha = \dim_R(V/\text{Im } \alpha) = \dim_R V - \dim_R \text{Im } \alpha.$$

Since $L_R(V,W) \subseteq L_R(V)$, we have

Proposition 2.2.7. *If* $\dim_R V$ *is finite and* k *is a cardinal number such that* $k \leq \dim_R V$*, then the following statements hold.*

(i) $K_R((V,W),k) = CI_R((V,W),k)$.

(ii) $K_R((V,W),k) = CI_R((V,W),k) = I_R((V,W), \dim_R V - k)$ *if* $\dim_R V - k \leq \dim_R W$.

However, these are not generally true if $\dim_R W$ is infinite. The following proposition also shows that the semigroups $K_R((V,W),k), CI_R((V,W),k)$ and $I_R((V,W),k)$ should be considered independently if $\dim_R W$ is infinite.

Proposition 2.2.8. *If* $\dim_R W$ *is infinite and* k *is a cardinal number with* $k \leq \dim_R V$, *then the following statements hold.*

(i) $CI_R((V,W),l) \neq K_R((V,W),k)$ *for every cardinal number* l *with* $\dim_R(V/W) < l \leq \dim_R V$.

(ii) $I_R((V,W),l) \neq K_R((V,W),k)$ *and* $I_R((V,W),l) \neq CI_R((V,W),k)$ *for every cardinal number* l *with* $l < \dim_R W$.

Proof. Let B be a basis of W and B' a basis of V extended from B. Since $\dim_R W$ is infinite, we can let B_1 and B_2 be disjoint subsets of B such that $|B_1| = |B_2| = |B|$ and $B_1 \cup B_2 = B$. Then there exists a bijection $\varphi : B_1 \to B$. Define $\alpha, \beta \in L_R(V,W)$ by

$$\alpha = \begin{pmatrix} B' \setminus B_1 & v \\ 0 & v\varphi \end{pmatrix}_{v \in B_1} \text{ and } \beta = \begin{pmatrix} B' \setminus B & v \\ 0 & v\varphi^{-1} \end{pmatrix}_{v \in B}.$$

Then by Proposition 2.2.2, Ker $\alpha = \langle B' \setminus B_1 \rangle \supseteq \langle B_2 \rangle$. First, we will show that $|B' \setminus B_1| = \dim_R V$.

Case 1: $\dim_R V = \dim_R W$. Then $\dim_R W = |B_2| \leq |B' \setminus B_1| \leq \dim_R V = \dim_R W$.

Case 2: $\dim_R V > \dim_R W$. Since $|B'| = |B' \smallsetminus B_1| + |B_1|$ and $|B_1|$ is infinite, by Proposition 2.2.5, $|B'| = \max\{|B' \smallsetminus B_1|, |B_1|\}$. By assumption, $|B_1| = \dim_R W < \dim_R V = |B'|$, this implies that $|B' \smallsetminus B_1| = |B'| = \dim_R V$.

Since $\dim_R \mathrm{Ker}\ \alpha = |B' \smallsetminus B_1| = \dim_R V$, $\alpha \in K_R((V,W),k)$. We also have $\dim_R(V/\mathrm{Im}\ \alpha) = \dim_R(V/W)$. This means $\alpha \notin CI_R((V,W),l)$ for every cardinal number l with $\dim_R(V/W) < l \leq \dim_R V$, so (i) is proved.

By Proposition 2.2.2 and Proposition 2.2.3, $\dim_R(V/\mathrm{Im}\ \beta) = |B' \smallsetminus B_1| = \dim_R V$, so $\beta \in CI_R((V,W),k)$. It is obvious from the definitions that $\dim_R \mathrm{Im}\ \alpha = \dim_R W = \dim_R \mathrm{Im}\ \beta$. We therefore have $\alpha \in K_R((V,W),k) \smallsetminus I_R((V,W),l)$ and $\beta \in CI_R((V,W),k) \smallsetminus I_R((V,W),l)$ for every cardinal number $l < \dim_R W$. Hence (ii) is proved. □

For a cardinal number $k < \dim_R V$, we define $K'_R((V,W),k), CI'_R((V,W),k)$ and $I'_R((V,W),k)$ which are subsets of $K_R((V,W),k)$, $CI_R((V,W),k)$ and $I_R((V,W),k)$ respectively, as follow :

$$K'_R((V,W),k) = \{\alpha \in L_R(V,W) \mid \dim_R \mathrm{Ker}\ \alpha > k\},$$

$$CI'_R((V,W),k) = \{\alpha \in L_R(V,W) \mid \dim_R(V/\mathrm{Im}\ \alpha) > k\},$$

$$I'_R((V,W),k) = \{\alpha \in L_R(V,W) \mid \dim_R \mathrm{Im}\ \alpha < k\} \text{ where } 0 < k \leq \dim_R W.$$

It is easy to prove that they are respectively subsemigroups of $K_R((V,W),k)$, $CI_R((V,W),k)$ and $I_R((V,W),k)$. All of them contain the zero map which is the zero element. Observe that if $k < \dim_R V$, then $K'_R((V,W),k) = K_R((V,W),k')$ and $CI'_R((V,W),k) = CI_R((V,W),k')$ where k' is the successor of k. Also, if $0 < k \leq \dim_R W$, k is a finite cardinal number and $\tilde{k}$ is the predecessor of k, then $I'_R((V,W),k) = I_R((V,W),\tilde{k})$. Similar to the previous semigroups, we let $K'_R(V,k), CI'_R(V,k)$ and $I'_R(V,k)$ denote $K'_R((V,W),k)$, $CI'_R((V,W),k)$ and $I'_R((V,W),k)$ when $V = W$.

For $\alpha \in L_R(V)$, let $F(\alpha) = \{v \in V \mid v\alpha = v\}$. It is easy to see that $F(\alpha)$ is a subspace of V. If $\alpha \in L_R(V,W)$, then $F(\alpha) \subseteq W$ and $F(\alpha)$ is also a subspace of W. Define

$$AM_R(V,W) = \{\alpha \in L_R(V,W) \mid \dim_R \operatorname{Ker} \alpha_{|W} < \infty\},$$

$$AE_R(V,W) = \{\alpha \in L_R(V,W) \mid \dim_R(W/(\operatorname{Im} \alpha_{|W})) < \infty\},$$

$$AI_R(V,W) = \{\alpha \in L_R(V,W) \mid \dim_R(W/F(\alpha)) < \infty\}.$$

If $V = W$, we let $AM_R(V) = AM_R(V,W)$, $AE_R(V) = AE_R(V,W)$ and $AI_R(V) = AI_R(V,W)$.

To show that $AM_R(V,W)$ and $AE_R(V,W)$ are subsemigroups of $L_R(V,W)$, the following facts given in [14] will be used. For all $\alpha, \beta \in L_R(W)$,

$$\dim_R \operatorname{Ker} \alpha\beta \leq \dim_R \operatorname{Ker} \alpha + \dim_R \operatorname{Ker} \beta,$$

$$\dim_R(W/\operatorname{Im} \alpha\beta) \leq \dim_R(W/\operatorname{Im} \alpha) + \dim_R(W/\operatorname{Im} \beta).$$

By Proposition 2.2.4, for all $\alpha, \beta \in L_R(V,W)$, $\alpha_{|W}\beta_{|W} = (\alpha\beta)_{|W}$ and $\alpha_{|W}, \beta_{|W} \in L_R(W)$, so we obtain that

$$\dim_R \operatorname{Ker} (\alpha\beta)_{|W} \leq \dim_R \operatorname{Ker} \alpha_{|W} + \dim_R \operatorname{Ker} \beta_{|W},$$

$$\dim_R(W/\operatorname{Im} (\alpha\beta)_{|W}) \leq \dim_R(W/\operatorname{Im} \alpha_{|W}) + \dim_R(W/\operatorname{Im} \beta_{|W}).$$

Hence both $AM_R(V,W)$ and $AE_R(V,W)$ are subsemigroups of $L_R(V,W)$.

Next, we will show that $AI_R(V,W)$ is also a subsemigroup of $L_R(V,W)$. Let $\alpha, \beta \in AI_R(V,W)$. Then $\dim_R(W/F(\alpha))$ and $\dim_R(W/F(\beta))$ are finite. Since $F(\alpha) \cap F(\beta) \subseteq F(\alpha\beta)$, it suffices to show that $\dim_R(W/(F(\alpha) \cap F(\beta)))$ is finite. Let B_1 be a basis of $F(\alpha) \cap F(\beta)$ and let $B_2 \subseteq F(\alpha) \smallsetminus B_1$ and $B_3 \subseteq F(\beta) \smallsetminus B_1$ be such that $B_1 \cup B_2$ and $B_1 \cup B_3$ are bases of $F(\alpha)$ and $F(\beta)$, respectively. To show that $B_1 \cup B_2 \cup B_3$ is linearly independent over R, let $u_1, u_2, \ldots, u_k \in B_1 \cup B_2$

and $v_1, v_2, \ldots, v_l \in B_3$ be distinct such that

$$\sum_{i=1}^{k} a_i u_i + \sum_{i=1}^{l} b_i v_i = 0$$

for some $a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_l \in R$. Then $\displaystyle\sum_{i=1}^{k} a_i u_i = -\sum_{i=1}^{l} b_i v_i \in F(\alpha) \cap F(\beta) = \langle B_1 \rangle$. Since $B_1 \cup B_3$ is linearly independent $b_i = 0$ for all $i = 1, 2, \ldots, l$, $\displaystyle\sum_{i=1}^{k} a_i u_i = 0$. This implies that $a_i = 0$ for all $i = 1, 2, \ldots, k$. Hence $B_1 \cup B_2 \cup B_3$ is linearly independent over R. Let $B_4 \subseteq W \smallsetminus (B_1 \cup B_2 \cup B_3)$ be such that $B_1 \cup B_2 \cup B_3 \cup B_4$ is a basis of W. It is easy to see that $\{\ v + F(\alpha) \mid v \in B_3 \cup B_4\ \}$ and $\{\ v + F(\beta) \mid v \in B_2 \cup B_4\ \}$ are bases of $W/F(\alpha)$ and $W/F(\beta)$, respectively. Since $\dim_R(W/F(\alpha))$ and $\dim_R(W/F(\beta))$ are finite, so do $|B_3 \cup B_4|$ and $|B_2 \cup B_4|$. Therefore $|B_2 \cup B_3 \cup B_4|$ is finite. Also, we can show that $\{\ v + (F(\alpha) \cap F(\beta)) \mid v \in B_2 \cup B_3 \cup B_4\ \}$ is a basis of $W/(F(\alpha) \cap F(\beta))$ which implies that $\dim_R(W/(F(\alpha) \cap F(\beta)))$ is finite.

Note that, if $\dim_R W$ is finite, then $AM_R(V, W) = AE_R(V, W) = AI_R(V, W) = L_R(V, W)$ which has the zero element. It follows that $AM_R(V, W), AE_R(V, W)$ and $AI_R(V, W)$ admit a ring structure when $\dim_R W$ is finite.

Proposition 2.2.9. *$AM_R(V, W)$, $AE_R(V, W)$ and $AI_R(V, W)$ have no zero element if and only if $\dim_R W$ is infinite.*

Proof. It remains to show the converse. Let $S(V, W)$ be one of the semigroups $AM_R(V, W)$, $AE_R(V, W)$ and $AI_R(V, W)$. Assume that $\dim_R W$ is infinite, B is a basis of W and B' is a basis of V extended from B. For each $u \in B$, we define $\alpha_u \in L_R(V, W)$ by

$$\alpha_u = \begin{pmatrix} (B' \smallsetminus B) \cup \{u\} & v \\ 0 & v \end{pmatrix}_{v \in B \smallsetminus \{u\}}.$$

Then $\alpha_u \in S(V,W)$ for every $u \in B$. If $\gamma \in L_R(V,W)$ is such that $\beta\gamma = \gamma$ for every $\beta \in S(V,W)$, then for every $u \in B$, $u\gamma = u(\alpha_u\gamma) = 0$, so $\gamma_{|W} = 0$. Thus $\gamma \notin S(V,W)$. This implies that $S(V,W)$ has no zero if and only if $\dim_R W$ is infinite. □

Moreover, when $\dim_R W$ is infinite we have that $AM_R(V,W)$, $AE_R(V,W)$ and $AI_R(V,W)$ are distinct semigroups.

Proposition 2.2.10. *If* $\dim_R W$ *is infinite, then* $AM_R(V,W) \neq AE_R(V,W)$, $AM_R(V,W) \neq AI_R(V,W)$ *and* $AE_R(V,W) \neq AI_R(V,W)$.

Proof. Let B be a basis of W and B' a basis of V extended from B. Since $\dim_R W$ is infinite, we can let B_1 and B_2 be disjoint subsets of B such that $|B_1| = |B_2| = |B|$ and $B_1 \cup B_2 = B$. Then there exists a bijection $\varphi : B_1 \to B$. Define $\alpha, \beta \in L_R(V,W)$ by

$$\alpha = \begin{pmatrix} B' \setminus B_1 & v \\ 0 & v\varphi \end{pmatrix}_{v \in B_1} \quad \text{and } \beta = \begin{pmatrix} B' \setminus B & v \\ 0 & v\varphi^{-1} \end{pmatrix}_{v \in B}.$$

By Proposition 2.2.2 and Proposition 2.2.3, $\dim_R(W/\text{Im } \alpha_{|W}) = \dim_R(W/\langle B \rangle) = 0$, so $\alpha \in AE_R(V,W)$. By such propositions, Ker $\alpha_{|W} = \langle B_2 \rangle$. Hence $\alpha \in AE_R(V,W) \setminus AM_R(V,W)$. Then we will show that $\alpha \notin AI_R(V,W)$. For each $v \in B_2, v \notin F(\alpha)$, this implies that $F(\alpha) \notin \{v + F(\alpha) \mid v \in B_2\} \subseteq W/F(\alpha)$. Since B_2 is linearly independent, so does $\{v + F(\alpha) \mid v \in B_2\}$ and we also have $|B_2| = |\{v + F(\alpha) \mid v \in B_2\}| \leq \dim_R(W/F(\alpha))$. Thus $\alpha \in AE_R(V,W) \setminus AI_R(V,W)$.

Next, we can see that Ker $\beta_{|W} = \{0\}$, so $\beta \in AM_R(V,W)$. By Proposition 2.2.2, Im $\beta_{|W} = \langle B_1 \rangle$. We can conclude that for each $v \in B_2, v \notin F(\beta)$. By the previous proof, we can show that $\beta \notin AI_R(V,W)$. Therefore $\beta \in AM_R(V,W) \setminus AI_R(V,W)$.

Then the proof is complete. □

By a *partial linear transformation of* V *into* W, we mean a linear transformation from a subspace of V into W. Let $PL_R(V,W)$ be the set of all partial transformations of V into W, that is

$$PL_R(V,W) = \{\alpha : U \to W \mid U \text{ is a subspace of } V \text{ and } \alpha \text{ is a linear transformation } \}.$$

Then $PL_R(V,W)$ is a semigroup under the composition of linear transformations, since for $\alpha, \beta \in PL_R(V,W)$,

$$\text{Dom } \alpha\beta = \{v \in \text{ Dom } \alpha \mid v\alpha \in \text{ Dom } \beta\},$$

$$v(\alpha\beta) = (v\alpha)\beta \text{ for all } v \in \text{ Dom } \alpha\beta.$$

In addition, the notation $PL_R(V)$ means $PL_R(V,W)$ if $V = W$.

In this thesis, elements of $PL_R(V,W)$ are usually written from linearly independent vectors. Then we denote, for linearly independent vectors $v_1, v_2, \ldots, v_n$ in V and vectors $w_1, w_2, \ldots, w_n$ in W, the notation

$$\alpha = \begin{pmatrix} v_1 & v_2 & \ldots & v_n \\ w_1 & w_2 & \ldots & w_n \end{pmatrix}$$

means the linear transformation α from the subspace $\langle v_1, v_2, \ldots, v_n \rangle$ into W and $v_i\alpha = w_i$ for all $i \in \{1, 2, \ldots, n\}$. If U is a subspace of V, let 1_U and U_0 denote the identity map on U and the zero map which its domain is U, respectively. Observe that

$$\{0\}_0\alpha = \{0\}_0 \text{ and } V_0\alpha = V_0 \text{ for all } \alpha \in PL_R(V,W).$$

It follows that if $\dim_R V > 0$, then $PL_R(V,W)$ does not have a zero.

We obviously see that if $V = W$, $L_R(V,W) = L_R(V)$ or we can say that $L_R(V,W)$ is defined from $L_R(V)$ in order to generalize $L_R(V)$. Similarly, all the semigroups that we have previously mentioned are defined from semigroups studied in [4], [1], [3] and [2]. Moreover, we can generalize their results.

In Chapter III, we deal with linear transformtion semigroups with zero. The purpose is to characterize when the semigroups $K_R(V,W)$, $CI_R(V,W)$ and $I_R(V,W)$ admit the structure of a semihyperring with zero. Moreover, the semigroups $K'_R(V,W)$, $CI'_R(V,W)$ and $I'_R(V,W)$ are also studied in the same matter.

In Chapter IV, we intend to deal with semigroups without zero. We provide the sufficient and necessary conditions for $AM_R(V,W)$, $AE_R(V,W)$ and $AI_R(V,W)$ to admit the structure of an AC semiring with zero. In addition, necessary conditions for $PL_R(V,W)$ to admit such the structure are provided.

We can also see from Chapter III and Chapter IV that main results shown by Kemprasit Y. and Chaopraknoi S. in [4], [1], [3] and [2] become our corollaries.

CHAPTER III

SEMIGROUPS ADMITTING THE STRUCTURE OF A SEMIHYPERRING WITH ZERO

First, we recall that V is a vector space over a division ring R, W is a subspace of V, $L_R(V,W)$ is the semigroup of all linear transformations from V into W under a composition and k is a cardinal number such that $k \le \dim_R V$. In this chapter, we deal with some linear transformation semigroups given in Chapter II as follow:

$$K_R((V,W),k) = \{\alpha \in L_R(V,W) \mid \dim_R \mathrm{Ker}\ \alpha \ge k\},$$
$$K'_R((V,W),k) = \{\alpha \in L_R(V,W) \mid \dim_R \mathrm{Ker}\ \alpha > k\} \text{ where } k < \dim_R V,$$
$$CI_R((V,W),k) = \{\alpha \in L_R(V,W) \mid \dim_R (V/\mathrm{Im}\ \alpha) \ge k\},$$
$$CI'_R((V,W),k) = \{\alpha \in L_R(V,W) \mid \dim_R (V/\mathrm{Im}\ \alpha) > k\} \text{ where } k < \dim_R V,$$
$$I_R((V,W),k) = \{\alpha \in L_R(V,W) \mid \dim_R \mathrm{Im}\ \alpha \le k\} \text{ where } k \le \dim_R W,$$
$$I'_R((V,W),k) = \{\alpha \in L_R(V,W) \mid \dim_R \mathrm{Im}\ \alpha < k\} \text{ where } 0 < k \le \dim_R W.$$

These semigroups contain the zero map. Moreover, the zero map is also the zero element of each semigroup.

3.1 The semigroups $K_R((V,W),k)$ and $K'_R((V,W),k)$

We shall provide some necessary conditions for $K_R((V,W),k)$ to admit the structure of a semihyperring with zero. Since $K_R((V,W),k') = K'_R((V,W),k)$ if k' is the successor of k, we also obtain some necessary conditions for $K'_R((V,W),k)$ to admit such a structure.

Theorem 3.1.1. *Let k be a cardinal number with $k \leq \dim_R V$. If $K_R((V,W),k)$ admits the structure of a semihyperring with zero, then one of the following statements holds.*

(i) $\dim_R V = k$ *and* $\dim_R V$ *is finite.*

(ii) $\dim_R(V/W) \geq k$.

Proof. Assume that $(K_R^0((V,W),k), \oplus, \cdot)$ is a semihyperring with zero. We will prove by contradiction. Then suppose that (i) and (ii) are false, so we have ($\dim_R V > k$ or $\dim_R V$ is infinite) and ($\dim_R(V/W) < k$). These equivalent to ($\dim_R V > k$ and $\dim_R(V/W) < k$) or ($\dim_R V$ is infinite and $\dim_R(V/W) < k$). Then either ($\dim_R(V/W) < k < \dim_R V$ where $\dim_R V$ is finite) or ($\dim_R(V/W) < k$ where $\dim_R V$ is infinite).

Case 1 : $\dim_R(V/W) < k < \dim_R V$ where $\dim_R V$ is finite. Since $\dim_R V$ is finite, $0 \leq \dim_R V - \dim_R W = \dim_R(V/W) < k$ which implies that $\dim_R W > \dim_R V - k > 0$. Then we can conclude that $k > 0$ and $\dim_R W > 0$. Let B be a basis of W and B' a basis of V extended from B. Since $\dim_R V$ is finite and $B \subseteq B'$,

$$|B' \setminus B| = |B'| - |B| = \dim_R V - \dim_R W < k.$$

We denote $k - (\dim_R V - \dim_R W)$ by n. Hence $n \in \mathbb{N}$ and $\dim_R W - n = \dim_R V - k > 0$, that is $|B| = \dim_R W > n$. Therefore we can choose distinct elements $w_1, w_2, \ldots, w_n$ from B such that $B \setminus \{w_1, w_2, \ldots, w_n\} \neq \varnothing$. Let $B_1 = (B' \setminus B) \cup \{w_1, w_2, \ldots, w_n\}$. Then

$$|B_1| = |B' \setminus B| + |\{w_1, w_2, \ldots, w_n\}| = k - n + n = k,$$

since $\dim_R V$ is finite and $(B' \setminus B) \cap \{w_1, w_2, \ldots, w_n\} = \varnothing$. Define

$$\alpha = \begin{pmatrix} B_1 & v \\ 0 & v \end{pmatrix}_{v \in B' \smallsetminus B_1} \text{ and } \beta = \begin{pmatrix} w_1 & B' \smallsetminus \{w_1\} \\ w_1 & 0 \end{pmatrix}.$$

We can see that $\dim_R \operatorname{Ker} \alpha = \dim_R \langle B_1 \rangle = k$ and $\operatorname{Im} \alpha = \langle B' \smallsetminus B_1 \rangle \subseteq \langle B \rangle = W$, so $\alpha \in K_R((V,W),k)$. Since $\dim_R \operatorname{Ker} \beta = \dim_R \langle B' \smallsetminus \{w_1\} \rangle = \dim_R V - 1 \geq k$ and $\operatorname{Im} \beta = \langle w_1 \rangle \subseteq W$, we have $\beta \in K_R((V,W),k)$. By definitions of α and β,

$$\alpha^2 = \alpha, \beta^2 = \beta, \alpha\beta = 0 \text{ and } \beta\alpha = 0,$$

which imply that $\alpha(\alpha \oplus \beta) = \{\alpha\}$ and $\beta(\alpha \oplus \beta) = \{\beta\}$. Next, let $\gamma \in \alpha \oplus \beta \subseteq K_R((V,W),k)$. Then $\alpha\gamma = \alpha$ and $\beta\gamma = \beta$. It is obvious that

$$\operatorname{Im} \alpha = \operatorname{Im} \alpha\gamma \subseteq \operatorname{Im} \gamma \text{ and } \operatorname{Im} \beta = \operatorname{Im} \beta\gamma \subseteq \operatorname{Im} \gamma.$$

Consequently, $B' \smallsetminus (B_1 \smallsetminus \{w_1\}) = (B' \smallsetminus B_1) \cup \{w_1\} \subseteq \operatorname{Im} \alpha \cup \operatorname{Im} \beta \subseteq \operatorname{Im} \gamma$. Thus $\dim_R \operatorname{Im} \gamma \geq |B' \smallsetminus (B_1 \smallsetminus \{w_1\})| = |B'| - |B_1 \smallsetminus \{w_1\}| = \dim_R V - (k-1)$. Since $\dim_R V$ is finite, $\dim_R \operatorname{Ker} \gamma = \dim_R V - \dim_R \operatorname{Im} \gamma \leq k-1$. Therefore $\gamma \notin K_R((V,W),k)$, a contradiction.

Case 2 : $\dim_R(V/W) < k$ where $\dim_R V$ is infinite. We clearly have $0 \leq \dim_R(V/W) < k$. If we assume that $\dim_R W < \dim_R V$, then by Proposition 2.2.6, $\dim_R V = \dim_R(V/W) < k$ which is a contradiction. Hence $\dim_R W = \dim_R V$. Let B be a basis of W and B' a basis of V extended from B. Since $\dim_R W$ is infinite, we can let B_1 and B_2 be disjoint subsets of B such that $|B_1| = |B_2| = |B|$ and $B_1 \cup B_2 = B$. Note that $B_2 \subseteq B' \smallsetminus B_1$ and $B_1 \subseteq B' \smallsetminus B_2$. Define

$$\alpha = \begin{pmatrix} B' \smallsetminus B_1 & v \\ 0 & v \end{pmatrix}_{v \in B_1} \text{ and } \beta = \begin{pmatrix} v & B' \smallsetminus B_2 \\ v & 0 \end{pmatrix}_{v \in B_2}.$$

Then Ker $\alpha = \langle B' \setminus B_1 \rangle \supseteq \langle B_2 \rangle$ and Im $\alpha = \langle B_1 \rangle \subseteq W$, imply that $\dim_R \text{Ker } \alpha = |B' \setminus B_1| \geq |B_2| = \dim_R W = \dim_R V \geq k$. Hence $\alpha \in K_R((V,W),k)$. Similarly, we have $\langle B_1 \rangle \subseteq \langle B' \setminus B_2 \rangle = \text{Ker } \beta$ and Im $\beta = \langle B_2 \rangle \subseteq W$. It follows that $\dim_R \text{Ker } \beta = |B' \setminus B_2| \geq |B_1| = \dim_R W = \dim_R V \geq k$. Hence $\beta \in K_R((V,W),k)$. It is easy to see that

$$\alpha^2 = \alpha, \beta^2 = \beta, \alpha\beta = 0 \text{ and } \beta\alpha = 0.$$

Then $\alpha(\alpha \oplus \beta) = \{\alpha\}$ and $\beta(\alpha \oplus \beta) = \{\beta\}$. Next, let $\gamma \in \alpha \oplus \beta \subseteq K_R((V,W),k)$. Then $\alpha\gamma = \alpha$ and $\beta\gamma = \beta$. Consequently,

$$v\gamma = (v\alpha)\gamma = v(\alpha\gamma) = v\alpha = v \text{ for every } v \in B_1,$$

$$v\gamma = (v\beta)\gamma = v(\beta\gamma) = v\beta = v \text{ for every } v \in B_2.$$

So Im $\gamma = W$ and $\gamma_{|W} = 1_W$. Let $T = \{x - x\gamma \mid x \in B' \setminus B\}$. Claim that Ker $\gamma \subseteq \langle T \rangle$. Let $y \in \text{Ker } \gamma$. Then $y = a_1s_1 + a_2s_2 + \ldots + a_ms_m + b_1t_1 + b_2t_2 + \ldots + b_nt_n$ for some $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n \in R$, $s_1, s_2, \ldots, s_m \in B$ and $t_1, t_2, \ldots, t_n \in B' \setminus B$. Thus

$$\begin{aligned} 0 &= y\gamma \\ &= (a_1s_1 + a_2s_2 + \ldots + a_ms_m)\gamma + (b_1t_1 + b_2t_2 + \ldots + b_nt_n)\gamma \\ &= a_1s_1 + a_2s_2 + \ldots + a_ms_m + (b_1t_1 + b_2t_2 + \ldots + b_nt_n)\gamma. \end{aligned}$$

So

$$a_1s_1 + a_2s_2 + \ldots + a_ms_m = -(b_1t_1 + b_2t_2 + \ldots + b_nt_n)\gamma.$$

Consequently,

$$\begin{aligned} y &= -(b_1t_1 + b_2t_2 + \ldots + b_nt_n)\gamma + b_1t_1 + b_2t_2 + \ldots + b_nt_n \\ &= b_1(t_1 - t_1\gamma) + b_2(t_2 - t_2\gamma) + \ldots + b_n(t_n - t_n\gamma), \end{aligned}$$

this implies that $y \in \langle T \rangle$. To show that T is linearly independent. Let $a_1, a_2, \ldots, a_n \in R$ and $x_1, x_2, \ldots, x_n$ be all distinct elements in $B' \smallsetminus B$ such that

$$a_1(x_1 - x_1\gamma) + a_2(x_2 - x_2\gamma) + \ldots + a_n(x_n - x_n\gamma) = 0.$$

Then $a_1x_1 + a_2x_2 + \ldots + a_nx_n = a_1(x_1\gamma) + a_2(x_2\gamma) + \ldots + a_n(x_n\gamma) \in \langle B' \smallsetminus B \rangle \cap \langle B \rangle = \{0\}$, hence $a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$. We therefore have $a_1 = a_2 = \ldots = a_n = 0$. This shows that T is linearly independent and $x - x\gamma \neq y - y\gamma$ for distinct elements $x, y \in B' \smallsetminus B$. Since $\text{Ker}\ \gamma \subseteq \langle T \rangle$, $\dim_R \text{Ker}\ \gamma \leq |T| = |B' \smallsetminus B| = \dim_R(V/W) < k$. This yields a contradiction.

Therefore the proof is complete. □

The following corollary providing some necessary conditions for $K'_R((V, W), k)$ to admit the structure of a semihyperring with zero are obtained directly from the previous theorem.

Corollary 3.1.2. *Let k be a cardinal number with $k < \dim_R V$. If $K'_R((V, W), k)$ admits the structure of a semihyperring with zero, then one of the following statements holds.*

(i) $\dim_R V - 1 = k$ *and* $\dim_R V$ *is finite.*

(ii) $\dim_R(V/W) \geq k'$ *where* k' *is the successor of* k.

Proof. Assume that $K'_R((V, W), k)$ admits the structure of a semihyperring with zero. Since $k < \dim_R V$, $k' \leq \dim_R V$ and $K'_R((V, W), k) = K_R((V, W), k')$. We have by Theorem 3.1.1 that either $\dim_R V$ is finite and $\dim_R V - 1 = k$ or $\dim_R(V/W) \geq k'$ hold. □

Moreover, the necessary conditions of some results mentioned in [4] become our special cases as follow.

Corollary 3.1.3. ([4]) *Let k be a cardinal number with $k \leq \dim_R V$. Then $K_R(V,k)$ admits the structure of a semihyperring with zero if and only if either*

(i) $\dim_R V = k$ *and* $\dim_R V$ *is finite or*

(ii) $k = 0$.

Corollary 3.1.4. ([4]) *Let k be a cardinal number with $k < \dim_R V$. Then $K'_R(V,k)$ admits the structure of a semihyperring with zero if and only if $k+1 = \dim_R V$ and $\dim_R V$ is finite.*

Remark 3.1.5. (i) Assume that $\dim_R V$ is finite. If k is a cardinal number such that $k \leq \dim_R(V/W)$, then $K_R((V,W),k) = L_R(V,W)$. Next, let k_1 be a cardinal number such that $\dim_R(V/W) \leq k_1 \leq \dim_R V$. Since $\dim_R V$ is finite, we have $\dim_R V - k_1 \leq \dim_R W$. Then let B be a basis of W, B' a basis of V extended from B and $B_1 \subseteq B$ such that $|B_1| = \dim_R V - k_1$. Define

$$\alpha = \begin{pmatrix} B' \smallsetminus B_1 & v \\ 0 & v \end{pmatrix}_{v \in B_1},$$

so $\dim_R \operatorname{Ker} \alpha = |B' \smallsetminus B_1| = \dim_R V - (\dim_R V - k_1) = k_1$. If k_2 is a cardinal number such that $\dim_R(V/W) \leq k_1 < k_2 \leq \dim_R V$, then $\alpha \in K_R((V,W),k_1) \smallsetminus K_R((V,W),k_2)$, implies that $K_R((V,W),k_1) \supset K_R((V,W),k_2)$.

For each cardinal number l,k such that $k \leq \dim_R(V/W)$ and $l < \dim_R(V/W)$, we can conclude that

$$\begin{aligned} L_R(V,W) &= K_R((V,W),k) = K'_R((V,W),l) \\ &\supset K_R((V,W),\dim_R(V/W)+1) = K'_R((V,W),\dim_R(V/W)) \\ &\supset K_R((V,W),\dim_R(V/W)+2) = K'_R((V,W),\dim_R(V/W)+1) \\ &\vdots \\ &\supset K_R((V,W),\dim_R(V)). \end{aligned}$$

(ii) Assume that $\dim_R V$ is infinite and $\dim_R V > \dim_R W$. Then we have $K_R((V,W),k) = L_R(V,W) = K'_R((V,W),l)$ for all cardinal numbers k,l such that $k \leq \dim_R V$ and $l < \dim_R V$.

(iii) Assume that $\dim_R V = \dim_R W$ is infinite and k_1, k_2 are cardinal numbers such that $k_1 < k_2 \leq \dim_R V$. We will show that $K_R((V,W),k_1) \supset K_R((V,W),k_2)$. Let B be a basis of W and B' a basis of V extended from B. Since $k_1 < \dim_R V = |B'| = |B|$, there exists $B_1 \subset B$ such that $|B_1| = k_1$ and by assumption, we can assume that B_1 have the property $|B' \smallsetminus B_1| = |B'|$. Let φ be a bijection from $B' \smallsetminus B_1$ to B and define $\alpha \in L_R(V,W)$ by

$$\alpha = \begin{pmatrix} B_1 & v \\ 0 & v\varphi \end{pmatrix}_{v \in B' \smallsetminus B_1},$$

so $\dim_R \operatorname{Ker} \alpha = |B_1| = k_1$. Thus $\alpha \in K_R((V,W),k_1) \smallsetminus K_R((V,W),k_2)$. This implies that $K_R((V,W),k_1) \supset K_R((V,W),k_2)$. Then we can conclude that

$$\begin{aligned} K_R((V,W),0) &\supset K_R((V,W),1) = K'_R((V,W),0) \\ &\supset K_R((V,W),2) = K'_R((V,W),1) \\ &\;\vdots \\ &\supset K_R((V,W),\dim_R V). \end{aligned}$$

3.2 The semigroups $CI_R((V,W),k)$ and $CI'_R((V,W),k)$

By Proposition 2.2.7, $K_R((V,W),k) = CI_R((V,W),k)$ for every cardinal number k with $k \leq \dim_R V$ if V is a finite dimensional vector space. However, it is also shown in Proposition 2.2.8 that if $\dim_R V$ is infinite, then $K_R((V,W),k) \neq CI_R((V,W),l)$ where k,l are cardinal numbers such that $\dim_R(V/W) < l \leq \dim_R V$ and $k \leq \dim_R V$. Then necessary conditions for $CI_R((V,W),k)$ to admit the structure of a symihyperring with zero can not be obtained from Theorem

3.1.1, so we also characterize when $CI_R((V,W),k)$ admits the structure of a semihyperring with zero.

Theorem 3.2.1. *Let k be a cardinal number with $k \leq \dim_R V$. Then $CI_R((V,W),k)$ admits the structure of a semihyperring with zero if and only if one of the following statements holds.*

(i) $\dim_R V = k$ *and* $\dim_R V$ *is finite.*

(ii) $\dim_R(V/W) \geq k$.

Proof. To prove sufficiency, first assume that $\dim_R V = k$ and $\dim_R V$ is finite. Let $\alpha \in CI_R((V,W),k)$. Since $\dim_R V$ is finite, $\dim_R V - \dim_R \operatorname{Im}\alpha = \dim_R(V/\operatorname{Im}\alpha) \geq k = \dim_R V$. This implies that $\dim_R \operatorname{Im}\alpha = 0$. We then have $CI_R((V,W),k) = \{0\}$ which admits a ring structure. Next, assume that $\dim_R(V/W) \geq k$. We shall show that $L_R(V,W) = CI_R((V,W),k)$. Let $\alpha \in L_R(V,W)$. Then $\dim_R(V/\operatorname{Im}\alpha) \geq \dim_R(V/W) \geq k$. So $\alpha \in CI_R((V,W),k)$. Hence $CI_R((V,W),k) = L_R(V,W)$ which admits a ring structure.

Conversely, assume that $CI_R((V,W),k)$ admits the structure of a semihyperring with zero . Suppose that (i) and (ii) are false. Then we have 2 cases which are the same as the proof of Theorem 3.1.1.

Case 1 : $\dim_R(V/W) < k < \dim_R V$ where $\dim_R V$ is finite. Then we have $K_R((V,W),k) = CI_R((V,W),k)$. By Theorem 3.1.1, they do not admit the structure of a semihyperring with zero, a contradiction.

Case 2 : $\dim_R(V/W) < k$ where $\dim_R V$ is infinite. We can see form case 2 in the proof of Theorem 3.1.1 that $\dim_R V = \dim_R W$ and there exist sets $B_1, B_2 \subseteq B \subseteq B' \subseteq V$ such that $|B_1| = |B_2| = |B|$, B is a basis of W and B' is a basis of V. Moreover, the following linear transformations from V to W are recalled,

$$\alpha = \begin{pmatrix} B' \setminus B_1 & v \\ 0 & v \end{pmatrix}_{v \in B_1} \quad \text{and } \beta = \begin{pmatrix} v & B' \setminus B_2 \\ v & 0 \end{pmatrix}_{v \in B_2} .$$

Since $\alpha \in L_R(V,W)$ and $\dim_R(V/\text{Im }\alpha) = \dim_R(V/\langle B_1 \rangle) \geq \dim_R(V/\langle B' \setminus B_2 \rangle) = |B' \setminus (B' \setminus B_2)| = |B_2| \geq k$. Hence $\alpha \in CI_R((V,W),k)$. Similarly, $\dim_R(V/\text{Im }\beta) = \dim_R(V/\langle B_2 \rangle) \geq \dim_R(V/\langle B' \setminus B_1 \rangle) = |B' \setminus (B' \setminus B_1)| = |B_1| \geq k$ and $\text{Im }\beta \subseteq W$. Hence $\beta \in CI_R((V,W),k)$. By the same case of the proof of Theorem 3.1.1, we have $\gamma \in CI_R((V,W),k)$ such that $\text{Im }\gamma = W$. Then $\dim_R(V/\text{Im }\gamma) = \dim_R(V/W) < k$, so this contradicts to $\gamma \in CI_R((V,W),k)$.

Therefore the proof is complete. □

The following corollary is obtained from Theorem 3.2.1.

Corollary 3.2.2. *Let k be a cardinal number with $k < \dim_R V$. Then $CI'_R((V,W),k)$ admits the structure of a semihyperring with zero if and only if one of the following statements holds.*

(i) $\dim_R V - 1 = k$ *and* $\dim_R V$ *is finite.*

(ii) $\dim_R(V/W) \geq k'$ *where k' is the successor of k.*

Proof. Note that if $\dim_R V$ is finite, then $k' = k+1$. Assume that $CI'_R((V,W),k)$ admits the structure of a semihyperring with zero. Since $k \geq 0$, $k' > 0$ and $CI'_R((V,W),k) = CI_R((V,W),k')$. We have by Theorem 3.2.1 that either $\dim_R V$ is finite and $\dim_R V - 1 = k$ or $\dim_R(V/W) \geq k'$ holds.

Conversely, assume that $\dim_R V - 1 = k$ and $\dim_R V$ is finite. Then $k' = \dim_R V$, and thus by Theorem 3.2.1, $CI_R((V,W),k')$ admits the structure of a semihyperring with zero. Since $CI'_R((V,W),k) = CI_R((V,W),k')$, we have that $CI'_R((V,W),k)$ admits the structure of a semihyperring with zero. Next, assume that $\dim_R(V/W) \geq k'$. Then by Theorem 3.2.1, $CI_R((V,W),k')$ admits the structure of a semihyperring with zero, so does $CI'_R((V,W),k)$. □

From the proof of Theorem 3.2.1 and Corollary 3.2.2, we can conclude that necessary conditions of those theorems are $CI_R((V,W),k) = L_R(V,W)$ or $\{0\}$ and $CI'_R((V,W),k) = L_R(V,W)$ or $\{0\}$, respectively. Hence the following corollaries are obtained directly.

Corollary 3.2.3. *Let k be a cardinal number with with $k \leq \dim_R V$. Then $CI_R((V,W),k)$ admits a hyperring [ring] structure if and only if one of the following statements hold.*

(i) $\dim_R V = k$ *and* $\dim_R V$ *is finite.*

(ii) $\dim_R(V/W) \geq k$.

Corollary 3.2.4. *Let k be a cardinal number with with $k < \dim_R V$. Then $CI'_R((V,W),k)$ admits a hyperring [ring] structure if and only if one of the following statements hold.*

(i) $\dim_R V - 1 = k$ *and* $\dim_R V$ *is finite.*

(ii) $\dim_R(V/W) \geq k'$ *where* k' *is the successor of* k.

In addition, if we set $V = W$ in Theorem 3.2.1 and Corollary 3.2.2, then some results mentioned in [4] become our special cases as follow.

Corollary 3.2.5. ([4]) *Let k be a cardinal number with $k \leq \dim_R V$. Then $CI_R(V,k)$ admits the structure of a semihyperring with zero if and only if either*

(i) $\dim_R V = k$ *and* $\dim_R V$ *is finite or*

(ii) $k = 0$.

Corollary 3.2.6. ([4]) *Let k be a cardinal number with $k < \dim_R V$. Then $CI'_R(V,k)$ admits the structure of a semihyperring with zero if and only if $k+1 = \dim_R V$ and $\dim_R V$ is finite.*

Remark 3.2.7. (i) Assume that $\dim_R V$ is finite. By Proposition 2.2.7, if k is a cardinal number such that $k \leq \dim_R V$, then $CI_R((V,W),k) = K_R((V,W),k)$. Then we have by Remark 3.1.5 that for each cardinal number l, k such that $k \leq \dim_R(V/W)$ and $l < \dim_R(V/W)$,

$$\begin{aligned} L_R(V,W) = CI_R((V,W),k) &= CI'_R((V,W),l) \\ &\supset CI_R((V,W),\dim_R(V/W)+1) = CI'_R((V,W),\dim_R(V/W)) \\ &\supset CI_R((V,W),\dim_R(V/W)+2) = CI'_R((V,W),\dim_R(V/W)+1) \\ &\vdots \\ &\supset CI_R((V,W),\dim_R(V)). \end{aligned}$$

(ii) Assume that $\dim_R V$ is infinite and $\dim_R V > \dim_R W$. Then $\dim_R(V/\text{Im }\alpha) \geq \dim_R(V/W) = \dim_R V$ for all $\alpha \in L_R(V,W)$. This implies that $CI_R((V,W),k) = L_R(V,W) = CI'_R((V,W),l)$ for all cardinal numbers k, l such that $k \leq \dim_R V$ and $l < \dim_R V$.

(iii) Assume that $\dim_R V = \dim_R W$ is infinite. If $\dim_R(V/W) \geq k$, then $CI_R((V,W),k) = L_R(V,W)$. Let k_1 be a cardinal number such that $\dim_R(V/W) \leq k_1 \leq \dim_R V$. Let B be a basis of W and B' a basis of V extended from B. Since $\dim_R W = \dim_R V$ is infinite and $\dim_R(V/W) \leq k_1$, there exists $B_1 \subseteq B$ such that $|B' \smallsetminus (B \smallsetminus B_1)| = k_1$ and $|B \smallsetminus B_1| = |B|$. Define $\alpha \in L_R(V,W)$ by

$$\alpha = \begin{pmatrix} B' \smallsetminus (B \smallsetminus B_1) & v \\ 0 & v \end{pmatrix}_{v \in B \smallsetminus B_1},$$

so $\dim_R(V/\mathrm{Im}\ \alpha) = |B' \smallsetminus (B \smallsetminus B_1)| = k_1$. Hence if k_2 is a cardinal number such that $k_1 < k_2 \leq \dim_R V$, then $\alpha \in CI_R((V,W),k_1) \smallsetminus CI_R((V,W),k_2)$ and $CI_R((V,W),k_1) \supset CI_R((V,W),k_2)$, respectively. Therefore for cardinal numbers k,l such that $\dim_R(V/W) \geq k$ and $\dim_R(V/W) > l$,

$$\begin{aligned} L_R(V,W) &= CI_R((V,W),k) = CI'_R((V,W),l) \\ &\supset CI_R((V,W),\dim_R(V/W)+1) = CI'_R((V,W),\dim_R(V/W)) \\ &\supset CI_R((V,W),\dim_R(V/W)+2) = CI'_R((V,W),\dim_R(V/W)+1) \\ &\vdots \\ &\supset CI_R((V,W),\dim_R(V)). \end{aligned}$$

3.3 The semigroups $I_R((V,W),k)$ and $I'_R((V,W),k)$

We have already shown in Proposition 2.2.8 if $\dim_R W$ is infinite, we then have

$$K_R((V,W),l) \neq I_R((V,W),k) \neq CI_R((V,W),l)$$

for any cardinal number k,l with $k < \dim_R W$ and $l \leq \dim_R V$. Contrasting between this section and previous sections in this chapter will assure that what we have mentioned above is true. In this section, we shall characterize when $I_R((V,W),k)$ and $I'_R((V,W),k)$ admit the structure of a semihyperring with zero.

Theorem 3.3.1. *Let k be a cardinal number with $k \leq \dim_R W$. Then $I_R((V,W),k)$ admits the structure of a semihyperring with zero if and only if one of the following statements holds.*

(i) $k = 0$.

(ii) $k = \dim_R W$.

(iii) *k is infinite.*

Proof. To prove sufficiency, assume (i), (ii) or (iii) holds. Since $I_R((V,W),0) = \{\alpha \in L_R(V,W) | \dim_R \text{Im } \alpha \leq 0\} = \{0\}$ and $I_R((V,W), \dim_R W) = \{\alpha \in L_R(V,W) | \dim_R \text{Im } \alpha \leq \dim_R W\} = L_R(V,W)$. Therefore if we have (i) or (ii), then $I_R((V,W),k)$ admits a ring structure. Next, we will show that if k is an infinite cardinal number, then $(I_R((V,W),k),+,\circ)$ is also a ring where $+$ is the usual addition of linear transformations and $\circ$ is a composition. Let $\alpha, \beta \in I_R((V,W),k)$. We know that Im $(\alpha+\beta) \subseteq$ Im $\alpha +$ Im β and Im $\beta =$ Im $(-\beta)$. Thus

$$\dim_R \text{Im } (\alpha - \beta) \leq \dim_R \text{Im } \alpha + \dim_R \text{Im } \beta \leq k + k = k.$$

Hence $I_R((V,W),k)$ is a subring of $L_R(V,W)$.

Conversely, assume that $(I_R^0((V,W),k),\oplus,\cdot)$ is a semihyperring with zero. To show that one of (i), (ii) and (iii) holds, suppose on the contrary that all of them are false. Then $0 < k < \dim_R W$ and k is finite. Let B be a basis of W, B' a basis of V extended from B and $B_1 \subseteq B$ such that $|B_1| = k$. Note that B_1 is not empty. Since $k < \dim_R W$, there exists an element $u \in B \smallsetminus B_1$. Define

$$\alpha = \begin{pmatrix} v & B' \smallsetminus B_1 \\ v & 0 \end{pmatrix}_{v \in B_1} \quad \text{and } \beta = \begin{pmatrix} u & B' \smallsetminus \{u\} \\ u & 0 \end{pmatrix}.$$

Then Im $\alpha = \langle B_1 \rangle \subseteq W$ and $\dim_R \text{Im } \alpha = |B_1| = k$. Hence $\alpha \in I_R((V,W),k)$. Similarly, Im $\beta = \langle u \rangle \subseteq W$ and $\dim_R \text{Im } \beta = 1 \leq k$, since $k > 0$. Hence $\beta \in I_R((V,W),k)$. It is clear that

$$\alpha^2 = \alpha, \beta^2 = \beta, \alpha\beta = 0 \text{ and } \beta\alpha = 0.$$

Thus $\alpha(\alpha \oplus \beta) = \{\alpha\}$ and $\beta(\alpha \oplus \beta) = \{\beta\}$. Next, let $\gamma \in \alpha \oplus \beta \subseteq I_R((V,W),k)$. Then $\alpha\gamma = \alpha$ and $\beta\gamma = \beta$. Consequently, for every $v \in B_1$,

$$v\gamma = (v\alpha)\gamma = v(\alpha\gamma) = v\alpha = v, \text{ and}$$

$$u\gamma = (u\beta)\gamma = u(\beta\gamma) = u\beta = u.$$

Therefore Im $\gamma \supseteq \langle B_1 \cup \{u\}\rangle$ which implies that $\dim_R \operatorname{Im} \gamma \geq |B_1 \cup \{u\}| = k+1 > k$, since k is finite. This contradicts the fact that $\gamma \in \alpha \oplus \beta \subseteq I_R((V,W),k)$.

Hence the theorem is proved. □

Corollary 3.3.2. *Let k be a cardinal number with $0 < k \leq \dim_R W$. Then $I'_R((V,W),k)$ admits the structure of a semihyperring with zero if and only if one of the following statements holds.*

(i) $k = 1$.

(ii) k *is infinite.*

Proof. We know that $I'_R((V,W),1) = I_R((V,W),0) = \{0\}$ which admits a ring structure. Next, assume that k is an infinite cardinal number. Then $k+k=k$. We shall show that $(I'_R((V,W),k),+,\circ)$ is a ring where $+$ is the usual addition of linear transformations. If $\alpha,\ \beta \in I'_R((V,W),k)$, then $\dim_R \operatorname{Im} \alpha < k$ and $\dim_R \operatorname{Im} \beta < k$, and hence

$$\dim_R \operatorname{Im} (\alpha - \beta) \leq \dim_R \operatorname{Im} \alpha + \dim_R \operatorname{Im} \beta < k + k = k.$$

Therefore the sufficiency is proved.

To prove necessity, suppose on the contrary that $1 < k$ and k is finite. Then $I'_R((V,W),k) = I_R((V,W),k-1)$ where $0 < k-1 < \dim_R W$ and $k-1$ is finite. It therefore follows from Theorem 3.3.1 that $I'_R((V,W),k)$ does not admit the structure of a semihyperring with zero. □

The following corollaries are direct consequences of Theorem 3.3.1 and Corollary 3.3.2.

Corollary 3.3.3. *Let k be a cardinal number with $k \leq \dim_R W$. Then $I_R((V,W),k)$ admit a hyperring [ring] structure if and only if one of the following statements holds.*

(i) $k = 0$.

(ii) $k = \dim_R W$.

(iii) k *is infinite.*

Corollary 3.3.4. *Let* k *be a cardinal number with* $0 < k \leq \dim_R W$. *Then* $I'_R((V, W), k)$ *admits a hyperring [ring] structure if and only if one of the following statements holds.*

(i) $k = 1$.

(ii) k *is infinite.*

Apart from two corollaries above, we also obtain some results mentioned in [4] directly from Theorem 3.2.1 and Corollary 3.2.2.

Corollary 3.3.5. ([4]) *Let* k *be a cardinal number with* $k \leq \dim_R V$. *Then* $I_R(V, k)$ *admits the structure of a semihyperring with zero if and only if one of the following statements hods.*

(i) $k = 0$.

(ii) $k = \dim_R V$.

(iii) k *is infinite.*

Corollary 3.3.6. ([4]) *Let* k *be a cardinal number with* $0 < k \leq \dim_R V$. *Then* $I'_R(V, k)$ *admits the structure of a semihyperring with zero if and only if either*

(i) $k = 1$ *or*

(ii) k *is infinite.*

Remark 3.3.7. Assume that k_1, k_2 are cardinal numbers such that $k_2 < k_1 \leq \dim_R W$. Claim that $I_R((V,W),k_1) \supset I_R((V,W),k_2)$. Let B be a basis of W and B' a basis of V extended of B. Since $0 < k_1 \leq \dim_R W$, there exists $B_1 \subseteq B$ such that $|B_1| = k_1$. Define $\alpha \in L_R(V,W)$ by

$$\alpha = \begin{pmatrix} B' \setminus B_1 & v \\ 0 & v \end{pmatrix}_{v \in B_1},$$

so $\dim_R \operatorname{Im} \alpha = |B_1| = k_1$. Hence $\alpha \in I_R((V,W),k_1) \setminus I_R((V,W),k_2)$. If $\dim_R W$ is infinite, then

$$\begin{aligned} I_R((V,W),0) = I'_R((V,W),1) &\subset I_R((V,W),1) = I'_R((V,W),2) \\ &\subset I_R((V,W),2) = I'_R((V,W),3) \\ &\vdots \\ &\subset I_R((V,W),\dim_R W). \end{aligned}$$

CHAPTER IV

SEMIGROUPS ADMITTING THE STRUCTURE OF AN AC SEMIRING WITH ZERO

In this chapter, we recall that V is a vector space over a division ring R, W is a subspace of V, $L_R(V,W)$ denote the set of all linear transformations $\alpha : V \to W$ and $F(\alpha) = \{v \in V \mid v\alpha = v\}$. The following linear transformation semigroups are considered.

$$AM_R(V,W) = \{\alpha \in L_R(V,W) \mid \dim_R \operatorname{Ker} \alpha_{|W} < \infty\},$$

$$AE_R(V,W) = \{\alpha \in L_R(V,W) \mid \dim_R(W/(\operatorname{Im} \alpha_{|W})) < \infty\},$$

$$AI_R(V,W) = \{\alpha \in L_R(V,W) \mid \dim_R(W/F(\alpha)) < \infty\},$$

$$PL_R(V,W) = \{\alpha : U \to W \mid U \text{ is a subspace of } V \text{ and } \alpha \text{ is a linear transformation }\}.$$

For the first and second sections, we let B be a basis of W and B' a basis of V containing B. The following notations will be used and fixed.

$$\overline{G_R(V,W)} = \{\alpha \in L_R(V,W) \mid \alpha_{|W} \in G_R(W) \text{ and } \alpha_{|_{B' \smallsetminus B}} = 0\}$$

where $G_R(W)$ is the set of all isomorphisms on W,

$$\overline{1}_W = \begin{pmatrix} v & B' \smallsetminus B \\ v & 0 \end{pmatrix}_{v \in B}.$$

If $u, w \in B$ are distinct, we define

$$\overline{(u,w)}_B = \begin{pmatrix} u & w & v & B' \smallsetminus B \\ w & u & v & 0 \end{pmatrix}_{v \in B \smallsetminus \{u,w\}} \quad \text{and}$$

$$\overline{(u \to w)}_B = \begin{pmatrix} u & v & B' \smallsetminus B \\ w & v & 0 \end{pmatrix}_{v \in B \smallsetminus \{u\}} .$$

We note here that $\overline{1}_W$, $\overline{(u,w)}_B$, $\overline{(u \to w)}_B \in AM_R(V,W) \cap AE_R(V,W) \cap AI_R(V,W)$ and $\overline{1}_W, \overline{(u,w)}_B \in \overline{G_R(V,W)} \subseteq AM_R(V,W) \cap AE_R(V,W) \cap AI_R(V,W)$. For the proof of main theorem, the properties

$$\overline{(u,w)}_B^2 = \overline{1}_W \text{ and } \overline{(u \to w)}_B^2 = \overline{(u \to w)}_B$$

are useful.

4.1 The semigroups $AM_R(V,W)$ and $AE_R(V,W)$

We have shown in Proposition 2.2.9 and Proposition 2.2.10 that $AM_R(V,W)$ and $AE_R(V,W)$ are distinct semigroups without zero if $\dim_R W$ is infinite. Otherwise, they admit a ring structure. The purpose is to characterize when $AM_R(V,W)$ and $AE_R(V,W)$ admit the structure of an AC semiring with zero, the following lemmas are required. For the first lemma, recall that $G_R(V)$ is the set of all isomorphisms on V, $L_R(V)$ be the semigroup of all linear transformations on V under a composition. In this section, if $\alpha \in L_R(V)$ and $a \in R$, we define $a\alpha \in L_R(V)$ by

$$v(a\alpha) = a(v\alpha)$$

for all $v \in V$.

Lemma 4.1.1. ([8]) *Let $\alpha \in L_R(V)$ and assume that that $\alpha\beta = \beta\alpha$ for all $\beta \in G_R(V)$. Then there exists $a \in C(R)$ such that $\alpha = a1_V$ where $C(R)$ is the center of R.*

Lemma 4.1.2. *Let $\alpha \in L_R(V,W)$ and assume that $\alpha\beta = \beta\alpha$ for all $\beta \in \overline{G_R(V,W)}$. Then there exists $a \in C(R)$ such that $\alpha = a\overline{1}_W$.*

Proof. First we will show that $\{\beta_{|W} \mid \beta \in \overline{G_R(V,W)}\} = G_R(W)$. Obviously, $\{\beta_{|W} \mid \beta \in \overline{G_R(V,W)}\} \subseteq G_R(W)$. Let $\gamma \in G_R(W)$. Define $\overline{\gamma} \in L_R(V,W)$ by

$$\overline{\gamma} = \begin{pmatrix} v & x \\ v\gamma & 0 \end{pmatrix}_{v \in B, x \in B' \smallsetminus B}.$$

We can see that $\overline{\gamma}_{|W} = \gamma$ and $\overline{\gamma} \in \overline{G_R(V,W)}$. By assumption and Proposition 2.2.4,

$$\alpha_{|W}\beta_{|W} = \beta_{|W}\alpha_{|W} \text{ for all } \beta \in \overline{G_R(V,W)}.$$

Since $\{\beta_{|W} \mid \beta \in \overline{G_R(V,W)}\} = G_R(W)$, $\alpha_{|W}\beta = \beta\alpha_{|W}$ for all $\beta \in G_R(W)$. By Lemma 4.1.1, $\alpha_{|W} = a1_W$ for some $a \in C(R)$. Let $y \in B' \smallsetminus B$ and $\beta \in \overline{G_R(V,W)}$. It follows from assumption that $y\alpha\beta = y\beta\alpha = 0\alpha = 0$. Thus $y\alpha \in \operatorname{Ker} \beta$. Since $y\alpha \in W$ and $\operatorname{Ker} \beta_{|W} = \{0\}$, $y\alpha = 0$. This shows that $\alpha = a\overline{1}_W$. □

Theorem 4.1.3. *Let $S(V,W)$ be $AM_R(V,W)$ or $AE_R(V,W)$. Then $S(V,W)$ admits the structure of an AC semiring with zero if and only if $\dim_R W$ is finite.*

Proof. As was mentioned, if $\dim_R W$ is finite, then $S(V,W) = L_R(V,W)$ admits a ring structure. Assume that $S(V,W)$ admits the structure of an AC semiring with zero. Then there is an operation $\oplus$ on $S^0(V,W)$ such that $(S^0(V,W), \oplus, \cdot)$ is an AC semiring with zero 0 where $\cdot$ is the operation on $S^0(V,W)$. Suppose on the contrary that $\dim_R W$ is infinite. Since $0 \notin S(V,W)$, so for $\alpha, \beta \in S^0(V,W), \alpha\beta = 0$ implies $\alpha = 0$ or $\beta = 0$. Let u, w be distinct elements of B. Define $\alpha \in L_R(V,W)$ by

$$\alpha = \begin{pmatrix} \{u,w\} \cup (B' \smallsetminus B) & v \\ 0 & v \end{pmatrix}_{v \in B \smallsetminus \{u,w\}}. \quad (1)$$

Then $\dim_R \operatorname{Ker} \alpha_{|W} = \dim_R\langle u, w\rangle = 2$ and $\dim_R(W/\operatorname{Im} \alpha_{|W}) = \dim_R(W/\langle B \smallsetminus \{ u, w \}\rangle) = |\{u, w\}| = 2$. We deduce that $\alpha \in S(V, W)$. It is clear that $\overline{(u, w)}_B\alpha = \alpha = \alpha\overline{(u, w)}_B$. Since $\beta(\overline{1}_W \oplus \overline{1}_W) = \beta \oplus \beta = (\overline{1}_W \oplus \overline{1}_W)\beta$ for every $\beta \in \overline{G_R(V, W)}$, we have by Lemma 4.1.2 that $\overline{1}_W \oplus \overline{1}_W = a\overline{1}_W$ for some $a \in C(R)$. If $a = 0$, then $\overline{1}_W \oplus \overline{1}_W$ is the zero map which does not contain in $S(V, W)$. Then $\overline{1}_W \oplus \overline{1}_W = 0$ and

$$0 = (\overline{1}_W \oplus \overline{1}_W)\alpha = \alpha \oplus \alpha = \alpha \oplus \alpha\overline{(u, w)}_B = \alpha(\overline{1}_W \oplus \overline{(u, w)}_B)$$

which imply that $\overline{1}_W \oplus \overline{(u, w)}_B = 0$ and then

$$\overline{1}_W = \overline{1}_W \oplus 0 = \overline{1}_W \oplus (\overline{1}_W \oplus \overline{(u, w)}_B) = (\overline{1}_W \oplus \overline{1}_W) \oplus \overline{(u, w)}_B = 0 \oplus \overline{(u, w)}_B = \overline{(u, w)}_B,$$

a contradiction. Then $a \neq 0$. From (1), we have

$$(\overline{1}_W \oplus \overline{(u, w)}_B)\alpha = \alpha \oplus \alpha = (\overline{1}_W \oplus \overline{1}_W)\alpha = (a\overline{1}_W)\alpha = a\alpha. \tag{2}$$

We have by (1) and (2) that

$$\begin{aligned} u(\overline{1}_W \oplus \overline{(u, w)}_B)\alpha &= u(a\alpha) = a(u\alpha) = 0, \\ w(\overline{1}_W \oplus \overline{(u, w)}_B)\alpha &= w(a\alpha) = a(w\alpha) = 0. \end{aligned} \tag{3}$$

Since $\overline{(u, w)}_B\overline{1}_W = \overline{(u, w)}_B$ and $\overline{(u, w)}_B^2 = \overline{1}_W$, we can deduce that

$$\overline{(u, w)}_B(\overline{1}_W \oplus \overline{(u, w)}_B) = \overline{1}_W \oplus \overline{(u, w)}_B$$

and

$$u(\overline{1}_W \oplus \overline{(u, w)}_B) = u\overline{(u, w)}_B(\overline{1}_W \oplus \overline{(u, w)}_B) = w(\overline{1}_W \oplus \overline{(u, w)}_B).$$

By (3), it is obtained that

$$u(\overline{1}_W \oplus \overline{(u, w)}_B) = w(\overline{1}_W \oplus \overline{(u, w)}_B) \in \operatorname{Ker} \alpha_{|W} = \langle u, w\rangle,$$

thus there exist $b, c \in R$ such that

$$u(\overline{1}_W \oplus \overline{(u, w)}_B) = w(\overline{1}_W \oplus \overline{(u, w)}_B) = bu + cw\ . \tag{4}$$

Next, we define $\gamma \in L_R(V,W)$ by

$$\gamma = \begin{pmatrix} \{u,w\} & v & B' \setminus B \\ u+w & v & 0 \end{pmatrix}_{v \in B \setminus \{u,w\}}. \tag{5}$$

Then $\text{Ker } \gamma_{|W} \subseteq \langle u,w \rangle = \text{Ker } \alpha_{|W}$ and $\text{Im } \alpha_{|W} = \langle B \setminus \{u,w\} \rangle \subseteq \text{Im } \gamma_{|W}$, so $\gamma \in S(V,W)$. Since $u\gamma\overline{(u,w)}_B = w\gamma\overline{(u,w)}_B = (u+w)\overline{(u,w)}_B = u+w$, then $\gamma\overline{(u,w)}_B = \gamma$, and hence

$$\gamma(\overline{1}_W \oplus \overline{(u,w)}_B) = \gamma \oplus \gamma = (\overline{1}_W \oplus \overline{1}_W)\gamma = a\gamma. \tag{6}$$

Therefore

$$\begin{aligned} 2bu + 2cw &= (u+w)(\overline{1}_W \oplus \overline{(u,w)}_B) && \text{from (4)} \\ &= w\gamma(\overline{1}_W \oplus \overline{(u,w)}_B) && \text{from (5)} \\ &= w(a\gamma) && \text{from (6)} \\ &= a(u+w) && \text{from (5)} \\ &= au + aw. \end{aligned}$$

Since u and w are linearly independent, $2b = 2c = a \neq 0$. Consequently, $\operatorname{char} R \neq 2$. Because $-\overline{1}_W \in S(V,W)$ and $\beta(\overline{1}_W \oplus (-\overline{1}_W)) = \beta \oplus (-\beta) = (\overline{1}_W \oplus (-\overline{1}_W))\beta$ for all $\beta \in \overline{G_R(V,W)}$, by Lemma 4.1.2, $\overline{1}_W \oplus (-\overline{1}_W) = a'\overline{1}_W$ for some $a' \in C(R)$. If $a' = 0$, then $\overline{1}_W \oplus (-\overline{1}_W)$ is the zero map contained in $S^0(V,W)$. It is obvious that $S(V,W)$ does not contain the zero map. Then $\overline{1}_W \oplus (-\overline{1}_W) = 0$ and so

$$0 = (\overline{1}_W \oplus (-\overline{1}_W))\alpha = \alpha \oplus (-\alpha) = \alpha \oplus (-\alpha\overline{(u,w)}_B) = \alpha(\overline{1}_W \oplus (-\overline{(u,w)}_B)).$$

We can conclude that $\overline{1}_W \oplus (-\overline{(u,w)}_B) = 0$, so $\overline{1}_W \oplus \overline{1}_W \oplus (-\overline{(u,w)}_B) = \overline{1}_W \oplus 0$.

Hence $-\overline{1}_W = -\overline{(u,w)}_B$, which is a contradiction. This shows that $a' \neq 0$. But

$$\begin{aligned} a'\overline{1}_W = & \ \overline{1}_W \oplus (-\overline{1}_W) \\ = & \ -\overline{1}_W(\overline{1}_W \oplus (-\overline{1}_W)) \ , \quad \text{since } \oplus \text{ is commutative} \\ = & \ -a'\overline{1}_W \\ = & \ -a'\overline{1}_W + V_0 \ , \end{aligned}$$

$2a'\overline{1}_W = V_0$, and thus $2a' = 0$ since $W \neq \{0\}$. We have a contradiction directly from the facts that $a' \neq 0$ and $\operatorname{char} R \neq 2$. □

Corollary 4.1.4. *Let $S(V,W)$ be $AM_R(V,W)$ or $AE_R(V,W)$. Then $S(V,W)$ admits a ring structure if and only if $\dim_R W$ is finite.*

Moreover, the results given in [1] become our special cases as follow.

Corollary 4.1.5. ([1]) *Let $S(V)$ be $AM_R(V)$ or $AE_R(V)$. Then $S(V)$ admits the structure of an AC semiring with zero if and only if $\dim_R V$ is finite.*

Corollary 4.1.6. ([1]) *Let $S(V)$ be $AM_R(V)$ or $AE_R(V)$. Then $S(V)$ admits a ring structure if and only if $\dim_R V$ is finite.*

4.2 The semigroup $AI_R(V,W)$

By Proposition 2.2.10, $AI_R(V,W)$ is different from $AM_R(V,W)$ and $AE_R(V,W)$ when $\dim_R W$ is infinite. Moreover, by Proposition 2.2.9, $AI_R(V,W)$ does not contain the zero. We shall show that $AI_R(V,W)$ admits the structure of an AC semiring with zero if and only if $\dim_R W$ is finite, the following lemma will be used.

Lemma 4.2.1. *Let $(AI_R^0(V,W), \oplus, \cdot)$ be an AC semiring with zero. If $\dim_R W$ is infinite, then $\overline{1}_W \oplus \overline{1}_W = \overline{1}_W$.*

Proof. Assume that $(AI_R^0(V,W),\oplus,\cdot)$ is an AC semiring with zero and $\dim_R W$ is infinite. Recall that $\overline{1}_W \in AI_R(V,W)$. We then have $\overline{1}_W \oplus \overline{1}_W \in AI_R^0(V,W)$. Since $\beta(\overline{1}_W \oplus \overline{1}_W) = (\overline{1}_W \oplus \overline{1}_W)\beta$ for all $\beta \in \overline{G_R(V,W)}$, by Lemma 4.1.2, there exists $a \in R$ such that $\overline{1}_W \oplus \overline{1}_W = a\overline{1}_W$. Suppose on the contrary that $a \neq 1$. It is obtained that $F(\overline{1}_W \oplus \overline{1}_W) = F(a\overline{1}_W) = \{0\}$. Hence we have $\dim_R(W/F(\overline{1}_W \oplus \overline{1}_W)) = \dim_R(W/\{0\}) = \dim_R W$. Since $\dim_R W$ is infinite, $\overline{1}_W \oplus \overline{1}_W \notin AI_R(V,W)$, implies that $\overline{1}_W \oplus \overline{1}_W = 0$. Let $u,w \in W$ be distinct elements. Next, recall α from the proof of Theorem 4.1.3 and we can see that $\dim_R(W/F(\alpha)) = \dim_R \langle u,w\rangle = 2$, so $\alpha \in AI_R(V,W)$. Note that $\overline{(u,w)}_B\alpha = \alpha = \alpha\overline{(u,w)}_B$. We have

$$0 = (\overline{1}_W \oplus \overline{1}_W)\alpha = \alpha \oplus \alpha = \alpha \oplus \alpha\overline{(u,w)}_B = \alpha(\overline{1}_W \oplus \overline{(u,w)}_B)$$

which imply that $\overline{1}_W \oplus \overline{(u,w)}_B = 0$ and then

$$\overline{1}_W = \overline{1}_W \oplus 0 = \overline{1}_W \oplus (\overline{1}_W \oplus \overline{(u,w)}_B) = (\overline{1}_W \oplus \overline{1}_W) \oplus \overline{(u,w)}_B = 0 \oplus \overline{(u,w)}_B = \overline{(u,w)}_B,$$

a contradiction. □

Theorem 4.2.2. $AI_R(V,W)$ *admits the structure of an AC semiring with zero if and only if* $\dim_R W$ *is finite.*

Proof. If $\dim_R W$ is finite, then $AI_R(V,W) = L_R(V,W)$ which admits a ring structure. Conversely, assume that $AI_R(V,W)$ admits the structure of an AC semiring with zero. Then there is an operation $\oplus$ on $AI_R^0(V,W)$ such that $(AI_R^0(V,W),\oplus,\cdot)$ is an AC semiring with zero 0 where $\cdot$ is the operation on $AI_R^0(V,W)$. To show $\dim_R W$ is finite, suppose on the contrary that $\dim_R W$ is infinite. By Lemma 4.2.1 , $\overline{1}_W \oplus \overline{1}_W = \overline{1}_W$. For every $\alpha \in AI_R^0(V,W)$, if $\alpha_{|_{B' \smallsetminus B}} = 0$, we get

$$\alpha \oplus \alpha = (\overline{1}_W \oplus \overline{1}_W)\alpha = \overline{1}_W\alpha = \alpha. \tag{1}$$

Recall the fact that $\overline{(u,w)}_B, \overline{(u \to w)}_B \in AI_R(V,W)$ for all distinct $u, w \in B$. Next, let u and w be fixed distinct elements of B. We have

$$\begin{aligned}
\overline{(u \to w)}_B^2 &= \overline{(u \to w)}_B = \overline{(w \to u)}_B \overline{(u \to w)}_B \\
&= \overline{(w \to u)}_B \overline{(u,w)}_B = \overline{(u,w)}_B \overline{(u \to w)}_B, \\
\overline{(w \to u)}_B^2 &= \overline{(w \to u)}_B = \overline{(u \to w)}_B \overline{(w \to u)}_B \\
&= \overline{(u \to w)}_B \overline{(u,w)}_B = \overline{(u,w)}_B \overline{(w \to u)}_B.
\end{aligned} \tag{2}$$

We therefore have from (2) that $\overline{(w \to u)}_B[\overline{1}_W \oplus \overline{(u \to w)}_B] = \overline{(w \to u)}_B \oplus \overline{(u \to w)}_B$ and $\overline{(u \to w)}_B[\overline{1}_W \oplus \overline{(w \to u)}_B] = \overline{(u \to w)}_B \oplus \overline{(w \to u)}_B$. Since $\oplus$ is commutative,

$$\overline{(w \to u)}_B[\overline{1}_W \oplus \overline{(u \to w)}_B] = \overline{(u \to w)}_B[\overline{1}_W \oplus \overline{(w \to u)}_B], \tag{3}$$

and for each $v \in B \smallsetminus \{u\}$,

$$\begin{aligned}
v[\overline{1}_W \oplus \overline{(u \to w)}_B] &= v\overline{(u \to w)}_B[\overline{1}_W \oplus \overline{(u \to w)}_B] \\
&= v[\overline{(u \to w)}_B \oplus \overline{(u \to w)}_B] \quad \text{from (2)} \\
&= v\overline{(u \to w)}_B = v \qquad \text{from (1).}
\end{aligned}$$

Let $u[\overline{1}_W \oplus \overline{(u \to w)}_B] = au + bw + \sum_{i=1}^{n} c_i v_i$ for some $a, b, c_1, c_2, \ldots, c_n \in R$ and distinct $v_1, v_2, \ldots, v_n \in B \smallsetminus \{u, w\}$. We therefore have

$$\begin{aligned}
u &= u\overline{(w \to u)}_B \\
&= u[\overline{(w \to u)}_B \oplus \overline{(w \to u)}_B] \qquad \text{from (1)} \\
&= u[\overline{1}_W \oplus \overline{(u \to w)}_B]\overline{(w \to u)}_B \quad \text{from (2)} \\
&= (au + bw + \sum_{i=1}^{n} c_i v_i)\overline{(w \to u)}_B \\
&= au + bu + \sum_{i=1}^{n} c_i v_i \\
&= (a+b)u + \sum_{i=1}^{n} c_i v_i
\end{aligned}$$

which implies that $a+b=1$ and $c_i=0$ for all $i=1,2,\ldots,n$. Consequently,

$$\begin{aligned} v[\bar{1}_W \oplus \overline{(u\to w)}_B] &= v && \text{if } v\in B\setminus\{u\}, \\ u[\bar{1}_W \oplus \overline{(u\to w)}_B] &= au+bw && \text{where } a+b=1. \end{aligned} \tag{4}$$

By interchanging between u and w, from (4), there are $a',b'\in R$ such that

$$\begin{aligned} v[\bar{1}_W \oplus \overline{(w\to u)}_B] &= v && \text{if } v\in B\setminus\{w\}, \\ w[\bar{1}_W \oplus \overline{(w\to u)}_B] &= a'u+b'w && \text{and } a'+b'=1. \end{aligned} \tag{5}$$

Case 1 : $a\neq 0$. Let $v_1,v_2,\ldots,v_m\in B\setminus\{u\}$ be distinct and let $d_0,d_1,\ldots,d_m\in R$ be such that $(d_0u+\sum_{i=1}^{m} d_iv_i)[\bar{1}_W \oplus \overline{(u\to w)}_B]=0$. Then from (4),

$$d_0au+d_0bw+\sum_{i=1}^{m} d_iv_i=0,$$

so $d_0a=0$. Since $a\neq 0$, we have $d_0=0$ which implies that $d_i=0$ for all $i\in\{1,2,\ldots,m\}$, hence Ker $(\bar{1}_W \oplus \overline{(u\to w)}_B)_{|W}=\{0\}$. This shows that

$$(\bar{1}_W \oplus \overline{(u\to w)}_B)_{|W} \text{ is a one-to-one map.} \tag{6}$$

Since

$$\begin{aligned} [\bar{1}_W \oplus \overline{(u\to w)}_B]^2 &= \bar{1}_W \oplus \overline{(u\to w)}_B \oplus \overline{(u\to w)}_B \oplus \overline{(u\to w)}_B^{\,2} \\ &= \bar{1}_W \oplus \overline{(u\to w)}_B \quad \text{from (1) and (2),} \end{aligned}$$

by Proposition 2.2.4,

$$\begin{aligned} (\bar{1}_W \oplus \overline{(u\to w)}_B)_{|W}(\bar{1}_W \oplus \overline{(u\to w)}_B)_{|W} &= ([\bar{1}_W \oplus \overline{(u\to w)}_B]^2)_{|W} \\ &= (\,\bar{1}_W \oplus \overline{(u\to w)}_B)_{|W}. \end{aligned}$$

It follows from (6) that $(\bar{1}_W \oplus \overline{(u\to w)}_B)_{|W}=1_W$ and for every $x\in B'\setminus B$,

$$\begin{aligned} x(\bar{1}_W \oplus \overline{(u\to w)}_B) &= x(\overline{(u,w)}_B\overline{(u,w)}_B \oplus \overline{(u,w)}_B\overline{(u\to w)}_B) \text{ from (2)} \\ &= x\overline{(u,w)}_B(\overline{(u,w)}_B \oplus \overline{(u\to w)}_B) \\ &= 0(\overline{(u,w)}_B \oplus \overline{(u\to w)}_B)=0. \end{aligned}$$

Consequently,

$$\overline{1}_W \oplus \overline{(u \to w)}_B = \overline{1}_W. \tag{7}$$

We therefore have

$$\begin{aligned}
\overline{1}_W &= \overline{(u,w)}_B^2 \\
&= [\overline{1}_W \overline{(u,w)}_B]^2 \\
&= [(\overline{1}_W \oplus \overline{(u \to w)}_B)\overline{(u,w)}_B]^2 && \text{from (7)} \\
&= [\overline{(u,w)}_B \oplus \overline{(w \to u)}_B]^2 && \text{from (2)} \\
&= \overline{(u,w)}_B^2 \oplus \overline{(u,w)}_B \overline{(w \to u)}_B \oplus \overline{(w \to u)}_B \overline{(u,w)}_B \oplus \overline{(w \to u)}_B^2 \\
&= \overline{1}_W \oplus \overline{(w \to u)}_B \oplus \overline{(u \to w)}_B \oplus \overline{(w \to u)}_B && \text{from (2)} \\
&= \overline{1}_W \oplus \overline{(u \to w)}_B \oplus \overline{(w \to u)}_B && \text{from (1)} \\
&= \overline{1}_W \oplus \overline{(w \to u)}_B && \text{from (7).}
\end{aligned}$$

Replace (7) and $\overline{1}_W = \overline{1}_W \oplus \overline{(w \to u)}_B$ in (3), we get $\overline{(u \to w)}_B = \overline{(w \to u)}_B$, a contradiction.

Case 2 : $b' \neq 0$. From (5) and interchanging between u and w, we obtain as case 1 that $\overline{1}_W \oplus \overline{(w \to u)}_B = \overline{1}_W = \overline{1}_W \oplus \overline{(u \to w)}_B$. This implies by (3) that $\overline{(u \to w)}_B = \overline{(w \to u)}_B$, a contradiction.

Case 3 : $a = 0 = b'$.

From (4) and (5), we have respectively that

$$\overline{1}_W \oplus \overline{(u \to w)}_B = \overline{(u \to w)}_B \text{ and } \overline{1}_W \oplus \overline{(w \to u)}_B = \overline{(w \to u)}_B. \tag{8}$$

Hence

$$\begin{aligned}
\overline{(u \to w)_B} &= \overline{(w \to u)_B}\,\overline{(u \to w)_B} && \text{from (2)} \\
&= \overline{(w \to u)_B}[\overline{1}_W \oplus \overline{(u \to w)_B}] && \text{from (8)} \\
&= \overline{(u \to w)_B}[\overline{1}_W \oplus \overline{(w \to u)_B}] && \text{from (3)} \\
&= \overline{(u \to w)_B}\,\overline{(w \to u)_B} && \text{from (8)} \\
&= \overline{(w \to u)_B} && \text{from (2)}
\end{aligned}$$

which is a contradiction.

Therefore the theorem is proved. □

Corollary 4.2.3. *$AI_R(V, W)$ admits a ring structure if and only if $\dim_R W$ is finite.*

Moreover, the results given in [3] become our special cases as follow.

Corollary 4.2.4. ([3]) *$AI_R(V)$ admits the structure of an AC semiring with zero if and only if $\dim_R V$ is finite.*

Corollary 4.2.5. ([3]) *$AI_R(V)$ admits a ring structure if and only if $\dim_R V$ is finite.*

4.3 The semigroup $PL_R(V, W)$

The purpose of this section is to characterize when $PL_R(V, W)$ admits the structure of an AC semiring with zero and the following lemmas will be used.

Lemma 4.3.1. *If $\dim_R W > 0$ and $(PL_R^0(V, W), \oplus, \cdot)$ is an AC semiring with zero, then the following statements are satisfied.*

(i) There exists $a \in C(R) \smallsetminus \{0\}$ such that $1_W \oplus (-1_W) = a1_W$.

(ii) If U is a subspace of W, then $U_0 \oplus U_0 = U_0$.

Proof. (i) Since $\dim_R W > 0$, so is $\dim_R V$. It follows from Chapter II that $PL_R(V,W)$ is a semigroup without zero. Thus $W_0 \neq 0$ and $\alpha\beta = 0$ implies $\alpha = 0$ or $\beta = 0$ for all $\alpha, \beta \in PL_R^0(V,W)$. By assumption, $1_W \oplus (-1_W) \in PL_R^0(V,W)$. Claim that $1_W \oplus (-1_W) \neq 0$. Suppose on the contrary that $1_W \oplus (-1_W) = 0$. Consequently,

$$\begin{aligned} W_0(W_0 \oplus \{0\}_0) &= W_0 \oplus W_0 \\ &= (1_W \oplus (-1_W))W_0 \\ &= 0. \end{aligned}$$

Since $W_0 \neq 0$, $W_0 \oplus \{0\}_0 = 0 = W_0 \oplus W_0$. Then we have

$$W_0 = W_0 \oplus 0 = W_0 \oplus (W_0 \oplus \{0\}_0) = (W_0 \oplus W_0) \oplus \{0\}_0 = 0 \oplus \{0\}_0 = \{0\}_0,$$

which contradicts to $\dim_R W > 0$. We conclude that $1_W \oplus (-1_W) \in PL_R(V,W)$. Since $W_0(W_0 \oplus \{0\}_0) = (1_W \oplus (-1_W))W_0$, we have $\text{Dom}(1_W \oplus (-1_W))$=W. It is obtained that $\alpha(1_W \oplus (-1_W)) = \alpha \oplus (-\alpha) = (1_W \oplus (-1_W))\alpha$ for all $\alpha \in G_R(W)$. By Lemma 4.1.1, $1_W \oplus (-1_W) = a1_W$ for some $a \in C(R)$. If $a = 0$, then $1_W \oplus (-1_W) = W_0$ and

$$\begin{aligned} \{0\}_0 = \{0\}_0 W_0 = \{0\}_0(1_W \oplus (-1_W)) \\ &= (1_W \oplus (-1_W))\{0\}_0 \\ &= W_0\{0\}_0 = W_0, \end{aligned}$$

a contradiction. Hence $a \neq 0$.

(ii) Let U be a subspace of W. By (i),

$$U_0 \oplus U_0 = U_0(1_W \oplus (-1_W)) = U_0(a1_W) = U_0.$$

Therefore the proof is complete. □

Lemma 4.3.2. *If* $\dim_R W > 0$ *and* $(PL_R^0(V,W), \oplus, \cdot)$ *is an AC semiring with zero, then* $charR = 2$.

Proof. By Lemma 4.3.1, $1_W \oplus (-1_W) = a1_W$ for some $a \in C(R) \smallsetminus \{0\}$. Then

$$\begin{aligned} a^2 1_W &= (a1_W)(a1_W) \\ &= (a1_W)(1_W \oplus (-1_W)) \\ &= (-a1_W)((-1_W) \oplus 1_W) \\ &= (-a1_W)(1_W \oplus (-1_W)) \ , \quad \text{since } \oplus \text{ is commutative} \\ &= (-a1_W)(a1_W) \quad = \quad -a^2 1_W, \end{aligned}$$

so $2a^2(1_W) = W_0$. Since $a^2 \neq 0$ and $\dim_R W > 0$, $2a^2 = 0$. This implies that $\operatorname{char} R = 2$. □

Now, the proof of required lemmas are complete. The following theorem is our main result.

Theorem 4.3.3. *If the semigroup $PL_R(V,W)$ admits the structure of an AC semiring with zero, then either*

(i) $\dim_R W = 0$ *or*

(ii) $\dim_R W = 1$ *and* $\operatorname{char} R = 2$.

Proof. Assume that $PL_R(V,W)$ admits the structure of an AC semiring with zero. Then there is an operation $\oplus$ on $PL_R^0(V,W)$ such that $(PL_R^0(V,W), \oplus, \cdot)$ is an AC semiring with zero. Suppose on the contrary that $\dim_R W > 1$ or ($\dim_R W > 0$ and $\operatorname{char} R \neq 2$). By Lemma 4.3.2, $\dim_R W > 0$ and $\operatorname{char} R = 2$ are impossible. Therefore $\dim_R W > 1$. It is obtained that $PL_R(V,W)$ does not have a zero and we have linearly independent elements $u, w \in W$. It is clear that $\begin{pmatrix} u \\ u \end{pmatrix} \oplus \begin{pmatrix} u \\ w \end{pmatrix} \in PL_R^0(V,W)$. Now, we consider 3 cases as follow.

Case 1: $\begin{pmatrix} u \\ u \end{pmatrix} \oplus \begin{pmatrix} u \\ w \end{pmatrix} = 0$. By Lemma 4.3.1 (ii),

$$0 = \left(\begin{pmatrix} u \\ u \end{pmatrix} \oplus \begin{pmatrix} u \\ w \end{pmatrix} \right) W_0 = \begin{pmatrix} u \\ 0 \end{pmatrix} \oplus \begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix} \in PL_R(V,W),$$

a contradiction.

Case 2: $\begin{pmatrix} u \\ u \end{pmatrix} \oplus \begin{pmatrix} u \\ w \end{pmatrix} = \{0\}_0$. By Lemma 4.3.1 (ii),

$$\{0\}_0 = \{0\}_0 W_0 = \left(\begin{pmatrix} u \\ u \end{pmatrix} \oplus \begin{pmatrix} u \\ w \end{pmatrix} \right) W_0 = \begin{pmatrix} u \\ 0 \end{pmatrix} \oplus \begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix},$$

a contradiction.

Case 3: $\begin{pmatrix} u \\ u \end{pmatrix} \oplus \begin{pmatrix} u \\ w \end{pmatrix} = \alpha$ for some $\alpha \in PL_R(V,W)$ with $\operatorname{Dom} \alpha \neq \{ 0 \}$.

Since

$$\begin{pmatrix} u \\ u \end{pmatrix} \left(\begin{pmatrix} u \\ u \end{pmatrix} \oplus \begin{pmatrix} u \\ w \end{pmatrix} \right) = \begin{pmatrix} u \\ u \end{pmatrix} \oplus \begin{pmatrix} u \\ w \end{pmatrix},$$

we have $\begin{pmatrix} u \\ u \end{pmatrix} \alpha = \alpha$. Then $\operatorname{Dom} \alpha = \operatorname{Dom} \left(\begin{pmatrix} u \\ u \end{pmatrix} \alpha \right) \subseteq \operatorname{Dom} \begin{pmatrix} u \\ u \end{pmatrix} = \langle u \rangle$, and thus $\dim_R \operatorname{Dom} \alpha \leq \dim_R \langle u \rangle = 1$. But $\operatorname{Dom} \alpha \neq \{0\}$, so $\operatorname{Dom} \alpha = \langle u \rangle$. Also, since

$$\begin{aligned}
\alpha \begin{pmatrix} u & w \\ w & u \end{pmatrix} &= \left(\begin{pmatrix} u \\ u \end{pmatrix} \oplus \begin{pmatrix} u \\ w \end{pmatrix} \right) \begin{pmatrix} u & w \\ w & u \end{pmatrix} \\
&= \begin{pmatrix} u \\ w \end{pmatrix} \oplus \begin{pmatrix} u \\ u \end{pmatrix} \\
&= \begin{pmatrix} u \\ u \end{pmatrix} \oplus \begin{pmatrix} u \\ w \end{pmatrix}, \quad \text{since } \oplus \text{ is commutative} \\
&= \alpha,
\end{aligned}$$

we have Im $\alpha \subseteq \langle u, w\rangle$. Then there are $a, b \in R$ such that $u\alpha = au + bw$. This implies that $\alpha = \begin{pmatrix} u \\ au+bw \end{pmatrix}$. Therefore

$$\begin{pmatrix} u \\ au+bw \end{pmatrix} = \alpha = \alpha \begin{pmatrix} u & w \\ w & u \end{pmatrix} = \begin{pmatrix} u \\ au+bw \end{pmatrix} \begin{pmatrix} u & w \\ w & u \end{pmatrix} = \begin{pmatrix} u \\ aw+bu \end{pmatrix}.$$

Since u and w are linearly independent, $a = b$. Thus $\alpha = \begin{pmatrix} u \\ a(u+w) \end{pmatrix}$. Now, we have

$$\begin{pmatrix} u \\ u \end{pmatrix} \oplus \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} u \\ a(u+w) \end{pmatrix}. \qquad (1)$$

Consequently,

$$\begin{aligned}
\begin{pmatrix} u \\ u+w \end{pmatrix} \oplus \begin{pmatrix} u \\ u+w \end{pmatrix} &= \left(\begin{pmatrix} u \\ u \end{pmatrix} \oplus \begin{pmatrix} u \\ w \end{pmatrix}\right) \begin{pmatrix} u & w \\ u+w & u+w \end{pmatrix} \\
&= \begin{pmatrix} u \\ a(u+w) \end{pmatrix} \begin{pmatrix} u & w \\ u+w & u+w \end{pmatrix} \quad \text{from (1)} \\
&= \begin{pmatrix} u \\ a(u+w+u+w) \end{pmatrix} \\
&= \begin{pmatrix} u \\ 2a(u+w) \end{pmatrix} \\
&= \begin{pmatrix} u \\ 0 \end{pmatrix} \quad \text{by Lemma 4.3.2.} \qquad (2)
\end{aligned}$$

Since u and w are linearly independent, $u + w \neq 0$, and so Ker $\begin{pmatrix} u \\ u+w \end{pmatrix} = \{0\}$.

Hence

$$\begin{aligned}
\{0\}_0 \oplus \{0\}_0 &= \begin{pmatrix} u \\ u+w \end{pmatrix} \{0\}_0 \oplus \begin{pmatrix} u \\ u+w \end{pmatrix} \{0\}_0 \\
&= \left(\begin{pmatrix} u \\ u+w \end{pmatrix} \oplus \begin{pmatrix} u \\ u+w \end{pmatrix} \right) \{0\}_0 \\
&= \begin{pmatrix} u \\ 0 \end{pmatrix} \{0\}_0 \qquad \text{from (2)} \\
&= \begin{pmatrix} u \\ 0 \end{pmatrix}
\end{aligned}$$

which contradicts to Lemma 4.3.1.

Therefore the main theorem is complete. □

Corollary 4.3.4. *If* $PL_R(V, W)$ *admits a ring structure, then* $\dim_R W = 0$.

Proof. Assume that $(PL_R^0(V, W), \oplus, \cdot)$ is a ring. Suppose on the contrary that $\dim_R W > 0$. By Theorem 4.3.3, $\dim_R W = 1$ and $\operatorname{char} R = 2$. By Lemma 4.3.1 (ii),

$$W_0 \oplus W_0 = W_0.$$

Since $(PL_R^0(V, W), \oplus)$ is a group, W_0 is a zero of $PL_R(V, W)$, a contradiction. □

If $V = W$, then we have $PL_R(V) = PL_R(V, W)$. Therefore the following corollary is obtained directly from Theorem 4.3.3.

Corollary 4.3.5. ([2]) *If* $PL_R(V)$ *admits the structure of an AC semiring with zero, then either*

(i) $\dim_R V = 0$ *or*

(ii) $\dim_R V = 1$ *and* $\mathit{char} R = 2$.

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- *Some Semihyperring Structures of Some Linear Transformation Semigroups* in The 14th National Graduate Research Conference, 10-11 September 2009

Attend

- The 13th Annual Meeting in Mathematics, 6-7 May 2008
- The 14th Annual Meeting in Mathematics, 5-6 March 2009