เซมิกรุปเรกูลาร์สตาร์และเซมิกรุปสตาร์เรกูลาร์

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REGULAR-STAR SEMIGROUPS AND STAR-REGULAR SEMIGROUPS

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บทคัดย่อ

ในวิทยานิพนธ์นี้ เราเปรียบเทียบคุณสมบัติต่าง ๆ ของเซมิกรุปเรกูลาร์สตาร์และเซมิกรุปสตาร์เรกูลาร์ให้เห็น จุดมุ่งหมายที่สำคัญของวิทยานิพนธ์นี้คือพิสูจน์ว่า สำหรับเซต X ใด ๆ เซมิกรุปของการแปลงบางส่วน เซมิกรุปของการแปลงเต็ม เซมิกรุปของการแปลงบางส่วนที่เกือบเป็นเอกลักษณ์หรือเซมิกรุปของการแปลงเต็มที่เกือบเป็นเอกลักษณ์บนเซต X จะเป็นเรกูลาร์สตาร์หรือสตาร์เรกูลาร์เมื่อและต่อเมื่อจำนวนสมาชิกของ X น้อยกว่าหรือเท่ากับ 1 ยิ่งไปกว่านั้นเราแนะนำคอนกรูเอนซ์ที่รู้จักกันทั่ว ๆ ไปซึ่งเป็นคอนกรูเอนซ์สตาร์บนบางเซมิกรุปที่เป็นเซมิกรุปสตาร์ เราพิสูจน์สิ่งต่อไปนี้ คอนกรูเอนซ์เซมิแลตติสที่เล็กที่สุดบนเซมิกรุปสตาร์ใด ๆ คอนกรูเอนซ์ที่แยกไอเดมโพเทนต์ที่ใหญ่ที่สุดบนเซมิกรุปออร์ธอดอกซ์ซึ่งเป็นเซมิกรุปสตาร์ คอนกรูเอนซ์กรุปที่เล็กที่สุดบนเซมิกรุปผกผันซึ่งเป็นเซมิกรุปสตาร์ คอนกรูเอนซ์ผกผันที่เล็กที่สุดบนเซมิกรุปออร์ธอดอกซ์ซึ่งเป็นเซมิกรุปสตาร์และคอนกรูเอนซ์ที่ใหญ่ที่สุดซึ่งอยู่ในความสัมพันธ์ของกรีน $\mathcal{H}$ ของเซมิกรุปเรกูลาร์ซึ่งเป็นเซมิกรุปสตาร์ต่างก็เป็นคอนกรูเอนซ์สตาร์

Thesis Title Regular-Star Semigroups and Star-Regular Semigroups

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ABSTRACT

In this thesis, various properties of regular-* semigroups and *-regular semigroups are compared. The main purpose of this thesis is to show that for any set X, the partial transformation semigroup, the full transformation semigroup, the semigroup of almost identical partial transformations or the semigroup of almost identical full transformations on the set X is regular-* or *-regular if and only if the cardinality of X is less than or equal to one. Moreover, we introduce well-known congruences which preserve * on some *-semigroups. The following are shown : The minimum semilattice congruence on any *-semigroup, the maximum idempotent-separating congruence on an orthodox semigroup which is a *-semigroup, the minimum group congruence on an inverse semigroup which is a *-semigroup, the minimum inverse congruence on an orthodox semigroup which is a *-semigroup and the maximum congruence contained in the Green's relation $\mathcal{H}$ of a regular semigroup which is a *-semigroup are all *-congruences.

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CONTENTS

INTRODUCTION

Let S be a semigroup. An element a of S is called an <u>idempotent</u> of S if $a^2 = a$. For a semigroup S, let E(S) denote the set of all idempotents of S, that is,

$$E(S) = \{a \in S \mid a^2 = a\}.$$

A semigroup S is a <u>semilattice</u> if $a^2 = a$ and $ab = ba$ for all elements a, b of S.

An element z of a semigroup S is called a <u>zero</u> of S if $xz = zx = z$ for all $x \in S$. An element e of a semigroup S is called an <u>identity</u> of S if $ex = xe = x$ for all $x \in S$. A zero and an identity of a semigroup are unique, if exist, and they are usually denoted by 0 and 1, respectively.

A semigroup S with zero 0 is called a <u>zero semigroup</u> if $ab = 0$ for all $a, b \in S$.

Let S be a semigroup, and let 1 be a symbol not representing any element of S. The notation $S \cup 1$ denotes the semigroup obtained by extending the binary operation on S to 1 by defining $1.1 = 1$ and $1.a = a.1 = a$ for all $a \in S$. For a semigroup S, the notation S^1 denotes the following semigroup :

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity,} \\ S \cup 1 & \text{if } S \text{ has no identity.} \end{cases}$$

Then for any element a of a semigroup S, $S^1a = Sa \cup \{a\}$, $aS^1 = aS \cup \{a\}$ and $S^1aS^1 = SaS \cup Sa \cup aS \cup \{a\}$.

A nonempty subset G of a semigroup S is a <u>subgroup</u> of S if it is a group under the same operation of S.

Let S be a semigroup with identity 1. An element a of S is called a <u>unit</u> of S if there exists $a' \in S$ such that $aa' = a'a = 1$. Let G be the set of all units of S, that is,

$$G = \{a \in S \mid aa' = a'a = 1 \text{ for some } a' \in S\}.$$

Then G is the greatest subgroup of S having 1 as its identity, and it is called <u>the group of units</u> or <u>the unit group</u> of the semigroup S.

An element a of a semigroup S is <u>regular</u> if a = axa for some $x \in S$. A semigroup S is <u>regular</u> if every element of S is regular.

In any semigroup S, if a, $x \in S$ such that a = axa, then ax and xa are idempotents of S. Hence, if S is a regular semigroup, then $E(S) \neq \phi$.

Let a be an element of a semigroup S. An element x of S is called an <u>inverse</u> of a if a = axa and x = xax. If a is a regular element of a semigroup S, then a = axa for some $x \in S$, and hence xax is an inverse of a. Therefore a semigroup S is regular if and only if every element of S has an inverse. A semigroup S is called an <u>inverse semigroup</u> if every element of S has a unique inverse, and the unique inverse of the element a of S is denoted by a^{-1}. A semigroup S is an inverse semigroup if and only if S is regular and any two idempotents of S commute [2, Theorem 1.17]. Hence, if S is an inverse semigroup, then E(S) is a semilattice. If S is an inverse semigroup, then for a, b $\in$ S, and e $\in$ E(S),

$$e^{-1} = e, \ (a^{-1})^{-1} = a \text{ and } (ab)^{-1} = b^{-1}a^{-1}$$

[2, Lemma 1.18].

Every group is an inverse semigroup and the identity of a group is its only idempotent.

An <u>orthodox semigroup</u> is a regular semigroup S such that E(S) is a subsemigroup of S. Then every inverse semigroup is an orthodox semigroup.

A nonempty subset A of a semigroup S is called a <u>left ideal</u> of S if $SA \subseteq A$. A right ideal of a semigroup S is defined dually. An <u>ideal</u> of a semigroup S is both a left ideal and a right ideal of S.

Let S be a semigroup. An arbitrary intersection of left ideals [right ideals] of S if nonempty, is a left ideal [right ideal] of S. An arbitrary intersection of ideals of S if nonempty, is an ideal of S.

Let A be a nonempty subset of a semigroup S. The <u>left ideal of S generated by A</u> is the intersection of all left ideals of S containing A. The right ideal of S generated by A is defined dually. The <u>ideal of S generated by A</u> is the intersection of all ideals of S containing A. A <u>principal left ideal</u> of S is the left ideal of S generated by a set of one element of S. A principal right ideal and a principal ideal of S are defined similarly. Then a left ideal [right ideal, ideal] A of S is principal if and only if $A = S^1a$

$[A = aS^1, A = S^1aS^1]$ for some $a \in S$, and we call A the <u>principal left ideal</u> [principal right ideal, principal ideal] of S generated by a. If a is a regular element of S, then $S^1a = Sa$, $aS^1 = aS$ and $S^1aS^1 = SaS$. If a, x are elements of S such that $a = axa$, then $Sa = Sxa$, $aS = axS$ and $SaS = SaxS = SxaS$. Hence, every principal left ideal, every principal right ideal and every principal ideal of a regular semigroup has an idempotent generator.

Let S be a semigroup. Define the relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$ on S as follows :

$$a\mathcal{L}b \iff S^1a = S^1b.$$

$$a\mathcal{R}b \iff aS^1 = bS^1.$$

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R}.$$

$$\mathcal{D} = \mathcal{L} \circ \mathcal{R},$$

that is, $\mathcal{D} = \{(a, b) \in S \times S \mid (a, c) \in \mathcal{L} \text{ and } (c, b) \in \mathcal{R} \text{ for some } c \in S\}$.

$$a\mathcal{J}b \iff S^1aS^1 = S^1bS^1.$$

The relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$ are called <u>Green's relations</u> on S. By [2, Lemma 2.1], $\mathcal{D} = \mathcal{R} \circ \mathcal{L}$. All the Green's relations of S are equivalence relations on S, $\mathcal{H} \subseteq \mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J}$ and $\mathcal{H} \subseteq \mathcal{R} \subseteq \mathcal{D} \subseteq \mathcal{J}$. Equivalent definitions of the Green's relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$ on a semigroup S are given as follow :

$$a\mathcal{L}b \iff a = xb, \; b = ya \text{ for some } x, y \in S^1.$$

$$a\mathcal{R}b \iff a = bx, \; b = ay \text{ for some } x, y \in S^1.$$

$$a\mathcal{J}b \iff a = xby, \; b = ras \text{ for some } x, y, r, s \in S^1.$$

If S is a regular semigroup and $a, b \in S$, then

$a\mathcal{L}b \iff a = xb, \quad b = ya$ for some $x, y \in S$,

$a\mathcal{R}b \iff a = bx, \quad b = ay$ for some $x, y \in S$,

and $a\mathcal{J}b \iff a = xby$ and $b = ras$ for some $x, y, r, s \in S$.

For a semigroup S, $a, x \in S$ such that $a = axa$, we have that $a\mathcal{L}xa$ and $a\mathcal{R}ax$.

For a semigroup S and for $a \in S$, let L_a denote the $\mathcal{L}$-class of S containing a, and let R_a, H_a, D_a and J_a denote similarly.

In a semigroup S, any $\mathcal{H}$-class of S contains at most one idempotent [2, Lemma 2.15], an $\mathcal{H}$-class of S containing an idempotent e of S is a subgroup of S [2, Theorem 2.16], and it is the greatest subgroup of S having e as its identity. Hence, every subgroup of a semigroup S is contained in H_e for some idempotent e of S. If a semigroup S has an identity 1, then H_1 is the unit group of S.

Let X be a set. A partial transformation of X is a map which its domain and its range are subsets of X. If α is a partial transformation of X, let $\Delta\alpha$ and $\nabla\alpha$ denote the domain and the range of α, respectively. The empty transformation of X is referred as a map with empty domain, and it is denoted by 0. Let T_X denote the set of all partial transformations of X including the empty transformation 0. For $\alpha, \beta \in T_X$, define the product $\alpha\beta$ as follows :
If $\nabla\alpha \cap \Delta\beta = \phi$, let $\alpha\beta = 0$. If $\nabla\alpha \cap \Delta\beta \neq \phi$, let $\alpha\beta$ be the composition map of $\alpha|_{(\nabla\alpha \cap \Delta\beta)\alpha^{-1}}$ (α restricted to $(\nabla\alpha \cap \Delta\beta)\alpha^{-1}$) and $\beta|_{(\nabla\alpha \cap \Delta\beta)}$. Then for $\alpha, \beta \in T_X$, $\Delta\alpha\beta = (\nabla\alpha \cap \Delta\beta)\alpha^{-1} \subseteq \Delta\alpha$ and $\nabla\alpha\beta = (\nabla\alpha \cap \Delta\beta)\beta \subseteq \nabla\beta$. Thus T_X is a semigroup and it is called the

partial transformation semigroup on the set X. Hence the empty transformation of X, 0, is the zero of T_X and the identity map on X which is denoted by 1 is the identity of the semigroup T_X. For any set X, the semigroup T_X is a regular semigroup. For $\alpha \in T_X$, α is an idempotent of T_X if and only if $\nabla\alpha \subseteq \Delta\alpha$ and $x\alpha = x$ for all $x \in \nabla\alpha$. Hence $E(T_X) = \{\alpha \in T_X \mid \nabla\alpha \subseteq \Delta\alpha \text{ and } x\alpha = x \text{ for all } x \in \nabla\alpha\}$.

An element $\alpha \in T_X$ is called a 1-1 partial transformation of X if α is a one-to-one map. Let I_X denote the set of all 1-1 partial transformations of X, that is,

$$I_X = \{\alpha \in T_X \mid \alpha \text{ is one-to-one}\}.$$

Then I_X is an inverse subsemigroup of T_X with identity 1 and zero 0, and it is called the symmetric inverse semigroup on the set X and for $\alpha \in I_X$, the inverse map α^{-1}, is the inverse of α in I_X, so $\Delta\alpha^{-1} = \nabla\alpha$, $\nabla\alpha^{-1} = \Delta\alpha$. For $\alpha \in I_X$, α is an idempotent of I_X if and only if α is the identity map on $\Delta\alpha$. Then

$$E(I_X) = \{\alpha \in I_X \mid \alpha \text{ is the identity map on } \Delta\alpha\}.$$

An element $\alpha \in T_X$ is called a full transformation of X if $\Delta\alpha = X$. Let $\mathcal{T}_X$ denote the set of all full transformations of X; that is,

$$\mathcal{T}_X = \{\alpha \in T_X \mid \Delta\alpha = X\}.$$

Then $\mathcal{T}_X$ is a regular subsemigroup of T_X with identity 1 and it is called the full transformation semigroup on the set X. Therefore,

$$E(\mathcal{T}_X) = \{\alpha \in \mathcal{T}_X \mid a\alpha = a \text{ for all } a \in \nabla\alpha\}.$$

For any set A, let $|A|$ denote the cardinality of A.

Let X be a set. A partial transformation α of X is said to be almost identical if there exists at most a finite number of elements x in the domain of α such that $x\alpha \neq x$. Therefore, a partial transformation α of X is almost identical if and only if the set $\{x \in \Delta\alpha \mid x\alpha \neq x\}$ is finite. Let

$$U_X = \{\alpha \in T_X \mid \alpha \text{ is almost identical}\},$$
$$V_X = \{\alpha \in \mathcal{T}_X \mid \alpha \text{ is almost identical}\}$$

and

$$W_X = \{\alpha \in I_X \mid \alpha \text{ is almost identical}\}.$$

It has been proved in [6, Proposition 1.1, Proposition 1.5] that U_X is a regular subsemigroup of T_X, V_X is a regular subsemigroup of $\mathcal{T}_X$ and W_X is an inverse subsemigroup of I_X. The semigroups U_X, V_X and W_X are called the semigroup of almost identical partial transformations on X, the semigroup of almost identical full transformations on X and the semigroup of almost identical 1-1 partial transformations on X; respectively.

Let S and T be semigroups and ψ a map from S into T. The map ψ is a homomorphism from S into T if

$$(ab)\psi = (a\psi)(b\psi)$$

for all a, b $\in$ S. A semigroup T is a homomorphic image of a semigroup S if there exists a homomorphism from S onto T.

Let S be a semigroup. A relation ρ on S is called left compatible if for a, b, c $\in$ S, $a\rho b$ implies $ca\rho cb$. A right compatibility is defined dually. An equivalence relation ρ on S is called a congruence on S if it is both left compatible and right compatible. An arbitrary intersection of congruences on a semigroup S is a congruence on S. If ρ is a congruence on a semigroup S, then the set

$$S/\rho = \{a\rho \mid a \in S\}$$

with the operation defined by

$$(a\rho)(b\rho) = (ab)\rho \qquad (a, b \in S)$$

is a semigroup, and it is called the quotient semigroup relative to the congruence ρ.

Let S be a semigroup and A an ideal of S. Then the relation ρ_A defined by

$$a\rho_A b \text{ if and only if } a, b \in A \text{ or } a = b \qquad (a, b \in S)$$

is a congruence on S and it is called the Rees congruence on S induced by A and S/ρ_A is the Rees quotient semigroup induced by A and it is denoted by S/A. Hence

$$a\rho_A = \begin{cases} \{a\} & \text{if } a \notin A, \\ A & \text{if } a \in A. \end{cases}$$

A congruence ρ on a semigroup S is called (i) a semilattice congruence on S if S/ρ is a semilattice, (ii) an inverse congruence on S if S/ρ is an inverse semigroup and (iii) a group congruence on S if S/ρ is a group.

A congruence ρ on a semigroup S is a semilattice congruence if and only if $a\rho a^2$ and $ab\rho ba$ for all a, b $\in$ S. Then an arbitrary intersection of semilattice congruences on a semigroup S is a semilattice congruence on S, and hence the intersection of all semilatice congruences on a semigroup S is the minimum semilattice congruence on S.

A congruence ρ on a semigroup S is an <u>idempotent-separating congruence</u> on S if each ρ-class of S contains at most one idempotent. Howie has proved in [6] that the maximum idempotent-separating congruence on any inverse semigroup exists. Generally, it has been shown by Meakin in [8] that the maximum idempotent-separating congruence on any orthodox semigroup exists.

Hall has shown in [4] that the minimum inverse congruence on an orthodox semigroup exists, and it has been proved by Munn in [9] that any inverse semigroup has a minimum group congruence.

An <u>involution</u> on a semigroup S is a map $a \longmapsto a^*$ of S into S such that

$$(a^*)^* = a, \quad (ab)^* = b^*a^*$$

for all a, b ∈ S. A <u>*-semigroup</u> is a semigroup with an involution.

An idempotent e of a *-semigroup is a <u>projection</u> if $e^* = e$.

A *-semigroup S is a <u>regular-* semigroup</u> if $a = aa^*a$ for all a ∈ S.

The involution * of a *-semigroup S is said to be <u>proper</u> if for all a, b ∈ S, $a^*a = a^*b = b^*a = b^*b$ implies a = b; or equivalently, for all a, b ∈ S, $aa^* = ab^* = ba^* = bb^*$ implies a = b. A <u>proper *-semigroup</u> is a *-semigroup with a proper involution. A *-semigroup S is a <u>*-regular semigroup</u> if S is proper and for each a ∈ S, there exists x ∈ S such that a = axa, x = xax, $(ax)^* = ax$, $(xa)^* = xa$.

A congruence ρ on a *-semigroup S is a *-<u>congruence</u> on S if for a, b $\in$ S, $a\rho b$ implies $a^*\rho b^*$.

Nordahl and Scheiblich have studied regular - * semigroups in [10], and *-regular semigroups have been studied by Drazin in [3]. In the first chapter, general properties of *-semigroups are studied. In particular, properties of regular-* semigroups and *-regular semigroups are compared. Many examples are given.

We characterize well-known transformation semigroups which are regular-* or *-regular in Chapter II. It is shown in this chapter that the partial transformation semigroup, the full transformation semigroup, the semigroup of almost identical partial transformations or the semigroup of almost identical full transformations on a set X is regular-* or *-regular if and only if the cardinality of X is less than or equal to 1.

In the last chapter, *-congruences on *-semigroups are studies. We characterize a Rees congruence which is a *-congruence on a *-semigroup. It is proved that for any ideal A of a *-semigroup S, the Rees congruence ρ_A of S is a *-congruence if and only if A* is contained in A. The following are proved in this chapter. The minimum semilattice congruence on any *-semigroup, the maximum idempotent-separating congruence on an orthodox semigroup which is a *-semigroup, the minimum group congruence on an inverse semigroup which is a *-semigroup, the minimum inverse congruence on an orthodox semigroup which is a *-semigroup and the maximum congruence contained in the

Green's relation $\mathcal{H}$ of a regular semigroup which is a *-semigroup are all *-congruences.