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และของ (R,S)-มอดูลการคูณทางซ้าย

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PRIMALITIES OF (R, S)-SUBMODULES OF (R, S)-MODULES AND OF LEFT MULTIPLICATION (R, S)-MODULES

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ธวัชชัย คำประภัสสร : ความเป็นเฉพาะของ (R,S)-มอดูลย่อยของ (R,S)-มอดูลและของ (R,S)-มอดูลการคูณทางซ้าย (PRIMALITIES OF (R,S)-SUBMODULES OF (R,S)-MODULES AND OF LEFT MULTIPLICATION (R,S)-MODULES) อ. ที่ปรึกษาวิทยานิพนธ์หลัก : ผศ.ดร.ศจี เพียรสกุล, อ.ที่ปรึกษาวิทยานิพนธ์ร่วม : Prof. Mark Edwin Hall, 74 หน้า.

ในงานวิจัยนี้ เราแนะนำแนวคิดของ (R,S)-มอดูล และ (R,S)-มอดูลการคูณทางซ้าย เรานิยามและศึกษา (R,S)-มอดูลย่อยเฉพาะอย่างเต็ม (R,S)-มอดูลย่อยเฉพาะอย่างร่วม และ (R,S)-มอดูลย่อย R-เฉพาะทางซ้าย เรายังได้แสดงว่า (R,S)-มอดูลย่อยเฉพาะอย่างเต็ม และ (R,S)-มอดูลย่อยเฉพาะอย่างร่วมของ (R,S)-มอดูลการคูณทางซ้ายเป็นสิ่งเดียวกัน ยิ่งไปกว่านั้น เราให้การอธิบายลักษณะของ(R,S)-มอดูลย่อยเฉพาะอย่างเต็ม (R,S)-มอดูลย่อยเฉพาะอย่างร่วม และ (R,S)-มอดูลย่อย R-เฉพาะทางซ้ายทั้งใน (R,S)-มอดูล และ (R,S)-มอดูลการคูณทางซ้าย

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THAWATCHAI KHUMPRAPUSSORN : PRIMALITIES OF (R, S)-SUBMODULES OF (R, S)-MODULES AND OF LEFT MULTIPLI CATION (R, S)-MODULES. ADVISOR : ASST. PROF. SAJEE PIANSKOOL, Ph.D. CO-ADVISOR : PROF. MARK EDWIN HALL, Ph.D., 74 pp.

In this research, the notions of (R, S)-modules and left multiplication (R, S)-modules are introduced. We define and investigate fully prime (R, S)-submodules, jointly prime (R, S)-submodules and left R-prime (R, S)-submodules. Fully prime (R, S)-submodules and jointly prime (R, S)-submodules of a left multiplication (R, S)-module are shown to be identical. Moreover, characterizations of fully prime (R, S)-submodules, jointly prime (R, S)-submodules and left R-prime (R, S)-submodules of an (R, S)-module and of a left multiplication (R, S)-module are provided.

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Synopsis

In this research, we introduce a new algebraic structure called an (R, S)-module that is a generalization of a bimodule. We also define and investigate concepts of primality for (R, S)-submodules. Characterizations of such (R, S)-submodules are obtained. Moreover, the notion of a left multiplication (R, S)-module is given. This leads us to define products of (R, S)-submodules.

Chapter I contains four sections. The first section is an introduction to (R, S)-modules. We obtain that every bimodule admits an (R, S)-module structure. Moreover, we give an example showing that (R, S)-modules need not be bimodules. The second and third sections present some properties of (R, S)-modules which will be used throughout this dissertation. Our inspiration for doing this research is provided in the last section.

In Chapter II, our aim is to investigate concepts of primality for (R, S)-submodules. There are many choices of ways to extend the concepts of prime submodules to prime (R, S)-submodules. In this dissertation, we present three possibly ways to define primality for (R, S)-submodules independently, namely fully prime (R, S)-submodules, jointly prime (R, S)-submodules and left R-prime (R, S)-submodules. Characterizations of each of these are provided.

In Chapter III, we introduce and study left multiplication (R, S)-modules. Some characterizations of left multiplication (R, S)-modules are given. For each (R, S)-submodule N of a left multiplication (R, S)-module M, we obtain that $(N : M)_R$ is the unique maximal ideal of R such that $N = (N : M)_R MS$. The existence of the ideal of the form $(N : M)_R$ allows us to define the product of two (R, S)-submodules of an arbitrary (R, S)-module. Moreover, characterizations of the fully prime, jointly prime and left R-prime (R, S)-submodules of a left multiplication (R, S)-module are obtained in terms of products of (R, S)-submodules.

In the final chapter, we summarize what we investigated in this research. In particular, we compare the results between the primalities of submodules of R-module structures and the primalities of (R, S)-submodules of (R, S)-module

structures in the first section. More results on left multiplication (R, S)-modules are provided in the second section. We conclude with the observation that the set of all (R, S)-submodules of a left multiplication (R, S)-module forms a semiring.

CHAPTER I
PRELIMINARIES

This chapter contains four sections. The first section introduces (R, S)-modules and (R, S)-submodules along with some examples of these. Specifically, the existence of (R, S)-module is provided. The second and the third sections discuss some basic propreties of (R, S)-modules which will be used in this dissertation. Moreover, isomorphism theorems for (R, S)-modules are proven. The last section contains brief results on prime submodules of a unital left R-module over a commutative ring with identity which are the inspiration for this research.

1.1 Introduction

In this dissertation, R and S will be arbitrary rings, not necessarily with 1s. The concept of (R, S)-module allows us to act on both sides of an abelian group M by a ring R on its left and a ring S on its right. A well-known algebraic structure which is concerned a ring R acting on only one side of an abelian group M is an R-module. We show by an example that an (R, S)-module structure may be different from a module structure. In fact, the notion of (R, S)-modules is a generalization of bimodules.

Our aim is to introduce a new algebraic structure called an (R, S)-module.

Definition 1.1.1. Let R and S be rings and M an abelian group under addition. We say that M is an (R, S)**-module** if there is a function $\_\cdot\_\cdot\_ : R \times M \times S \to M$, called an (R, S)**-module action (of rings R and S on M)**, satisfying the following properties: for all $r, r_1, r_2 \in R$, $s, s_1, s_2 \in S$, and $m, n \in M$,

(i) $r \cdot (m + n) \cdot s = r \cdot m \cdot s + r \cdot n \cdot s$

(ii) $(r_1 + r_2) \cdot m \cdot s = r_1 \cdot m \cdot s + r_2 \cdot m \cdot s$

(iii) $r \cdot m \cdot (s_1 + s_2) = r \cdot m \cdot s_1 + r \cdot m \cdot s_2$

(iv) $r_1 \cdot (r_2 \cdot m \cdot s_1) \cdot s_2 = (r_1 r_2) \cdot m \cdot (s_1 s_2)$.

We usually abbreviate $r \cdot m \cdot s$ as rms.

Note in particular that every abelian group M admits the trivial (R, S)-module action $rms = 0$ for all $r \in R$, $s \in S$ and $m \in M$ so that, according to the definition, every abelian group is an (R, S)-module for every pair of rings R and S.

From here on, whenever we write "let M be an (R, S)-module" we mean that we are considering a fixed (R, S)-module action on M where R and S are any rings.

Before giving other examples of (R, S)-modules, we state the obvious definition of an (R, S)-submodule.

Definition 1.1.2. Let M be an (R, S)-module. An (R, S)**-submodule** of M is a subgroup N of M which is closed under the (R, S)-module action of the rings on N, that is, $rns \in N$ for all $r \in R$, $s \in S$ and $n \in N$.

Examples

(1) It is obvious that a ring R is an (R, R)-module via the usual multiplication on the ring R. Especially, $\mathbb{Z}$ is a $(\mathbb{Z}, \mathbb{Z})$-module and $(\mathbb{Z}, \mathbb{Z})$-submodules of $\mathbb{Z}$ are of the form $n\mathbb{Z}$ where $n \in \mathbb{Z}$.

Furthermore, let A and B be subrings of a ring R. Then R is an (A, B)-module via the usual multiplication on R. In particular, we obtain a general example that $\mathbb{Z}$ is an $(a\mathbb{Z}, b\mathbb{Z})$-module for all $a, b \in \mathbb{Z}$ and $(a\mathbb{Z}, b\mathbb{Z})$-submodules of $\mathbb{Z}$ are of form $n\mathbb{Z}$ where $n \in \mathbb{Z}$.

(2) Let A be a ring. For any positive integers m and n, $M_{m\times n}(A)$ is an $(M_m(A), M_n(A))$-module.

(3) Let R be a ring and n a positive integer. One can show that R^n is an (R, R)-module by componentwise addition and multiplication. Moreover, if $I_1, I_2, \ldots, I_n$ are ideals of R, then

$$\{(x_1, x_2, \ldots, x_n) \mid x_i \in I_i \text{ for all } i\}$$

and

$$\{(x_1, x_2, \ldots, x_n) \in R^n \mid x_1 + x_2 + \cdots + x_n = 0\}$$

are (R, R)-submodules of R^n.

Note that R^n is also a left R-module. If $I_1, I_2, \ldots, I_n$ are left ideals of R, then the set $J = \{(x_1, x_2, \ldots, x_n) \mid x_i \in I_i \text{ for all } i\}$ is a submodule of R^n. We observe that being one-sided ideals $I_1, I_2, \ldots, I_n$ is not enough to make the set J be an (R, R)-submodule of R^n.

Recall that R is a left R-module. In fact, I is a left ideal of R if and only if I is a submodule of R. We see that there is a relation between submodules and left ideals. Analogously, of course, we can replace "left" by "right" everywhere.

This suggested that we investigate a relation between (R, R)-submodules and ideals. It is easy to see that every ideal of R is an (R, R)-submodule of R. At this point, one difference between (R, R)-modules and R-modules occurs. That is an (R, R)-submodule of R need not be an ideal of the ring R. We demonstrate this by the following example.

(4) Let $SU_4(R)$ be the ring of all 4×4 strictly upper triangular matrices over a ring R, i.e.,

$$SU_4(R) = \left\{ \begin{bmatrix} 0 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{bmatrix} \;\middle|\; a, b, c, d, e, f \in R \right\}.$$

Then $SU_4(R)$ is an $(SU_4(R), SU_4(R))$-module. Moreover, let

$$N = \left\{ \begin{bmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \;\middle|\; x, y \in R \right\}.$$

Then N is an $(SU_4(R), SU_4(R))$-submodule of $SU_4(R)$. We can see that N is not an ideal of the ring $SU_4(R)$ because it is not a left (also not a right) ideal of $SU_4(F)$ as follows:

$$SU_4(R)N = NSU_4(R) = \left\{ \begin{bmatrix} 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \;\middle|\; x \in R \right\} \not\subseteq N.$$

(5) Let X be an infinite set and $R = \{A \mid A \text{ is a finite subset of } X\}$. Define $A \oplus B = (A \backslash B) \cup (B \backslash A)$ and $A \odot B = A \cap B$ for any $A, B \in R$. Then $(R, \oplus, \odot)$ is a commutative ring without identity and $R^2 = R$. (If R had an identity K, then $A = A \cap K \subseteq K$ for all $A \in R$ so that $x \in K$ for all $x \in X$. This would imply that $X = K \in R$ which is a contradiction.)

Next, let $S = \{A \subseteq X \mid A \text{ is finite or } X \backslash A \text{ is finite}\}$. Then $(S, \oplus, \odot)$ is a commutative ring with identity.

Finally, $(\wp(X), \oplus)$, where $\wp(X)$ is the collection of all subsets of X, is an abelian group. It is clear that $\wp(X)$ can be made into an (R, S)-module by defining the (R, S)-module action by $(A, Y, B) \mapsto A \cap Y \cap B$.

(6) Let R and S be any rings. An abelian group M is called a **bimodule over** (R, S) if M is a left R-module and a right S-module and $r(ms) = (rm)s$ for all $r \in R$, $s \in S$ and $m \in M$ (see [5] and [8].) Then every bimodule admits an (R, S)-module structure.

Let R be a commutative ring and M a left R-module. Then, in fact, M is automatically a right R-module by $(m, r) \mapsto rm$. This implies that M satisfies $r(ms) = (rm)s$ for all $r, s \in R$ and $m \in M$. Hence M is a bimodule over (R, R). Thus M is also an (R, R)-module. That is, if M is a left (right) module over a commutative ring R, then M is an (R, R)-module.

In particular, every abelian group is a $(\mathbb{Z}, \mathbb{Z})$-module.

(7) Let R and S be any rings. We show that (R, S)-modules need not be bimodules over (R, S) because the definition of a bimodule requires separate left module and right module actions; however, this requirement is omitted in our definition of an (R, S)-module.

Let A be a ring. Then

$$R = \left\{ \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \;\middle|\; x, y \in A \right\} \quad \text{and} \quad S = \left\{ \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} \;\middle|\; x, y \in A \right\}$$

are noncommutative rings without identity under usual matrix addition and multiplication. Next, let

$$M_1 = \left\{ \begin{bmatrix} x & 0 \\ y & z \end{bmatrix} \;\middle|\; x, y, z \in A \right\} \quad \text{and} \quad M_2 = \left\{ \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \;\middle|\; x, y, z \in A \right\}.$$

Then M_1 and M_2 are also (R, S)-modules under usual matrix addition and multiplication.

Clearly, M_1 is a right S-module but is not a left R-module. Similarly, M_2 is a left R-module but is not a right S-module. Hence M_1 and M_2 are not bimodules over (R, S).

Let $E(R)$ be the set of all idempotents of R and $C(R)$ be the center of R. Recall from [5] that if $\alpha \in E(R) \cap C(R)$, then α is called a **central idempotent** of R.

Lemma 1.1.3. *Let M be an (R, S)-module.*

(i) *If $\alpha \in E(S)$, then M is a left R-module by $(r, m) \mapsto rm\alpha$.*

(ii) *If $\beta \in E(R)$, then M is a right S-module by $(m, s) \mapsto \beta ms$.*

Proof. (i) Asuume that $\alpha \in E(S)$. We denote the image of (r, m) by $r \cdot m$. It is clear that $r \cdot (m + n) = r \cdot m + r \cdot n$ and $(r + s) \cdot m = r \cdot m + s \cdot m$ for all $r, s \in R$ and $m, n \in M$. Since α is an idempotent of S, $r \cdot (s \cdot m) = (rs) \cdot m$ for all $r, s \in R$ and $m \in M$.

(ii) The proof is similar to (i). $\square$

As a result of Lemma 1.1.3, an (R, S)-module structure admits a module structure if an idempotent of one of the rings R and S exists.

Proposition 1.1.4. *Let M be an (R, S)-module. Assume that both R and S contain central idempotents. Then there are left R-module and right S-module* *structures on M such that $r \cdot (m \cdot s) = (r \cdot m) \cdot s$ for all $r \in R$, $m \in M$ and $s \in S$. In the other word, an (R, S)-module structure admits a bimodule structure over (R, S) if central idempotents of both rings exist.*

Proof. It follows from Lemma 1.1.3 that M forms a left R-module and a right S-module. Now, let $r \in R$, $m \in M$ and $s \in S$. Then for $\alpha \in E(S) \cap C(S)$ and $\beta \in E(R) \cap C(R)$, we see that $r \cdot (m \cdot s) = r \cdot (\beta ms) = r(\beta ms)\alpha = (r\beta)m(s\alpha) = (\beta r)m(\alpha s) = \beta(rm\alpha)s = (rm\alpha) \cdot s = (r \cdot m) \cdot s$. $\square$

(8) Let M be an (R, S)-module such that $RMS = M$. The sets

$$I = \{r \in R \mid rMS = 0\} \text{ and } J = \{s \in S \mid RMs = 0\}$$

are ideals of R and S, respectively. Then M is an $(R/I, S/J)$-module by defining the (R, S)-module action of the quotient rings R/I and S/J on M as follows :

$$(r + I)m(s + J) = rms \text{ for all } r \in R,\ m \in M \text{ and } s \in S.$$

We show only that the above action is well-defined. Let $r, r' \in R$, $m \in M$ and $s, s' \in S$ be such that $r+I = r'+I$ and $s+J = s'+J$. Hence $r-r' \in I$ and $s'-s \in J$. This implies that $(r-r')MS = 0 = RM(s'-s)$. Then $(r-r')ms' = 0 = rm(s'-s)$. Hence $rms - r'ms' = rms - r'ms' + rm(s' - s) = rms - r'ms' + rms' - rms = (r - r')ms' = 0$. Therefore $rms = r'ms'$.

The universal property of tensor products yields two ways of constructions of (R, S)-modules. Let M be an (R, S)-module. For each $r \in R$ and $s \in S$, define $\varphi_{(r,s)} : M \to M$ by $\varphi_{(r,s)}(m) = rms$ for all $m \in M$. Let $End(M)$ be the set of all group homomorphisms of M under addition. Then the following lemma is obtained.

Lemma 1.1.5. *Let $r, r_1, r_2 \in R$, $s, s_1, s_2 \in S$ and $z \in \mathbb{Z}$. Then*

(i) $\varphi_{(r,s)} \in End(M)$.

(ii) $\varphi_{(r_1+r_2,s)} = \varphi_{(r_1,s)} + \varphi_{(r_2,s)}$.

(iii) $\varphi_{(r,s_1+s_2)} = \varphi_{(r,s_1)} + \varphi_{(r,s_2)}$.

(iv) $\varphi_{(r,zs)} = \varphi_{(zr,s)}$.

Proof. (i) Let $x, y \in M$. Then $\varphi_{(r,s)}(x+y) = r(x+y)s = rxs + rys = \varphi_{(r,s)}(x) + \varphi_{(r,s)}(y)$. Hence $\varphi_{(r,s)} \in End(M)$.

(ii)–(iv) are obvious. □

Let M be a right R-module, N a left R-module and L an abelian group. Recall from [5] that a map $\varphi : M \times N \to L$ is called **R-balanced** or **middle linear with respect to** R if

$$\varphi(m_1 + m_2, n) = \varphi(m_1, n) + \varphi(m_2, n)$$
$$\varphi(m, n_1 + n_2) = \varphi(m, n_1) + \varphi(m, n_2)$$
$$\varphi(m, rn) = \varphi(mr, n)$$

for all $m, m_1, m_2 \in M, n, n_1, n_2 \in N$ and $r \in R$.

Recall from [5] that the **opposite** of a ring $(S, +, \cdot)$ is the ring $(S, +, *)$ whose multiplication $*$ is defined by $a * b = b \cdot a$. The opposite of a ring $(S, +, \cdot)$ is denoted by S^{op}.

Let R and S be rings. Recall that $R \otimes S$ is the tensor product of R and S over $\mathbb{Z}$. The map $i : R \times S \to R \otimes S$ defined by $i(r, s) = r \otimes s$ is a $\mathbb{Z}$-balanced. By the universal property of tensor products, the following theorem is obtained immediately.

Theorem 1.1.6. *Let R and S be rings and M an abelian group and $i : R \times S^{op} \to R \otimes S^{op}$ be the $\mathbb{Z}$-balanced map defined above.*

(i) *If $\phi : R \otimes S^{op} \to End(M)$ is a ring homomorphism, then the composite $\phi \circ i$ is an (R, S)-module action on M.*

(ii) *If $\_\cdot\_\cdot\_ : R \times M \times S \to M$ is an (R, S)-module action on M, then there is a unique ring homomorphism $\phi : R \otimes S^{op} \to End(M)$ such that $r \cdot m \cdot s = \phi(r \otimes s)(m)$ for all $r \in R$, $m \in M$ and $s \in S$.*

These results establish a bijection

$$\{(R, S)\text{-module actions on } M\} \longleftrightarrow \{\text{ring homomorphisms } \phi : R \otimes S^{op} \to End(M)\}.$$

Proof. The proof of (i) follows immediately from the properties of the map i given above. For (ii), define $\theta : R \times S^{op} \to End(M)$ by $\theta(r,s) = \varphi_{(r,s)}$ for all $r \in R$ and $s \in S$. Lemma 1.1.5 yields that θ is a $\mathbb{Z}$-balanced map. It follows from the universal property of tensor products that there is a unique ring homomorphism $\phi : R \otimes S^{op} \to End(M)$ satisfying $\theta = \varphi \circ i$. From the definitions of θ, $\varphi_{(r,s)}$ and i, it follows that $r \cdot m \cdot s = \phi(r \otimes s)(m)$ for all $r \in R, s \in S, m \in M$. □

Proposition 1.1.7. *Let M be an (R,S)-module. For each $r \in R$, define $\varphi_r : M \times S \to M$ by $\varphi_r(m,s) = rms$ for all $m \in M$ and $s \in S$.*

(i) *For all $r \in R$, φ_r is a $\mathbb{Z}$-balanced.*

(ii) *For all $r_1, r_2 \in R$, $\varphi_{r_1} + \varphi_{r_2} = \varphi_{r_1+r_2}$.*

(iii) *$\{\varphi_r \mid r \in R\}$ is a group under the usual addition of functions.*

(iv) *There is a group monomorphism from $\{\varphi_r \mid r \in R\}$ to $\{\phi : M \otimes_{\mathbb{Z}} S \to M \mid \phi$ is a group homomorphism$\}$.*

Proof. The proofs of (i)–(iii) are straightforward.

(iv) There exists a unique group homomorphism $\phi_r : M \otimes_{\mathbb{Z}} S \to M$ for each $r \in R$, by the universal property of tensor products, such that $\phi_r(m \otimes s) = rms$ for all $m \in M$ and $s \in S$. Consequently, the map sending such φ_r to ϕ_r is a group monomorphism as desired. □

Theorem 1.1.8. *Let M be a left R-module and N a right S-module. There is an (R,S)-module action on $L = M \otimes_{\mathbb{Z}} N$ such that $r(m \otimes n)s = (rm) \otimes (ns)$ for all $r \in R$, $s \in S$ and $n \in N$.*

Proof. Let $L = M \otimes_{\mathbb{Z}} N$. For each $r \in R$ and $s \in S$ define $\beta_{r,s} : M \times N \to L$ by $\beta_{r,s}(m,n) = (rm) \otimes (ns)$ for all $m \in M$ and $n \in N$. It is easy to check that each $\beta_{r,s}$ is a $\mathbb{Z}$-balanced map. Thus, for each $r \in R$ and $s \in S$ there exists a unique homomorphism $\phi_{r,s} : L \to L$ such that $\phi_{r,s} \circ \eta = \beta_{r,s}$, where $\eta : M \times N \to M \otimes_{\mathbb{Z}} N$ is the canonical $\mathbb{Z}$-balanced map defined by $\eta(m,n) = m \otimes n$.

Now, let $r, r' \in R$ and $s, s' \in S$. For all $m \in M$ and $n \in N$, we obtain that

$$\begin{aligned}(\phi_{r,s} + \phi_{r',s})(\eta(m,n)) &= \phi_{r,s}(\eta(m,n)) + \phi_{r',s}(\eta(m,n)) \\ &= \beta_{r,s}(m,n) + \beta_{r',s}(m,n) \\ &= (rm) \otimes (ns) + (r'm) \otimes (ns) \\ &= (rm + r'm) \otimes (ns) \\ &= ((r + r')m) \otimes (ns) \\ &= \beta_{r+r',s}(m,n).\end{aligned}$$

That is $(\phi_{r,s} + \phi_{r',s}) \circ \eta = \beta_{r+r',s}$. But $\phi_{r+r',s}$ is the unique homomorphism such that $\phi_{r+r',s} \circ \eta = \beta_{r+r',s}$, so $\phi_{r,s} + \phi_{r',s} = \phi_{r+r',s}$. Similar arguments show that $\phi_{r,s} + \phi_{r,s'} = \phi_{r,s+s'}$ and $\phi_{r,s} \circ \phi_{r',s'} = \phi_{rr',s's}$.

Define an action of R and S on L by $r \cdot x \cdot s = \phi_{r,s}(x)$ for all $r \in R$, $x \in L$ and $s \in S$. For $r, r' \in R$, $s, s' \in S$ and $x, y \in L$, it follows that

$$\begin{aligned}&r \cdot (x + y) \cdot s = \phi_{r,s}(x + y) = \phi_{r,s}(x) + \phi_{r,s}(y) = r \cdot x \cdot s + r \cdot y \cdot s, \\ &(r + r') \cdot x \cdot s = \phi_{r+r'}(x) = \phi_{r,s}(x) + \phi_{r',s}(x) = r \cdot x \cdot s + r' \cdot x \cdot s, \\ &r \cdot x \cdot (s + s') = \phi_{r,s+s'}(x) = \phi_{r,s}(x) + \phi_{r,s'}(x) = r \cdot x \cdot s + r \cdot x \cdot s', \text{ and} \\ &r \cdot (r' \cdot x \cdot s') \cdot s = \phi_{r,s}(\phi_{r',s'}(x)) = \phi_{rr',s's}(x) = (rr') \cdot x \cdot (s's).\end{aligned}$$

Hence this is an (R, S)-module action on L.

Finally, $r(m \otimes n)s = \phi_{r,s}(\eta(m,n)) = \beta_{r,s}(m,n) = (rm) \otimes (ns)$ for all $r \in R$, $s \in S$, $m \in M$ and $n \in N$.

$\square$

1.2 Some Properties of (R, S)-Modules

This section contains some of the background material which will be used throughout this dissertation. We study analogously the basic points in the module theory. As usual, let 0_R and 0_S be the zero elements of rings R and S, respectively, and 0 the zero element of M. All of our inspiration begins.

Proposition 1.2.1. *Let M be an (R, S)-module and $r \in R$, $s \in S$ $m \in M$ and $k \in \mathbb{Z}$. Then*

(i) $0 = r0s = rm0_S = 0_R ms$.

(ii) $(-r)ms = r(-m)s = rm(-s) = -rms$.

(iii) $k(rms) = (kr)ms = r(km)s = rm(ks)$.

Proof. These are straightforward. $\square$

Let M be an (R,S)-module. Then for any nonempty subsets X, Y and Z of R, M and S, respectively, define

$$XYZ = \left\{ \sum_{\text{finite}} x_i y_i z_i \;\middle|\; x_i \in X, y_i \in Y \text{ and } z_i \in Z \text{ for all } i \right\}.$$

Proposition 1.2.2. *Let M be an (R,S)-module and X_1, X_2 be nonempty subsets of R, Y a nonempty subset of M and Z_1, Z_2 be nonempty subsets of S. Then $X_1(X_2YZ_1)Z_2 = (X_1X_2)Y(Z_1Z_2)$.*

Proof. Let $x \in X_1(X_2YZ_1)Z_2$. Then

$$\begin{aligned} x &= \sum_{i=1}^{n} x_{1i}(u_i)z_{2i} \\ &= \sum_{i=1}^{n} x_{1i} \left(\sum_{j=1}^{k} x_{i2j} y_j z_{i1j} \right) z_{2i} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{k} x_{1i}(x_{i2j} y_j z_{i1j}) z_{2i} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{k} (x_{1i} x_{i2j}) y_j (z_{i1j} z_{2i}) \\ &\in (X_1X_2)Y(Z_1Z_2) \end{aligned}$$

where $x_{1i} \in X_1, x_{i2j} \in X_2$, $z_{i1j} \in Z_1, z_{2i} \in Z_2$, $y_j \in Y$, $u_i \in X_2YZ_1$ and k, n are positive integers. Conversely, let $x \in (X_1X_2)Y(Z_1Z_2)$. Then

$$x = \sum_{i=1}^{n} r_i y_i s_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{k} x_{i1j} x_{i2j} \right) y_i \left(\sum_{p=1}^{q} z_{i1p} z_{i2p} \right)$$

$$= \sum_{i=1}^{n} \sum_{p=1}^{q} \sum_{j=1}^{k} (x_{i1j} x_{i2j}) y_i (z_{i1p} z_{i2p}) = \sum_{i=1}^{n} \sum_{p=1}^{q} \sum_{j=1}^{k} x_{i1j}(x_{i2j} y_i z_{i1p}) z_{i2p} \in X_1(X_2YZ_1)Z_2$$

where $x_{i1j} \in X_1, x_{i2j} \in X_2$, $y_i \in Y$, $z_{i1p}, z_{i2p} \in Z_2$ and q, k, n are positive integers.

$\square$

Proposition 1.2.3. *Let M be an (R,S)-module.*

(i) *The intersection of any collection of (R,S)-submodules of M is an (R,S)-submodule.*

(ii) *The sum of two (R,S)-submodules of M is an (R,S)-submodule.*

(iii) *If I is a left ideal of R, J a right ideal of S and Y a nonempty subset of M, then IYJ is an (R,S)-submodule.*

(iv) *rMs is a subgroup of M for all $r \in R$ and $s \in S$.*

(v) *rMs is an (R,S)-submodule of M for all $r \in C(R)$ and $s \in C(S)$.*

Proof. The proofs of (i)–(iii) are straightforward.
(iv) Let $r \in R$ and $s \in S$. Clearly, $0 \in rMs$. Next, let $x, y \in rMs$. Then $x = ras$ and $y = rbs$ for some $a, b \in M$. Hence $x - y = ras - rbs = r(a-b)s \in rMs$. Therefore rMs is a subgroup of M.
(v) This follows from the fact that $R(rMs)S = r(RMS)s \subseteq rMs$ for all $r \in C(R)$ and $s \in C(S)$. $\square$

The (R,S)-**submodule of** M **generated by** $X \subseteq M$ is the intersection of all (R,S)-submodules of M containing X, and is denoted by $\langle X \rangle$.

Proposition 1.2.4. *Let M be an (R,S)-module. Then for any subset X of M, $\langle X \rangle = \mathbb{Z}X + RXS$. In particular, $\langle a \rangle = \mathbb{Z}a + RaS$ for all $a \in M$.*

Proof. Let $X \subseteq M$. It is clear that $\mathbb{Z}X + RXS \subseteq \langle X \rangle$. Since $\mathbb{Z}X + RXS$ is an (R,S)-submodule containing X, we have $\langle X \rangle \subseteq \mathbb{Z}X + RXS$. Hence $\langle X \rangle = \mathbb{Z}X + RXS$. $\square$

Corollary 1.2.5. *Let M be an (R,S)-module satisfying $a \in RaS$ for all $a \in M$. Then $\langle X \rangle = RXS$ for all $X \subseteq M$.*

Proof. This is straightforward. $\square$

The following proposition is a major tool for characterizing the primalities of (R,S)-submodules. Its proof is simple and is therefore omitted.

Proposition 1.2.6. *Let N be an (R,S)-submodule of M and X and Y nonempty subsets of R and S, respectively. If M satisfies $a \in RaS$ for all $a \in M$, then the following properties hold.*

(i) *(a) If $(RX)MS \subseteq N$, then $XMS \subseteq N$.*

(b) $XMS \subseteq (XR)MS$.

(ii) *(a) If $RM(YS) \subseteq N$, then $RMY \subseteq N$.*

(b) $RMY \subseteq RM(SY)$.

(iii) *$W \subseteq RWS$ for all subsets W of M. Moreover, equality holds if W is an (R,S)-submodule of M.*

We would like to point out that Proposition 1.2.6 (i)(b) and (ii)(b) are also valid if the condition "$a \in RaS$ for all $a \in M$" is replaced by "$RMS = M$".

Proposition 1.2.7. *Let M be an (R,S)-module.*

(i) *If $\{ I_x \mid x \in \Lambda \}$ is a collection of ideals of R, then $\displaystyle\sum_{x\in\Lambda}(I_xMS) = \left(\sum_{x\in\Lambda} I_x\right) MS$.*

(ii) *If $\{ J_x \mid x \in \Lambda \}$ is a collection of ideals of S, then $\displaystyle\sum_{x\in\Lambda}(RMJ_x) = RM\left(\sum_{x\in\Lambda} J_x\right)$.*

Proof. (i) First, since $I_y \subseteq \displaystyle\sum_{x\in\Lambda} I_x$ for all $y \in \Lambda$, we have $I_yMS \subseteq \left(\displaystyle\sum_{x\in\Lambda} I_x\right) MS$ for all $y \in \Lambda$. Then $\displaystyle\sum_{x\in\Lambda}(I_xMS) \subseteq \left(\sum_{x\in\Lambda} I_x\right) MS$.

Conversely, let $r \in \displaystyle\sum_{x\in\Lambda} I_x$ and $m \in M$ and $s \in S$. Then for each $x \in \Lambda$, there exists $r_x \in I_x$ such that $r = \displaystyle\sum_{x\in\Lambda} r_x$. Therefore $rms = \left(\displaystyle\sum_{x\in\Lambda} r_x\right) ms = \displaystyle\sum_{x\in\Lambda} r_xms \in \displaystyle\sum_{x\in\Lambda} I_xMS$.

(ii) The proof is similar to (i). $\square$

Next, we introduce two particular nonempty subsets of R and S which play important roles in this research. Let N be an (R,S)-submodule of an (R,S)-

module M. Define the following sets

$$(N : M)_R = \{r \in R \mid rMS \subseteq N\}$$
$$[N : M]_S = \{s \in S \mid RMs \subseteq N\}.$$

Proposition 1.2.8. *Let N be an (R, S)-submodule of an (R, S)-module M. The followings hold.*

(i) *$(N : M)_R$ is a subgroup of R under addition.*

(ii) *$[N : M]_S$ is a subgroup of S under addition.*

(iii) *If $S^2 = S$, then $(N : M)_R$ is an ideal of R.*

(iv) *If $R^2 = R$, then $[N : M]_S$ is an ideal of S.*

(v) *If $RMS = M$, then $(N : M)_R$ and $[N : M]_S$ are ideals of R and S, respectively.*

Proof. (i) Since $0_RMS = 0 \subseteq N$, $0_R \in (N : M)_R$. Let $a, b \in (N : M)_R$. Then $aMS \subseteq N$ and $bMS \subseteq N$. Hence $(a - b)MS \subseteq aMS + bMS \subseteq N$. This shows that $(N : M)_R$ is a subgroup of R under addition.

(iii) Assume that $S^2 = S$. Let $r \in R$ and $a \in (N : M)_R$. Then

$$(ra)MS = (ra)MSS = r(aMS)S \subseteq N$$

and

$$(ar)MS = (ar)MSS = a(rMS)S \subseteq aMS \subseteq N.$$

Hence $ar, ra \in (N : M)_R$. Therefore, $(N : M)_R$ is an ideal of R.

(v) Assume that $RMS = M$. Let $r \in R$ and $a \in (N : M)_R$. Then

$$\begin{aligned}(ra)MS &= (ra)(RMS)S \\ &= (raR)M(SS) \\ &= (raR)(RMS)(SS) \\ &= (raRR)M(SSS)\end{aligned}$$

$$
\begin{aligned}
&\subseteq (raR)M(SSS)\\
&= r(aRMSS)S\\
&= r(aMS)S\\
&\subseteq N
\end{aligned}
$$

and

$$
\begin{aligned}
(ar)MS &= (ar)(RMS)S\\
&= (arR)M(SS)\\
&\subseteq (aR)M(SS)\\
&= a(RMS)S\\
&= aMS\\
&\subseteq N.
\end{aligned}
$$

Therefore $ar, ra \in (N : M)_R$. This implies that $(N : M)_R$ is an ideal of R.

The proofs of (ii), (iv) and the rest of (v) are similar to above. □

1.3 (R, S)-Homomorphisms of (R, S)-Modules

Homomorphisms of each algebraic structure such as rings are major tools to study whether any two of them are identical. It is natural to study homomorphisms of (R, S)-modules.

Definition 1.3.1. Let R and S be rings and let M and N be (R, S)-modules. A map $\varphi : M \to N$ is called an (R, S)**-homomorphism** if it satisfies

(i) $\varphi(a + b) = \varphi(a) + \varphi(b)$

(ii) $\varphi(ras) = r\varphi(a)s$

for all $a, b \in M$, $r \in R$ and $s \in S$.

The standard terminology, an (R, S)-homomorphism φ is an (R, S)**-monomorphism** [respectively, **epimorphism**, **isomorphism**] if it is injective [respectively, surjective, bijective]. Moreover, M is **isomorphic to** N if there is an (R, S)-isomorphism from M onto N, denoted by $M \cong N$.

Proposition 1.3.2. *Let $\varphi : M \to N$ be an (R,S)-homomorphism. Then*

(i) $\varphi(0) = 0$.

(ii) $\varphi(-a) = -\varphi(a)$ *for all* $a \in M$.

(iii) $ker\varphi = \{m \in M \mid \varphi(m) = 0\}$ *is an* (R,S)*-submodule of* M.

(iv) $im\varphi = \{\varphi(m) \mid m \in M\}$ *is an* (R,S)*-submodule of* N.

(v) *If* K *is an* (R,S)*-submodule of* N*, then* $\varphi^{-1}(K) = \{m \in M \mid \varphi(m) \in K\}$ *is an* (R,S)*-submodule of* M.

Proof. The proof is straightforward. □

Let N be an (R,S)-submodule of an (R,S)-module M. Then $(M/N, \oplus)$ is an abelian group where $M/N = \{m+N \mid m \in M\}$ and $(a+N)\oplus(b+N) = (a+b)+N$ for all $a, b \in M$. Moreover, M/N is an (R,S)-module by $(r, m+N, s) \longmapsto rms+N$. This (R,S)-module is called the **quotient** (R,S)**-module of** M **by** N.

Theorem 1.3.3. ***Isomorphism Theorem for*** (R,S)***-Modules***

(i) ***The First Isomorphism Theorem***

If $\varphi : M \to N$ *is an* (R,S)*-homomorphism, then* $M/ker\varphi \cong im\varphi$.

(ii) ***The Second Isomorphism Theorem***

If A *and* B *are* (R,S)*-submodules of an* (R,S)*-module* M*, then* $(A+B)/B \cong A/(A \cap B)$.

(iii) ***The Third Isomorphism Theorem***

If A *and* B *are* (R,S)*-submodules of an* (R,S)*-module* M *with* $A \subseteq B$*, then* $(M/A)/(B/A) \cong M/B$.

Proof. (i) By the first isomorphism theorem for groups, the map $\hat{\varphi} : M/ker\varphi \to im\varphi$ defined by $\hat{\varphi}(m + ker\varphi) = \varphi(m)$ is a group isomorphism. We only need to show that $\hat{\varphi}$ preserves the module action. For each $r \in R$, $m \in M$ and $s \in S$, we obtain that $\hat{\varphi}(r\cdot(m+ker\varphi)\cdot s) = \hat{\varphi}(rms+ker\varphi) = \varphi(rms) = r\varphi(m)s = r\cdot\hat{\varphi}(m+ker\varphi)\cdot s$.

(ii) and (iii) follow from applying the first Isomorphism Theorem. □

Lemma 1.3.4. *Let $f : M \to K$ be an (R, S)-homomorphism.*

(i) *For each left ideal X of R, (R, S)-submodule Y of K and right ideal Z of S, $Xf^{-1}(Y)Z \subseteq f^{-1}(XYZ)$.*

(ii) *For subsets A and B of M, if $f(A) \subseteq f(B)$, then $A \subseteq B + kerf$.*

(iii) *Assume further that f is an (R, S)-epimorphism.*

(a) *If $a \in RaS$ for all $a \in M$, then $a \in RaS$ for all $a \in K$.*

(b) *If P is a proper (R, S)-submodule of M containing $kerf$, then $f(P)$ is a proper (R, S)-submodule of K.*

(c) *If L is a proper (R, S)-submodule of K, then $f^{-1}(L)$ is a proper (R, S)-submodule of M.*

Proof. (i) Let X be a left ideal of R, Y an (R, S)-submodule of K and Z a right ideal of S. Let $x \in X, w \in f^{-1}(Y)$ and $z \in Z$. Then $f(w) \in Y$. Since f is an (R, S)-homomorphism, $f(xwz) = xf(w)z \in XYZ$. Hence $xwz \in f^{-1}(XYZ)$.

(ii) If $A = \varnothing$, then it is done. Now, let $a \in A$. Then $f(a) \in f(B)$. So $f(a) = f(b)$ for some $b \in B$. It implies that $a - b \in kerf$. Hence $a = b + (a - b) \in B + kerf$.

(iii) These are obvious.

□

1.4 Motivation

There are various definitions of prime submodules of modules, see [1], [2] and [11]. We would like to extend such definitions to (R, S)-modules. Almost all authors investigate prime submodules of a unital module over a commutative ring with identity, so we will focus on this case as our motivation.

Definition 1.4.1. [1] Let R be a commutative ring with identity and M be a unital left R-module. A proper submodule N of M is called **prime** if for all $r \in R$ and $m \in M$, $rm \in N$ implies $rM \subseteq N$ or $m \in N$.

Let R be a commutative ring with identity and M a unitary left R-module. Let N be a submodule of M. Then $(N : M) = \{r \in R \mid rM \subseteq N\}$ is an ideal of R, see [1].

Z. El-Bast and P. Smith [6] gave a characterization of prime submodules of a unital module as follows.

Theorem 1.4.2. *[6] Let M be a left R-module and P a proper submodule of M. Then the following statements are equivalent.*

(i) *P is a prime submodule of M.*

(ii) *$(P : M)$ is a prime ideal of R.*

(iii) *$P = AM$ for some prime ideal A of R such that $(0 : M) \subseteq A$.*

Multiplication modules play an important role in studying prime submodules. One can define the product of two submodules of a multiplication module and then use this to charaterize its prime submodules.

Definition 1.4.3. [1] Let R be a commutative ring with identity. A unitary left R-module M is said to be a **multiplication module** if for each submodule N of M, there exists an ideal I of R such that $N = IM$. Such an ideal I is called a **presentation** of N.

Z. El-Bast and P. Smith [6] investigated multiplication modules and examined in particular when modules are multiplication modules. Some of their results are as follows.

Let R be a commutative ring with identity and M a unitary left R-module. Moreover, let N be a submodule of M and P a maximal ideal of R. El-Bast and Smith defined $T_P(M) = \{m \in M \mid (1-p)m = 0 \text{ for some } p \in P\}$. Then M is called **P-cyclic** if there exist $p \in P$ and $m \in M$ such that $(1-p)M \subseteq Rm$.

They also gave a characterization of multiplication modules as follows.

Theorem 1.4.4. *[6] Let R be a commutative ring with identity and M a unital left R-module. Then the following statements are equivalent.*

(i) *M is a multiplication module.*

(ii) *For any submodule N of M, $N = (N : M)M$.*

(iii) *For each $m \in M$, there exists an ideal I of R such that $Rm = IM$.*

(iv) *For every maximal ideal P of R, $M = T_P(M)$ or M is P-cyclic.*

Other studies on characterizations of multiplication modules have been provided. E. Kim and C. Choi [9] introduced the ideal $\sum_{m \in M}(Rm : M)$. This ideal has been fully taken advantage of to give necessary and sufficient conditions for a module to be a multiplication module.

Theorem 1.4.5. *[9] Let R be a commutative ring with identity and M a unital left R-module. A left R-module M is a multiplication module if and only if $Rm' = \left(\sum_{m \in M}(Rm : M)\right) m'$ for all $m' \in M$.*

The definition of a multiplication module is a clever idea because it allows us to define the product of two submodules. This notion was given by R. Ameri [1] as follows.

Definition 1.4.6. [1] Let M be multiplication module. Let N and K be submodules of M such that $N = IM$ and $K = JM$ for some ideals I and J of R. The **product** of N and K, denoted by NK, is defined by IJM.

Note that for all $m, n \in M$, mn means the product of the submodules Rm and Rn of M. R. Ameri showed that the product of two submodules is well-defined and does not depend on the presentation ideals of each submodule. Moreover, he also used the product of two submodules to characterize prime submodules of a multiplication module.

Theorem 1.4.7. *[1] Let M be a multiplication module. Then the product of submodules N and K of M is independent of presentations of N and K.*

Theorem 1.4.8. *[1] Let M be a multiplication module and N a proper submodule of M. Then the following statements are equivalent.*

(i) *N is prime.*

(ii) *For any submodule U and V of M, $UV \subseteq N$ implies $U \subseteq N$ or $V \subseteq N$.*

(iii) *For each $a, b \in M$, $ab \subseteq N$ implies $a \in N$ or $b \in N$.*

R.J. Nezhad and M.H. Naderi [10] emphasized that the definition of the product of elements in a multiplication module is useful. They introduced some special submodules of a multiplication module and used their tools to characterize prime submodules of a multiplication module as follows.

Definition 1.4.9. [10] Let M be a multiplication module and N and L submodules of M. The **residual** of L by N in M is $(L :_M N) = \{m \in M \mid mn \subseteq L$ for all $n \in N\}$.

Theorem 1.4.10. *[10] Let M be a multiplication module and L be a proper submodule of M. Then the following statements are equivalent.*

(i) *L is prime.*

(ii) *For every submodule N of M, if $N \nsubseteq L$, then $(L :_M N) = L$.*

All of the above results are stepping stones to study a new algebraic structure, namely the (R, S)-module. Moreover, the definition of prime submodules as given in Definition 1.4.1 is a guideline for us to define a prime (R, S)-submodule as well as a left multiplication (R, S)-module. We focus very much on providing a definition of prime (R, S)-submodules that is more general.

CHAPTER II

PRIMALITIES OF (R, S)-SUBMODULES

The notion "prime"appears almost all algebraic structures and has been studied in various incarnations such as prime integers in the natural numbers, prime ideals of semigroups, prime ideals of rings and prime submodules of modules. In Chapter I, we gave a new structure namely, the (R, S)-module. It is reasonable to study "prime" (R, S)-submodules of (R, S)-modules. However, there are many choices of ways to extend the concept of prime submodules to "prime" (R, S)-submodules.

An (R, S)-module M is composed of three important ingredients, which are two arbitrary rings R and S and an abelian group M. The first thought about primality for a proper (R, S)-submodule P of M is that every component of any (R, S)-submodule of the form INJ contained in P can be considered separately. To be precise, for each left ideal I of R, (R, S)-submodule N of M and right ideal J of S, $INJ \subseteq P$ implies $IMS \subseteq P$ or $N \subseteq P$ or $RMJ \subseteq P$. We see that each component, I (from R), N (from M) and J (from S), is considered separately and none of them depends on the others. In this situation, such a proper (R, S)-submodule P of M is called a fully prime (R, S)-submodule of M.

The second thought for primality pays attention to the significant mutuality of the ring R on the left and the ring S on the right of M. For each left ideal I of R, (R, S)-submodule N of M and right ideal J of S, $INJ \subseteq P$ implies $IMJ \subseteq P$ or $N \subseteq P$. The togetherness of I and J is the concept of this type, where we see that the components I and J come together ($INJ \subseteq P$) and have to go together ($IMJ \subseteq P$). This kind of an (R, S)-submodule P of M is called a jointly prime (R, S)-submodule of M.

For the last thought regarding primality, we pay attention to the ring R greatly while the ring S is fixed. For each pair of ideal I and J of R, $IJMSS \subseteq P$ implies $IMS \subseteq P$ or $JMS \subseteq P$. Although the ring R plays a major role, certain properties of the ring S cannot be omitted. Such a proper (R, S)-submodule P of M is called a left R-prime (R, S)-submodule of M.

However, there may be other notions of primality of (R, S)-submodules that are equally reasonable. For this dissertation, we investigate three natural ways of being prime for (R, S)-submodules, namely, fully prime (R, S)-submodules in the first section, jointly prime (R, S)-submodules in the second section and left R-prime (R, S)-submodules at the end.

2.1 Fully Prime (R, S)-Submodules

In this section, we first give the definition of fully prime (R, S)-submodules along with some examples. Our main goal is to provide various characterizations of fully prime (R, S)-submodules. Relationships between fully prime (R, S)-submodules and prime ideals of the rings R and S are also studied.

Definition 2.1.1. A proper (R, S)-submodule P of M is called **fully prime** if for each left ideal I of R, right ideal J of S and (R, S)-submodule N of M,

$$INJ \subseteq P \text{ implies } IMS \subseteq P \text{ or } N \subseteq P \text{ or } RMJ \subseteq P.$$

Example 2.1.2. Let $r, s \in \mathbb{Z}^+ \backslash \{1\}$. Recall that $\mathbb{Z}$ is an $(r\mathbb{Z}, s\mathbb{Z})$-module and $(r\mathbb{Z}, s\mathbb{Z})$-submodules of $\mathbb{Z}$ are of the form $d\mathbb{Z}$ for some $d \in \mathbb{Z}$. Then it is easy to see that $\{0\}$, $(rs)\mathbb{Z}$, $r\mathbb{Z}$, $s\mathbb{Z}$ and $p\mathbb{Z}$, where p is a prime integer, are fully prime $(r\mathbb{Z}, s\mathbb{Z})$-submodules of $\mathbb{Z}$.

However, $(rsk)\mathbb{Z}$ is not a fully prime $(r\mathbb{Z}, s\mathbb{Z})$-submodule of $\mathbb{Z}$ for all $k \in \mathbb{Z} \backslash \{0, 1, -1\}$ because if $I = r\mathbb{Z}$, $N = k\mathbb{Z}$ and $J = s\mathbb{Z}$, then $INJ \subseteq (rsk)\mathbb{Z}$ but $IMS \nsubseteq (rsk)\mathbb{Z}$ and $N \nsubseteq (rsk)\mathbb{Z}$ and $RMJ \nsubseteq (rsk)\mathbb{Z}$.

Example 2.1.2 shows that prime ideals of the ring $\mathbb{Z}$ and fully prime $(r\mathbb{Z}, s\mathbb{Z})$-submodules of the $(r\mathbb{Z}, s\mathbb{Z})$-module $\mathbb{Z}$ are different even though ideals of $\mathbb{Z}$ as a ring $(r\mathbb{Z}, s\mathbb{Z})$-submodules of $\mathbb{Z}$ as an $(r\mathbb{Z}, s\mathbb{Z})$-module as. In other words, there exists an ideal I of $\mathbb{Z}$ such that I is not a prime ideal of $\mathbb{Z}$ but I is a fully prime $(r\mathbb{Z}, s\mathbb{Z})$-submodule of $\mathbb{Z}$.

Example 2.1.3. Let A be a commutative ring with identity 1. We set

$$R = \left\{ \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \middle| \, x, y \in A \right\}, \qquad S = \left\{ \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} \middle| \, x, y \in A \right\}$$

and

$$M = \left\{ \begin{bmatrix} x & 0 \\ y & z \end{bmatrix} \middle| \, x, y, z \in A \right\}.$$

Recall that M is an (R, S)-module under the usual matrix addition and multiplication. Then

$$P = \left\{ \begin{bmatrix} x & 0 \\ ay & bz \end{bmatrix} \middle| \, x, y, z \in A \right\}$$

is a fully prime (R, S)-submodule of M for all $a, b \in A$ where $(a, b) \neq (1, 1)$.

Proof. It is clear that $RMS \subseteq \left\{ \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \middle| \, x \in A \right\}$ and P is a proper (R, S)-submodule of M. Since for any $x \in A$,

$$\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

we have $RMS = \left\{ \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \middle| \, x \in A \right\} \subseteq P$. This implies that P is a fully prime (R, S)-submodule of M. $\square$

We give a necessary condition for an (R, S)-submodule to be a fully prime (R, S)-submodule.

Theorem 2.1.4. *Let P be a proper (R, S)-submodule of M. If P satisfies the condition that for all $r \in R$, $m \in M$ and $s \in S$,*

$$rms \in P \text{ implies } rMS \subseteq P \text{ or } m \in P \text{ or } RMs \subseteq P, \tag{2.1}$$

then P is a fully prime (R, S)-submodule.

Furthermore, if R and S are commutative rings and $RMS = M$, then the converse is valid, i.e., if P is a fully prime (R, S)-submodule, then the condition (2.1) *holds for all $r \in R$, $m \in M$ and $s \in S$.*

Proof. Assume that the condition (2.1) holds for all $r \in R$, $m \in M$ and $s \in S$. Let I be a left ideal of R, J a right ideal of S and N an (R,S)-submodule of M such that $INJ \subseteq P$, $N \nsubseteq P$ and $RMJ \nsubseteq P$. To show that $IMS \subseteq P$, let $\alpha \in I$. Moreover, let $n \in N \backslash P$ and $\beta \in J$ be such that $RM\beta \nsubseteq P$. Then $\alpha n \beta \in P$. Since P satisfies the condition (2.1), $\alpha MS \subseteq P$. This shows that $IMS \subseteq P$. Therefore P is fully prime.

Next, assume that R and S are commutative rings and $RMS = M$. Let P be a fully prime (R,S)-submodule and $r \in R$, $m \in M$ and $s \in S$ be such that $rms \in P$. Since R and S are commutative rings, $(rR)\langle m\rangle(Ss) = (rR)(\mathbb{Z}m + RmS)(Ss) \subseteq P$. Since P is fully prime, $(rR)MS \subseteq P$ or $m \in P$ or $RM(Ss) \subseteq P$. We consider the case $(rR)MS \subseteq P$. Then $rMS = (rR)MSS \subseteq (rR)MS \subseteq P$ since $RMS = M$. Similarly, if $RM(Ss) \subseteq P$, then $RMs \subseteq P$. Hence the condition (2.1) holds. $\square$

A characterization of (R,S)-submodules satisfying the condition (2.1) is given by making use of Proposition 1.2.6.

Theorem 2.1.5. *Let M be an (R,S)-module satisfying $a \in RaS$ for all $a \in M$ and P a proper (R,S)-submodule of M. Then P satisfies the condition (2.1) if and only if for all left ideals I of R, elements $m \in M$ and right ideals J of S,*

$$ImJ \subseteq P \textit{ implies } IMS \subseteq P \textit{ or } m \in P \textit{ or } RMJ \subseteq P.$$

Proof. First, assume that P satisfies the condition (2.1). Let I be a left ideal of R, $m \in M$ and J a right ideal of S such that $ImJ \subseteq P$. Assume further that $m \notin P$ and $RMJ \nsubseteq P$. Then there exists $\beta \in J$ such that $RM\beta \nsubseteq P$. Thus, for each $\alpha \in I$, we obtain that $\alpha m \beta \in P$ so that $\alpha MS \subseteq P$ by condition (2.1). This shows that $IMS \subseteq P$.

Conversely, assume that $ImJ \subseteq P$ implies $IMS \subseteq P$ or $m \in P$ or $RMJ \subseteq P$ for every left ideal I of R, element $m \in M$ and right ideal J of S . Let $r \in R$, $m \in M$ and $s \in S$ be such that $rms \in P$. Then $(Rr)m(sS) = R(rms)S \subseteq P$. This implies that $(Rr)MS \subseteq P$ or $m \in P$ or $RM(sS) \subseteq P$. Applying Proposition 1.2.6 yields that $rMS \subseteq P$ or $m \in P$ or $RMs \subseteq P$. Hence P satisfies the condition (2.1). $\square$

The following result gives many alternative methods to characterize fully prime (R,S)-submodules. We observe that the definition of fully prime (R,S)-submodules considers only left ideals of R and right ideals of S. The condition $a \in RaS$ for all $a \in M$ is an important tool that allows us to switch "left ideal I of R "to "right ideal I of R ". Similarly, under the same assumption, "right ideal J of S "can be switched to "left ideal J of S ". Hence characterization of fully prime (R,S)-submodules may be easier because left ideals or right ideals of either ring can be considered.

Theorem 2.1.6. *Let M be an (R,S)-module satisfying the property that $a \in RaS$ for all $a \in M$ and let P be a proper (R,S)-submodule of M. Then the following statements are equivalent.*

(i) *P is fully prime.*

(ii) *For all right ideals I of R, $m \in M$ and left ideals J of S,*

$$ImJ \subseteq P \text{ implies } IMS \subseteq P \text{ or } m \in P \text{ or } RMJ \subseteq P.$$

(iii) *For all right ideals I of R, (R,S)-submodules N of M and left ideals J of S,*

$$INJ \subseteq P \text{ implies } IMS \subseteq P \text{ or } N \subseteq P \text{ or } RMJ \subseteq P.$$

(iv) *For all left ideals I of R, $m \in M$ and right ideals J of S,*

$$(IR)m(SJ) \subseteq P \text{ implies } IMS \subseteq P \text{ or } m \in P \text{ or } RMJ \subseteq P.$$

(v) *For all $a \in R, m \in M$ and $b \in S$,*

$$(aR)m(Sb) \subseteq P \text{ implies } aMS \subseteq P \text{ or } m \in P \text{ or } RMb \subseteq P.$$

Proof. **(i) $\Rightarrow$ (ii)** Assume (i). Let I be a right ideal of R, $m \in M$ and J a left ideal of S such that $ImJ \subseteq P$. Then $(RI)(RmS)(JS) \subseteq P$. By (i), we have $(RI)MS \subseteq P$ or $RmS \subseteq P$ or $RM(JS) \subseteq P$. It follows from Proposition 1.2.6 that $IMS \subseteq P$ or $m \in P$ or $RMJ \subseteq P$.

(ii) $\Rightarrow$ (iii) Assume (ii). Let I be a right ideal of R, J a left ideal of S and N an (R,S)-submodule of M such that $INJ \subseteq P$. Suppose that $N \nsubseteq P$ and $RMJ \nsubseteq P$. Let $n \in N \backslash P$. Then $InJ \subseteq INJ \subseteq P$ so that $IMS \subseteq P$ from (ii).

(iii) $\Rightarrow$ (iv) Assume (iii). Let I be a left ideal of R, $m \in M$ and J a right ideal of S such that $(IR)m(SJ) \subseteq P$. Then $(IR)(RmS)(SJ) \subseteq P$. We obtain from (iii) that $(IR)MS \subseteq P$ or $RmS \subseteq P$ or $RM(SJ) \subseteq P$. By Proposition 1.2.6, we have $IMS \subseteq P$ or $m \in P$ or $RMJ \subseteq P$.

(iv) $\Rightarrow$ (i) Assume (iv). Let I be a left ideal of R, J a right ideal of S and N an (R,S)-submodule of M such that $INJ \subseteq P$. Suppose that $N \nsubseteq P$ and $RMJ \nsubseteq P$. Let $n \in N \backslash P$. Then $(IR)n(SJ) \subseteq P$. By (iv), we have $IMS \subseteq P$.

(ii) $\Rightarrow$ (v) This is obtained from (ii) and Proposition 1.2.6.

(v) $\Rightarrow$ (iii) Assume (v). Let I be a right ideal of R, J a left ideal of S and N an (R,S)-submodule of M such that $INJ \subseteq P$. Suppose that $N \nsubseteq P$ and $RMJ \nsubseteq P$. Let $n \in N \backslash P$ and $b \in J$ with $RMb \nsubseteq P$. To show that $IMS \subseteq P$, let $a \in I$. Then $(aR)n(Sb) \subseteq P$. By (v), we have $aMS \subseteq P$. This implies that $IMS \subseteq P$. □

If the condition $a \in RaS$ for all $a \in M$ is replaced by $RMS = M$, then one-sided ideals of R and S can be replaced by ideals of R and S, respectively, in order to verify fully prime (R,S)-submodules.

Theorem 2.1.7. *Let M be an (R,S)-module satisfying $RMS = M$ and P a proper (R,S)-submodule of M. Then P is fully prime if and only if for all ideals I of R, ideals J of S and (R,S)-submodules N of M, $INJ \subseteq P$ implies $IMS \subseteq P$ or $N \subseteq P$ or $RMJ \subseteq P$.*

Proof. $(\Rightarrow)$ This part follows from the definition.

$(\Leftarrow)$ Let I be a left ideal of R, J a right ideal of S and N an (R,S)-submodule of M such that $INJ \subseteq P$. Then $(IR)N(SJ) \subseteq P$. Since IR is an ideal of R and SJ is an ideal of S, we have $(IR)MS \subseteq P$ or $N \subseteq P$ or $RM(SJ) \subseteq P$. If $(IR)MS \subseteq P$, then $IMS = I(RMS)S = (IR)M(SS) \subseteq (IR)MS \subseteq P$. Similarly, if $RM(SJ) \subseteq P$, then $RMJ \subseteq P$. This shows that $IMS \subseteq P$ or $N \subseteq P$ or $RMJ \subseteq P$. Hence P is fully prime. □

Theorem 1.4.2 described a relationship between the primality of a submodule P of a module M and the primality of the ideal $(P : M)$ of R. This suggests that we look for a connection between fully prime (R, S)-submodules of (R, S)-modules and prime ideals of rings.

Recall that the additive subgroups $(N : M)_R$ of R and $[N : M]_S$ of S are defined by

$$(N : M)_R = \{r \in R \mid rMS \subseteq N\} \text{ and } [N : M]_S = \{s \in S \mid RMs \subseteq N\}$$

for an (R, S)-submodule N of the (R, S)-module M. Both of $(N : M)_R$ and $[N : M]_S$ may not necessarily be ideals. However, if $S^2 = S$, then $(N : M)_R$ is an ideal of R, and similarly, if $R^2 = R$, then $[N : M]_S$ is an ideal of S.

Theorem 2.1.8. *Let P be an (R, S)-submodule of M such that $(P : M)_R$ and $[P : M]_S$ are proper ideals of R and S, respectively. If P is fully prime, then $(P : M)_R$ and $[P : M]_S$ are prime ideals of R and S, respectively.*

Proof. Let A and B be ideals of R such that $AB \subseteq (P : M)_R$. Then $(AB)MS \subseteq P$. Hence $A(BMS)S = (AB)MS^2 \subseteq (AB)MS \subseteq P$, so $AMS \subseteq P$ or $BMS \subseteq P$ or $RMS \subseteq P$. If $RMS \subseteq P$, then $(P : M)_R = R$, contrary to $(P : M)_R$ a proper ideal of R, so $RMS \nsubseteq P$. Thus $AMS \subseteq P$ or $BMS \subseteq P$. If $AMS \subseteq P$, then $A \subseteq (P : M)_R$ and if $BMS \subseteq P$, then $B \subseteq (P : M)_R$, so $A \subseteq (P : M)_R$ or $B \subseteq (P : M)_R$. Hence $(P : M)_R$ is a prime ideal of R. Similarly, we can show that $[P : M]_R$ is a prime ideal of S. $\square$

We have seen before that maximality implies primality. For example,

(i) in a commutative ring with non-zero identity, any maximal ideal is a prime ideal;

(ii) in a unital left module over a commutative ring with non-zero identity, any maximal submodule is a prime submodule.

From those results, we observe that commmutativity and identity are impotant for both structures. Unfortunately, commmutativity and identity are not required in all (R,S)-modules. However, we can prove that every maximal (R,S)-submodule of M is fully prime if the condition "$RMS = M$"is added.

Proposition 2.1.9. *Let M be an (R,S)-module such that $RMS = M$. Then every maximal (R,S)-submodule of M is a fully prime (R,S)-submodule of M.*

Proof. Let K be a maximal (R,S)-submodule of M. We prove that K is fully prime by using Theorem 2.1.7. Let I be an ideal of R, N an (R,S)-submodule of M and J an ideal of S such that $INJ \subseteq K$. Assume that $N \nsubseteq K$ and $RMJ \nsubseteq K$. Then $RMJ + K = M = N + K$. Therefore

$$IMJ = I(N+K)J = INJ + IKJ \subseteq K.$$

Moreover,

$$IMS = I(RMJ+K)S = (IR)M(JS) + IKS \subseteq IMJ + IKS \subseteq K.$$

The proof is complete. $\square$

We give an example that a fully prime (R,S)-submodule need not be maximal.

Example 2.1.10. Note that $\mathbb{Z}$ is a $(\mathbb{Z},\mathbb{Z})$-module such that $\mathbb{Z}^3 = \mathbb{Z}$ and 0 is a fully prime $(\mathbb{Z},\mathbb{Z})$-submodule of $\mathbb{Z}$ but 0 is not a maximal $(\mathbb{Z},\mathbb{Z})$-submodule of $\mathbb{Z}$.

A homomorphism is an algebraic structure-preserving map. One might ask whether an (R,S)-homomorphism preserves fully prime (R,S)-submodules. The answer is "NO" in general. See the following example.

Example 2.1.11. Recall that $\mathbb{Z}$ is a $(\mathbb{Z},\mathbb{Z})$-module. The $(\mathbb{Z},\mathbb{Z})$-homomorphism $g : \mathbb{Z} \to \mathbb{Z}$ defined by $g(x) = 2x$ does not preserve fully prime (R,S)-submodules, since $2\mathbb{Z}$ is a fully prime $(\mathbb{Z},\mathbb{Z})$-submodule of $\mathbb{Z}$ but $g(2\mathbb{Z}) = 4\mathbb{Z}$ is not a fully prime $(\mathbb{Z},\mathbb{Z})$-submodule of $\mathbb{Z}$.

Next, we show that (R,S)-epimorphisms do preserve fully prime (R,S)-submodules.

Proposition 2.1.12. *Let $f : M \to K$ be an (R,S)-homomorphism.*

(i) *If L is a fully prime (R,S)-submodule of K, then $f^{-1}(L)$ is a fully prime (R,S)-submodule of M containing ker f or $f^{-1}(L) = M$.*

(ii) *If P is a fully prime (R,S)-submodule of M containing ker f and f is onto, then $f(P)$ is a fully prime (R,S)-submodule of K.*

Proof. (i) Let L be a fully prime (R,S)-submodule of K such that $f^{-1}(L) \neq M$. To show that $f^{-1}(L)$ is a fully prime (R,S)-submodule of M containing ker f, let I be a left ideal of R, N an (R,S)-submodule of M and J a right ideal of S be such that $INJ \subseteq f^{-1}(L)$. Then $If(N)J = f(INJ) \subseteq f(f^{-1}(L)) \subseteq L$. Since L is a fully prime (R,S)-submodule of K, $IKS \subseteq L$ or $f(N) \subseteq L$ or $RKJ \subseteq L$. If $IKS \subseteq L$, then it implies that $IMS = If^{-1}(K)S \subseteq f^{-1}(IKS) \subseteq f^{-1}(L)$. If $f(N) \subseteq L$, then $N \subseteq f^{-1}(L)$. If $RKJ \subseteq L$, then $RMJ = Rf^{-1}(K)J \subseteq f^{-1}(RKJ) \subseteq f^{-1}(L)$. Hence $IMS \subseteq f^{-1}(L)$ or $N \subseteq f^{-1}(L)$ or $RMJ \subseteq f^{-1}(L)$. Since $0 \in L$, ker $f = f^{-1}(0) \subseteq f^{-1}(L)$. Therefore $f^{-1}(L)$ is a fully prime (R,S)-submodule of M containing ker f.

(ii) Let P be a fully prime (R,S)-submodule of M containing ker f and f is onto. To show that $f(P)$ is a fully prime (R,S)-submodule of K, let I be a left ideal of R, N an (R,S)-submodule of K and J a right ideal of S such that $INJ \subseteq f(P)$. Then $f^{-1}(N)$ is an (R,S)-submodule of M. Since f is an (R,S)-homomorphism, $f(If^{-1}(N)J) = If(f^{-1}(N))J \subseteq INJ \subseteq f(P)$. Then $If^{-1}(N)J \subseteq P+$ ker $f \subseteq P$ by Lemma 1.3.4 (ii). Since P is a fully prime (R,S)-submodule of M, we obtain that $IMS \subseteq P$ or $f^{-1}(N) \subseteq P$ or $RMJ \subseteq P$. Note that f is onto. If $IMS \subseteq P$, then it implies that $IKS = If(M)S = f(IMS) \subseteq f(P)$. If $f^{-1}(N) \subseteq P$, then $N \subseteq f(P)$. If $RMJ \subseteq P$, then $RKJ = Rf(M)J = f(RMJ) \subseteq f(P)$. Therefore $IKS \subseteq f(P)$ or $N \subseteq f(P)$ or $RKJ \subseteq f(P)$. Hence $f(P)$ is a fully prime (R,S)-submodule of K. $\square$

Theorem 2.1.13. *Let $f : M \to K$ be an (R,S)-epimorphism. Then there exists a one-to-one correspondence between the set of all fully prime (R,S)-submodules of M containing ker f and the set of all fully prime (R,S)-submodules of K.*

Proof. To complete this proof, we define a function ϕ by $\phi(P) = f(P)$ for all fully prime (R,S)-submodules P of M containing ker f. The above work shows that ϕ is a one-to-one correspondence between the set of all fully prime (R,S)-submodules of M containing ker f and the set of all fully prime (R,S)-submodules of K since f onto implies $f(f^{-1}(L)) = L$ for all (R, S)-submodules L of K and $f^{-1}(f(P)) = P$ for any (R, S)-submodule P of M with ker $f \subseteq P$. $\square$

Let $f : M \to K$ be an (R, S)-homomorphism. If f is an (R, S)-monomorphism, then every (R, S)-submodule of M contains ker f. We observe from Theorem 2.1.13 that if f is an (R, S)-isomorphism, then the set of all fully prime (R, S)-submodules of M and the set of all fully prime (R, S)-submodules of K are isomorphic. The following shows that if f is not an (R, S)-monomorphism, there may be an (R, S)-submodule which does not contain kef f.

Example 2.1.14. We know that $\mathbb{Z}_6$ is a $(\mathbb{Z}_6, \mathbb{Z}_6)$-module. A $(\mathbb{Z}_6, \mathbb{Z}_6)$-homomorphism g from $\mathbb{Z}_6$ into $\mathbb{Z}_6$ defined by $g(x) = 2x$ for all $x \in \mathbb{Z}_6$ is not a $(\mathbb{Z}_6, \mathbb{Z}_6)$-monomorphism and $\{0, 2, 4\}$ is a $(\mathbb{Z}_6, \mathbb{Z}_6)$-submodule of $\mathbb{Z}_6$ not containing ker $g = \{0, 3\}$.

2.2 Jointly Prime (R, S)-Submodules

Another way to motivate the idea of "prime" (R, S)-submodules is to observe that the structure of (R, S)-modules consists of two rings R and S on both sides of an abelian group M in which R and S always come together. We introduce jointly prime (R, S)-submodules under the concept of "together" that is "Whatever comes together, should go together." It will be seen that jointly prime (R, S)-submodules are generalizations of fully prime (R, S)-submodules.

Definition 2.2.1. A proper (R, S)-submodule P of M is called **jointly prime** if for each left ideal I of R, and right ideal J of S, and (R, S)-submodule N of M,

$$INJ \subseteq P \text{ implies } IMJ \subseteq P \text{ or } N \subseteq P.$$

Clearly, every fully prime (R, S)-submodule is jointly prime. Moreover, if either R or S is commutative, it can be shown that jointly prime (R, S)-submodules and fully prime (R, S)-submodules are identical.

Theorem 2.2.2. *Let M be an (R,S)-module. If R or S are commutative rings, then fully prime (R,S)-submodules and jointly prime (R,S)-submodules are the same.*

Proof. It is sufficient to consider the case where R is commutative since the case where S is commutative is nearly identical. Thus, assume that R is commutative. It is enough to show that a jointly prime (R,S)-submodule is fully prime. Let P be a jointly prime (R,S)-submodule, I a left ideal of R, N an (R,S)-submodule of M and J a right ideal of S such that $INJ \subseteq P$. Since P is jointly prime, $IMJ \subseteq P$ or $N \subseteq P$. Clearly, we only need to consider the case $IMJ \subseteq P$. Since R is commutative, $I(RMJ)S = R(IMJ)S \subseteq P$. Again, from the fact that P is jointly prime, $IMS \subseteq P$ or $RMJ \subseteq P$. Hence P is fully prime. □

Properties of jointly prime (R,S)-submodules are studied in the same manner as those of fully prime (R,S)-submodules.

Theorem 2.2.3. *Let P be a proper (R,S)-submodule of M. If P satisfies the property that for all $r \in R$, $s \in S$ and $m \in M$*

$$rms \in P \text{ implies } rMs \subseteq P \text{ or } m \in P, \tag{2.2}$$

then P is a jointly prime (R,S)-submodule.

Furthermore, if R and S are commutative rings and $RMS = M$, then the converse also holds, i.e., if P is a jointly prime (R,S)-submodule of M, then the condition (2.2) holds for all $r \in R$, $s \in S$ and all $m \in M$.

Proof. Assume that the condition (2.2) holds for all $r \in R$, $s \in S$ and all $m \in M$. Let I be a left ideal of R, J a right ideal of S and N an (R,S)-submodule of M such that $INJ \subseteq P$ and $N \nsubseteq P$. To show that $IMJ \subseteq P$, let $\alpha \in I$ and $\beta \in J$ and let $n \in N \backslash P$. Then $\alpha n \beta \in P$. Since P satisfies the condition (2.2), $\alpha M \beta \subseteq P$. This shows that $\alpha M \beta \subseteq P$ for all $\alpha \in I$ and $\beta \in J$. Hence $IMJ \subseteq P$. Therefore P is jointly prime.

Now, assume that R and S are commutative rings and $RMS = M$. Let P be jointly prime. Then P is also fully prime. By Theorem 2.1.4, P satisfies the condition (2.1). Then the condition (2.2) follows. □

A characterization of a proper (R, S)-submodule satisfying the condition (2.2) is given.

Theorem 2.2.4. *Let M be an (R, S)-module satisfying $a \in RaS$ for all $a \in M$ and P a proper (R, S)-submodule of M. Then P satisfies the condition* (2.2) *if and only if for all left ideals I of R, elements $m \in M$ and right ideals J of S,*

$$ImJ \subseteq P \text{ implies } IMJ \subseteq P \text{ or } m \in P.$$

Proof. First, assume that P satisfies the condition (2.2). Let I be a left ideal of R, $m \in M$ and J a right ideal of S such that $ImJ \subseteq P$. Assume that $m \notin P$. To show that $IMJ \subseteq P$, let $a \in I$ and $b \in J$. Then $amb \in P$. Since P is a jointly prime and $m \notin P$, $aMb \subseteq P$. This implies that $IMJ \subseteq P$.

Conversely, assume that for all left ideals I of R, elements $m \in M$ and right ideals J of S, $ImJ \subseteq P$ implies $IMJ \subseteq P$ or $m \in P$. Let $rms \in P$. Then $(Rr)m(sS) \subseteq P$. Hence $(Rr)M(sS) \subseteq P$ or $m \in P$. Then $rMs \subseteq P$ or $m \in P$ by Proposition 1.2.6. Hence P satisfies the condition (2.2). □

Characterizations of jointly prime (R, S)-submodules are also obtained in the same manner as those of fully prime (R, S)-submodules.

Theorem 2.2.5. *Let M be an (R, S)-module satisfying $a \in RaS$ for all $a \in M$ and P a proper (R, S)-submodule of M. The following statments are equivalent:*

(i) *P is jointly prime.*

(ii) *For all right ideals I of R, elements $m \in M$, and left ideals J of S,*

$$ImJ \subseteq P \text{ implies } IMJ \subseteq P \text{ or } m \in P.$$

(iii) *For all right ideals I of R, (R, S)-submodules N of M and left ideals J of S,*

$$INJ \subseteq P \text{ implies } IMJ \subseteq P \text{ or } N \subseteq P.$$

(iv) *For all left ideals I of R, elements $m \in M$ and right ideals J of S,*

$$(IR)m(SJ) \subseteq P \text{ implies } IMJ \subseteq P \text{ or } m \in P.$$

(v) *For all* $a \in R, m \in M$ *and* $b \in S$,

$$(aR)m(Sb) \subseteq P \text{ implies } aMb \subseteq P \text{ or } m \in P.$$

Proof. **(i) ⇒ (ii)** Assume (i). Let I be a right ideal of R, $m \in M$ and J a left ideal of S such that $ImJ \subseteq P$. Then $(RI)(RmS)(JS) \subseteq P$. By (i), we have $(RI)M(JS) \subseteq P$ or $RmS \subseteq P$. Applying Proposition 1.2.6 leads to $IMJ \subseteq P$ or $m \in P$.

(ii) ⇒ (iii) Assume (ii). Let I be a right ideal of R, N an (R, S)-submodule of M and J a left ideal of S such that $INJ \subseteq P$. Suppose that $N \not\subseteq P$. Let $n \in N \backslash P$. Then $InJ \subseteq INJ \subseteq P$. By (ii), we have $IMJ \subseteq P$.

(iii) ⇒ (iv) Assume (iii). Let I be a left ideal of R, $m \in M$ and J a right ideal of S such that $(IR)m(SJ) \subseteq P$. Then $(IR)(RmS)(SJ) \subseteq P$. By (iii), we have $(IR)M(SJ) \subseteq P$ or $RmS \subseteq P$. We conclude from Proposition 1.2.6 that $IMJ \subseteq P$ or $m \in P$.

(iv) ⇒ (i) Assume (iv). Let I be a left ideal of R, J a right ideal of S and N an (R, S)-submodule of M such that $INJ \subseteq P$. Suppose that $N \not\subseteq P$. Let $n \in N \backslash P$. Then $(IR)n(SJ) \subseteq P$. By (iv), we have $IMJ \subseteq P$.

(ii) ⇒ (v) It is obtained from (ii) and Proposition 1.2.6.

(v) ⇒ (iii) Assume (v). Let I be a right ideal of R, J a left ideal of S and N an (R, S)-submodule of M such that $INJ \subseteq P$. Suppose that $N \not\subseteq P$. Let $n \in N \backslash P$. To show that $IMJ \subseteq P$, let $a \in I$ and $b \in J$. Then $(aR)n(Sb) \subseteq P$. By (v), we have $aMb \subseteq P$. This implies that $IMJ \subseteq P$. □

Theorem 2.2.6. *Let* M *be an* (R, S)*-module satisfying* $RMS = M$ *and* P *a proper* (R, S)*-submodule of* M*. Then* P *is jointly prime if and only if for all ideals* I *of* R*, ideals* J *of* S *and* (R, S)*-submodules* N *of* M*,* $INJ \subseteq P$ *implies* $IMJ \subseteq P$ *or* $N \subseteq P$*.*

Proof. (⇒) This part follows from the definition.

($\Leftarrow$) Let I be a left ideal of R, J a right ideal of S and N an (R, S)-submodule of M such that $INJ \subseteq P$. Then $(IR)N(SJ) \subseteq P$. Since IR is an ideal of R and SJ is an ideal of S, we obtain that $(IR)M(SJ) \subseteq P$ or $N \subseteq P$. Then $IMJ \subseteq P$ or $N \subseteq P$ since $RMS = M$. Hence P is jointly prime. □

Between fully prime (R, S)-submodules and jointly prime (R, S)-submodules, neither is clearly a better definition than the others. However, the condition "$RMS = M$" gives a characterization of fully and jointly prime (R, S)-submodules by considering ideals of R and S. See Theorems 2.1.7 and 2.2.6.

Moreover, the condition "$RMS = M$" also implies that a fully prime (R, S)-submodule and a jointly prime (R, S)-submodule are identical.

Theorem 2.2.7. *Let M be an (R, S)-module such that $RMS = M$. Then fully prime and jointly prime (R, S)-submodules coincide.*

Proof. It is enough to show that every jointly prime (R, S)-submodule is fully prime. Let P be a jointly prime (R, S)-submodule of M. We show that P is fully prime by applying Theorem 2.1.7. Let I be an ideal of R, N an (R, S)-submodule of M and J an ideal of S such that $INJ \subseteq P$. Since P is jointly prime, $IMJ \subseteq P$ or $N \subseteq P$. We only consider the case $IMJ \subseteq P$. Then $I(RMJ)S \subseteq IMJ \subseteq P$. Again, $IMS \subseteq P$ or $RMJ \subseteq P$. Hence P is fully prime. □

If P is a jointly prime (R, S)-submodule of M, then the proper ideals $(P : M)_R$ and $[P : M]_S$ are prime ideals of R and of S, respectively.

Theorem 2.2.8. *Let P be an (R, S)-submodule of M such that $(P : M)_R$ and $[P : M]_S$ are proper ideals of R and S, respectively. If P is a jointly prime (R, S)-submodule of M, then $(P : M)_R$ and $[P : M]_S$ are prime ideals of R and S, respectively.*

Proof. Assume that P is a jointly prime (R, S)-submodule of M. Let A and B be ideals of R such that $AB \subseteq (P : M)_R$. Then $A(BMS)S = (AB)M(SS) \subseteq (AB)MS \subseteq P$. Since P is jointly prime, $AMS \subseteq P$ or $BMS \subseteq P$. Hence $A \subseteq (P : M)_R$ or $B \subseteq (P : M)_R$. This shows that $(P : M)_R$ is a prime ideal of R.

Similarly, we can also show that $[P : M]_S$ is a prime ideal of S. □

The following example shows that the converse of Theorem 2.2.8 is not valid in general, regardless of whether the rings R and S are identical or not.

Example 2.2.9. (1) We know that $6\mathbb{Z}$ is a $(2\mathbb{Z}, 2\mathbb{Z})$-submodule of $\mathbb{Z}$. Note that $(6\mathbb{Z} : \mathbb{Z})_{2\mathbb{Z}} = 6\mathbb{Z} = [6\mathbb{Z} : \mathbb{Z}]_{2\mathbb{Z}}$. It is easy to check that $6\mathbb{Z}$ is a prime ideal of $2\mathbb{Z}$. However, it follows from the definition that $6\mathbb{Z}$ is not a jointly prime $(2\mathbb{Z}, 2\mathbb{Z})$-submodule of $\mathbb{Z}$.

(2) Now consider $6\mathbb{Z}$ as a $(2\mathbb{Z}, 4\mathbb{Z})$-submodule of $\mathbb{Z}$. Then $(6\mathbb{Z} : \mathbb{Z})_{2\mathbb{Z}} = 6\mathbb{Z}$ and $[6\mathbb{Z} : \mathbb{Z}]_{4\mathbb{Z}} = 12\mathbb{Z}$. As above, $6\mathbb{Z}$ is a prime ideal of $2\mathbb{Z}$ and it is easy to check that $12\mathbb{Z}$ is a prime ideal of $4\mathbb{Z}$. However, it follows from the definition again that $6\mathbb{Z}$ is not a jointly prime $(2\mathbb{Z}, 4\mathbb{Z})$-submodule of $\mathbb{Z}$.

Note from Example 2.2.9 (1) and (2) that the converse of Theorem 2.2.8 may be false whether the ring R and S are identical or not.

We see in Proposition 2.1.9 that every maximal (R, S)-submodule is fully prime under the condition "$RMS = M$". In contrast, the following result suggests that jointly prime is a more natural definition than fully prime.

Proposition 2.2.10. *Every maximal (R, S)-submodule of an (R, S)-module M is jointly prime.*

Proof. Let K be a maximal (R, S)-submodule of M. Let I be a left ideal of R, N an (R, S)-submodule of M and J a right ideal of S such that $INJ \subseteq K$ and $N \not\subseteq K$. Then $M = N + K$. Thus $IMJ = I(N + K)J = INJ + IKJ \subseteq K$. Hence K is a jointly prime (R, S)-submodule. $\square$

In fact, jointly prime (R, S)-submodules need not be maximal.

Example 2.2.11. Let $r, s \in \mathbb{Z}^+ \backslash \{1\}$. Then $\mathbb{Z}$ is an $(r\mathbb{Z}, s\mathbb{Z})$-module. We can see that $(rs)\mathbb{Z}$ is an example of a jointly prime $(r\mathbb{Z}, s\mathbb{Z})$-submodule that is not maximal.

Proposition 2.2.12. *Let $f : M \to K$ be an (R, S)-homomorphism.*

(i) *If L is a jointly prime (R, S)-submodule of K, then $f^{-1}(L)$ is a jointly prime (R, S)-submodule of M containing $\ker f$ or $f^{-1}(L) = M$.*

(ii) *If P is a jointly prime (R,S)-submodule of M containing ker f and f is onto, then $f(P)$ is a jointly prime (R,S)-submodule of K.*

Proof. (i) Let L be a jointly prime (R,S)-submodule of K such that $f^{-1}(L) \neq M$. To show that $f^{-1}(L)$ is a jointly prime (R,S)-submodule of M containing ker f, let I be a left ideal of R, N an (R,S)-submodule of M and J a right ideal of S such that $INJ \subseteq f^{-1}(L)$. Then $If(N)J = f(INJ) \subseteq f(f^{-1}(L)) \subseteq L$. Since L is a jointly prime (R,S)-submodule of K, $IKJ \subseteq L$ or $f(N) \subseteq L$. If $IKJ \subseteq L$, then $IMJ = If^{-1}(K)J \subseteq f^{-1}(IKJ) \subseteq f^{-1}(L)$. If $f(N) \subseteq L$, then $N \subseteq f^{-1}(f(N)) \subseteq f^{-1}(L)$. Since $0 \in L$, ker $f = f^{-1}(0) \subseteq f^{-1}(L)$. It implies that $f^{-1}(L)$ is a jointly prime (R,S)-submodule of M containing ker f.

(ii) Let P be a jointly prime (R,S)-submodule of M containing ker f. To show that $f(P)$ is a jointly prime (R,S)-submodule of K, let I be a left ideal of R, N an (R,S)-submodule of K and J a right ideal of S such that $INJ \subseteq f(P)$. Then $f^{-1}(N)$ is an (R,S)-submodule of M. Since f is an (R,S)-homomorphism, $f(If^{-1}(N)J) = I(f(f^{-1}(N)))J \subseteq INJ \subseteq f(P)$. Thus $If^{-1}(N)J \subseteq P+$ker $f \subseteq P$. Since P is jointly prime, $IMJ \subseteq P$ or $f^{-1}(N) \subseteq P$. Note that f is onto. If $IMJ \subseteq P$, then $IKJ = If(M)J = f(IMJ) \subseteq f(P)$. If $f^{-1}(N) \subseteq P$, then $N = f(f^{-1}(N)) \subseteq f(P)$. Hence $IKJ \subseteq f(P)$ or $N \subseteq f(P)$. This shows that $f(P)$ is a jointly prime (R,S)-submodule of K. □

Theorem 2.2.13. *Let $f : M \to K$ be an (R,S)-epimorphism. Then there exists a one-to-one correspondence between the set of all jointly prime (R,S)-submodules of M containing ker f and the set of all jointly prime (R,S)-submodules of K.*

Proof. In fact, we have shown that there is a one-to-one correspondence between the set of all jointly prime (R,S)-submodules of M containing ker f and the set of all jointly prime (R,S)-submodules of K since f onto implies $f(f^{-1}(L)) = L$ for all (R,S)-submodules L of K and $f^{-1}(f(P)) = P$ for any (R,S)-submodule P of M with ker $f \subseteq P$. □

In some views, the concept of a module over a ring is a generalization of the notion of a ring where that ring is considered as a module over itself. Many theories of modules consist of extending as many as possible of the desirable properties of rings to modules. For example, if R is any ring and I is any left ideal of R, then I is a left submodule of R over R. Analogously, of course, right ideals are right submodules. However, the results may be different when R is considered as an (R, R)-module.

Throughout the rest of this section, let R be a ring and consider R as an (R, R)-module. It is clear that A is an (R, R)-submodule of R if and only if A is a subgroup of R under addition and $RAR \subseteq R$. We see that the property $RAR \subseteq R$ is not enough to imply that A is an ideal of the ring R. Note that example (4) on pages 3 and 4 shows that an (R,R)-submodule of R need not be an ideal of R.

We show that there are no difference between fully prime and jointly prime as (R, R)-submodules.

Proposition 2.2.14. *Let R be a ring. Then fully prime and jointly prime (R, R)-submodules of R are the same.*

Proof. It is enough to show that every jointly prime (R, R)-submodule of R is fully prime. Let P be a jointly prime (R, R)-submodule of R. Let I be a left ideal of R, N an (R, R)-submodule of R and J a right ideal of R such that $INJ \subseteq P$. Since P is jointly prime, $IRJ \subseteq P$ or $N \subseteq P$. Assume that $IRJ \subseteq P$. Then $I(RRJ)R \subseteq IR(JR) \subseteq IRJ \subseteq P$. Again, since P is jointly prime, we have $IRR \subseteq P$ or $RRJ \subseteq P$. Thus $IRR \subseteq P$ or $N \subseteq P$ or $RRJ \subseteq P$. This shows that P is fully prime. □

It is possible to study the relationship between prime ideals and fully (jointly) prime (R, R)-submodules. First of all, we obtain that fully (jointly) prime (R, R)-submodules may not be prime ideals.

Example 2.2.15. Choose $R = 2\mathbb{Z}$. We can see that $8\mathbb{Z}$ is not a prime ideal of $2\mathbb{Z}$ but $8\mathbb{Z}$ is a fully prime (R, R)-submodule since $R^3 \subseteq 8\mathbb{Z}$.

Proof. It is clear that $8\mathbb{Z}$ is both an ideal of $2\mathbb{Z}$ and a $(2\mathbb{Z}, 2\mathbb{Z})$-submodule of $2\mathbb{Z}$. Since $(4\mathbb{Z})(2\mathbb{Z}) \subseteq 8\mathbb{Z}$ but $4\mathbb{Z} \nsubseteq 8\mathbb{Z}$ and $2\mathbb{Z} \nsubseteq 8\mathbb{Z}$, it follows that $8\mathbb{Z}$ is not a prime ideal of $2\mathbb{Z}$. Obviously, $8\mathbb{Z}$ is a fully prime $(2\mathbb{Z}, 2\mathbb{Z})$-submodule of $2\mathbb{Z}$ since $(2\mathbb{Z})^3 \subseteq 8\mathbb{Z}$. □

Later, we investigate when prime ideals and fully (jointly) prime (R, R)-submodules are the same. The following is one answer.

Proposition 2.2.16. *Let R be a ring with non-zero identity. Then*

(i) *ideals of the ring R and (R, R)-submodules of the (R, R)-module R are the same;*

(ii) *prime ideals, fully prime (R, R)-submodules and jointly prime (R, R)-submodules are the same.*

Proof. (i) It is enough to show that (R, R)-submodules of R are ideals of R. Let I be an (R, R)-submodule of R. Clearly, I is a subgroup of R under addition. Since R is a ring with identity, $I \subseteq IR$ and $I \subseteq RI$. Then $RI \subseteq RIR \subseteq I$ and $IR \subseteq RIR \subseteq I$. Hence I is an ideal of R.

(ii) By Proposition 2.2.14, fully prime and jointly prime (R, R)-submodules are the same. It suffices to show that prime ideals and fully prime (R, R)-submodules are identical. Let P be a prime ideal of R, I and J ideals of R and N an (R, R)-submodule of R such that $INJ \subseteq P$. By (i), N is an ideal of R. Since P is a prime ideal of R, we have $I \subseteq P$ or $N \subseteq P$ or $J \subseteq P$. Hence $IRR \subseteq P$ or $N \subseteq P$ or $RRJ \subseteq P$. This means that P is a fully prime (R, R)-submodule of R.

Next, let P be a fully prime (R, R)-submodule of R and A and B ideals of R such that $AB \subseteq P$. Then $ARB \subseteq P$. Since P is a fully prime (R, R)-submodule of R, we obtain that $A = ARR \subseteq P$ or $B = RRB \subseteq P$. Hence P is a prime ideal of R. □

2.3 Left R-Prime (R, S)-Submodules

The third way to study the concept of "primality" for (R, S)-submodules is to give extreme importance to one of the rings R and S. Without loss of generality, we focus on the ring R.

Definition 2.3.1. A proper (R,S)-submodule Q of M is called **left R-prime** if for all ideals I and J of R, $(IJ)MSS \subseteq Q$ implies $IMS \subseteq Q$ or $JMS \subseteq Q$.

Note that right S-prime (R,S)-submodules can be defined and studied analogously. It is clear that all fully and jointly prime (R,S)-submodules are left R-prime. The converse does not hold in general.

Example 2.3.2. Clearly, $\mathbb{Z}$ is a $(2\mathbb{Z},2\mathbb{Z})$-module and $6\mathbb{Z}$ is a proper $(2\mathbb{Z},2\mathbb{Z})$-submodule of $\mathbb{Z}$. It is easy to see that $6\mathbb{Z}$ is left $2\mathbb{Z}$-prime but is not a jointly prime $(2\mathbb{Z},2\mathbb{Z})$-submodule.

In Theorem 2.2.2, we showed that if R or S is a commutative ring, then fully prime and jointly prime (R,S)-submodules are the same. In contrast, Example 2.3.2 shows that commutativity of the rings R or S is not enough to conclude that left R-prime and jointly prime coincide.

In a ring R, for each subset X of a ring R, let $(X)_t$ be the two-sided ideal of R generated by X, $(X)_l$ the left ideal of R generated by X and $(X)_r$ the right ideal of R generated by X.

Lemma 2.3.3 is basic knowledge in ring theory. We review it in order to use this result later.

Lemma 2.3.3. *Let R be a ring.*

(i) *If X is a subset of R, then*

  (a) $(X)_t = \mathbb{Z}X + RX + XR + RXR.$

  (b) $(X)_l = \mathbb{Z}X + RX.$

  (c) $(X)_r = \mathbb{Z}X + XR.$

(ii) *If X is a subring of R, then*

  (a) $(X)_t = X + RX + XR + RXR.$

  (b) $(X)_l = X + RX.$

  (c) $(X)_r = X + XR.$

(iii) *If X is a left [right] ideal of R, then*

(a) $(X)_t = X + XR$. *[$X + RX$]*

(b) $(X)_l = X$. *[$X + RX$]*

(c) $(X)_r = X + XR$. *[X]*

Applying Lemma 2.3.3 yields the following result.

Lemma 2.3.4. *Let Q be a proper (R, S)-submodule of M.*

(i) *Let A and B be left (right) ideals of R. Then $[(A)_t(B)_t]MSS \subseteq Q$ if $(AB)MS \subseteq Q$. Moreover, if $S^2 = S$, then $(AB)MS \subseteq Q$ if and only if $[(A)_t(B)_t]MS \subseteq Q$.*

(ii) *Let A and B be right ideals of R. Then $[(A)_l(B)_l]MSS \subseteq Q$ if $(AB)MS \subseteq Q$. Moreover, if $S^2 = S$, then $(AB)MS \subseteq Q$ if and only if $[(A)_l(B)_l]MS \subseteq Q$.*

(iii) *Let A be a right ideal of R and B a left ideal of R. Then $[A(B)_r]MSS \subseteq Q$ if $(AB)MS \subseteq Q$. Moreover, if $S^2 = S$, then $(AB)MS \subseteq Q$ if and only if $[A(B)_r]MS \subseteq Q$.*

Proof. (i) Let A and B be left ideals of R. Assume that $(AB)MS \subseteq Q$. Note that $(A)_t = A + AR$ and $(B)_t = B + BR$. Then

$$\begin{aligned}
[(A)_t(B)_t]MSS &= (A + AR)(B + BR)MSS \\
&\subseteq (AB + ABR)MSS \\
&\subseteq (AB)MSS + (ABR)MSS \\
&\subseteq (AB)MS \\
&\subseteq Q.
\end{aligned}$$

Next, if $S^2 = S$, then $(AB)MS \subseteq [(A)_t(B)_t]MS$.

In the rest case, let A and B be right ideals of R. Assume that $(AB)MS \subseteq Q$.

Note that $(A)_t = A + RA$ and $(B)_t = B + RB$. Then

$$\begin{aligned}[(A)_t(B)_t]MSS &= (A+RA)(B+RB)MSS \\ &\subseteq (AB+RAB)MSS \\ &\subseteq (AB)MSS + (RAB)MSS \\ &\subseteq (AB)MS + R(ABMS)S \\ &\subseteq Q + RQS \\ &\subseteq Q.\end{aligned}$$

Next, if $S^2 = S$, then $(AB)MS \subseteq [(A)_t(B)_t]MS$.

(ii) Let A and B be right ideals of R. Assume that $(AB)MS \subseteq Q$. Then $(A)_l = (A)_t$ and $(B)_l = (B)_t$, so this part follows from (i).

(iii) Assume that $(AB)MS \subseteq Q$. Note that $(B)_r = B + BR$. Then

$$\begin{aligned}[A(B)_r]MSS &= A(B+BR)MSS \\ &\subseteq (AB+ABR)MSS \\ &\subseteq Q.\end{aligned}$$

Next, if $S^2 = S$, then $(AB)MS \subseteq [A(B)_r]MS$. □

Let Q be a proper (R,S)-submodule of an (R,S)-module M satisfying the condition that for all $a, b \in R$,

$$(ab)MSS \subseteq Q \text{ implies } aMS \subseteq Q \text{ or } bMS \subseteq Q. \tag{2.3}$$

The condition (2.3) is a major tool to help characterize left R-prime (R,S)-submodules. We provide Lemma 2.3.5 as a characterization of the condition (2.3) and use it to characterize left R-prime (R,S)-submodules in Theorem 2.3.6.

Lemma 2.3.5. *Let Q be a proper (R,S)-submodule of M. Then the following statements are equivalent.*

(i) *For all $a, b \in R$, $(ab)MSS \subseteq Q$ implies $aMS \subseteq Q$ or $bMS \subseteq Q$.*

(ii) *For all nonempty subsets X and Y of R,*

$$(XY)MSS \subseteq Q \textit{ implies } XMS \subseteq Q \textit{ or } YMS \subseteq Q.$$

Proof. **(i) $\Rightarrow$ (ii)** Assume (i). Let X and Y be nonempty subsets of R such that $(XY)MSS \subseteq Q$ and $YMS \nsubseteq Q$. Then there exists $b \in Y$ such that $bMS \nsubseteq Q$. Let $a \in X$. Then $(ab)MSS \subseteq Q$. Thus $aMS \subseteq Q$ by assumption and the fact that $bMS \nsubseteq Q$. This shows that $aMS \subseteq Q$ for all $a \in X$. It follows that $XMS \subseteq Q$.

(ii) $\Rightarrow$ (i) This is obvious. □

Theorem 2.3.6. *If Q is a proper (R, S)-submodule of M satisfying the condition* (2.3), *then Q is a left R-prime (R, S)-submodule of M.*

Furthermore, if R is commutative and $RMS = M$, then the converse also holds, i.e., if Q is a left R-prime (R, S)-submodule of M, then Q satisfies the condition (2.3)*.*

Proof. Assume that Q satisfies the condition (2.3). It automatically implies from Lemma 2.3.5 that Q is left R-prime.

Conversely, assume further that R is commutative and $RMS = M$ and Q is a left R-prime (R, S)-submodule. Let $a, b \in R$ be such that $(ab)MSS \subseteq Q$. Since R is commutative and $RMS = M$, we obtain that $(aRbR)MSS \subseteq Q$. Since Q is left R-prime, $(aR)MS \subseteq Q$ or $(bR)MS \subseteq Q$. By Proposition 1.2.6, $aMS \subseteq Q$ or $bMS \subseteq Q$. Hence Q satisfies condition (2.3). □

The following example shows that the commutativity is necessary for the converse.

Example 2.3.7. Let R be the ring of all $n \times n$ matrices over a division ring. Then R has no proper ideals. Moreover, R is an (R, R)-module and 0 is a left R-prime submodule of R. Unfortunately, 0 does not satisfy the condition (2.3).

Next, we characterize left R-prime (R, S)-submodules in other ways.

Theorem 2.3.8. *Let M be an (R, S)-module such that $S^2 = S$ and Q a proper (R, S)-submodule of M. The following statments are equivalent.*

(i) *Q is left R-prime.*

(ii) *For all left ideals I and J of R,*

$$(IJ)MS \subseteq Q \text{ implies } IMS \subseteq Q \text{ or } JMS \subseteq Q.$$

(iii) *For all right ideals I and J of R,*

$$(IJ)MS \subseteq Q \text{ implies } IMS \subseteq Q \text{ or } JMS \subseteq Q.$$

(iv) *For all right ideals I and left ideals J of R,*

$$(IJ)MS \subseteq Q \text{ implies } IMS \subseteq Q \text{ or } JMS \subseteq Q.$$

Proof. This follows from Lemma 2.3.4. □

We seem to have forgotten one equivalent statement in the above theorem: the case where I is a left ideal and J is a right ideal. In fact we have not forgotten anything. As the following example shows, this case is not equivalent to the others. As an aside we give an example providing that 0 in Example 2.3.7 is left R-prime but there are a left ideal I of R and a right ideal J of R such that $(IJ)MR = 0$ but $IMR \neq 0$ and $JMR \neq 0$.

Example 2.3.9. Let R be the ring of all $n \times n$ matrices over a division ring. Moreover, let $I = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)_l$ and $J = \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)_r$. It is easy to show that $(IJ)MR = 0$ but $IMR \neq 0$ and $JMR \neq 0$.

Corollary 2.3.10. *Let M be an (R, S)-module such that $S^2 = S$ and Q a proper (R, S)-submodule of M. The following statments are equivalent.*

(i) *Q is left R-prime.*

(ii) *For all $a, b \in R$, $(a)_l(b)_lMS \subseteq Q$ implies $aMS \subseteq Q$ or $bMS \subseteq Q$.*

(iii) *For all $a, b \in R$, $(a)_r(b)_rMS \subseteq Q$ implies $aMS \subseteq Q$ or $bMS \subseteq Q$.*

(iv) *For all $a, b \in R$, $(a)_r(b)_lMS \subseteq Q$ implies $aMS \subseteq Q$ or $bMS \subseteq Q$.*

Proof. This follows from Theorem 2.3.8 □

Theorem 2.3.11. *Let M be an (R,S)-module such that $a \in RaS$ for all $a \in M$ and Q a proper (R,S)-submodule of M. The following statments are equivalent.*

(i) *Q is left R-prime.*

(ii) *For all left ideals I and right ideals J of R,*

$$(IRJ)MS \subseteq Q \text{ implies } IMS \subseteq Q \text{ or } JMS \subseteq Q.$$

(iii) *For all $a, b \in R$,*

$$(a)_l R(b)_r MS \subseteq Q \text{ implies } aMS \subseteq Q \text{ or } bMS \subseteq Q.$$

Proof. **(i) $\Rightarrow$ (ii)** Assume (i). Let I be a left ideal and J a right ideal of R such that $(IRJ)MS \subseteq Q$. Then $(IR)(RJ)MSS \subseteq Q$. Since IR and RJ are ideals of R and Q is left R-prime, $(IR)MS \subseteq Q$ or $(RJ)MS \subseteq Q$. Proposition 1.2.6 yields that $IMS \subseteq Q$ or $JMS \subseteq Q$.

(ii) $\Rightarrow$ (iii) This is obvious.

(iii) $\Rightarrow$ (i) Assume (iii). Let I and J be ideals of R such that $IJMSS \subseteq Q$. Assume that $JMS \nsubseteq Q$. Let $b \in J$ be such that $bMS \nsubseteq Q$. Then $(b)_r MS \nsubseteq Q$. Let $a \in I$. Then $(a)_l R(b)_r MSS \subseteq (IRJ)MSS \subseteq (IJ)MSS \subseteq Q$ and $(a)_l R(b)_r MS = (a)_l R(b)_r (RMS)S = ((a)_l R(b)_r R)MSS \subseteq (a)_l R(b)_r MSS$. By (iii), $aMS \subseteq Q$. This shows that $aMS \subseteq Q$ for all $a \in R$. Hence $IMS \subseteq Q$. Therefore Q is a left R-prime (R,S)-submodule of M. □

The following results are obtained in the same way as the analogous ones for fully and jointly prime (R,S)-submodules.

Proposition 2.3.12. *Every maximal (R,S)-submodule of M is left R-prime.*

Proof. This is obtained from the fact that every maximal (R,S)-submodule of M is jointly prime and every jointly prime (R,S)-submodule is left R-prime. □

Example 2.3.13. 0 is a left $\mathbb{Z}$-prime submodule of $\mathbb{Z}$ as a $(\mathbb{Z},\mathbb{Z})$-module but is not a maximal $(\mathbb{Z},\mathbb{Z})$-submodule of $\mathbb{Z}$.

Theorem 2.3.14. *Let Q be a proper (R,S)-submodule of M such that $(Q:M)_R$ is a proper ideal of R. If Q is left R-prime, then $(Q:M)_R$ is a prime ideal of R.*

Moreover, if $S^2 = S$, then Q is left R-prime if and only if $(Q:M)_R$ is a prime ideal of R.

Proof. This is clear. □

Example 2.3.15. Note that $\mathbb{Z}$ is a $(\mathbb{Z}, 2\mathbb{Z})$-module. Moreover, $(4\mathbb{Z}:\mathbb{Z})_{\mathbb{Z}} = 2\mathbb{Z}$ is a prime ideal of $\mathbb{Z}$ but $4\mathbb{Z}$ is not a left $\mathbb{Z}$-prime $(\mathbb{Z}, 2\mathbb{Z})$-sumodule.

Proposition 2.3.16. *Let $f : M \to K$ be an (R,S)-epimorphism.*

(i) *If L is a left R-prime (R,S)-submodule of K, then $f^{-1}(L)$ is a left R-prime (R,S)-submodule of M containing ker f.*

(ii) *If P is a left R-prime (R,S)-submodule of M containing ker f, then $f(P)$ is a left R-prime (R,S)-submodule of K.*

Proof. (i) Let L be a left R-prime (R,S)-submodule of K. To show that $f^{-1}(L)$ is a left R-prime (R,S)-submodule of M, let I and J be ideals of R such that $(IJ)MSS \subseteq f^{-1}(L)$. Then $f(IJMSS) \subseteq f(f^{-1}(L)) \subseteq L$. That is, $IJKSS \subseteq L$. Since L is a left R-prime (R,S)-submodule of K, we have $IKS \subseteq P$ or $JKS \subseteq P$. Thus $IMS = If^{-1}(K)S \subseteq f^{-1}(IKS) \subseteq f^{-1}(L)$ or $JMS = If^{-1}(K)S \subseteq f^{-1}(JKS) \subseteq f^{-1}(L)$. This implies that $f^{-1}(L)$ is a left R-prime (R,S)-submodule of M.

(ii) Let P be a left R-prime (R,S)-submodule of M containing ker f. To show that $f(P)$ is a left R-prime (R,S)-submodule of K, let I and J be ideals of R such that $(IJ)KSS \subseteq f(P)$. Then $f(IJMSS) = IJf(M)SS = IJKSS \subseteq f(P)$. Hence $IJMSS \subseteq P+$ ker $f \subseteq P$. Since P is a left R-prime (R,S)-submodule of M, we have $IMS \subseteq P$ or $JMS \subseteq P$. Therefore $IKS = If(M)S = f(IMS) \subseteq f(P)$ or $JKS = Jf(M)S = f(JMS) \subseteq f(P)$. This means that $f(P)$ is a left R-prime (R,S)-submodule of K. □

Theorem 2.3.17. *Let $f : M \to K$ be an (R,S)-epimorphism. Then there exists a one-to-one correspondence between the set of all left R-prime (R,S)-submodules of M containing ker f and the set of all left R-prime (R,S)-submodules of K.*

Proof. This follows from Proposition 2.3.16 □

We have now finished all of the concepts of primality for (R, S)-modules that are studied in this dissertation. We see that left R-prime is a generalization of both fully prime and jointly prime. Of course, we gave an example showing that both fully prime and jointy prime (R, S)-submodules may not be left R-prime.

Later, in the study of "maximality", we saw that maximality automatically implies primality only for jointly prime and left R-prime (R, S)-submodules while we show this fact holds for fully prime under the assumption that $RMS = M$.

Furthermore, we investigated how an (R, S)-homomorphisms preserve primality of (R, S)-submodules. Let $f : M \to K$ be an (R, S)-homomorphism. We concluded that if P is a fully (jointy) prime (R, S)-submodule of K and $f^{-1}(P) \neq M$, then $f^{-1}(P)$ is a fully (jointly) prime (R, S)-submodule of M containing $\ker f$. In contrast, $f^{-1}(P)$ may not be left R-prime even though P is left R-prime and $f^{-1}(P) \neq M$. See the following example.

Example 2.3.18. Recall that $\mathbb{Z}_6$ is a $(\mathbb{Z}_6, \mathbb{Z}_6)$-module. The map $f : \mathbb{Z}_6 \to \mathbb{Z}_6$ defined by $f(x) = 3x$ for all $x \in \mathbb{Z}_6$ is a $(\mathbb{Z}_6, \mathbb{Z}_6)$-homomorphism. Then $\{0, 2, 4\}$ is left $\mathbb{Z}_6$-prime but $f^{-1}(\{0, 2, 4\}) = \{0\}$ is not left $\mathbb{Z}_6$-prime.

CHAPTER III
LEFT MULTIPLICATION (R, S)-MODULES

Multiplication modules have been studied for a long time. The notion of multiplication modules leads to the idea of the product of two submodules as given by R. Ameri, see [1]. The product of two submodules is used as a tool to characterize prime submodules of multiplication modules, see [1] and [9]. In this chapter, we introduce a left multiplication (R, S)-module analogous to a multiplication module. The product of two (R, S)-submodules of an arbitrary left multiplication (R, S)-module is also given so that characterizations of its fully prime, jointly prime and left R-prime (R, S)-submodules are obtained.

3.1 Left Multiplication (R, S)-Modules

Recall that a unital left R-module M, where R is a commutative ring with identity, is a multiplication module if for each submodule N of M, there exists an ideal I of R such that $N = IM$. The structure of an (R, S)-module consists of two arbitrary rings R and S which need not be commutative or contain identity. In order to define a left multiplication (R, S)-module analogously to a multiplication module, for each (R, S)-submodule N of M, it is possible to choose an ideal of R or S. In this dissertation, we consider choosing an ideal I of R in order to write $N = IMS$.

Definition 3.1.1. Let R and S be rings and M an (R, S)-module. Then M is called a **left multiplication (R, S)-module** provided that for each (R, S)-submodule N of M, there exists an ideal I of R such that $N = IMS$.

Example 3.1.2. Recall that $\mathbb{Z}_6$ is a $(\mathbb{Z}_6, \mathbb{Z}_6)$-module. We know from Proposition 2.2.16 that ideals of $\mathbb{Z}_6$ and $(\mathbb{Z}_6, \mathbb{Z}_6)$-submodules of $\mathbb{Z}_6$ are identical. Note that the list of all ideals of $\mathbb{Z}_6$ and $(\mathbb{Z}_6, \mathbb{Z}_6)$-submodules of $\mathbb{Z}_6$ is $\{0\}$, $\{0, 2, 4\}$ and $\{0, 3\}$ and that

$$\{0\} = \{0\}\mathbb{Z}_6\mathbb{Z}_6$$
$$\{0, 3\} = \{0, 3\}\mathbb{Z}_6\mathbb{Z}_6$$
$$\{0, 2, 4\} = \{0, 2, 4\}\mathbb{Z}_6\mathbb{Z}_6.$$

This shows that $\mathbb{Z}_6$ is a left multiplication $(\mathbb{Z}_6, \mathbb{Z}_6)$-module.

The crucial observation is the following.

Proposition 3.1.3. *If M is a left multiplication (R, S)-module, then $RMS = M$.*

Proof. Assume that M is a left multiplication (R, S)-module. Since M is an (R, S)-submodule of M, there is an ideal I of R such that $M = IMS$. Then $M = IMS \subseteq RMS \subseteq M$ so that $RMS = M$. □

The converse of Proposition 3.1.3 is not valid in general.

Example 3.1.4. Recall that $\mathbb{Q}$ is a $(\mathbb{Z}, \mathbb{Z})$-module and $\mathbb{Q} = \mathbb{Z}\mathbb{Q}\mathbb{Z}$. Note also that $\mathbb{Z}$ is a $(\mathbb{Z}, \mathbb{Z})$-submodule of $\mathbb{Q}$ but $\mathbb{Z} \neq I\mathbb{Q}\mathbb{Z}$ for all ideals I of $\mathbb{Z}$. This means that $\mathbb{Q}$ is not a left multiplication $(\mathbb{Z}, \mathbb{Z})$-module.

If M is a left multiplication (R, S)-module and N is an (R, S)-submodule of M, then N can also be written as $(N : M)_R MS$.

Proposition 1.2.8 obtains that $RMS = M$ implies that $(N : M)_R$ is an ideal of R for all (R, S)-submodules N of M. Hence we obtain a characterization of a left multiplication (R, S)-module as follows.

Proposition 3.1.5. *Let M be an (R, S)-module. Then M is a left multiplication (R, S)-module if and only if $N = (N : M)_R MS$ for all (R, S)-submodule N of M.*

Proof. $(\Rightarrow)$ Assume that M is a left multiplication (R, S)-module and let N be an (R, S)-submodule of M. Then $N = IMS$ for some ideal I of R. Clearly, $I \subseteq (N : M)_R$. This implies that $N = IMS \subseteq (N : M)_R MS \subseteq N$. Hence $N = (N : M)_R MS$.

($\Leftarrow$) Assume that $N = (N : M)_R MS$ for all (R, S)-submodule N of M. It implies that $RMS = M$. As a result of Proposition 1.2.8, $(N : M)_R$ is an ideal of R. Hence M is a left multiplication (R, S)-module. □

Another characterization of left multiplication (R, S)-modules is given.

Proposition 3.1.6. *Let M be an (R, S)-module. Then M is a left multiplication (R, S)-module if and only if for each $m \in M$ there exists an ideal I of R such that $\langle m \rangle = IMS$.*

Proof. ($\Rightarrow$) This is clear.

($\Leftarrow$) Let N be an (R, S)-submodule of M. Note that for each $n \in N$ there exists an ideal I_n of R such that $\langle n \rangle = I_n MS$. Then $I = \sum_{n \in N} I_n$ is an ideal of R satisfying $N = IMS$. □

If M is a left multiplication (R, S)-module and N is an (R, S)-submodule of M, then there may be many ideals I of R such that $N = IMS$; that is, I is not uniquely determined by N. Fortunately, we can recover uniqueness by choosing the maximal ideal J such that $N = JMS$.

Proposition 3.1.7. *Let M be a left multiplication (R, S)-module. For any ideal K of R and (R, S)-submodule N of M, $KMS \subseteq N$ implies $K \subseteq (N : M)_R$.*

Consequently, for each (R, S)-submodule N of M, $(N : M)_R$ is a unique maximal ideal of R such that $N = (N : M)_R MS$.

Proof. Let N be an (R, S)-submodule of M. Then $N = (N : M)_R MS$. It follows immediately that $(N : M)_R$ is the unique maximal ideal of R such that $N = (N : M)_R MS$. □

Proposition 3.1.8. *Let M be a left multiplication (R, S)-module and I an ideal of R. Assume that R is a commutative ring. Then $M = IMS$ if and only if $m \in ImS$ for all $m \in M$.*

Proof. ($\Rightarrow$) Assume that $M = IMS$. Let $m \in M$. Then $\langle m \rangle = JMS$ for some ideal J of R.

Thus

$$\langle m\rangle = JMS = J(IMS)S = I(JMS)S = I(\langle m\rangle)S = I(\mathbb{Z}m + RmS)S$$
$$= \mathbb{Z}(ImS) + I(RmS)S \subseteq ImS.$$

Hence $m \in ImS$.

($\Leftarrow$) This is obvious. $\square$

Let M be a left multiplication (R, S)-module. It is clear that $RMS = M$. Moreover, if R is a commutative ring, then $m \in RmS$ for all $m \in M$. However, if the assumption "M is a left multiplication (R, S)-module" is omitted, then the result of Proposition 3.1.8 may be fail. That is, there is an ideal I of R and $m \in M$ such that $M = IMS$ and $m \notin RmS$.

Example 3.1.9. $\mathbb{Q}$ is a $(\mathbb{Z}, 2\mathbb{Z})$-module but $\mathbb{Q}$ is not a left multiplication $(\mathbb{Z}, 2\mathbb{Z})$-module because $\mathbb{Z}$ is a $(\mathbb{Z}, 2\mathbb{Z})$-submodule of $\mathbb{Q}$ and $\mathbb{Z} \neq I\mathbb{Q}(2\mathbb{Z})$ for all ideal I of $\mathbb{Z}$. However, $\mathbb{Q} = \mathbb{Z}\mathbb{Q}(2\mathbb{Z})$ and there exists $x \in \mathbb{Q}$ such that $x \notin \mathbb{Z}x(2\mathbb{Z})$.

Recall that fully prime and jointly prime (R, S)-submodules may not be the same if M is an (R, S)-module. However, they are identical if M is a left multiplication (R, S)-module because $RMS = M$ in this case, see also Theorem 2.2.7.

Proposition 3.1.10. *Let M be a left multiplication (R, S)-module. Then fully prime and jointly prime (R, S)-submodules are identical.*

Let M be a left multiplication (R, S)-module. For each $x \in M$, denote the ideal I_x by $(\langle x\rangle : M)_R$. We obtain the form of $\langle x\rangle$ for each $x \in M$ as follows.

Proposition 3.1.11. *Let M be a left multiplication (R, S)-module. Assume that R is a commutative ring. Then $\langle x\rangle = \left(\sum_{y\in M} I_y\right)xS$ for all $x \in M$.*

Proof. Note that for each $x \in M$, $\langle x\rangle = I_xMS$. By Proposition 1.2.7, we have $M = \sum_{x\in M}\langle x\rangle = \sum_{x\in M}(I_xMS) = (\sum_{x\in M} I_x)MS$. By Proposition 3.1.8, we obtain $x \in (\sum_{y\in M} I_y)xS$ for all $x \in M$. Let $x \in M$. Then $(\sum_{y\in M} I_y)xS \subseteq \langle x\rangle$. Since $(\sum_{y\in M} I_y)xS$ is an (R, S)-submodule containing x, $\langle x\rangle \subseteq (\sum_{y\in M} I_y)xS$. Therefore $\langle x\rangle = (\sum_{y\in M} I_y)xS$ for all $x \in M$. $\square$

Since each (R,S)-submodule of a left multiplication (R,S)-module is associated with a unique ideal of R, and we have concepts of primalities in both cases, it is natural to ask whether the primality of one implies the primality of the other. The following result provides one answer to that question.

Proposition 3.1.12. *Let M be a left multiplication (R,S)-module and P a proper (R,S)-submodule of M. If P is left R-prime, then $(P:M)_R$ is a prime ideal of R.*

Furthermore, if $S^2 = S$, then the converse is true as well, i.e., if $(P:M)_R$ is a prime ideal of R, then P is left R-prime.

Proof. Assume that P is left R-prime. Since $RMS = M$, the ideal $(P:M)_R$ is proper. Let A and B be ideals of R such that $AB \subseteq (P:M)_R$. Then $A(BMS)S \subseteq P$. Since P is left R-prime, $AMS \subseteq P$ or $BMS \subseteq P$. By Proposition 3.1.7, $A \subseteq (P:M)_R$ or $B \subseteq (P:M)_R$. Hence $(P:M)_R$ is a prime ideal of R.

Next, assume that $S^2 = S$ and $(P:M)_R$ is a prime ideal of R. Let I and J be ideals of R such that $(IJ)MSS \subseteq P$. Then $IJ \subseteq (P:M)_R$ from the fact that $S^2 = S$ and Proposition 3.1.7. Since $(P:M)_R$ is a prime ideal of R, $I \subseteq (P:M)_R$ or $J \subseteq (P:M)_R$. Then $IMS \subseteq P$ or $JMS \subseteq P$. Hence P is left R-prime. $\square$

Proposition 3.1.13. *Let M be a left multiplication (R,S)-module and P a proper (R,S)-submodule of M. If P is jointly prime, then $(P:M)_R$ is a prime ideal of R.*

Furthermore, if R is commutative and $S^2 = S$, then the converse holds, i.e., if $(P:M)_R$ is a prime ideal of R, then P is jointly prime.

Proof. If P is jointly prime, then P is left R-prime so that the result follows.

Conversely, assume that R is a commutative ring, $S^2 = S$ and $(P:M)_R$ is a prime ideal of R. Let I be an ideal of R, J an ideal of S and N an (R,S)-submodule of M such that $INJ \subseteq P$. Let K and L be ideals of R such that $N = KMS$ and $IMJ = LMS$. Then

$$
\begin{aligned}
R(INJ)S &= (RI)N(JS) \\
&= (RI)KMS(JS) \\
&= (RIK)M(SJS)
\end{aligned}
$$

$$
\begin{aligned}
&= (IK)RMS(JS) \\
&= (IK)M(JS) \\
&= (KI)M(JS) \\
&= K(IMJ)S \\
&= (KL)M(SS) \\
&= (KL)MS.
\end{aligned}
$$

Thus $(KL)MS \subseteq P$. By Proposition 3.1.7, $KL \subseteq (P : M)_R$. Since $(P : M)_R$ is a prime ideal of R, $K \subseteq (P : M)_R$ or $L \subseteq (P : M)_R$. Hence $IMJ \subseteq P$ or $N \subseteq P$. Therefore P is jointly prime. $\square$

Compare the converse of the proofs of Proposition 3.1.12 and Proposition 3.1.13, the commutativity of the ring R is needed in Proposition 3.1.13 but not in Proposition 3.1.12. This may be a strong reason that a left R-prime (R, S)-submodule is a generalization of a jointly prime (R, S)-submodule.

In addition, if R is commutative, then jointly prime (R, S)-submodules and fully prime (R, S)-submodules coincide. Hence we can conclude that the converses of the two parts of Proposition 2.2.8 hold in a left multiplication (R, S)-module where R is commutative and $S^2 = S$.

Theorem 3.1.14. *Let P be a proper (R, S)-submodule of a left multiplication (R, S)-module M and $S^2 = S$. If R is commutative, then the following statements are equivalent.*

(i) *P is a fully prime (R, S)-submodule.*

(ii) *P is a jointly prime (R, S)-submodule.*

(iii) *P is a left R-prime (R, S)-submodule.*

(iv) *$(P : M)_R$ is a prime ideal of R.*

The existence of the ideal of the form $(N : M)_R$ allows us to define the product of two (R, S)-submodules of an arbitrary (R, S)-module.

Definition 3.1.15. Let N and K be (R,S)-submodules of a left multiplication (R,S)-module of M. The **product** of N and K, denoted by NK, is defined by

$$(N:M)_R(K:M)_RMSS.$$

Clearly, the product NK is an (R,S)-submodule of M and is contained in $N \cap K$.

Proposition 3.1.16. *Let N and K be (R,S)-submodules of a left multiplication (R,S)-module M. If R is commutative, then $NK = (IJ)M(SS)$ for any ideals I and J of R such that $N = IMS$ and $K = JMS$.*

Proof. Let $N = IMS$ and $K = JMS$ where I and J are ideals of R. Note that $N = (N:M)_RMS$ and $K = (K:M)_RMS$. Thus

$$\begin{aligned} NK = (N:K)_R(K:M)_RMSS &= (N:M)_R[(K:M)_RMS]S \\ &= (N:M)_R[JMS]S \\ &= [(N:M)_RJ]M(SS) \\ &= [J(N:M)_R]M(SS) \\ &= J[(N:M)_RMS]S \\ &= J(IMS)S \\ &= (JI)M(SS) \\ &= (IJ)M(SS). \end{aligned}$$

The proof is complete. □

3.2 Jointly and Left R-Prime (R,S)-Submodules of Left Multiplication (R,S)-Modules

Left multiplication (R,S)-modules allow us to characterize jointly prime (R,S)-submodules in term of the product of two (R,S)-submodules.

Proposition 3.2.1. *Let P be a proper (R, S)-submodule of a left multiplication (R, S)-module M. If P is a jointly prime (R, S)-submodule, then for all (R, S)-submodules U and V of M,*

$$UV \subseteq P \quad \textit{implies} \quad U \subseteq P \textit{ or } V \subseteq P. \tag{3.1}$$

Furthermore, if R is commutative and $S^2 = S$, then the converse is true as well, i.e., if P satisfies the condition (3.1)*, then P is a jointly prime (R, S)-submodule.*

Proof. Assume that P is a jointly prime (R, S)-submodule. Let U and V be (R, S)-submodules of M such that $UV \subseteq P$. Then $(U : M)_R[(V : M)_R MS]S \subseteq P$. Since P is jointly prime, $U = (U : M)_R MS \subseteq P$ or $V = (V : M)_R MS \subseteq P$.

Conversely, assume that R is commutative, $S^2 = S$ and the condition (3.1) holds. By Theorem 3.1.13, it is enough to show that $(P : M)_R$ is a prime ideal of R. Let A and B be ideals of R such that $AB \subseteq (P : M)_R$. Then $(AB)MSS \subseteq (AB)MS \subseteq (P : M)_R MS = P$. Then $(AMS)(BMS) = (AB)MSS \subseteq P$. From condition (3.1) we obtain that $AMS \subseteq P$ or $BMS \subseteq P$. By Proposition 3.1.7, $A \subseteq (P : M)_R$ or $B \subseteq (P : M)_R$. Therefore, $(P : M)_R$ is a prime ideal of R, as desired. □

Corollary 3.2.2. *Let P be a proper (R, S)-submodule of a left multiplication (R, S)-module M. If P is a jointly prime (R, S)-submodule, then for each $a, b \in M$,*

$$\langle a \rangle \langle b \rangle \subseteq P \textit{ implies } a \in P \textit{ or } b \in P. \tag{3.2}$$

Furthermore, if R is commutative and $S^2 = S$, then the converse holds, i.e., if P satisfies the condition (3.2)*, then P is a jointly prime (R, S)-submodule.*

Proof. If P is a jointly prime (R, S)-submodule, then the validity of the condition (3.2) for any $a, b \in M$ follows from Proposition 3.2.1.

Next, assume that R is commutative, $S^2 = S$ and the condition (3.2) holds. Let U and V be (R, S)-submodules of M such that $UV \subseteq P$. Suppose that $U \nsubseteq P$ and $V \nsubseteq P$. Then there exist $u \in U \backslash P$ and $v \in V \backslash P$. Note that $\langle u \rangle \langle v \rangle \subseteq UV \subseteq P$, and thus the condition (3.2) yields $u \in P$ or $v \in P$, which is a contradiction. Hence $U \subseteq P$ or $V \subseteq P$. Therefore P is a jointly prime (R, S)-submodule. $\square$

Another characterization of jointly prime (R, S)-submodules of left multiplication (R, S)-module is given. However, the following lemma is needed.

For (R, S)-submodules N and K of a left multiplication (R, S)-module M, define $(K :_M N) = \{m \in M \mid \langle m\rangle\langle n\rangle \subseteq K \text{ for all } n \in N\}$.

Lemma 3.2.3. *Let M be a left multiplication (R, S)-module and let N and K be (R, S)-submodules of M. Then the followings hold.*

(i) *For all $m \in M$, $m \in (K :_M N)$ if and only if $\langle m\rangle N \subseteq K$.*

(ii) *$(K :_M N)$ is an (R, S)-submodule of M.*

(iii) *If $N \subseteq K$, then $(K :_M N) = M$. Moreover, if K is a jointly prime (R, S)-submodule of M, then $N \subseteq K$ if and only if $(K :_M N) = M$.*

Proof. (i) Let $m \in M$.
($\Rightarrow$) We prove this by contrapositive. Assume that $\langle m\rangle N \nsubseteq K$. It follows that $(\langle m\rangle : M)_R(N : M)_R MSS \nsubseteq K$. Hence $(\langle m\rangle : M)_R(ays)S \nsubseteq K$ for some $a \in (N : M)_R$, $y \in M$ and $s \in S$. It is clear that $ays \in N$. Then

$$\begin{aligned}\langle m\rangle\langle ays\rangle &= (\langle m\rangle : M)_R(\langle ays\rangle : M)_R MSS \\ &= (\langle m\rangle : M)_R((\langle ays\rangle : M)_R MS)S \\ &= (\langle m\rangle : M)_R\langle ays\rangle S.\end{aligned}$$

Clearly, $(\langle m\rangle : M)_R(ays)S \subseteq (\langle m\rangle : M)_R\langle ays\rangle S$. Since $(\langle m\rangle : M)_R(ays)S \nsubseteq K$, we obtain that $(\langle m\rangle : M)_R\langle ays\rangle S \nsubseteq K$. Hence $\langle m\rangle\langle ays\rangle \nsubseteq K$. Therefore $m \notin (K :_M N)$.

($\Leftarrow$) This follows from the fact that for all $n \in M$, $\langle m\rangle\langle n\rangle \subseteq \langle m\rangle N \subseteq K$.

(ii) Clearly, $0 \in (K :_M N)$. Let $a, b \in (K :_M N)$, $r \in R$ and $s \in S$. Then

$$\langle a - b\rangle N \subseteq (\langle a\rangle + \langle b\rangle)N \subseteq \langle a\rangle N + \langle b\rangle N \subseteq K, \text{and}$$

$$\langle ras\rangle N \subseteq \langle a\rangle N \subseteq K.$$

Hence $a - b \in (K :_M N)$ and $ras \in (K :_M N)$. Therefore $(K :_M N)$ is an (R, S)-submodule of M.

(iii) Assume that $N \subseteq K$. Then $\langle m\rangle N \subseteq N \subseteq K$ for any $m \in M$. Hence $m \in (K :_M N)$.

On the other hand, assume that K is a jointly prime (R, S)-submodule of M and $(K : N)_M = M$. To show that $N \subseteq K$, let $n \in N$ and $m \in M\backslash K$. Then $\langle m\rangle\langle n\rangle \subseteq K$ because $m \in (K :_M N)$. Since K is a jointly prime (R, S)-submodule of M and $m \in M\backslash K$, we obtain that $n \in K$. □

Theorem 3.2.4. *Let P be a proper (R, S)-submodule of a left multiplication (R, S)-module M. If P is a jointly prime (R, S)-submodule of M, then for all (R, S)-submodule N of M, $N \nsubseteq P$ implies $P = (P :_M N)$.*

Furthermore, if R is commutative and $S^2 = S$, then the converse is also true, i.e., if for all (R, S)-submodule N of M, $N \nsubseteq P$ implies $P = (P :_M N)$, then P is a jointly prime (R, S)-submodule of M.

Proof. Assume that P is a jointly prime (R, S)-submodule of M. Let N be an (R, S)-submodule of M such that $N \nsubseteq P$. It is clear that $P \subseteq (P :_M N)$. Let $m \in (P :_M N)$. By Lemma 3.2.3, $\langle m\rangle N \subseteq P$. Since P is a jointly prime (R, S)-submodule of M and $N \nsubseteq P$, we have $m \in P$. Hence $P = (P :_M N)$.

Next, assume that R is commutative, $S^2 = S$ and if for all (R, S)-submodule N of M, $N \nsubseteq P$ implies $P = (P :_M N)$. Let $a, b \in M$ be such that $\langle a\rangle\langle b\rangle \subseteq P$. Suppose that $b \notin P$. Then $\langle b\rangle \nsubseteq P$. It implies that $P = (P :_M \langle b\rangle)$. Since $\langle a\rangle\langle b\rangle \subseteq P$, $a \in (P :_M \langle b\rangle) = P$. Hence P is a jointly prime (R, S)-submodule of M. □

The following result provides other relationship between (R, S)-submodules of N and the ideals $(N : M)_R$ of R.

Proposition 3.2.5. *Let N and L be (R, S)-submodules of a left multiplication (R, S)-module M. Then the followings hold.*

(i) *$N \subseteq L$ if and only if $(N : M)_R \subseteq (L : M)_R$.*

(ii) *If Λ is an arbitrary family of (R,S)-submodules of M, then*

$$(\bigcap_{N\in\Lambda} N : M)_R = \bigcap_{N\in\Lambda}(N : M)_R.$$

Proof. (i) It is clear that $(N : M)_R \subseteq (L : M)_R$ if $N \subseteq L$. Moreover, Proposition 3.1.5 yields the other direction.

(ii) Let Λ be an arbitrary family of (R,S)-submodule of M and $r \in R$. Then

$$\begin{aligned} r \in (\bigcap_{N\in\Lambda} N : M)_R &\Longleftrightarrow rMS \subseteq \bigcap_{N\in\Lambda} N \\ &\Longleftrightarrow rMS \subseteq N \text{ for all } N \in \Lambda \\ &\Longleftrightarrow r \in (N : M)_R \text{ for all } N \in \Lambda \\ &\Longleftrightarrow r \in \bigcap_{N\in\Lambda}(N : M)_R. \end{aligned}$$

□

Let M be a left multiplication (R,S)-module. We observe from Proposition 3.2.5 (i) that there is an order preserving between (R,S)-submodules of M and ideals of R under inclusion. For (R,S)-submodules N and K of M with $N \subseteq K$, we study a relationship between jointly prime (R,S)-submodules lying between N and K and prime ideals lying between the ideals $(N : M)_R$ and $(K : M)_R$.

Proposition 3.2.6. *Let N and K be (R,S)-submodules of a left multiplication (R,S)-module M. Then there is a one-to-one function preserving the inclusion-order between the set of jointly prime (R,S)-submodules of M lying between N and K and the set of prime ideals of R lying between $(N : K)_R$ and $(K : M)_R$.*

Proof. The map φ sending P to $(P : M)_R$, where P is a jointly prime (R,S)-submodule with $N \subseteq P \subseteq K$, is clearly a one-to-one function. Proposition 3.2.5 (i) ensures that $(N : M)_R \subseteq (P : M)_R \subseteq (K : M)_R$ for such P and also φ preserves the inclusion-order. Moreover, Proposition 3.1.12 yields that the ideal $(P : M)_R$ is a prime ideal. The proof is complete. □

Proposition 3.2.1 gives a characterization of jointly prime (R,S)-submodules involving products of two (R,S)-submodules. However, this holds under two assumptions, i.e., R is a commutative ring and $S^2 = S$. We also know that every jointly prime (R,S)-submodule is left R-prime. Theorem 3.2.7 confirms us that a left R-prime (R,S)-submodule is a generalization of a jointly prime (R,S)-submodule.

Theorem 3.2.7. *Let Q be a proper (R,S)-submodule of a left multiplication (R,S)-module M. If Q is a left R-prime (R,S)-submodule of M, then for each (R,S)-submodules U and V of M,*

$$UV \subseteq Q \text{ implies } U \subseteq Q \text{ or } V \subseteq Q. \tag{3.3}$$

Furthermore, if R is commutative, then the converse is also true, i.e., if the condition (3.3) *holds, then Q is left R-prime.*

Proof. Firstly, assume that Q is a left R-prime (R,S)-submodule and let U and V be (R,S)-submodules of M such that $UV \subseteq Q$. Then $(U:M)_R(V:M)_RMSS \subseteq Q$. Thus $U = (U:M)_RMS \subseteq Q$ or $V = (V:M)_RMS \subseteq Q$ since Q is a left R-prime (R,S)-submodule.

Conversely, assume that R is a commutative ring and the condition (3.3) holds. Let I and J be ideals of R such that $(IJ)MSS \subseteq Q$. Then $(IMS)(JMS) = (IJ)MSS \subseteq H$. By condition (3.3), $IMS \subseteq Q$ or $JMS \subseteq Q$. Hence Q is left R-prime. □

Corollary 3.2.8. *Let Q be a proper (R,S)-submodule of a left multiplication (R,S)-module M. If Q is a left R-prime (R,S)-submodule of M, then for each $a,b \in M$,*

$$\langle a\rangle\langle b\rangle \subseteq Q \text{ implies } a \in Q \text{ or } b \in Q. \tag{3.4}$$

Furthermore, if R is commutative, then the converse is true as well, i.e., if the condition (3.4) *holds, then Q is left R-prime.*

Proof. Assume that Q is a left R-prime (R,S)-submodule. By Theorem 3.2.7, the condition (3.4) holds.

Conversely, assume that R is commutative and the condition (3.4) holds. Let U and V be (R,S)-submodules of M such that $UV \subseteq Q$. Suppose that $U \nsubseteq Q$

and $V \nsubseteq Q$. Let $a \in U \backslash Q$ and $b \in V \backslash Q$. Then $\langle a \rangle \langle b \rangle \subseteq UV \subseteq Q$. By the condition (3.4), $a \in Q$ or $b \in Q$ which is a contradiction. Hence $U \subseteq Q$ or $V \subseteq Q$. Therefore Q is left R-prime. □

Note that $NK \subseteq N$ and $NK \subseteq K$ for each (R, S)-submodules N and K of a left multiplication (R, S)-module.

Theorem 3.2.9. *Let M be a left multiplication (R, S)-module. Consider the following two statements.*

(i) *Every proper (R, S)-submodule of M is left R-prime.*

(ii) *For all (R, S)-submodules N and K of M, $NK = N$ or $NK = K$.*

Then (i) implies (ii). Moreover, if R is commutative, then (ii) implies (i).

Proof. **(i) $\Rightarrow$ (ii)** Assume that (i) holds. Let N and K be (R, S)-submodules of M. Then NK is an (R, S)-submodule of M. If $NK = M$, then $N = M = K$. Assume that $NK \neq M$. Then NK is left R-prime. This implies that $N \subseteq NK$ or $K \subseteq NK$. Hence $NK = N$ or $NK = K$.

Assume further that R is commutative.

(ii) $\Rightarrow$ (i) Assume that (ii) holds. Let Q be a proper (R, S)-submodule of M. To show that Q is left R-prime, let U and V be (R, S)-submodules of M such that $UV \subseteq Q$. Then $UV = V$ or $UV = U$. Hence $U \subseteq Q$ or $V \subseteq Q$. Therefore Q is left R-prime. □

The following corollary is obtained immediately.

Corollary 3.2.10. *Let M be a left multiplication (R, S)-module whose every proper (R, S)-submodule is left R-prime. Then $N^2 = N$ for all (R, S)-submodule N of M.*

Lemma 3.2.11. *Every proper ideal of a ring R is prime if and only if for any ideals I and J of R, $IJ = I$ or $IJ = J$.*

Proof. It is essentially the same as the proof of Theorem 3.2.9. □

Lemma 3.2.11 is a useful tool in order to obtain the following Corollary.

Corollary 3.2.12. *Let R be a ring whose every proper ideal is prime and $S^2 = S$. Then every proper (R, S)-submodule of M is left R-prime.*

Proof. Let Q be a proper (R, S)-submodule of M. To show that Q is left R-prime, let I and J be ideals of R such that $IJMSS \subseteq Q$. It obviously implies from Lemma 3.2.11 and $S^2 = S$ that $IMS \subseteq Q$ or $JMS \subseteq Q$. $\square$

Note that M in Corollary 3.2.12 needs not be a left multiplication (R, S)-module.

The result of Theorem 3.2.9 is a conclusion that an (R, S)-module M has at most one maximal (R, S)-submodule if every proper (R, S)-submodule of M is left R-prime.

Theorem 3.2.13. *Let M be a left multiplication (R, S)-module with every proper (R, S)-submodule of M is left R-prime. Then M has at most one maximal (R, S)-submodule.*

Proof. Suppose that M_1 and M_2 are maximal (R, S)-submodules of M. By Theorem 3.2.9, $M_1M_2 = M_1$ or $M_1M_2 = M_2$, without loss of generality, we assume that $M_1M_2 = M_1$. Then $M_1M_2 \subseteq M_2$ so that $M_1 \subseteq M_2$. But M_1 is maximal and M_2 is a proper (R, S)-submodule of M, so $M_1 = M_2$. Thus M has at most one maximal (R, S)-submodule. $\square$

In a studying of ring theory, a subset of a ring is called **multiplicatively closed** if it is closed under multiplication. For commutative rings, the complement of a prime ideal is an especially important example of a multiplicatively closed set. In [8], an ideal P of a commutative ring R is prime if and only if the complement $R\backslash P$ is multiplicatively closed. In this dissertation, we introduce the concept of closed sets of a left multiplication (R, S)-module and give a characterization of left R-prime (R, S)-submodules in the mood of closed set.

Definition 3.2.14. For each nonempty subset C of a left multiplication (R, S)-module, we call C a **closed set** if $\langle a\rangle\langle b\rangle \cap C \neq \varnothing$ for all $a, b \in C$.

Theorem 3.2.15. *Let Q be a proper (R, S)-submodule of a left multiplication (R, S)-module M. If Q is left R-prime, then $M\backslash Q$ is a closed set. Moreover, if R is commutative, then Q is left R-prime if and only if $M\backslash Q$ is a closed set.*

Proof. Firstly, assume that Q is left R-prime and let $a, b \in M\backslash Q$. Then $\langle a\rangle\langle b\rangle \nsubseteq Q$. Hence $\langle a\rangle\langle b\rangle \cap (M\backslash Q) \neq \varnothing$.

Next, assume that R is commutative and $M\backslash Q$ is a closed set. We prove that Q is R-prime by showing that the condition (3.4) holds. Let $a, b \in M$ be such that $a \notin Q$ and $b \notin Q$. Since $M\backslash Q$ is a closed set, $\langle a\rangle\langle b\rangle \cap (M\backslash Q) \neq \varnothing$. Hence $\langle a\rangle\langle b\rangle \nsubseteq Q$. □

Next result needs the following lemma.

Lemma 3.2.16. *Let M be a left multiplication (R, S)-module and A, B, C and D (R, S)-submodules of M. If R is a commutative ring, then*

(i) $A(B + C) = AB + AC$

(ii) $(A + B)(C + D) = AC + AD + BC + BD$.

Proof. This is staightforward. □

Theorem 3.2.17. *Let A be an (R, S)-submodule of a left multiplication (R, S)-module M and C a closed set in M such that $A \cap C = \varnothing$. Then there exists an (R, S)-submodule K of M which is maximal with respect to the property that $A \subseteq K$ and $K \cap C = \varnothing$.*

Furthermore, if R is commutative, then K is a left R-prime (R, S)-submodule of M.

Proof. Let Ω be the set of all (R, S)-submodules B of M such that $A \subseteq B$ and $B \cap C = \varnothing$. We can see that $A \in \Omega$. It implies from Zorn's Lemma that Ω has a maximal element, say K. Note that $K \neq M$.

Assume that R is commutative. Suppose for contradiction that K is not left R-prime. Then $M\backslash K$ is not a closed set. Let $a, b \in M\backslash K$ be such that $\langle a\rangle\langle b\rangle \cap (M\backslash K) = \varnothing$. Then $\langle a\rangle\langle b\rangle \subseteq K$. Since K is maximal in Ω, $K + \langle a\rangle$ and

$K + \langle b \rangle$ are not in Ω. There are $s, t \in C$ such that $s \in K + \langle a \rangle$ and $t \in K + \langle b \rangle$. Since C is a closed set, $\langle s \rangle \langle t \rangle \cap C \neq \varnothing$. Hence $\langle s \rangle \langle t \rangle \subseteq (K + \langle a \rangle)(K + \langle b \rangle) \subseteq K$ by Lemma 3.2.16. Therefore $K \cap C \neq \varnothing$ which is a contradiction. $\square$

Theorem 3.2.18. *Let $N_1, N_2, ..., N_k$ be (R, S)-submodules of a left multiplication (R, S)-module M and Q a left R-prime (R, S)-submodule of M. Consider the following statements.*

(i) *$N_j \subseteq Q$ for some j with $1 \leq j \leq k$.*

(ii) $\bigcap\limits_{i=1}^{k} N_i \subseteq Q$.

(iii) $\prod\limits_{i=1}^{k} N_i \subseteq Q$.

Then (i), (ii) and (iii) are equivalent.

Proof. (i)$\Rightarrow$ (ii) This is obvious.

(ii)$\Rightarrow$ (iii) For each $i = 1, 2, \ldots, k$, there exists an ideal $(N_i : M)_R$ of R such that $N_i = (N_i : M)_R MS$. Then for each $j = 1, 2, \ldots, k$ and $i = 1, 2, \ldots, k$, let $S_i = S$,

$$\begin{aligned}\prod_{i=1}^{k} N_i &= \left(\prod_{i=1}^{k} (N_i : M)_R\right) M \left(\prod_{i=1}^{k} S_i\right)\\ &\subseteq (N_j : M)_R MS_j\\ &= N_j\end{aligned}$$

This implies that $\prod\limits_{i=1}^{k} N_i \subseteq \bigcap\limits_{i=1}^{k} N_i \subseteq Q$.

(iii)$\Rightarrow$ (i) This follows from Theorem 3.2.7. $\square$

Corollary 3.2.19. *Let $N_1, N_2, ..., N_k$ be (R, S)-submodules of a left multiplication (R, S)-module M. If $\bigcap\limits_{i=1}^{k} N_i$ is left R-prime, then $\bigcap\limits_{i=1}^{k} N_i = N_j$ for some j with $1 \leq j \leq k$.*

Proof. Let $Q = \bigcap\limits_{i=1}^{k} N_i$. The proof is complete by Theorem 3.2.18. $\square$

3.3 Maximal and Minimal (R, S)-Submodules of Left Multiplication (R, S)-Modules

Let M be a left multiplication (R, S)-module. For each (R, S)-submodule N of M, we know that $(N : M)_R$ is the unique maximal ideal corresponding to N such that $N = (N : M)_R MS$. Comparisons between N and $(N : M)_R$ are provided. One of the results shows that if N is a jointly prime (R, S)-submodule of M, then $(N : M)_R$ is a prime ideal of R.

On the other hand, let A be an ideal of R. Then AMS is an (R, S)-submodule of M. A natural question to ask is that if A is a prime ideal of R, then is AMS a jointly prime (R, S)-submodule of M? The following example shows that it may not be.

Example 3.3.1. We know that $\mathbb{Z}$ is a $(\mathbb{Z}, 2\mathbb{Z})$-module and $2\mathbb{Z}$ is a prime ideal of $\mathbb{Z}$. However, $4\mathbb{Z} = (2\mathbb{Z})\mathbb{Z}(2\mathbb{Z})$ is not a jointly prime $(\mathbb{Z}, 2\mathbb{Z})$-submodule of $\mathbb{Z}$, as can be seen by choosing $I = 3\mathbb{Z}$, $N = 6\mathbb{Z}$ and $J = 2\mathbb{Z}$.

We see that the comparison between the primality of A and the primality of AMS may not be appropriate. Another direction that we consider is to compare the maximality of A and the maximality of AMS.

Proposition 3.3.2. *Let M be a left multiplication (R, S)-module and K a maximal ideal of R. The followings hold.*

(i) *$M \neq KMS$ if and only if there is an element $x \in M \backslash KMS$ such that $M = KMS + \langle x \rangle$.*

Moreover, if $M \neq KMS$, then $M = KMS + \langle x \rangle$ for all $x \in M \backslash KMS$.

(ii) *If $M \neq KMS$, then KMS is a maximal (R, S)-submodule of M.*

(iii) *$(KMS : M)_R = K$ if and only if KMS is a maximal (R, S)-submodule of M.*

(iv) *$(KMS : M)_R = R$ if and only if $KMS = M$.*

Proof. (i) ($\Rightarrow$) Assume that $M \neq KMS$. Then there exists $x \in M \backslash KMS$. We have $\langle x \rangle = AMS$ for some ideal A of R. It is clear that $A \nsubseteq K$. Thus $K + A = R$. Therefore $M = RMS = (K + A)MS = KMS + AMS = KMS + \langle x \rangle$.

($\Leftarrow$) This is clear.

(ii) Assume that $M \neq KMS$. Let N be an (R, S)-submodule of M such that $KMS \subset N \subseteq M$. Let $n \in N \backslash KMS$. By (i), $M = KMS + \langle n \rangle \subseteq N \subseteq M$. Hence $N = M$. This shows that KMS is a maximal (R, S)-submodule of M.

(iii) First, assume that $(KMS : M)_R = K$. Suppose for contradiction that $KMS = M$. Then $K = (KMS : M)_R = (M : M)_R = R$ which is a contradiction. Hence $KMS \neq M$. It follows from (ii) that KMS is a maximal (R, S)-submodule of M.

Conversely, assume that KMS is a maximal (R, S)-submodule of M. It is clear that $KMS \neq M$ and $(KMS : M)_R \neq R$. Since $K \subseteq (KMS : M)_R \subset R$ and K is a maximal ideal of R, we have $K = (KMS : M)_R$.

(iv) This is clear. $\square$

Proposition 3.3.3. *Let M be a left multiplication (R, S)-module such that $M \neq KMS$ for all maximal ideals K of R. Then there is a one-to-one function between the set of all maximal ideals of R and the set of all maximal (R, S)-submodules of M.*

Proof. Define a map φ by $\varphi(K) = KMS$ for all maximal ideals K of R. It follows from Proposition 3.3.2 (ii) and (iii) that φ is a well-defined and one-to-one function and $\varphi(K)$ is a maximal (R, S)-submodule of M for all maximal ideals K of R. $\square$

Proposition 3.3.4. *Let M be a left multiplication (R, S)-module and P an (R, S)-submodule of M.*

(i) *If $(P : M)_R$ is a maximal ideal of R, then P is a maximal (R, S)-submodule of M.*

(ii) *If $(P : M)_R$ is a minimal ideal of R, then P is a minimal (R, S)-submodule of M.*

Proof. (i) Assume that $(P : M)_R$ is a maximal ideal of R. Let N be an (R, S)-submodule of M such that $P \subseteq N \subseteq M$. Then $(P : M)_R \subseteq (N : M)_R \subseteq R$. By the maximality of $(P : M)_R$, $(P : M)_R = (N : M)_R$ or $(N : M)_R = R$. Hence $P = N$ or $N = M$.

(ii) Assume that $(P : M)_R$ is a minimal ideal of R. Let N be an (R, S)-submodule of M such that $0 \subseteq N \subseteq P$. Then $0 \subseteq (N : M)_R \subseteq (P : M)_R$. By the minimality of $(P : M)_R$, $0 = (N : M)_R$ or $(N : M)_R = (P : M)_R$. Hence $0 = N$ or $N = P$. $\square$

CHAPTER IV
CONCLUSION

In this chapter, we summarize our overall results. The first section concentrates on the similarities and differences between the primality of submodules of R-module structures and the primality of (R, S)-submodules of (R, S)-module structures. Moreover, we investigate the form of jointly prime $(\mathbb{Z}, \mathbb{Z})$-submodules of $\mathbb{Z}$ and also jointly prime $(r\mathbb{Z}, s\mathbb{Z})$-submodules of $\mathbb{Z}$, where $r, s \in \mathbb{Z}^+$. In the last section, we take advantage of results from left multiplication (R, S)-modules to determine when a $(b\mathbb{Z}, c\mathbb{Z})$-module $a\mathbb{Z}$ is a left multiplication $(b\mathbb{Z}, c\mathbb{Z})$-module. The form of jointly prime $(\mathbb{Z}, \mathbb{Z})$-submodules of a left multiplication $(\mathbb{Z}, \mathbb{Z})$-module $a\mathbb{Z}$ where $a \in \mathbb{Z}^+$ is given. We close the chapter by giving a conclusion that the set of all (R, S)-submodules of a multiplication (R, S)-module forms a commutative semiring.

4.1 Primalities of (R, S)-Submodules

In this section, we compare results between the primality of submodules of R-module structures and the primality of (R, S)-submodules of (R, S)-module structures. We present the similarities and differences between both structures by giving examples and stating properties.

First, note that $\mathbb{Q}$ is both a $\mathbb{Z}$-module and a $(\mathbb{Z}, \mathbb{Z})$-module. Theorem 2.2.2 tell us that fully and jointly prime $(\mathbb{Z}, \mathbb{Z})$-submodules are identical. The following result shows that 0 is both the only prime submodule of $\mathbb{Q}$ as a $\mathbb{Z}$-module and the only jointly prime $(\mathbb{Z}, \mathbb{Z})$-submodule of $\mathbb{Q}$ as a $(\mathbb{Z}, \mathbb{Z})$-module. This result is the first similarity.

Remark 4.1.1. *Recall that $\mathbb{Q}$ can be considered as a $\mathbb{Z}$-module as well as a $(\mathbb{Z}, \mathbb{Z})$-module. The followings hold.*

(i) 0 *is the only prime submodule of* $\mathbb{Q}$ *as a* $\mathbb{Z}$*-module.*

(ii) 0 *is the only jointly prime* $(\mathbb{Z}, \mathbb{Z})$*-submodule of* $\mathbb{Q}$ *as a* $(\mathbb{Z}, \mathbb{Z})$*-module.*

Proof. (ii) It is clear that 0 is a jointly prime $(\mathbb{Z}, \mathbb{Z})$-submodule of $\mathbb{Q}$ as a $(\mathbb{Z}, \mathbb{Z})$-module. Let N be a jointly prime $(\mathbb{Z}, \mathbb{Z})$-submodule of $\mathbb{Q}$ and $x \in N$. Then $x = \frac{a}{b} = a \cdot \frac{1}{b} \cdot 1$ for some $a \in \mathbb{Z}$ and $b \in \mathbb{Z}\backslash\{0\}$. By Theorem 2.2.3, N is jointly prime implies $a\mathbb{Q} \subseteq N$ or $\frac{1}{b} \in N$. Suppose that $\frac{1}{b} \in N$. Then $1 \in N$. Hence $\mathbb{Z} \subseteq N$. Let $p \in \mathbb{Z}$ and $q \in \mathbb{Z}\backslash\{0\}$. Then $q \cdot \frac{p}{q} \cdot 1 = p \in N$. By Theorem 2.2.3 again, $q\mathbb{Q} \subseteq N$ or $\frac{p}{q} \in N$. If $q\mathbb{Q} \subseteq N$, then $\mathbb{Q} = N$ which is also a contradiction. If $\frac{p}{q} \in N$, then it means that $\mathbb{Q} = N$ which is a contradiction. Hence $\frac{1}{b} \notin N$, so that $a\mathbb{Q} \subseteq N$. Moreover, if $a \neq 0$, then $\mathbb{Q} = a\mathbb{Q} \subseteq N$ which is a contradiction. Thus $a = 0$, i.e., $x = 0$. This shows that $N = \{0\}$.

(i) This can be proved similarly to (ii). □

It is clear that all prime submodules of the left $\mathbb{Z}$-module $\mathbb{Z}$ are of the form $p\mathbb{Z}$ where $p = 0$ or p is a prime integer. Now, let us characterize jointly prime $(\mathbb{Z}, \mathbb{Z})$-submodules of $\mathbb{Z}$ and also jointly prime $(r\mathbb{Z}, s\mathbb{Z})$-submodules of $\mathbb{Z}$ where $r, s \in \mathbb{Z}^+$.

Remark 4.1.2. *Let* $r, s \in \mathbb{Z}^+$ *and* $p \in \mathbb{Z}_0^+\backslash\{1\}$*. Then* $p\mathbb{Z}$ *is a jointly prime* $(r\mathbb{Z}, s\mathbb{Z})$*-submodule of* $\mathbb{Z}$ *if and only if* $p = 0$*,* p *is a prime integer or* $p \mid rs$*.*

Proof. ($\Rightarrow$) Assume that $p\mathbb{Z}$ is a jointly prime $(r\mathbb{Z}, s\mathbb{Z})$-submodule of $\mathbb{Z}$. Suppose that $p \neq 0$ and p is not a prime integer. Then $p = mn$ for some integers $m, n > 1$. Then $(rm\mathbb{Z})(n\mathbb{Z})(s\mathbb{Z}) = (rsp\mathbb{Z}) \subseteq p\mathbb{Z}$. Since $p\mathbb{Z}$ is jointly prime and $p \nmid n$, $(rm\mathbb{Z})(\mathbb{Z})(s\mathbb{Z}) \subseteq p\mathbb{Z}$. Note that $(r\mathbb{Z})(m\mathbb{Z})(s\mathbb{Z}) = (rm\mathbb{Z})(\mathbb{Z})(s\mathbb{Z}) \subseteq p\mathbb{Z}$. Since $p\mathbb{Z}$ is jointly prime and $p \nmid m$, we have $rs\mathbb{Z} = (r\mathbb{Z})(\mathbb{Z})(s\mathbb{Z}) \subseteq p\mathbb{Z}$. Hence $p \mid rs$.

($\Leftarrow$) If $p = 0$ or p is a prime integer or $p \mid rs$, then it is clear that $p\mathbb{Z}$ is a jointly prime $(r\mathbb{Z}, s\mathbb{Z})$-submodule of $\mathbb{Z}$. □

We conclude that prime ideals of $\mathbb{Z}$, prime submodules of the left $\mathbb{Z}$-module $\mathbb{Z}$ and jointly prime $(\mathbb{Z}, \mathbb{Z})$-submodules of $\mathbb{Z}$ are all identical and of the form $p\mathbb{Z}$

where $p = 0$ or p is a prime integer. Moreover, there are more jointly prime $(r\mathbb{Z}, s\mathbb{Z})$-submodules of $\mathbb{Z}$ if r and s are positive integers greater than 1.

Second, recall a well-known result from [8] that if R is a commutative ring with $R^2 = R$, then every maximal ideal of R is a prime ideal of R. Moreover, we see in [2] that if M is a unital left R-module over any ring R with identity, then every maximal submodule of M is a prime submodule of M. The following is an example showing that maximality may not imply primality in general.

Example 4.1.3. Let R be the matrix ring over the integers such that

$$R = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} \;\middle|\; x, y \in \mathbb{Z} \right\}$$

and let

$$P = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & 0 & 2y \\ 0 & 0 & 0 \end{bmatrix} \;\middle|\; x, y \in \mathbb{Z} \right\}.$$

Then P is a maximal ideal of R satisfying $R^2 \subseteq P$. Hence P is not a prime ideal of R. Moreover, it is clear that $R^2 \neq R$.

Notice from the previous example that there is an ideal of R containing R^2 that is not a prime ideal of R. In contrast, every (R, R)-submodule of R containing R^3 is a fully prime (R, R)-submodule of R. See the following for the general case.

Remark 4.1.4. *Let M be an (R, S)-module. Every proper (R, S)-submodule P of M such that $RMS \subseteq P$ is a fully prime (R, S)-submodule of M.*

Remark 4.1.4 also yields that if $RMS \neq M$, then fully prime, jointly prime and left R-prime (R, S)-submodules always exist.

Maximality does not imply primality in both of rings and of modules. In the same manner, proving that maximal (R, S)-submodules are fully prime also requires some conditions, see Proposition 2.1.9. However, in the structure of (R, S)-modules, there are two types of primality for (R, S)-submodules, namely, jointly prime and left R-prime, and maximal (R, S)-submodules are naturally both jointly prime and left R-prime immediately, see Proposition 2.2.10 and Proposition 2.3.4.

Finally, fully prime and jointly prime (R, S)-submodules are quite similar. It is easy to see that every fully prime (R, S)-submodule is jointly prime. In fact, it is not clear that jointly prime (R, S)-submodules differ from fully prime (R, S)-submodules. However, we have identified conditions that make jointly prime (R, S)-submodules be fully prime, see Theorem 2.2.2, Theorem 2.2.7 and Proposition 2.2.14.

Remark 4.1.5. (i) *If M is an (R, S)-module where R or S is a commutative ring, then fully prime and jointly prime (R, S)-submodules are the same.*

(ii) *If M is an (R, S)-module such that $RMS = M$, then fully prime and jointly prime (R, S)-submodules are identical.*

(iii) *If R is a ring, then fully prime and jointly prime (R, R)-submodules of R are the same.*

4.2 Left Multiplication (R, S)-Modules

We discuss, in this section, results regarding left multiplication (R, S)-modules. An (R, S)-module M is a left multiplication (R, S)-module if for each (R, S)-submodule N of M there exists an ideal I of R such that $N = IMS$. Proposition 3.1.5 (i) obtains that, in fact, $N = (N : M)_R MS$ for all (R, S)-submodules N of M. In particular, $M = (M : M)_R MS = RMS$.

Remark 4.2.1. *If M is a left multiplication (R, S)-module, then $M = RMS$.*

We can see that the condition $M \neq RMS$ is a tool to verify that M is not a left multiplication (R, S)-module.

For each $a, b, c \in \mathbb{Z}$, we know that $a\mathbb{Z}$ is a $(b\mathbb{Z}, c\mathbb{Z})$-module. The contrapositive of Remark 4.2.1 helps us to study when a $(b\mathbb{Z}, c\mathbb{Z})$-module $a\mathbb{Z}$ is a left multiplication $(b\mathbb{Z}, c\mathbb{Z})$-module.

Remark 4.2.2. *Let $a, b, c \in \mathbb{Z}_0^+$. Then $a\mathbb{Z}$ is a left multiplication $(b\mathbb{Z}, c\mathbb{Z})$-module if and only if $b = c = 1$.*

Proof. ($\Rightarrow$) We prove the contrapositive. Without loss of generality, assume that $b > 1$. Then $(b\mathbb{Z})(a\mathbb{Z})(c\mathbb{Z}) = abc\mathbb{Z} \neq a\mathbb{Z}$. By Remark 4.2.1, $a\mathbb{Z}$ is not a left multiplication $(b\mathbb{Z}, c\mathbb{Z})$-module.

($\Leftarrow$) Assume that $b = c = 1$. Let N be a $(\mathbb{Z}, \mathbb{Z})$-submodule of $a\mathbb{Z}$. Then $N = ak\mathbb{Z}$ for some $k \in \mathbb{Z}$. Then $N = ak\mathbb{Z} = (k\mathbb{Z})(a\mathbb{Z})(\mathbb{Z})$. Hence $a\mathbb{Z}$ is a left multiplication $(\mathbb{Z}, \mathbb{Z})$-module. $\square$

Theorem 3.1.14 provides that jointly prime and left $\mathbb{Z}$-prime $(\mathbb{Z}, \mathbb{Z})$-submodules of $a\mathbb{Z}$ are the same. The following gives a characterization of jointly prime $(\mathbb{Z}, \mathbb{Z})$-submodules of $a\mathbb{Z}$ where $a \in \mathbb{Z}^+$. Recall that all $(\mathbb{Z}, \mathbb{Z})$-submodules of $a\mathbb{Z}$ are of the form $ak\mathbb{Z}$ for some integer k. It can be shown that $(ak\mathbb{Z} : a\mathbb{Z})_{\mathbb{Z}} = k\mathbb{Z}$ for all $k \in \mathbb{Z}$.

Remark 4.2.3. *Let $a \in \mathbb{Z}^+$. For each $p \in \mathbb{Z}_0^+\backslash\{1\}$, $ap\mathbb{Z}$ is a jointly prime $(\mathbb{Z}, \mathbb{Z})$-submodule of $a\mathbb{Z}$ if and only if $p = 0$ or p is a prime integer.*

Proof. Let $p \in \mathbb{Z}_0^+\backslash\{1\}$.

($\Rightarrow$) Assume that $ap\mathbb{Z}$ is a jointly prime $(\mathbb{Z}, \mathbb{Z})$-submodule of $a\mathbb{Z}$ and $p \neq 0$. To show that p is a prime integer, let $k_1, k_2 \in \mathbb{Z}^+$ be such that $p \mid k_1k_2$. Then $(ak_1\mathbb{Z})$ and $(ak_2\mathbb{Z})$ are $(\mathbb{Z}, \mathbb{Z})$-submodules of $a\mathbb{Z}$. We consider their products. Thus, $(ak_1\mathbb{Z})(ak_2\mathbb{Z}) = (ak_1\mathbb{Z} : a\mathbb{Z})_{\mathbb{Z}}(ak_2\mathbb{Z} : a\mathbb{Z})_{\mathbb{Z}}(a\mathbb{Z})\mathbb{Z}\mathbb{Z} = (k_1\mathbb{Z})(k_2\mathbb{Z})(a\mathbb{Z}) = ak_1k_2\mathbb{Z} \subseteq ap\mathbb{Z}$. Hence $ak_1\mathbb{Z} \subseteq ap\mathbb{Z}$ or $ak_2\mathbb{Z} \subseteq ap\mathbb{Z}$. This implies that $p \mid k_1$ or $p \mid k_2$. Therefore p is a prime integer.

($\Leftarrow$) If $p = 0$, then it is clear that 0 is a jointly prime $(\mathbb{Z}, \mathbb{Z})$-submodule of $a\mathbb{Z}$. Assume that p is a prime integer. To show that $ap\mathbb{Z}$ is a jointly prime $(\mathbb{Z}, \mathbb{Z})$-submodule of $a\mathbb{Z}$, let U and V be $(\mathbb{Z}, \mathbb{Z})$-submodules of $a\mathbb{Z}$ such that $UV \subseteq ap\mathbb{Z}$. Note that $U = ak_1\mathbb{Z}$, $V = ak_2\mathbb{Z}$ and $UV = ak_1k_2\mathbb{Z}$ for some $k_1, k_2 \in \mathbb{Z}$. Thus $ak_1k_2\mathbb{Z} \subseteq ap\mathbb{Z}$. Hence $p \mid k_1k_2$. Since p is a prime integer, $p \mid k_1$ or $p \mid k_2$. This implies that $U = ak_1\mathbb{Z} \subseteq ap\mathbb{Z}$ or $V = ak_2\mathbb{Z} \subseteq ap\mathbb{Z}$. Hence $ap\mathbb{Z}$ is a jointly prime $(\mathbb{Z}, \mathbb{Z})$-submodule of $a\mathbb{Z}$. $\square$

Ihis is a good place to point out that the condition "$M = RMS$" plays an important role in this research. We studied properties of fully prime and jointly prime (R, S)-submodules of arbitary (R, S)-modules in Chapter II. Proposition 2.1.9 emphasizes the importance of "$M = RMS$" in primality, i.e., fully prime and jointly prime are identical on an (R, S)-modules M provided $M = RMS$.

Later, we know from Chapter III that if M is a left multiplication (R, S)-module, then $M = RMS$ so that fully prime and jointly prime (R, S)-submodules are identical. However, the condition "$M = RMS$" is a weaker assumption than the condition "M is a left multiplication (R, S)-module" which still implies that fully prime and jointly prime are identical.

Furthermore, let N be an (R, S)-submodule of a left multiplication (R, S)-module M. Proposition 1.2.8 again provides that the condition "$M = RMS$" is a weaker assumption than the condition "M is a left multiplication (R, S)-module" in order to obtain that $(N : M)_R$ is an ideal of R.

For our second point, we see from Section 3.2 that results for jointly prime and left R-prime (R, S)-submodules are different. However, those different results can be proven to be similar to one another. Here, we give an idea of how to do so. Let P be an (R, S)-submodule of M. First we recall condition (3.2) from page 53 which says that for each $a, b \in M$, $\langle a \rangle \langle b \rangle \subseteq P$ implies $a \in P$ or $b \in P$. Corollary 3.2.2 and Corollary 3.2.8 provide that if P is jointly prime or left R-prime, then the condition (3.2) holds. By the condition (3.2) and the same proof as for Theorem 3.2.4, we obtain the necessary part of Theorem 3.2.4 in the case of left R-prime (R, S)-submodules. For the converse, Theorem 3.2.4 works under assumption that R is commutative and $S^2 = S$. We obtain from Theorem 3.1.14 that jointly prime and left R-prime are the same under these assumptions. Therefore we can write Theorem 3.2.4 from the point of view of left R-prime (R, S)-submodules as follows.

Remark 4.2.4. *Let P be a proper (R, S)-submodule of a left multiplication (R, S)-module M. If P is a left R-prime (R, S)-submodule of M, then for all (R, S)-submodules N of M, $N \nsubseteq P$ implies $P = (P :_M N)$.*

Furthermore, if R is commutative and $S^2 = S$, then the converse is true as well, i.e., if for all (R, S)-submodules N of M, $N \nsubseteq P$ implies $P = (P :_M N)$, then P is a left R-prime (R, S)-submodule of M.

Following the same line of thought, we will not mention about studying properties of jointly prime (R, S)-submodules of a left multiplication (R, S)-module. To obtain more general results, we investigated properties of left R-prime (R, S)-submodules of a left multiplication (R, S)-module, see Theorem 3.2.9, Theorem 3.2.13, Theorem 3.2.15, Theorem 3.2.17 and Theorem 3.2.18. However, it is easy to get the same results in the case of jointly prime (R, S)-submodules by adding the assumption $S^2 = S$.

As our last point, the notion of left multiplication (R, S)-modules permits us to define the product of two (R, S)-submodules, see Proposition 3.1.7 and Definition 3.1.15. Let N and K be (R, S)-submodules of a left multiplication (R, S)-module M. Recall that the sum and product of N and K are, respectively,

$$N + K = \{n + k \mid n \in N \text{ and } k \in K\}$$

$$N \cdot K = (N : M)_R (K : M)_R MSS.$$

We can see that $N + K$ and $N \cdot K$ (or simply, NK) are (R, S)-submodules of M.

We start with an obvious observation.

Remark 4.2.5. *Let M be a left multiplication (R, S)-module and A, B, C and D be (R, S)-submodules of M. The following hold.*

(i) $A + (B + C) = (A + B) + C$.

(ii) *If I and J are ideals of R, then $IMS + JMS = (I + J)MS$.*

(iii) $(A \cap B) + (A \cap C) \subseteq A \cap (B + C)$.

(iv) *If $A \subseteq C$ and $B \subseteq D$, then $AB \subseteq CD$.*

(v) *$AB \subseteq A$ and $AB \subseteq B$.*

Proof. Parts (i)–(iii) and (v) are clear.

(iv) Assume that $A \subseteq C$ and $B \subseteq D$. Then $(A : M)_R \subseteq (C : M)_R$ and $(B : M)_R \subseteq (D : M)_R$ by Proposition 3.2.5. Hence $(A : M)_R(B : M)_R \subseteq (C : M)_R(D : M)_R$. This implies that $(A : M)_R(B : M)_R MSS \subseteq (C : M)_R(D : M)_R MSS$. That is, $AB \subseteq CD$. □

Remark 4.2.6. *Let M be a left multiplication (R, S)-module and A, B, C and D be (R, S)-submodules of M. Assume that R is a commutative ring. The followings hold.*

(i) *If $S^2 = S$, then $A(BC) = (AB)C$.*

(ii) *$AM = A = MA$.*

(iii) *$A(B + C) = AB + AC$.*

(iv) *$(A + B)(A \cap B) \subseteq AB$.*

Proof. This is clear. □

Let R be a commutative ring and $S^2 = S$. Then we conclude from Remark 4.2.5 and Remark 4.2.6 that the set $\mathcal{N}$ of all (R, S)-submodules of a nonzero left multiplication (R, S)-module M satisfies the following

(i) $(\mathcal{N}, +)$ is a commutative monoid with identity 0,

(ii) $(\mathcal{N}, \cdot)$ is a commutative monoid with identity $M \neq 0$,

(iii) The multiplication is left and right distributive over the addition, and

(iv) $A0 = 0 = 0A$ for all $A \in \mathcal{N}$.

In other words, $(\mathcal{N}, +, \cdot)$ forms a commutative semiring, see [7].

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