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SUPERSETS OF WAVELET SETS

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ในวิทยานิพนธ์นี้เราพิจารณาเซตพื้นฐานสองซึ่งเป็นเซตที่คลุมเซต $\mathbb{R}$ โดยการเลื่อนขนานด้วยจำนวนเต็มและการเปลี่ยนขนาดด้วยไดแอดิก และเซตนี้ตัดตัวเองอย่างมากสองครั้งโดยการเลื่อนขนานและการเปลี่ยนขนาด เราได้รับเงื่อนไขจำเป็นและเงื่อนไขเพียงพอบางเงื่อนไขสำหรับเซตพื้นฐานสอง S ที่บรรจุเซตเวฟเลต ผลลัพธ์หลักซึ่งอยู่ในรูปของความสัมพันธ์ระหว่างเซตย่อยซึ่งถูกสร้างอย่างชัดแจ้ง A และ B ของ S กับเซตย่อย T_2 และ D_2 ของ S ซึ่งตัดตัวเองสองครั้งโดยการเลื่อนขนานและการเปลี่ยนขนาดตามลำดับ คือ (1) ถ้า $A \cup B \not\subseteq T_2 \cap D_2$ แล้ว S ไม่บรรจุเซตเวฟเลต และ (2) ถ้า $A \cup B \subseteq T_2 \cap D_2$ แล้วทุก ๆ เซตย่อยซึ่งเป็นเซตเวฟเลตของ S ต้องเป็นเซตย่อย $S \setminus (A \cup B)$ และ ถ้า $S \setminus (A \cup B)$ สอดคล้องกับเงื่อนไขอย่างอ่อนเงื่อนไขหนึ่งแล้ว จะมีเซตเวฟเลต ซึ่งเป็นเซตย่อยของ $S \setminus (A \cup B)$

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In this thesis, we consider a two-basic set which is defined as a set whose both integral translations and dyadic dilations cover $\mathbb{R}$ and that intersects itself at most twice translationally and dilationally. We obtain some necessary conditions and some sufficient conditions for a two-basic set S to contain a wavelet set. The main results, which are in terms of the relationship between two explicitly constructed subsets A and B of S and two subsets T_2 and D_2 of S intersecting itself exactly twice translationally and dilationally respectively, are that (1) if $A \cup B \not\subseteq T_2 \cap D_2$ then S does not contain a wavelet set; and that (2) if $A \cup B \subseteq T_2 \cap D_2$ then every wavelet subset of S must be a subset of $S \setminus (A \cup B)$ and if $S \setminus (A \cup B)$ satisfies a "weak" condition then there exists a wavelet subset of $S \setminus (A \cup B)$.

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CONTENTS

CHAPTER I

INTRODUCTION

The notion of "wavelet sets" was introduced by Dai and Larson in [7]. Wavelet sets have been constructed mainly to show the existence of wavelet functions. Since then, because of their tiling and fractal-like properties, wavelet sets have received considerable interest in their own right [1, 2, 3, 21]. However, many natural questions remain open. One of them asks whether the frequency support of a given wavelet function must contain a wavelet set [17, 20]. This question is still open even in the simplest one-dimensional case with dilation factor 2 and integral translation. In [20], Rzeszotnik and Speegle answered the question partially by proving that if the Fourier support of a wavelet function is "small enough," then it is either a wavelet set or a union of two wavelet sets. However, to the best of our knowledge, there are no other results that would point to an answer to the open problem in a more general setting. This problem therefore seems to be more difficult than expected. In an attempt to gain a better insight of the geometry of supersets of wavelet sets, we set out to find necessary conditions and sufficient conditions for a "small" set to contain a wavelet set without assuming that the set is the Fourier support of a wavelet. Here, a "small" set is a 2-*basic set*, that is, it covers $\mathbb{R}$ by integral translation and dyadic dilation but intersects itself at most twice both translationally and dilationally. We hope that this approach will allow a generalization to sets which are not necessarily "small." In fact, it is mentioned in [19] that Qing Gu has an unpublished example which shows that the techniques used in [20] do not extend to "larger" sets. A remark is worth mentioning here. Under the assumption that S is small, it is proved in [20] that if S is the Fourier support of a wavelet function then $T_2(S) = D_2(S)$ where $T_2(S)$ ($D_2(S)$) is the part of S which intersects some of its translation (dilations). We show, under the same assumption, that if $T_2(S) = D_2(S)$ and a weak condition holds, then S contains a wavelet set. We then conjecture that the weak condition is always valid and therefore S contains a wavelet set whenever $T_2(S) = D_2(S)$, a weaker assumption than S being the Fourier support of a wavelet.

A part of this thesis is about wavelet set construction. Since ad hoc examples wavelet sets in $\mathbb{R}^d$ were introduced, many methods to construct various classes of wavelet sets have been proposed. Most of these methods [1, 2, 3, 4, 5, 8, 9, 14, 16, 21] rely on cut-and-paste technique where an initial set tiling the space by translations is chosen. Recursively, part of the set which intersects itself by dilations is then moved out by translations. These construction do not allow us to choose a priori a set of which the to-be-constructed is a subset. The novelty of our construction of wavelet sets is that one can, under some assumption, construct a wavelet set by carving out parts of a given "small" set without any pasting.

In Chapter 2, we begin by introducing the concept of holding almost everywhere on sets. Next, we recall about wavelets and wavelet sets. In the last section, we give a few properties of supersets of wavelet set.

In Chapter 3, we study geometric properties of a 2-basic set containing a wavelet set. We start with the definition of a 2-basic set and provide a characterization of a 2-basic set to contain a wavelet set. Next, we introduce two functions for a procedure to construct two sequences of subsets of a 2-basic set whose union never intersects with wavelet subsets of a given 2-basic set. Moreover, we obtain necessary conditions and sufficient conditions for a 2-basic set containing a wavelet set in terms of this union. We discuss a 2-basic set with the empty union.

In Chapter 4, we provide a few examples of constructing wavelet sets from 2-basic sets.

Finally, in Chapter 5 we summarize the results of our work and discuss possibilities for future studies.

CHAPTER II

PRELIMINARIES

In this chapter we present some basic definitions and theorems used in this thesis. We assume that the reader has a basic understanding of real analysis or measure theory. First of All, we introduce the notation in this manuscript. All sets in here are Lebesgue measurable subsets of the real line $\mathbb{R}$; λ denotes Lebesgue measure on $\mathbb{R}$; $\mathbb{Z}$ and $\mathbb{N}$ are the set of all integers and the set of all positive integers, respectively. Furthermore, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$ and $\mathbb{N}_0^\infty = \mathbb{N}_0 \cup \{\infty\}$. The characteristic function of a measurable set E is denoted by $\mathbb{1}_E$. In the first section, we extend some set relations used extensively in later chapters to almost everywhere relations. Next, we recall some definitions and theorems on wavelets and wavelet sets. Finally, we will present our basic result.

2.1 Concepts of holding almost everywhere on sets

We will work with Lebesgue measure λ on $\mathbb{R}$, although it is possible to use other measure spaces. We start by introducing definition of being subset and equality of sets almost everywhere.

Definition 2.1.1. *A measurable set E is a **subset** of a measurable set F **almost everywhere**, denoted by $E \underset{\text{a.e.}}{\subseteq} F$ or $F \underset{\text{a.e.}}{\supseteq} E$, if $\mathbb{1}_E \leq \mathbb{1}_F$* a.e.

Definition 2.1.2. *Two measurable sets E and F are* ***equal almost everywhere****, denoted by $E \underset{\text{a.e.}}{=} F$, if $E \underset{\text{a.e.}}{\subseteq} F$ and $F \underset{\text{a.e.}}{\subseteq} E$.*

Obviously, $\underset{\text{a.e.}}{=}$ is an equivalence relation on the set of Lebesgue measurable subsets of $\mathbb{R}$. For convenience in using subset almost everywhere, we will give a characterization of subset almost everywhere by difference between sets.

Proposition 2.1.3. *Let E and F be measurable subsets of $\mathbb{R}$. Then*

(a) $E \underset{\text{a.e.}}{\subseteq} F$ if and only if $\lambda(E \smallsetminus F) = 0$, and

(b) $E \underset{\text{a.e.}}{=} F$ if and only if $\lambda(E \triangle F) = 0$ where $E \triangle F = (E \smallsetminus F) \cup (F \smallsetminus E)$.

Proof. To prove (a), we assume that $E \underset{\text{a.e.}}{\subseteq} F$. Then $\mathbb{1}_E \leq \mathbb{1}_F$ a.e., and so $1 - \mathbb{1}_E \geq 1 - \mathbb{1}_F$ a.e. By a property of a characteristic function, we have $\mathbb{1}_{E^c} \geq \mathbb{1}_{F^c}$ a.e. Thus $\mathbb{1}_E \cdot \mathbb{1}_{E^c} \geq \mathbb{1}_E \cdot \mathbb{1}_{F^c}$ a.e., which implies that $\mathbb{1}_{E \cap E^c} \geq \mathbb{1}_{E \cap F^c}$ a.e. Now, we get that $\mathbb{1}_{E \smallsetminus F} = 0$ a.e. Hence $\lambda(E \smallsetminus F) = 0$. Conversely, we assume that $\lambda(E \smallsetminus F) = 0$. Thus $\mathbb{1}_{E \smallsetminus F} = 0$ a.e. By properties of a characteristic function, $\mathbb{1}_E - \mathbb{1}_{E \cap F} = 0$ a.e. Because $\mathbb{1}_{E \cap F} \leq \mathbb{1}_F$, we obtain that $\mathbb{1}_E \leq \mathbb{1}_F$ a.e.

Consequently, by (a), the statement (b) is true. □

The property being subset almost everywhere shares many properties with being subset, such as reflexivity, anti-symmetry, transitivity, etc. Using the equality almost everywhere, we can define a partition almost everywhere, τ-congruence almost everywhere and δ-congruence almost everywhere as follow:

Definition 2.1.4. *Let E be a measurable subset of $\mathbb{R}$. A countable collection $\mathcal{P}$ of measurable subsets of E is said to be a* ***partition*** *of E* ***almost everywhere*** *or a* ***partition*** *of E a.e. if $\bigcup \mathcal{P} \underset{\text{a.e.}}{=} E$ and $A \cap B \underset{\text{a.e.}}{=} \varnothing$ whenever $A, B \in \mathcal{P}$ with $A \neq B$.*

Remark. *Clearly, if $\mathcal{P}$ is a partition of a set E a.e., then $\mathcal{P}' = \{C \setminus N : C \in \mathcal{P}\}$ is a partition of a set E a.e. satisfying $C \cap D = \varnothing$ whenever $C, D \in \mathcal{P}'$ with $C \neq D$, where $N = \bigcup_{\substack{A,B\in\mathcal{P}\\ A\neq B}} (A \cap B)$. Throughout this thesis, we will assume that every partition almost everywhere satisfying the above property.*

Definition 2.1.5. *Two sets E and F are* τ**-congruent almost everywhere**, *denoted by $E \underset{\text{a.e.}}{\overset{\tau}{\sim}} F$, if there are partitions $\{E_k : k \in \mathbb{Z}\}$ and $\{F_k : k \in \mathbb{Z}\}$ of E and F a.e., respectively, and a sequence $\{n_k : k \in \mathbb{Z}\} \subseteq \mathbb{Z}$ such that $E_k \underset{\text{a.e.}}{=} F_k + n_k$, for all $k \in \mathbb{Z}$. Two sets E and F are* δ**-congruent almost everywhere**, *denoted by $E \underset{\text{a.e.}}{\overset{\delta}{\sim}} F$, if there are partitions $\{E_k : k \in \mathbb{Z}\}$ and $\{F_k : k \in \mathbb{Z}\}$ of E and F a.e., respectively, and a sequence $\{j_k : k \in \mathbb{Z}\} \subseteq \mathbb{Z}$ such that $E_k \underset{\text{a.e.}}{=} 2^{j_k} F_k$, for all $k \in \mathbb{Z}$.*

2.2 Wavelets and wavelet sets

In this section we briefly review the definition of wavelets and wavelet sets. We begin this section with the Fourier transform of a function. For $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, the Fourier transform $\widehat{f}$ of f is defined by

$$\widehat{f}(s) = \int_{\mathbb{R}} f(t) e^{-2\pi i s t} \mathrm{d}t, \quad s \in \mathbb{R}.$$

By the well-known Parseval-Plancherel theorem, the Fourier transform can be extended to an isometry from $L^2(\mathbb{R})$ onto itself. Now, we will recall the definition of a wavelet and its characterization.

Definition 2.2.1. *A function $\psi \in L^2(\mathbb{R})$ is said to be a* **single dyadic orthonormal wavelet** *or* **wavelet** *if the collection $\{\psi_{j,n} : j, n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$, where each $\psi_{j,n}$ is defined by*

$$\psi_{j,n}(t) = 2^{\frac{j}{2}} \psi(2^j t - n), \quad \textit{for } t \in \mathbb{R}.$$

Theorem 2.2.2. *Let $\psi \in L^2(\mathbb{R})$. Then $\{\psi_{j,n} : j, n \in \mathbb{Z}\}$ is an orthonormal set if and only if*

$$\sum_{n\in\mathbb{Z}} |\widehat{\psi}(s+n)|^2 = 1 \qquad \text{a.e. } on\ \mathbb{R} \qquad (2.1)$$

$$and\ for\ all \quad j \geq 1 \qquad \sum_{n\in\mathbb{Z}} \widehat{\psi}(2^j(s+n))\overline{\widehat{\psi}(s+n)} = 0 \qquad \text{a.e. } on\ \mathbb{R}. \qquad (2.2)$$

Theorem 2.2.3. *Let $\psi \in L^2(\mathbb{R})$ be such that $\|\psi\|_{L^2(\mathbb{R})} = 1$. Then ψ is a wavelet in $L^2(\mathbb{R})$ if and only if*

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j s)|^2 = 1 \qquad \text{a.e. } on\ \mathbb{R} \quad (2.3)$$

$$and\ for\ all \quad n \in 2\mathbb{Z}+1 \qquad \sum_{j=0}^{\infty} \widehat{\psi}(2^j s)\overline{\widehat{\psi}(2^j(s+n))} = 0 \qquad \text{a.e. } on\ \mathbb{R}. \quad (2.4)$$

The proofs of both theorems above can be found in [12]. Now, we come to the definition of a wavelet set.

Definition 2.2.4. *A measurable subset W of $\mathbb{R}$ is a* **wavelet set** *or* **MSF set** *if $\mathbb{1}_W = \widehat{\psi}$ where ψ is a wavelet in $L^2(\mathbb{R})$. Such a wavelet ψ is called an* **s-element wavelet** *or* **minimally supported frequency wavelet**.

We recall here a geometric characterization of a wavelet set. For a proof of the following theorem, see [7] and [15].

Theorem 2.2.5. *A measurable subset W of $\mathbb{R}$ is a wavelet set if and only if*

$$\{W + n : n \in \mathbb{Z}\} \text{ is a partition of } \mathbb{R} \text{ a.e. and} \qquad (2.5)$$

$$\{2^j W : j \in \mathbb{Z}\} \text{ is a partition of } \mathbb{R} \text{ a.e.} \qquad (2.6)$$

Remark. *In particular, if $\mathbb{R} = \dot{\bigcup}_{n\in\mathbb{Z}} W + n$ and $\mathbb{R} \setminus \{0\} = \dot{\bigcup}_{j\in\mathbb{Z}} 2^j W$ where $\dot{\bigcup}$ denotes the disjoint union, then W is called a* **regularized** *wavelet set. The term* *was first used in* [15]. *Furthermore, if* W *satisfies the properties* (2.5) *and* (2.6), *then we say that* W **tiles** $\mathbb{R}$ *by* **integral translation** *and* **dyadic dilation**, *respectively. Since every wavelet set satisfies the property* (2.5), *it must have Lebesgue measure* 1.

The fact that a wavelet ψ satisfies equation (2.1) and (2.3), implies that the Fourier support K of ψ covers $\mathbb{R}$ by both integral translations and dyadic dilations, that is, $\bigcup_{n\in\mathbb{Z}} K - n \underset{\text{a.e.}}{=} \mathbb{R}$ and $\bigcup_{j\in\mathbb{Z}} 2^{-j}K \underset{\text{a.e.}}{=} \mathbb{R}$, respectively. Thus it is interesting to ask: "Must the support of the Fourier transform of such a wavelet function contain a wavelet set?" This open problem was posed by Larson, see [17]. In the next section we will give some properties of supersets of wavelet sets.

2.3 Necessary conditions for supersets of wavelet sets

In this section, we introduce two measurable functions and discuss some of their properties which will be useful in our work. Finally, we give our basic results. We begin our discussion of supersets of wavelet sets by first defining the following two functions. Let S be a measurable subset of $\mathbb{R}$. We define $\tau_S : \mathbb{R} \to \mathbb{N}_0^\infty$ and $\delta_S : \mathbb{R} \to \mathbb{N}_0^\infty$ by

$$\tau_S(x) := \sum_{n\in\mathbb{Z}} \mathbb{1}_S(x+n) \quad \text{and} \quad \delta_S(x) := \sum_{j\in\mathbb{Z}} \mathbb{1}_S(2^j x)$$

for each $x \in \mathbb{R}$. It is clear that $\tau_S(x) = \sum_{n\in\mathbb{Z}} \mathbb{1}_{S-n}(x)$ and $\delta_S(x) = \sum_{j\in\mathbb{Z}} \mathbb{1}_{2^{-j}S}(x)$ for each $x \in \mathbb{R}$. Set $T_0(S) := \tau_S^{-1}(0)$ and $D_0(S) := \delta_S^{-1}(0)$, and, for each $m \in \mathbb{N}^\infty$, let $T_m(S) := S \cap \tau_S^{-1}(m)$ and $D_m(S) := S \cap \delta_S^{-1}(m)$. Obviously, $T_m(S)$ and $D_m(S)$ are measurable sets for all $n \in \mathbb{N}_0^\infty$. When the context is clear, we shall write T_m and D_m for $T_m(S)$ and $D_m(S)$, respectively.

Noticeably, $\tau_S(x)$ counts the number of times that $x \in S - n$ and similarly for $\delta_S(x)$. Furthermore, $\tau_S(x+k) = \tau_S(x)$ and $\delta_S(2^i x) = \delta_S(x)$ for all $k, i \in \mathbb{Z}$ and for all $x \in \mathbb{R}$. The following proposition summarizes several properties of τ_S and δ_S in term of T_m and D_m, respectively.

Proposition 2.3.1. *Let S be a measurable subset of $\mathbb{R}$. Then*

(a) $S = \dot{\bigcup}_{m \in \mathbb{N}^\infty} T_m$ *and* $S = \dot{\bigcup}_{m \in \mathbb{N}^\infty} D_m$,

(b) $\mathbb{R} = T_0 \dot{\cup} (\bigcup_{n \in \mathbb{Z}} S + n)$ *and* $\mathbb{R} = D_0 \dot{\cup} (\bigcup_{j \in \mathbb{Z}} 2^j S)$,

(c) $\bigcup_{n \in \mathbb{Z}} S + n = T_1 \dot{\cup} (\bigcup_{n \neq 0} S + n)$ *and* $\bigcup_{j \in \mathbb{Z}} 2^j S = D_1 \dot{\cup} (\bigcup_{j \neq 0} 2^j S)$,

(d) $\tau_S^{-1}(m) = \bigcup_{n \in \mathbb{Z}} T_m + n$ *and* $\delta_S^{-1}(m) = \bigcup_{j \in \mathbb{Z}} 2^j D_m$.

Proof. The results (a), (b), and (c) are obvious from the definition of τ_S and δ_S. And the result (d) follows from the fact that, for all $k, i \in \mathbb{Z}$ and for all $x \in \mathbb{R}$, $\tau_S(x+k) = \tau_S(x)$ and $\delta_S(2^i x) = \delta_S(x)$. □

Our next proposition provides geometric characterizations of a measurable set S by using both functions τ_S and δ_S. Its proof is easy, and will be omitted.

Proposition 2.3.2. *Let S be a measurable subset of $\mathbb{R}$. Then*

(a) $\tau_S \geq 1$ a.e. *if and only if* $\bigcup_{n \in \mathbb{Z}} S + n \underset{\text{a.e.}}{=} \mathbb{R}$;

(b) $\delta_S \geq 1$ a.e. *if and only if* $\bigcup_{j \in \mathbb{Z}} 2^j S \underset{\text{a.e.}}{=} \mathbb{R}$;

(c) $\tau_S \leq 1$ a.e. *if and only if for all* $n \in \mathbb{Z}$, $(S+n) \cap S \underset{\text{a.e.}}{=} \varnothing$;

(d) $\delta_S \leq 1$ a.e. *if and only if for all* $j \in \mathbb{Z}$, $2^j S \cap S \underset{\text{a.e.}}{=} \varnothing$.

Remark. *Consequently, by Proposition 2.3.2, we obtain that* W *is a wavelet set if and only if* $\tau_W = 1$ a.e. *and* $\delta_W = 1$ a.e.

Proposition 2.3.3. *Let* S *be a measurable subset of* $\mathbb{R}$ *and* A, B *measurable subsets of* S *with* $A \cap B = \varnothing$ *and* $S = A \cup B$. *Then* $\tau_S = \tau_A + \tau_B$ *and* $\delta_S = \delta_A + \delta_B$.

Proof. It follows from the fact that $\mathbb{1}_S = \mathbb{1}_A + \mathbb{1}_B$. □

Some necessary conditions for a set to contain a wavelet set are listed below.

Theorem 2.3.4. *Let* S *be a measurable subset of* $\mathbb{R}$ *such that* S *contains a wavelet set* W. *Then* S *have the following properties:*

(a) $\lambda(S \cap (-1, 0)) > 0$ *and* $\lambda(S \cap (0, 1)) > 0$;

(b) $T_1 \cup D_1 \underset{\text{a.e.}}{\subseteq} W$, *and hence* $\lambda(T_1 \cup D_1) \leq 1$;

(c) $T_1 \cup D_1$ *does not contain* $(-\delta, 0)$ *and* $(0, \delta)$ *for all* $\delta > 0$;

(d) $\tau_{D_1} \leq 1$ a.e. *and* $\delta_{T_1} \leq 1$ a.e.

Proof. Obviously, the results (c) and (d) follow readily from the result (b), that is, $T_1 \cup D_1 \underset{\text{a.e.}}{\subseteq} W$. It remains to show the results (a) and (b). To prove the result (a) by contradiction, suppose that $\lambda(S \cap (0, 1)) = 0$. Since W is a subset of S, $\lambda((0, 1) \cap 2^k W) = 0$ for all $k \geq 0$. Consequently, by W being a wavelet set, $(0, 1) \underset{\text{a.e.}}{\subseteq} \bigcup_{k>0} 2^{-k} W$. For each $k \in \mathbb{N}$, let $A_k = (0, 1) \cap 2^{-k} W$. It is easy to check that $\bigcup_{k \in \mathbb{N}} A_k \underset{\text{a.e.}}{=} (0, 1)$. Recall the fact that $\lambda(W) = 1$ and W has to cover both $\mathbb{R}^+$ and $\mathbb{R}^-$ by dyadic dilation. It follows that $\lambda(W \cap \mathbb{R}^+) := c < 1$. By definition of A_k, we obtain that $2^k A_k \underset{\text{a.e.}}{\subseteq} W \cap \mathbb{R}^+$ for all $k \in \mathbb{N}$. It implies that $2^k \lambda(A_k) = \lambda(2^k A_k) \leq \lambda(W \cap \mathbb{R}^+) = c$, that is, $\lambda(A_k) \leq \frac{c}{2^k}$ for all $k \in \mathbb{N}$.

Hence $1 = \lambda(0,1) = \lambda(\bigcup_{k\in\mathbb{N}} A_k) \leq \sum_{k\in\mathbb{N}} \lambda(A_k) \leq \sum_{k\in\mathbb{N}} \frac{c}{2^k} = c < 1$, which is a contradiction. Similarly, we can prove that $\lambda(S \cap (-1,0)) > 0$.

Now, we will prove that $T_1 \underset{\text{a.e.}}{\subseteq} W$. Since W is a subset of S, we get that $\mathbb{R} \underset{\text{a.e.}}{=} \bigcup_{n\in\mathbb{Z}} S+n$. Then $T_1 \underset{\text{a.e.}}{=} \mathbb{R} \smallsetminus (\bigcup_{n\neq 0} S+n) \subseteq \mathbb{R} \smallsetminus (\bigcup_{n\neq 0} W+n) \underset{\text{a.e.}}{=} W$, which is a consequence of Proposition 2.3.1(c). Similarly, $D_1 \underset{\text{a.e.}}{\subseteq} W$ follows. Therefore, $T_1 \cup D_1 \underset{\text{a.e.}}{\subseteq} W$, and hence $\lambda(T_1 \cup D_1) \leq \lambda(W) \leq 1$, we get the required result (b). □

Example 2.3.5. *These are some examples of sets that do not contain a wavelet set.*

(a) $S := \left[-\frac{1}{2}, -\frac{1}{4}\right) \cup [2,4)$ *does not contain a wavelet set because* $S \cap (0,1) = \varnothing$.

(b) $S := \left[-1, -\frac{1}{2}\right) \cup \left[\frac{3}{4}, 2\right)$ *does not contain a wavelet set because* $\lambda(T_1 \cup D_1) = \frac{5}{4}$ *where* $T_1 = \left[\frac{3}{2}, \frac{7}{4}\right)$ *and* $D_1 = \left[-1, -\frac{1}{2}\right) \cup \left[1, \frac{3}{2}\right)$.

In the problem in the previous section, Rzeszotnik and Speegle answered the problem partially in [20]. It is proved that if ψ is a wavelet and $K = \operatorname{supp}\widehat{\psi}$ satisfying $1 \leq \tau_K \leq 2$ a.e. and $1 \leq \delta_K \leq 2$ a.e., then K contains a wavelet set. In the next chapter, we will study the geometric properties of sets satisfying both properties above but which are not necessary the Fourier support of a wavelet.

CHAPTER III

TWO-BASIC SUPERSETS OF WAVELET SETS

In this chapter we investigate necessary conditions and sufficient conditions for a 2-basic set to contain a wavelet set. In the first section, we define a 2-basic set and provide a simple characterization of a 2-basic set containing a wavelet set. Consequently, we obtain some necessary conditions for a 2-basic set to contain a wavelet set. Section 2 introduces two functions, τ_2 and δ_2, and prove several properties of both functions that are useful in the next section. In section 3, we provide a procedure to construct two sequences of subsets of a given 2-basic set whose union never intersect wavelet subsets of this 2-basic set. Importantly, we get a useful necessary condition for a 2-basic set to contain a wavelet set in terms of this union. In section 4, we present some sufficient conditions for a 2-basic set S containing a wavelet set in terms of the union in Section 3. In the last section, we discuss a 2-basic set for which the above union is an empty set.

3.1 Two-basic sets and characterizations of two basic supersets of wavelet sets

We begin this section with the definition of a 2-basic set and give examples of 2-basic sets which do not contain a wavelet set.

Definition 3.1.1. *Let S be a measurable subset of $\mathbb{R}$. We say that S is a* **two-basic set** *or a* **2-basic set** *if $1 \leq \tau_S \leq 2$ a.e. and $1 \leq \delta_S \leq 2$ a.e., that is,*

(i) $\mathbb{R} \underset{\text{a.e.}}{=} \bigcup_{n\in\mathbb{Z}} S + n$ *and* $S \underset{\text{a.e.}}{=} T_1 \cup T_2$, *and*

(ii) $\mathbb{R} \underset{\text{a.e.}}{=} \bigcup_{j\in\mathbb{Z}} 2^j S$ *and* $S \underset{\text{a.e.}}{=} D_1 \cup D_2$.

Remark. *From the definition of a 2-basic set, we will mention a few notices.*

(a) In [6], S is said to be a **basic set** *if* $S = \bigcup_{m=1}^{M} T_m = \bigcup_{m=1}^{M} D_m$, *for some M.*

(b) One can easily come up with sets which cover $\mathbb{R}$ translationally and dilationally, but do not contain a wavelet set. For instance, $S = [-2,-1] \cup [1,2]$ tiles $\mathbb{R}$ by dilation and covers $\mathbb{R}$ by translation but $\{S+n\}_{n\in\mathbb{Z}}$ are not disjoint. Since any proper subsets of S cannot cover $\mathbb{R}$ by dyadic dilation, S does not contain a wavelet set. Other examples are $[-1,-\frac{1}{2}) \cup [\frac{3}{4},2)$, $[-\frac{1}{4},-\frac{1}{8}) \cup [\frac{15}{16},2)$, etc.

(c) There are plenty of examples of wavelet functions whose supports are 2-basic. For instance, the Meyer wavelet [18] has Fourier support in $\left[-\frac{4}{3},-\frac{1}{3}\right] \cup \left[\frac{1}{3},\frac{4}{3}\right]$ which is obviously 2-basic. Another set of examples are wavelets whose Fourier supports are $\left[-\frac{4}{3}\alpha,-\frac{1}{3}\alpha\right) \cup \left[1-\frac{2}{3}\alpha, 2-\frac{2}{3}\alpha\right)$, $0 < \alpha \leq 1$, see [13].

Theorem 3.1.2. *If S is a 2-basic set, then there exists a 2-basic set $\tilde{S}$ such that $S \underset{\text{a.e.}}{=} \tilde{S}$, $\tau_{\tilde{S}} \leq 2$ and $\delta_{\tilde{S}} \leq 2$ on $\mathbb{R}$.*

Proof. Assume that S is a 2-basic set, that is, $T_1(S)\dot{\cup}T_2(S) \underset{\text{a.e.}}{=} S \underset{\text{a.e.}}{=} D_1(S)\dot{\cup}D_2(S)$ and $\bigcup_{n\in\mathbb{Z}} S + n \underset{\text{a.e.}}{=} \mathbb{R} \underset{\text{a.e.}}{=} \bigcup_{j\in\mathbb{Z}} 2^j S$. Put $\tilde{S} = [(T_1(S)\dot{\cup}T_2(S)) \cap (D_1(S)\dot{\cup}D_2(S))]$. Clearly, $\tilde{S} \subseteq S$. Moreover, $\tilde{S} \underset{\text{a.e.}}{=} S$ because $T_1(S)\dot{\cup}T_2(S) \underset{\text{a.e.}}{=} S \underset{\text{a.e.}}{=} D_1(S)\dot{\cup}D_2(S)$. Then $\tilde{S}$ is a 2-basic set. To show that $\tau_{\tilde{S}}(x) \leq 2$ for all $x \in \mathbb{R}$, let $x \in \mathbb{R}$. If $x \notin \tilde{S} - n$ for all $n \in \mathbb{Z}$, then we are done. Suppose that there exists an integer

n_0 such that $x \in \tilde{S} - n_0$. Thus $x + n_0 \in \tilde{S}$, and so $x + n_0 \in T_1(S) \dot{\cup} T_2(S)$. Hence $\tau_S(x + n_0) \leq 2$. Since $\tilde{S} \subseteq S$, we get that $\tau_{\tilde{S}}(x) = \tau_{\tilde{S}}(x + n_0) \leq \tau_S(x + n_0) \leq 2$. Similarly, we can prove that $\delta_{\tilde{S}}(x) \leq 2$ for all $x \in \mathbb{R}$. □

Remark. *We know that if S_1 and S_2 are 2-basic sets such that $S_1 \underset{\text{a.e.}}{=} S_2$ and S_1 contains a wavelet set, then S_2 contains a wavelet set. By the preceding theorem, we will only study 2-basic sets S satisfying $T_1 \cup T_2 = S = D_1 \cup D_2$.*

Next, we provide an easy and natural characterization of a 2-basic set containing a wavelet set in terms of the existence of a subset of S whose complement in T_2 (D_2) is τ-congruent (δ-congruent) to itself. We prove this characterization by using the following two lemmas.

Lemma 3.1.3. *Let S be a measurable subset of $\mathbb{R}$ such that $S = T_1 \dot{\cup} T_2$ and $\mathbb{R} \underset{\text{a.e.}}{=} \bigcup_{n \in \mathbb{Z}} S + n$. There exists a measurable subset W of S such that $\{W + n : n \in \mathbb{Z}\}$ is a partition of $\mathbb{R}$ a.e. if and only if there is a measurable subset R of S such that $R \underset{\text{a.e.}}{\subseteq} T_2$ and $T_2 \setminus R \underset{\text{a.e.}}{\overset{\tau}{\sim}} R$. In either case, we have $W = S \setminus R$ $(R = S \setminus W)$.*

Proof. To prove the sufficient condition, we shall show that the set $W = S \setminus R$ tiles $\mathbb{R}$ by integral translation. From $\tau_S \leq 2$, it is easy to verify that $\tau_W = \tau_S - \tau_R$. We show only that $\tau_R = \mathbb{1}_{\tau_S^{-1}(2)}$ a.e. That is, we shall prove that $\bigcup_{n \in \mathbb{Z}}(R - n) \underset{\text{a.e.}}{=} \tau_S^{-1}(2)$ and $(R - n) \cap R \underset{\text{a.e.}}{=} \varnothing$ for all $n \in \mathbb{Z}$. Since $T_2 \setminus R \underset{\text{a.e.}}{\overset{\tau}{\sim}} R$, $T_2 \underset{\text{a.e.}}{\subseteq} \bigcup_{n \in \mathbb{Z}}(R - n)$, and so $\tau_S^{-1}(2) = \bigcup_{n \in \mathbb{Z}}(T_2 - n) \underset{\text{a.e.}}{\subseteq} \bigcup_{n \in \mathbb{Z}}(R - n)$. By the property $R \underset{\text{a.e.}}{\subseteq} T_2$, we have $\bigcup_{n \in \mathbb{Z}}(R - n) \underset{\text{a.e.}}{\subseteq} \bigcup_{n \in \mathbb{Z}}(T_2 - n)$. Hence $\tau_S^{-1}(2) \underset{\text{a.e.}}{=} \bigcup_{n \in \mathbb{Z}}(R - n)$. To show that $(R - n) \cap R \underset{\text{a.e.}}{=} \varnothing$ for all $n \in \mathbb{Z}$ by contradiction, suppose that there exists a nonzero integer n_0 such that $\lambda((R - n_0) \cap R) > 0$. From $T_2 \setminus R \underset{\text{a.e.}}{\overset{\tau}{\sim}} R$, there exists a nonzero integer n_1 such that $\lambda((R - n_0) \cap R \cap ((T_2 \setminus R) - n_1)) > 0$. Obviously, $n_1 \neq n_0$. Set

$P = (R-n_0)\cap R\cap((T_2\setminus R)-n_1)$. Then $x, x+n_0, x+n_1 \in S$ for all $x \in P$, and hence $\tau_S(x) \geq 3$ for all $x \in P$, which is impossible. Conversely, set $R = S \setminus W$. Since $T_1 \underset{\text{a.e.}}{\subseteq} W$, we get that $R \underset{\text{a.e.}}{\subseteq} T_2$. We will show that $T_2 \setminus R \underset{\text{a.e.}}{\overset{\tau}{\sim}} R$. Set $G = T_2 \setminus R$. Then $G = W \cap T_2$ and $W \underset{\text{a.e.}}{=} T_1 \cup G$. Since $(T_1 - n) \cap R \underset{\text{a.e.}}{\subseteq} \tau_S^{-1}(1) \cap \tau_S^{-1}(2) = \varnothing$ for all $n \in \mathbb{Z}$, we get that $\bigcup_{n\in\mathbb{Z}}(T_1 - n) \cap R \underset{\text{a.e.}}{=} \varnothing$, and so $R \underset{\text{a.e.}}{\subseteq} \bigcup_{n\in\mathbb{Z}} G - n$. For each $n \in \mathbb{Z}$, we let $R_n = (G - n) \cap R$. We see that $\{R_n : n \in \mathbb{Z}\}$ is a partition of R a.e. For each $n \in \mathbb{Z}$, let $G_n = R_n + n$. Finally, we claim that $\{G_n : n \in \mathbb{Z}\}$ is a partition of G a.e. Since $\bigcup_{n\in\mathbb{Z}} S + n = T_1\dot{\cup}(\bigcup_{n\neq 0} S + n)$, $G \subseteq \bigcup_{n\neq 0} S + n$. From $S = T_1\dot{\cup}T_2$ and $G \cap (\bigcup_{n\in\mathbb{Z}} T_1 + n) = \varnothing$, we obtain that $G \subseteq \bigcup_{n\neq 0} T_2 + n$. Since $T_2 \underset{\text{a.e.}}{=} G \cup R$ and $W + n \cap W \underset{\text{a.e.}}{=} \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$, we have $G \underset{\text{a.e.}}{\subseteq} \bigcup_{n\neq 0} R + n$. Hence $G \underset{\text{a.e.}}{=} G \cap (\bigcup_{n\in\mathbb{Z}} R + n) = \bigcup_{n\in\mathbb{Z}}(G \cap (R + n)) = \bigcup_{n\in\mathbb{Z}}(G - n \cap R) + n = \bigcup_{n\in\mathbb{Z}} R_n + n = \bigcup_{n\in\mathbb{Z}} G_n$, and this proves the result. □

Lemma 3.1.4. *Let S be a measurable subset of $\mathbb{R}$ such that $S = D_1\dot{\cup}D_2$ and $\mathbb{R} \underset{\text{a.e.}}{=} \bigcup_{j\in\mathbb{Z}} 2^j S$. There exists a measurable subset W of S such that $\{2^j W : j \in \mathbb{Z}\}$ is a partition of $\mathbb{R}$ a.e. if and only if there is a measurable subset R of S such that $R \underset{\text{a.e.}}{\subseteq} D_2$ and $D_2 \setminus R \underset{\text{a.e.}}{\overset{\delta}{\sim}} R$. In either case, we have $W = S \setminus R$ ($R = S \setminus W$).*

Proof. To prove the sufficient condition, we will show that the set $W = S \setminus R$ tiles $\mathbb{R}$ by dyadic dilation. From $\delta_S \leq 2$, it is easy to check that $\delta_W = \delta_S - \delta_R$. We show only that $\delta_R = \mathbb{1}_{\delta_S^{-1}(2)}$ a.e. That is, we shall prove that $\bigcup_{j\in\mathbb{Z}} 2^j R \underset{\text{a.e.}}{=} \delta_S^{-1}(2)$ and $2^j R \cap R \underset{\text{a.e.}}{=} \varnothing$ for all $j \in \mathbb{Z}$. Since $D_2 \setminus R \underset{\text{a.e.}}{\overset{\delta}{\sim}} R$, it follows that $D_2 \underset{\text{a.e.}}{\subseteq} \bigcup_{n\in\mathbb{Z}}(R - n)$, and so $\delta_S^{-1}(2) = \bigcup_{j\in\mathbb{Z}} 2^j D_2 \underset{\text{a.e.}}{\subseteq} \bigcup_{j\in\mathbb{Z}} 2^j R$. By the property of R that $R \underset{\text{a.e.}}{\subseteq} D_2$, we obtain that $\bigcup_{j\in\mathbb{Z}} 2^j R \underset{\text{a.e.}}{\subseteq} \bigcup_{j\in\mathbb{Z}} 2^j D_2$. Then $\delta_S^{-1}(2) \underset{\text{a.e.}}{=} \bigcup_{j\in\mathbb{Z}} 2^j R$. To show that $2^j R \cap R \underset{\text{a.e.}}{=} \varnothing$ for all $j \in \mathbb{Z}$ by contradiction, suppose that there exists a nonzero integer j_0 such that $\lambda(2^{-j_0} R \cap R) > 0$. From $D_2 \setminus R \underset{\text{a.e.}}{\overset{\delta}{\sim}} R$, there exists a nonzero integer j_1 such that $\lambda(2^{-j_0}R \cap R \cap 2^{-j_1}(D_2 \setminus R)) > 0$. Clearly, $j_1 \neq j_0$. Put $P = 2^{-j_0}R \cap R \cap 2^{-j_1}(D_2 \setminus R)$. Then $x, 2^{j_0}x, 2^{j_1}x \in S$ for all $x \in P$, and hence $\delta_S(x) \geq 3$ for all $x \in P$, which is a contradiction. Conversely, set $R = S \setminus W$. Since $D_1 \underset{\text{a.e.}}{\subseteq} W$, we get that $R \underset{\text{a.e.}}{\subseteq} D_2$. We will show that $D_2 \setminus R \underset{\text{a.e.}}{\overset{\delta}{\sim}} R$. Set $H = D_2 \setminus R$. Then $H = W \cap D_2$ and $W \underset{\text{a.e.}}{=} D_1 \cup H$. Since $2^{-j}D_1 \cap R \underset{\text{a.e.}}{\subseteq} \delta_S^{-1}(1) \cap \delta_S^{-1}(2) = \varnothing$ for all $j \in \mathbb{Z}$, we get that $(\bigcup_{j\in\mathbb{Z}} 2^{-j}D_1) \cap R \underset{\text{a.e.}}{=} \varnothing$, and so $R \underset{\text{a.e.}}{\subseteq} \bigcup_{n\in\mathbb{Z}} 2^{-j}H$. For each $j \in \mathbb{Z}$, we let $R_j = 2^{-j}H \cap R$. We see that $\{R_j : j \in \mathbb{Z}\}$ is a partition of R a.e. For each $j \in \mathbb{Z}$, let $H_j = 2^j R_j$. Finally, we claim that $\{H_j : j \in \mathbb{Z}\}$ is a partition of H a.e. Since $\bigcup_{j\in\mathbb{Z}} 2^j S = D_1 \dot\cup (\bigcup_{j\neq 0} 2^j S)$, $H \subseteq \bigcup_{j\neq 0} 2^j S$. From $S = D_1 \dot\cup D_2$ and $H \cap (\bigcup_{j\in\mathbb{Z}} 2^j D_1) = \varnothing$, we obtain that $H \subseteq \bigcup_{j\neq 0} 2^j D_2$. Since $D_2 \underset{\text{a.e.}}{=} H \cup R$ and $2^j W \cap W \underset{\text{a.e.}}{=} \varnothing$ for all $j \in \mathbb{Z} \setminus \{0\}$, we have $H \underset{\text{a.e.}}{\subseteq} \bigcup_{j\neq 0} 2^j R$. Hence $H \underset{\text{a.e.}}{=} H \cap (\bigcup_{j\in\mathbb{Z}} 2^j R) = \bigcup_{j\in\mathbb{Z}} (H \cap 2^j R) = \bigcup_{j\in\mathbb{Z}} 2^j(2^{-j}H \cap R) = \bigcup_{j\in\mathbb{Z}} 2^j R_j = \bigcup_{j\in\mathbb{Z}} H_j$, and this proves the result. □

Theorem 3.1.5. *Let S be a 2-basic set. There exists a wavelet subset W of S if and only if there is a measurable subset R of S such that $R \underset{\text{a.e.}}{\subseteq} T_2 \cap D_2$, $T_2 \setminus R \underset{\text{a.e.}}{\overset{\tau}{\sim}} R$, and $D_2 \setminus R \underset{\text{a.e.}}{\overset{\delta}{\sim}} R$. In either case, we have $W = S \setminus R$ ($R = S \setminus W$).*

Proof. This result is an obvious consequence of Lemma 3.1.3 and Lemma 3.1.4. □

The following results are immediate consequences of the previous theorem.

Corollary 3.1.6. *Let S be a 2-basic subset of $\mathbb{R}$ such that S contains a wavelet set W. Then we have the following properties*

(a) $\tau_S^{-1}(1) \cup \operatorname{supp}(\tau_{T_2\cap D_2}) \underset{\text{a.e.}}{=} \mathbb{R}$, *and*

(b) $\delta_S^{-1}(1) \cup \operatorname{supp}(\delta_{T_2\cap D_2}) \underset{\text{a.e.}}{=} \mathbb{R}$.

Proof. By Theorem 3.1.5, there exists a measurable subset $R = S\backslash W$ of S such that $R \underset{\text{a.e.}}{\subseteq} T_2 \cap D_2$, $T_2\backslash R \underset{\text{a.e.}}{\overset{\tau}{\sim}} R$, and $D_2\backslash R \underset{\text{a.e.}}{\overset{\delta}{\sim}} R$. It is enough to show that

$$\tau_S^{-1}(1) \cup \operatorname{supp}(\tau_R) \underset{\text{a.e.}}{=} \mathbb{R} \text{ and } \delta_S^{-1}(1) \cup \operatorname{supp}(\delta_R) \underset{\text{a.e.}}{=} \mathbb{R}.$$

Now, we claim that $\mathbb{R} \underset{\text{a.e.}}{=} \operatorname{supp}(\delta_R) \cup \delta_S^{-1}(1)$. From the facts that $\bigcup_{j\in\mathbb{Z}} 2^j D_1 = \delta_S^{-1}(1)$ and $\bigcup_{j\in\mathbb{Z}} 2^j D_2 \underset{\text{a.e.}}{=} \cup_{j\in\mathbb{Z}} 2^j R = \operatorname{supp}(\delta_R)$, it implies that

$$\begin{aligned} \mathbb{R} &\underset{\text{a.e.}}{=} \bigcup_{j\in\mathbb{Z}} 2^j S \\ &\underset{\text{a.e.}}{=} (\bigcup_{j\in\mathbb{Z}} 2^j D_1) \cup (\bigcup_{j\in\mathbb{Z}} 2^j D_2) \\ &\underset{\text{a.e.}}{=} \delta_S^{-1}(1) \cup \operatorname{supp}(\delta_R). \end{aligned}$$

This proves the result (b). A similar proof shows that $\tau_S^{-1}(1) \cup \operatorname{supp}(\tau_R) \underset{\text{a.e.}}{=} \mathbb{R}$. □

Lemma 3.1.7. *Let S be a 2-basic set. Then*

(a) $\tau_{T_1\cup D_2} \geq 1$ a.e. *if and only if* $\tau_S^{-1}(1) \cup \operatorname{supp}(\tau_{T_2\cap D_2}) \underset{\text{a.e.}}{=} \mathbb{R}$*; and*

(b) $\delta_{D_1\cup T_2} \geq 1$ a.e. *if and only if* $\delta_S^{-1}(1) \cup \operatorname{supp}(\delta_{T_2\cap D_2}) \underset{\text{a.e.}}{=} \mathbb{R}$.

Proof. We shall only prove (a), as a similar proof works for (b). To show the only if part of (a), we assume that $\tau_{T_1\cup D_2} \geq 1$ a.e. From the fact that $T_1 \cup (D_2 \cap T_2) = (T_1 \cup D_2) \cap (T_1 \cup T_2) = T_1 \cup D_2$, we obtain that $\tau_{T_1\cup(D_2\cap T_2)} \geq 1$ a.e. Then $\mathbb{R} \underset{\text{a.e.}}{=} \bigcup_{n\in\mathbb{Z}}(T_1 \cup (D_2 \cap T_2)) + n = (\bigcup_{n\in\mathbb{Z}} T_1 + n) \cup (\bigcup_{n\in\mathbb{Z}}(D_2 \cap T_2) + n)$. Since $\bigcup_{n\in\mathbb{Z}} T_1 + n = \tau_S^{-1}(1)$ and $\bigcup_{n\in\mathbb{Z}}(D_2 \cap T_2) + n = \operatorname{supp}(D_2 \cap T_2)$, we get that $\mathbb{R} \underset{\text{a.e.}}{=} \tau_S^{-1}(1) \cup \operatorname{supp}(D_2 \cap T_2)$. By the above argument, the converse holds. □

Corollary 3.1.8. *Let S be a 2-basic subset of $\mathbb{R}$ such that S contains a wavelet set W. Then we have the following properties*

$$\tau_{T_1\cup D_2} \geq 1 \text{ a.e. } \textit{and } \delta_{D_1\cup T_2} \geq 1 \text{ a.e.}$$

Proof. It follows from Corollary 3.1.6 and Lemma 3.1.7. □

The converse of the previous Corollary is not true as there exist 2-basic sets satisfying the above conditions that do not contain a wavelet set, as shown in the next example.

Example 3.1.9. *In this example we show that the converse of the previous corollary is not true. Let* $S = \left[-1,-\frac{1}{2}\right) \cup \left[\frac{1}{2},\frac{3}{4}\right) \cup [1,2)$. *Then we have*

$$T_1 = \left[\frac{7}{4},2\right), \qquad T_2 = \left[-1,-\frac{1}{2}\right) \cup \left[\frac{1}{2},\frac{3}{4}\right) \cup \left[1,\frac{7}{4}\right),$$
$$D_1 = \left[-1,-\frac{1}{2}\right) \cup \left[\frac{3}{2},2\right), \quad \text{and} \quad D_2 = \left[\frac{1}{2},\frac{3}{4}\right) \cup \left[1,\frac{3}{2}\right).$$

Thus

$$T_1 \cup D_2 = \left[\frac{1}{2},\frac{3}{4}\right) \cup \left[1,\frac{3}{2}\right) \cup \left[\frac{7}{4},2\right)$$

and

$$D_1 \cup T_2 = \left[-1,-\frac{1}{2}\right) \cup \left[\frac{1}{2},\frac{3}{4}\right) \cup [1,2).$$

It implies that $\tau_{T_1 \cup D_2} \geq 1$ *a.e. and* $\delta_{D_1 \cup T_2} \geq 1$ *a.e. Moreover, we can find*

$$T_1 \cup D_1 = \left[-1,-\frac{1}{2}\right) \cup \left[\frac{3}{2},2\right).$$

It is clear that $\lambda(T_1 \cup D_1) = 1$ *and* $T_1 \cup D_1$ *does not cover* $\mathbb{R}$ *by dilation. Finally, we claim that* S *does not contain a wavelet set. Assume that* S *contains a wavelet set* W. *By Theorem 2.3.4 (b), we have* $T_1 \cup D_1 \subseteq W$. *Since* $T_1 \cup D_1$ *does not cover* $\mathbb{R}$ *by dilation, it implies that* $\lambda(W \setminus (T_1 \cup D_1)) > 0$. *Hence* $\lambda(W) > 1$ *which is a contradiction. This proves our claim.*

3.2 τ_2 and δ_2 functions

In this section, we introduce two functions and discuss their properties which will be useful in obtaining a necessary condition for a 2-basic set to contain a wavelet set. We begin this section by considering the properties of T_2 and D_2.

By the definition of T_2 and D_2, for each $x \in T_2$ there exists a unique nonzero integer n such that $x + n \in T_2$; and for each $x \in D_2$ there exists a unique nonzero integer j such that $2^j x \in D_2$. Thus we can define a map $\tau_2 : T_2 \to T_2$ by

$$\tau_2(x) = x + n$$

where n is the nonzero integer such that $x + n \in T_2$ and a map $\delta_2 : D_2 \to D_2$ by

$$\delta_2(x) = 2^j x$$

where j is the nonzero integer such that $2^j x \in D_2$. Obviously, $\tau_2^2 = id_{T_2}$ and $\delta_2^2 = id_{D_2}$, and hence τ_2 and δ_2 are bijections. Furthermore, $\tau_2^{-1} = \tau_2$ and $\delta_2^{-1} = \delta_2$.

We shall study properties of functions τ_2 and δ_2. First of all, we will give partitions of T_2 and D_2. For each $n, j \in \mathbb{Z} \setminus \{0\}$, let

$$P_n = \{x \in T_2 : x + n \in T_2\} \text{ and } M_j = \{x \in D_2 : 2^j x \in D_2\}.$$

Clearly, $\{P_n : n \neq 0\}$ and $\{M_j : j \neq 0\}$ are partitions of T_2 and D_2, respectively. Furthermore, $P_n = T_2 \cap (T_2 - n)$ and $M_j = D_2 \cap 2^{-j} D_2$ are measurable sets for all $n, j \in \mathbb{Z} \setminus \{0\}$. It implies that $P_n + n = P_{-n}$ and $2^j M_j = M_{-j}$ for all $n, j \in \mathbb{Z} \setminus \{0\}$.

Lemma 3.2.1. *Let $n, j \in \mathbb{Z} \setminus \{0\}$. Then*

(a) if A is a subset of P_n, then $\tau_2(A) = A + n$ is a subset of P_{-n}, and

(b) if B is a subset of M_j, then $\delta_2(B) = 2^j B$ is a subset of M_{-j}.

Proof. (a) Assume that A is a subset of P_n. To show that $\tau_2(A) \subseteq A+n$, let $y \in \tau_2(A)$. Then there exist $x \in A$ and a nonzero integer n_1 such that $y = \tau_2(x) = x + n_1 \in T_2$. By assumption, it implies that $x \in P_n$, and so $x + n \in T_2$. By definition of T_2, we obtain that $n = n_1$. Thus $y = x + n \in A + n$. Hence $\tau_2(A) \subseteq A + n$. Now, let $y \in A + n$. Then there exists $x \in A$ such that $y = x + n$. By assumption, we have $x \in P_n$, which implies that $x + n \in T_2$. Since $n \neq 0$, $y = x + n = \tau_2(x) \in \tau_2(A)$. This then implies that $A + n \subseteq \tau_2(A)$. Therefore, $\tau_2(A) = A + n$. Obviously, $A + n$ is a subset of P_{-n}.

The proof of (b) is similar to (a) and thus is omitted. □

Proposition 3.2.2. *The image of a measurable subset of T_2 under τ_2 is a measurable set. Furthermore, $\lambda(\tau_2(E)) = \lambda(E)$ for all measurable subset E of T_2.*

Proof. Let E be a measurable subset of T_2. We recall that $\{P_n : n \neq 0\}$ are measurable partitions of T_2. For each $n \in \mathbb{Z} \setminus \{0\}$, let $E_n = E \cap P_n$. Then $\{E_n : n \neq 0\}$ are measurable partitions of E, that is, $E = \dot{\bigcup}_{n\neq 0} E_n$. By Lemma 3.2.1 (a), we obtain that $\tau_2(E) = \bigcup_{n\neq 0} \tau_2(E_n) = \bigcup_{n\neq 0} E_n + n$ is measurable. Since τ_2 is injective, we get that $\{\tau_2(E_n) : n \neq 0\}$ are measurable partitions of $\tau_2(E)$. Since Lebesgue measure is translation invariant, we have

$$\begin{aligned} \lambda(\tau_2(E)) &= \lambda(\dot{\bigcup_{n\neq 0}} \tau_2(E_n)) \\ &= \sum_{n\neq 0} \lambda(E_n + n) \\ &= \sum_{n\neq 0} \lambda(E_n) \\ &= \lambda(\dot{\bigcup_{n\neq 0}} E_n) \\ &= \lambda(E). \end{aligned}$$

This proves the proposition. □

Remark. *From the preceding proposition, we obtain that τ_2^{-1} is a measurable function in sense that every inverse image of measurable sets is measurable. Indeed, τ_2 is also measurable because $\tau_2^{-1} = \tau_2$. Moreover, τ_2 and τ_2^{-1} preserve measure.*

Theorem 3.2.3. *Let A and B be measurable subsets of $\mathbb{R}$ such that $A \cup B \underset{\text{a.e.}}{\subseteq} T_2$ and $A \cap B = \varnothing$. Then $A \underset{\text{a.e.}}{\overset{\mathcal{T}}{\sim}} B$ if and only if $\tau_2(A \cap T_2) \underset{\text{a.e.}}{=} B$.*

Proof. Assume that $A \underset{\text{a.e.}}{\overset{\mathcal{T}}{\sim}} B$. Then there exist a partition $\{A_k : k \in \mathbb{Z}\}$ of A a.e. and a sequence $\{n_k : k \in \mathbb{Z}\}$ of integers such that $\{A_k + n_k : k \in \mathbb{Z}\}$ is a partition a.e. of B. Without loss of generality we assume that $n_k \neq 0$ for all $k \in \mathbb{Z}$ because $A \cap B = \varnothing$. Since $A \cup B \underset{\text{a.e.}}{\subseteq} T_2$, we obtain that $A_k, A_k + n_k \underset{\text{a.e.}}{\subseteq} T_2$ for all $k \in \mathbb{Z}$. From $A_k \underset{\text{a.e.}}{=} A_k \cap T_2$ and $A_k + n_k \underset{\text{a.e.}}{\subseteq} T_2$ for all $k \in \mathbb{Z}$, which implies that $A_k \cap T_2 \underset{\text{a.e.}}{\subseteq} T_2 - n_k$ for all $k \in \mathbb{Z}$. Thus $A_k \cap T_2 \underset{\text{a.e.}}{=} A_k \cap T_2 \cap (T_2 - n_k)$ for all $k \in \mathbb{Z}$. By Lemma 3.2.1 (a) and as τ_2 preserves sets of measure zero, we obtain that

$$\begin{aligned}
\tau_2(A_k \cap T_2) &\underset{\text{a.e.}}{=} \tau_2(A_k \cap T_2 \cap (T_2 - n_k)) \\
&\underset{\text{a.e.}}{=} (A_k \cap T_2) + n_k \cap T_2 \\
&\underset{\text{a.e.}}{=} A_k + n_k \cap T_2 \\
&\underset{\text{a.e.}}{=} A_k + n_k
\end{aligned}$$

for all $k \in \mathbb{Z}$. Hence $\tau_2(A \cap T_2) \underset{\text{a.e.}}{=} B$. Conversely, suppose that $\tau_2(A \cap T_2) \underset{\text{a.e.}}{=} B$. We recall that $\{P_n : n \neq 0\}$ are measurable partitions of T_2. For each $n \in \mathbb{Z} \setminus \{0\}$, let $A_n = A \cap P_n$. Then $A \cap T_2 = \dot{\bigcup}_{n \neq 0} A_n$. Moreover, $\dot{\bigcup}_{n \neq 0}(A_n + n) = \dot{\bigcup}_{n \neq 0} \tau_2(A_n) = \tau_2(A \cap T_2) \underset{\text{a.e.}}{=} B$. Thus $A \underset{\text{a.e.}}{\overset{\mathcal{T}}{\sim}} B$. □

Proposition 3.2.4. *The image of a measurable subset of D_2 under δ_2 is a measurable set. Moreover, if F is a measurable subset of D_2 such that $\lambda(F) = 0$, then $\lambda(\delta_2(F)) = 0$.*

Proof. Let F be a measurable subset of D_2. We recall that $\{M_j : j \neq 0\}$ are measurable partitions of D_2. For each $j \in \mathbb{Z} \setminus \{0\}$, let $F_j = F \cap M_j$. Then $\{F_j : j \neq 0\}$ are measurable partitions of F, that is, $F = \dot{\bigcup}_{j \neq 0} F_j$. By Lemma 3.2.1 (b), we obtain that $\delta_2(F) = \bigcup_{j \neq 0} \delta_2(F_j) = \bigcup_{j \neq 0} 2^j F_j$ is measurable. Since δ_2 is injective, we get that $\{\delta_2(F_j) : j \neq 0\}$ are measurable partitions of $\delta_2(F)$. Now, we suppose that F has measure zero. Then $\lambda(F_j) = 0$ for all $j \neq 0$. By a property of Lebesgue measure , we have

$$
\begin{aligned}
\lambda(\delta_2(F)) &= \lambda(\dot{\bigcup_{j \neq 0}} \delta_2(F_j)) \\
&= \sum_{n \neq 0} \lambda(2^j F_j) \\
&= \sum_{n \neq 0} 2^j \lambda(F_j) \\
&= 0.
\end{aligned}
$$

This proves the proposition. □

Remark. *By the previous proposition, it implies that δ_2^{-1} is a measurable function in sense that every inverse image of measurable sets is measurable. Since $\delta_2^{-1} = \delta_2$, we obtain that δ_2 is also measurable. Moreover, even though δ_2 and δ_2^{-1} obviously do not preserve measure, they preserve sets of measure zero.*

Theorem 3.2.5. *Let A and B be measurable subsets of $\mathbb{R}$ such that $A \cup B \underset{\text{a.e.}}{\subseteq} D_2$ and $A \cap B = \varnothing$. Then $A \underset{\text{a.e.}}{\overset{\delta}{\sim}} B$ if and only if $\delta_2(A \cap D_2) \underset{\text{a.e.}}{=} B$.*

Proof. Assume that $A \underset{\text{a.e.}}{\overset{\delta}{\sim}} B$. Then there exist a partition $\{A_k : k \in \mathbb{Z}\}$ of A a.e. and a sequence $\{j_k : k \in \mathbb{Z}\}$ of integers such that $\{2^{j_k} A_k : k \in \mathbb{Z}\}$ is a partition of B a.e. Without loss of generality we assume that $j_k \neq 0$ for all $k \in \mathbb{Z}$ because $A \cap B = \varnothing$. Since $A \cup B \underset{\text{a.e.}}{\subseteq} D_2$, we obtain that $A_k, 2^{j_k} A_k \underset{\text{a.e.}}{\subseteq} D_2$ for all $k \in \mathbb{Z}$. From

$A_k \underset{\text{a.e.}}{=} A_k \cap D_2$ and $2^{j_k} A_k \underset{\text{a.e.}}{\subseteq} D_2$ for all $k \in \mathbb{Z}$, which implies that $A_k \cap D_2 \underset{\text{a.e.}}{\subseteq} 2^{-j_k} D_2$ for all $k \in \mathbb{Z}$. Thus $A_k \cap D_2 \underset{\text{a.e.}}{=} A_k \cap D_2 \cap 2^{-j_k} D_2$ for all $k \in \mathbb{Z}$. By Lemma 3.2.1 (b) and as δ_2 preserves sets of measure zero, we obtain that

$$\begin{aligned} \delta_2(A_k \cap D_2) &\underset{\text{a.e.}}{=} \delta_2(A_k \cap D_2 \cap 2^{-j_k} D_2) \\ &\underset{\text{a.e.}}{=} 2^{j_k}(A_k \cap D_2) \cap D_2 \\ &\underset{\text{a.e.}}{=} 2^{j_k} A_k \cap D_2 \\ &\underset{\text{a.e.}}{=} 2^{j_k} A_k \end{aligned}$$

for all $k \in \mathbb{Z}$. Hence $\delta_2(A \cap D_2) \underset{\text{a.e.}}{=} B$. Conversely, suppose that $\delta_2(A \cap D_2) \underset{\text{a.e.}}{=} B$. We recall that $\{M_j : j \neq 0\}$ are measurable partitions of D_2. For each $j \in \mathbb{Z} \setminus \{0\}$, we let $A_j = A \cap M_j$. Then $A \cap D_2 = \dot{\bigcup}_{j \neq 0} A_j$. Moreover, $\dot{\bigcup}_{j \neq 0} 2^j A_j = \dot{\bigcup}_{j \neq 0} \delta_2(A_j) = \delta_2(A \cap D_2) \underset{\text{a.e.}}{=} B$. Thus $A \underset{\text{a.e.}}{\overset{\delta}{\sim}} B$. □

Remark. *Let S be a 2-basic set which contains a wavelet set W and $R = S \setminus W$. By Theorem 3.1.5, it implies that $T_2 \setminus R \underset{\text{a.e.}}{\overset{\tau}{\sim}} R$, and $D_2 \setminus R \underset{\text{a.e.}}{\overset{\delta}{\sim}} R$. Then we have $\tau_2(T_2 \setminus R) \underset{\text{a.e.}}{=} R$, $\tau_2(T_2 \cap R) \underset{\text{a.e.}}{=} T_2 \setminus R$, $\delta_2(D_2 \setminus R) \underset{\text{a.e.}}{=} R$, and $\delta_2(D_2 \cap R) \underset{\text{a.e.}}{=} D_2 \setminus R$ as consequences of Theorem 3.2.3 and Theorem 3.2.5. We will use these results in the next section.*

3.3 Two test sequences of subsets of two-basic sets

Notice that if S is a 2-basic set containing a wavelet set W, then $T_2 \setminus D_2 = T_2 \cap D_1$ is contained in W, and so the image of $T_2 \setminus D_2$ under τ_2 must be disjoint from W. Similarly, we get that $\delta_2(D_2 \setminus T_2)$ must be disjoint from W. From this concept, we will construct two sequences of subsets of a 2-basic set S which will be useful in deriving a sufficient condition for a 2-basic set to contain a wavelet set.

Definition 3.3.1. *Assume that S is a 2-basic set. Set*

$$A_0 = \tau_2(T_2 \setminus D_2) \text{ and } B_0 = \delta_2(D_2 \setminus T_2).$$

For each $n \in \mathbb{N}$, let

$$A_n = \tau_2(T_2 \cap \delta_2(D_2 \cap A_{n-1})) \text{ and } B_n = \delta_2(D_2 \cap \tau_2(T_2 \cap B_{n-1})).$$

Put $A = \bigcup_{n \in \mathbb{N}_0} A_n$ *and* $B = \bigcup_{n \in \mathbb{N}_0} B_n$.

The following properties are immediate consequences of the construction sequences.

(a) $A \subseteq T_2$ and $B \subseteq D_2$.

(b) If there exists $n_1 \in \mathbb{N}_0$ such that $A_{n_1} = \varnothing$ (or $A_{n_1} \underset{\text{a.e.}}{=} \varnothing$), then $A_n = \varnothing$ (or $A_n \underset{\text{a.e.}}{=} \varnothing$) for all $n \geq n_1$.

(c) If there exists $n_1 \in \mathbb{N}_0$ such that $B_{n_1} = \varnothing$ (or $B_{n_1} \underset{\text{a.e.}}{=} \varnothing$), then $B_n = \varnothing$ (or $B_n \underset{\text{a.e.}}{=} \varnothing$) for all $n \geq n_1$.

(d) If $T_2 \underset{\text{a.e.}}{\subseteq} D_2$, then $A \underset{\text{a.e.}}{=} \varnothing$.

(e) If $D_2 \underset{\text{a.e.}}{\subseteq} T_2$, then $B \underset{\text{a.e.}}{=} \varnothing$.

(f) $\tau_2(T_2 \cap \delta_2(D_2 \cap A)) = \bigcup_{n \in \mathbb{N}} A_n$.

(g) $\delta_2(D_2 \cap \tau_2(T_2 \cap B)) = \bigcup_{n \in \mathbb{N}} B_n$.

Proposition 3.3.2. *Let S be a 2-basic set. Then, for all $n, m \in \mathbb{N}_0$ with $n \neq m$,*

$$A_n \cap A_m = \varnothing \text{ and } B_n \cap B_m = \varnothing.$$

Proof. To show that $A_n \cap A_m = \varnothing$ for all $n, m \in \mathbb{N}_0$ with $n \neq m$, let $n, m \in \mathbb{N}_0$ be such that $n \neq m$. Without loss of generality we assume that $n < m$. If $n = 0$, then

$$\begin{aligned} A_0 \cap A_m &= \tau_2(T_2 \smallsetminus D_2) \cap \tau_2(T_2 \cap \delta_2(D_2 \cap A_{m-1})) \\ &= \tau_2((T_2 \smallsetminus D_2) \cap T_2 \cap \delta_2(D_2 \cap A_{m-1})) \\ &= \tau_2(\varnothing) \\ &= \varnothing, \end{aligned}$$

since τ_2 is an injection and $\delta_2(D_2 \cap A_{m-1}) \subseteq D_2$. Now, we suppose that $0 < n$. Since $m - n > 0$, by the above argument, we can prove that

$$A_0 \cap A_{m-n} = \varnothing.$$

Applying δ_2 to the above equation, we have

$$\delta_2(D_2 \cap A_0) \cap \delta_2(D_2 \cap A_{m-n}) = \varnothing.$$

We take τ_2 on the previous equation, we get that

$$\tau_2(T_2 \cap \delta_2(D_2 \cap A_0)) \cap \tau_2(T_2 \cap \delta_2(D_2 \cap A_{m-n})) = \varnothing,$$

which implies that $A_1 \cap A_{m-n+1} = \varnothing$. Repeating this process n times, we obtain that $A_n \cap A_m = \varnothing$. A similar proof shows that $B_n \cap B_m = \varnothing$ for all $n, m \in \mathbb{N}_0$ with $n \neq m$. □

The next theorem explains that A and B are bad parts of the 2-basic set S in the sense that every wavelet subset of S does not intersect $A \cup B$.

Theorem 3.3.3. *Let S be a 2-basic set. If S contains a wavelet set W, then $A \cup B \underset{\text{a.e.}}{\subseteq} T_2 \cap D_2$ and $W \underset{\text{a.e.}}{\subseteq} S \smallsetminus (A \cup B)$.*

Proof. Assume that S contains a wavelet set W. By Theorem 3.1.5, we obtain that there exists a measurable subset $R = S \setminus W$ of S such that $R \underset{\text{a.e.}}{\subseteq} T_2 \cap D_2$, $T_2 \setminus R \underset{\text{a.e.}}{\overset{\tau}{\sim}} R$, and $D_2 \setminus R \underset{\text{a.e.}}{\overset{\delta}{\sim}} R$. It implies that $\tau_2(T_2 \setminus R) \underset{\text{a.e.}}{=} R$, $\tau_2(T_2 \cap R) \underset{\text{a.e.}}{=} T_2 \setminus R$, $\delta_2(D_2 \setminus R) \underset{\text{a.e.}}{=} R$, and $\delta_2(D_2 \cap R) \underset{\text{a.e.}}{=} D_2 \setminus R$.

First, we will claim that $A \cup B \underset{\text{a.e.}}{\subseteq} R$. It suffices to prove that $A_n, B_n \underset{\text{a.e.}}{\subseteq} R$ for all $n \in \mathbb{N}_0$. We prove by using the above properties of R and the fact that both τ_2 and δ_2 preserve sets of measure zero. Since the proofs are similar, we shall only give a proof for A_n by induction on n. When $n = 0$, we have

$$A_0 = \tau_2(T_2 \setminus D_2) \underset{\text{a.e.}}{\subseteq} \tau_2(T_2 \setminus R) \underset{\text{a.e.}}{=} R,$$

because $T_2 \setminus D_2 \underset{\text{a.e.}}{\subseteq} T_2 \setminus R$. Now, suppose that $A_n \underset{\text{a.e.}}{\subseteq} R$ for $n \geq 0$. Applying δ_2 to $A_n \underset{\text{a.e.}}{\subseteq} R$, we have

$$\delta_2(D_2 \cap A_n) \underset{\text{a.e.}}{\subseteq} \delta_2(D_2 \cap R) \underset{\text{a.e.}}{=} D_2 \setminus R.$$

Taking τ_2 to $\delta_2(D_2 \cap A_n) \underset{\text{a.e.}}{\subseteq} D_2 \setminus R$, it implies that

$$A_{n+1} = \tau_2(T_2 \cap \delta_2(D_2 \cap A_n)) \underset{\text{a.e.}}{\subseteq} \tau_2(T_2 \cap (D_2 \setminus R)) \underset{\text{a.e.}}{\subseteq} \tau_2(T_2 \setminus R) \underset{\text{a.e.}}{=} R.$$

This proves our claim. Consequently, $A \cup B \underset{\text{a.e.}}{\subseteq} T_2 \cap D_2$ and $W \underset{\text{a.e.}}{\subseteq} S \setminus A \cup B$. $\square$

Remark. *Theorem 3.3.3 says that if a 2-basic set S contains a wavelet set, then $A \underset{\text{a.e.}}{\subseteq} D_2$ and $B \underset{\text{a.e.}}{\subseteq} T_2$. It is unknown whether the converse is true or not as we cannot find an example of a 2-basic sets S for which $A \underset{\text{a.e.}}{\subseteq} D_2$ and $B \underset{\text{a.e.}}{\subseteq} T_2$ but S has no wavelet subset.*

In general, a 2-basic set S may not satisfy $A \cup B \underset{\text{a.e.}}{\subseteq} T_2 \cap D_2$. In lieu of the fact that $A \subseteq T_2$ and $B \subseteq D_2$, it means that it may not be the case that $A \underset{\text{a.e.}}{\subseteq} D_2$ and $B \underset{\text{a.e.}}{\subseteq} T_2$ as the next example shows.

Example 3.3.4. *We will give some examples for which $A \cap D_1 \neq \varnothing$ or $B \cap T_1 \neq \varnothing$.*

(a) Let $S = [-2, -\frac{3}{2}) \cup [-\frac{3}{4}, -\frac{1}{4}) \cup [\frac{1}{2}, 1)$. Then we have

$$T_1 = \left[-2, -\frac{7}{4}\right) \cup \left[\frac{3}{4}, 1\right), \qquad T_2 = \left[-\frac{7}{4}, -\frac{3}{2}\right) \cup \left[-\frac{3}{4}, -\frac{1}{4}\right) \cup \left[\frac{1}{2}, \frac{3}{4}\right),$$
$$D_1 = \left[\frac{1}{2}, 1\right), \qquad \text{and} \quad D_2 = \left[-2, -\frac{3}{2}\right) \cup \left[-\frac{3}{4}, -\frac{1}{4}\right).$$

Thus we can find

$$A_0 = \left[-\frac{1}{2}, -\frac{1}{4}\right), \qquad A_1 = \left[-\frac{7}{4}, -\frac{3}{2}\right) \cup \left[-\frac{3}{4}, -\frac{1}{2}\right),$$
$$A_2 = \left[\frac{9}{16}, \frac{3}{4}\right), \qquad A_n = \varnothing \quad \text{for all } n \geq 3, \text{ and}$$
$$B_0 = \left[-\frac{1}{2}, -\frac{7}{16}\right), \qquad B_n = \varnothing \quad \text{for all } n \geq 1.$$

Hence $A = [-\frac{7}{4}, -\frac{3}{2}) \cup [-\frac{3}{4}, -\frac{1}{4}) \cup [\frac{9}{16}, \frac{3}{4})$ and $B = [-\frac{1}{2}, -\frac{7}{16})$. We see that $B \cap T_1 = \varnothing$ but $A \cap D_1 = [\frac{9}{16}, \frac{3}{4}) \neq \varnothing$. This implies that $\lambda(A \setminus D_2) > 0$, and so Theorem 3.3.3 implies that S does not contain a wavelet set. Furthermore, this example shows that there is a 2-basic set S for which $\tau_{T_1 \cup D_2} \geq 1$ a.e. and $\delta_{D_1 \cup T_2} \geq 1$ a.e. but S has no a wavelet subset.

(b) $S = [-\frac{5}{4}, -1) \cup [-\frac{1}{2}, -\frac{1}{4}) \cup [\frac{1}{4}, \frac{1}{2}) \cup [1, \frac{5}{4})$. Then

$$T_1 = S, \qquad D_1 = \left[-\frac{1}{2}, -\frac{5}{16}\right) \cup \left[\frac{5}{16}, \frac{1}{2}\right),$$
$$T_2 = \varnothing, \quad \text{and} \quad D_2 = \left[-\frac{5}{4}, -1\right) \cup \left[-\frac{5}{16}, -\frac{1}{4}\right) \cup \left[1, \frac{5}{4}\right) \cup \left[\frac{1}{4}, \frac{5}{16}\right).$$

Obviously, $A = \varnothing$ and $B = D_2$. Then $A \cap D_1 = \varnothing$ and $B \cap T_1 = D_2 \neq \varnothing$. Indeed, S does not contain a wavelet set since $\lambda(S \setminus (A \cup B)) = \frac{3}{8} < 1$.

(c) $S = [-1, -\frac{1}{2}) \cup [\frac{3}{4}, \frac{7}{4})$. Then

$$T_1 = \left[\frac{3}{4}, 1\right) \cup \left[\frac{3}{2}, \frac{7}{4}\right), \qquad T_2 = \left[-1, -\frac{1}{2}\right) \cup \left[1, \frac{3}{2}\right),$$

$$D_1 = \left[-1, -\frac{1}{2}\right) \cup \left[\frac{7}{8}, \frac{3}{2}\right), \quad \textit{and} \quad D_2 = \left[\frac{3}{4}, \frac{7}{8}\right) \cup \left[\frac{3}{2}, \frac{7}{4}\right).$$

From $T_2 \cap D_2 = \varnothing$, we get that $A = T_2$ and $B = D_2$. Moreover $A \cap D_1 \neq \varnothing$ and $B \cap T_1 \neq \varnothing$. Since $\lambda(T_1 \cup D_1) = \frac{3}{2} > 1$, S does not contain a wavelet set. It is interesting that any set of the form $S = \left[-1, -\frac{1}{2}\right) \cup \left[\frac{1}{2} + \varepsilon, \frac{3}{2} + \varepsilon\right)$ does not contain a wavelet set for various values of $\varepsilon \in \left(0, \frac{1}{4}\right]$.

Now, we study several properties of A and B which will be useful in proving some sufficient conditions for a 2-basic set to contain a wavelet set in the next section.

Lemma 3.3.5. *Let S be a 2-basic set. Then, for all $n, m \in \mathbb{N}_0$,*

$$A_n \cap \tau_2(T_2 \cap B_m) = \varnothing \textit{ and } B_m \cap \delta_2(D_2 \cap A_n) = \varnothing.$$

Proof. First, we shall prove that $A_n \cap \tau_2(T_2 \cap B_m) \neq \varnothing$ for all $n, m \in \mathbb{N}_0$. From the fact that $B_m \subseteq D_2$ for all $m \in \mathbb{N}_0$, we have

$$A_0 \cap \tau_2(T_2 \cap B_m) = \tau_2((T_2 \setminus D_2) \cap T_2 \cap B_m) = \varnothing$$

for all $m \in \mathbb{N}_0$. Since $A_n \subseteq T_2$ for all $n \in \mathbb{N}_0$, we get that

$$\begin{aligned} A_n \cap \tau_2(T_2 \cap B_0) &= \tau_2(T_2 \cap \delta_2(D_2 \cap A_{n-1}) \cap B_0) \\ &= \tau_2(T_2 \cap \delta_2(D_2 \cap A_{n-1} \cap (D_2 \setminus T_2))) \\ &= \varnothing \end{aligned}$$

for all $n \in \mathbb{N}$. Fix $n, m \in \mathbb{N}_0$.

If $n \leq m$, then $m - n \in \mathbb{N}_0$. Thus we have

$$A_0 \cap \tau_2(T_2 \cap B_{m-n}) = \varnothing.$$

Applying δ_2 to the preceding equation, we get that

$$\delta_2(D_2 \cap A_0) \cap \delta_2(D_2 \cap \tau_2(T_2 \cap B_{m-n})) = \varnothing,$$

which implies that

$$\delta_2(D_2 \cap A_0) \cap B_{m-n+1} = \varnothing.$$

We take τ_2 to the above equation, we obtain that

$$\tau_2(T_2 \cap \delta_2(D_2 \cap A_0)) \cap \tau_2(T_2 \cap B_{m-n+1}) = \varnothing$$

It is implies that

$$A_1 \cap \tau_2(T_2 \cap B_{m-n+1}) = \varnothing.$$

Continuing this process n times, we have $A_n \cap \tau_2(T_2 \cap B_m) = \varnothing$.

If $m < n$, then $n - m \in \mathbb{N}$. We start from the equation

$$A_{n-m} \cap \tau_2(T_2 \cap B_0) = \varnothing.$$

Taking δ_2 to the above equation, we obtain that

$$\delta_2(D_2 \cap A_{n-m}) \cap B_1 = \varnothing.$$

We apply τ_2 to the previous equation, we get that

$$A_{n-m+1} \cap \tau_2(T_2 \cap B_1) = \varnothing.$$

Repeating this process m time, we have $A_n \cap \tau_2(T_2 \cap B_m) = \varnothing$.

Similar arguments show that $B_m \cap \delta_2(D_2 \cap A_n) = \varnothing$ for all $n, m \in \mathbb{N}_0$. □

Lemma 3.3.6. *Let S be a 2-basic set with the sets A and B defined in Definition 3.3.1. Then* *(a)* $A \cap \tau_2(T_2 \cap B) = \varnothing$,

(b) $B \cap \delta_2(D_2 \cap A) = \varnothing$,

(c) $B \cap \tau_2(A) = \varnothing$,

(d) $A \cap \delta_2(B) = \varnothing$.

Proof. We shall prove (a) and (c), as (b) and (d) can be verified in a similar manner. By the definition of A and B,

$$\begin{aligned} A \cap \tau_2(T_2 \cap B) &= A \cap \bigcup_{m=0}^{\infty} \tau_2(T_2 \cap B_m) \\ &= \bigcup_{m=0}^{\infty} A \cap \tau_2(T_2 \cap B_m) \\ &= \bigcup_{m=0}^{\infty} \bigcup_{n=0}^{\infty} A_n \cap \tau_2(T_2 \cap B_m), \end{aligned}$$

which, by Lemma 3.3.5, is a countable union of empty sets, hence empty. Next,

$$\begin{aligned} B \cap \tau_2(A) &= B \cap [(T_2 \setminus D_2) \cup \bigcup_{n=1}^{\infty} T_2 \cap \delta_2(D_2 \cap A_{n-1})] \\ &= [B \cap (T_2 \setminus D_2)] \cup [B \cap \bigcup_{n=0}^{\infty} T_2 \cap \delta_2(D_2 \cap A_n)] \\ &= T_2 \cap \bigcup_{n=0}^{\infty} B \cap \delta_2(D_2 \cap A_n) \\ &= T_2 \cap \bigcup_{n=0}^{\infty} \bigcup_{m=0}^{\infty} B_m \cap \delta_2(D_2 \cap A_n) \end{aligned}$$

which is empty by Lemma 3.3.5. The proof is complete. □

Lemma 3.3.7. *Let S be a 2-basic set with the sets A and B defined in Definition 3.3.1.*

(a) If $A \underset{\text{a.e.}}{\subseteq} D_2$, then $A \cap \delta_2(D_2 \cap A) \underset{\text{a.e.}}{=} \varnothing$, and $A \cap \tau_2(A) \underset{\text{a.e.}}{=} \varnothing$.

(b) If $B \underset{\text{a.e.}}{\subseteq} T_2$*, then* $B \cap \tau_2(T_2 \cap B) \underset{\text{a.e.}}{=} \varnothing$*, and* $B \cap \delta_2(B) \underset{\text{a.e.}}{=} \varnothing$.

Proof. (a) Assume that $A \underset{\text{a.e.}}{\subseteq} D_2$. It is easy to verify that

$$A \cap \delta_2(D_2 \cap A) = \bigcup_{m=0}^{\infty} \bigcup_{n=0}^{\infty} A_n \cap \delta_2(D_2 \cap A_m).$$

Then it suffices to show that $A_n \cap \delta_2(D_2 \cap A_m) \underset{\text{a.e.}}{=} \varnothing$ for all $n, m \in \mathbb{N}_0$. Fix $n, m \in \mathbb{N}_0$. If $n = 0$, then

$$\begin{aligned} A_0 \cap \delta_2(D_2 \cap A_m) &= A_0 \cap T_2 \cap \delta_2(D_2 \cap A_m) \\ &= \tau_2(T_2 \smallsetminus D_2) \cap \tau_2(\tau_2(T_2 \cap \delta_2(D_2 \cap A_m))) \\ &= \tau_2((T_2 \smallsetminus D_2) \cap \tau_2(T_2 \cap \delta_2(D_2 \cap A_m))) \\ &= \tau_2((T_2 \smallsetminus D_2) \cap A_{m+1}) \\ &\underset{\text{a.e.}}{=} \varnothing. \end{aligned}$$

which is a consequence from $A_0 \subseteq T_2$ and $A_{m+1} \underset{\text{a.e.}}{\subseteq} D_2$. Now, assume that $n \neq 0$. By an argument similar to the case $n = 0$, we obtain that

$$A_0 \cap \delta_2(A_{m+n} \cap D_2) \underset{\text{a.e.}}{=} \varnothing.$$

We apply δ_2 to the above equation, it implies that

$$\delta_2(D_2 \cap A_0) \cap A_{m+n} \underset{\text{a.e.}}{=} \varnothing.$$

Applying τ_2 to the previous equation, we get that

$$\tau_2(T_2 \cap \delta_2(D_2 \cap A_0)) \cap T_2 \cap \delta_2(D_2 \cap A_{m+n-1}) \underset{\text{a.e.}}{=} \varnothing,$$

which implies that $A_1 \cap \delta_2(D_2 \cap A_{m+n-1}) \underset{\text{a.e.}}{=} \varnothing$. By continuing process n time, we have $A_n \cap \delta_2(D_2 \cap A_m) \underset{\text{a.e.}}{=} \varnothing$. Then $A \cap \delta_2(D_2 \cap A) \underset{\text{a.e.}}{=} \varnothing$.

Consequently,

$$\begin{aligned}
A \cap \tau_2(A) &= A \cap \bigcup_{n=0}^{\infty} \tau_2(A_n) \\
&= A \cap [(T_2 \setminus D_2) \cup \bigcup_{n=0}^{\infty} T_2 \cap \delta_2(D_2 \cap A_n)] \\
&= A \cap [(T_2 \setminus D_2) \cup (T_2 \cap \delta_2(D_2 \cap \bigcup_{n=0}^{\infty} A_n))] \\
&= A \cap [(T_2 \setminus D_2) \cup (T_2 \cap \delta_2(D_2 \cap A))] \\
&= [A \cap (T_2 \setminus D_2)] \cup [A \cap T_2 \cap \delta_2(D_2 \cap A)] \\
&\underset{\text{a.e.}}{=} \varnothing
\end{aligned}$$

follows from $A \underset{\text{a.e.}}{\subseteq} D_2$ and $A \cap \delta_2(D_2 \cap A) \underset{\text{a.e.}}{=} \varnothing$.

(b) can be proved by similar arguments. □

Proposition 3.3.8. *Let S be a 2-basic set for which the sets A and B defined in Definition 3.3.1 satisfy $A \underset{\text{a.e.}}{\subseteq} D_2$ and $B \underset{\text{a.e.}}{\subseteq} T_2$. Then*

(a) $(A \cup B) \cap \delta_2(D_2 \cap (A \cup B)) \underset{\text{a.e.}}{=} \varnothing$ and $(A \cup B) \cap \tau_2(T_2 \cap (A \cup B)) \underset{\text{a.e.}}{=} \varnothing$,

(b) $\tau_2(T_2 \setminus (A \cup B)) \underset{\text{a.e.}}{=} A \cup B$ if and only if $\delta_2(D_2 \setminus (A \cup B)) \underset{\text{a.e.}}{=} A \cup B$.

Proof. (a) It is easily seen that

$$\begin{aligned}
(A \cup B) \cap \delta_2(D_2 \cap (A \cup B)) &= (A \cup B) \cap [\delta_2(D_2 \cap A) \cup \delta_2(B)] \\
&= [(A \cup B) \cap \delta_2(D_2 \cap A)] \cup [(A \cup B) \cap \delta_2(B)] \\
&= [(A \cap \delta_2(D_2 \cap A)] \cup [B \cap \delta_2(D_2 \cap A)] \\
&\quad \cup [A \cap \delta_2(B)] \cup [B \cap \delta_2(B)].
\end{aligned}$$

By Lemma 3.3.7 and Lemma 3.3.6, it implies that

$$(A \cup B) \cap \delta_2(D_2 \cap (A \cup B)) \underset{\text{a.e.}}{=} \varnothing.$$

The set equation $(A \cup B) \cap \tau_2(T_2 \cap (A \cup B)) \underset{\text{a.e.}}{=} \varnothing$ can be proved very similarly under the assumption that $A \underset{\text{a.e.}}{\subseteq} D_2$ and $B \underset{\text{a.e.}}{\subseteq} T_2$.

(b) Suppose $\delta_2(D_2 \setminus (A \cup B)) \underset{\text{a.e.}}{=} A \cup B$. Applying δ_2, we get that

$$D_2 \setminus (A \cup B) \underset{\text{a.e.}}{=} \delta_2(D_2 \cap A) \cup (D_2 \setminus T_2) \cup (D_2 \cap \tau_2(T_2 \cap B)).$$

We take complement in D_2 and obtain

$$A \cup B \underset{\text{a.e.}}{=} D_2 \cap T_2 \cap \delta_2(D_2 \cap A)^c \cap \tau_2(T_2 \cap B)^c.$$

Taking complement in T_2, we have

$$T_2 \setminus (A \cup B) \underset{\text{a.e.}}{=} (T_2 \setminus D_2) \cup (T_2 \cap \delta_2(D_2 \cap A)) \cup \tau_2(T_2 \cap B).$$

We apply τ_2 and get

$$\tau_2(T_2 \setminus (A \cup B)) \underset{\text{a.e.}}{=} A_0 \cup \bigcup_{n=1}^{\infty} A_n \cup B \underset{\text{a.e.}}{=} A \cup B.$$

The converse is also true as the above arguments are valid in the opposite direction.

□

3.4 Sufficient conditions

In this section, we provide sufficient condition that, given a 2-basic set S, there exists a subset R satisfying the properties in Theorem 3.1.5, which implies that S contains a wavelet set. By Theorem 3.3.3, we know that $A \cup B \subseteq T_2 \cap D_2$ if S contains a wavelet set. Indeed, $A \cup B \subseteq R$. We will give some conditions on measure which imply that $A \cup B$ satisfies the same properties as R in Theorem 3.1.5. We start by recalling a lemma of Gröchenig and Madych

In [11] Gröchenig and Madych verified the next lemma which, along with the generalized lemma that follows, is useful in proving our main theorem.

Lemma 3.4.1. [11] *Suppose Q is a measurable subset of $\mathbb{R}$ such that $\bigcup_{k\in\mathbb{Z}}(Q+k) \underset{\text{a.e.}}{=} \mathbb{R}$. Then the followings are equivalent*

(a) $Q \cap (Q+k) \underset{\text{a.e.}}{=} \varnothing$ *whenever k is a nonzero element in $\mathbb{Z}$.*

(b) $\lambda(Q) = 1$.

Lemma 3.4.2. *Suppose Q is a measurable subset of $\mathbb{R}$. Then any two of the following statements imply the other.*

(a) $\bigcup_{k\in\mathbb{Z}}(Q+k) \underset{\text{a.e.}}{=} \mathbb{R}$.

(b) $Q \cap (Q+k) \underset{\text{a.e.}}{=} \varnothing$ *whenever k is a nonzero element in $\mathbb{Z}$.*

(c) $\lambda(Q) = 1$.

Proof. From Lemma 3.4.1, it follows that the two statements (a) and (b) imply the statement (c), and the two statements (a) and (c) imply the statement (b). Finally, we will show that the two statements (b) and (c) imply the statement (a). Assume that the two statements (b) and (c) hold. Without loss of generality we assume that $Q \cap (Q+k) = \varnothing$ whenever k is a nonzero element in $\mathbb{Z}$. We will prove by contradiction. Suppose that $\lambda(\mathbb{R} \setminus (\bigcup_{k\in\mathbb{Z}} Q + k)) > 0$. Since $[0,1)$ covers $\mathbb{R}$ by translation, we get that $\lambda([0,1) \setminus (\bigcup_{k\in\mathbb{Z}} Q + k)) > 0$, which implies that $\lambda([0,1) \cap \bigcup_{k\in\mathbb{Z}}(Q+k)) < 1$. Thus

$$\begin{aligned} 1 &= \lambda(Q \cap \mathbb{R}) \\ &= \lambda(\bigcup_{k\in\mathbb{Z}}(Q \cap [k, k+1))) \\ &= \sum_{k\in\mathbb{Z}} \lambda(Q \cap [k, k+1)) \\ &= \sum_{k\in\mathbb{Z}} \lambda((Q-k) \cap [0,1)) \end{aligned}$$

$$\begin{aligned} &= \sum_{k\in\mathbb{Z}} \lambda((Q+k)\cap[0,1)) \\ &= \lambda(\bigcup_{k\in\mathbb{Z}}((Q+k)\cap[0,1))) \\ &= \lambda([0,1)\cap\bigcup_{k\in\mathbb{Z}}(Q+k)) < 1, \end{aligned}$$

which is a contradiction. The proof is complete. □

Observe that if $S \smallsetminus (A \cup B)$ contains a wavelet set, then $\lambda(S \smallsetminus (A \cup B)) \geq 1$. Thus $\lambda(S \smallsetminus (A \cup B)) \geq 1$ is a necessary condition but certainly not a sufficient condition. The next theorem gives a sufficient condition for $S \smallsetminus (A \cup B)$ to be a wavelet set.

Proposition 3.4.3. *Let S be a 2-basic set with the sets A and B defined in Definition 3.3.1. If $A \cup B \underset{\text{a.e.}}{\subseteq} T_2 \cap D_2$ and $\lambda(S \smallsetminus (A \cup B)) = 1$, then $S \smallsetminus (A \cup B)$ is a wavelet set. In fact, it is the only wavelet subset of S up to measure zero.*

Proof. Assume that $A \cup B \underset{\text{a.e.}}{\subseteq} T_2 \cap D_2$ and $\lambda(S \smallsetminus (A \cup B)) = 1$. Then we have

$$(A \cup B) \cap \tau_2(T_2 \cap (A \cup B)) \underset{\text{a.e.}}{=} \varnothing$$

which follows from $A \cup B \underset{\text{a.e.}}{\subseteq} T_2 \cap D_2$ and Proposition 3.3.8. It implies that $\tau_2(T_2 \cap (A \cup B)) \underset{\text{a.e.}}{\subseteq} T_2 \smallsetminus (A \cup B)$. Thus

$$A \cup B \underset{\text{a.e.}}{\subseteq} \tau_2(T_2 \smallsetminus (A \cup B)),$$

which implies that $A \cup B \underset{\text{a.e.}}{\subseteq} \bigcup_{n\in\mathbb{Z}}(T_2 \smallsetminus (A \cup B)) + n \underset{\text{a.e.}}{\subseteq} \bigcup_{n\in\mathbb{Z}}(S \smallsetminus (A \cup B)) + n$. Hence $S \underset{\text{a.e.}}{\subseteq} \bigcup_{n\in\mathbb{Z}}(S \smallsetminus (A \cup B)) + n$, and so $\mathbb{R} \underset{\text{a.e.}}{=} \bigcup_{n\in\mathbb{Z}}(S \smallsetminus (A \cup B)) + n$. By the assumption $\lambda(S\setminus(A \cup B)) = 1$ and Lemma 3.4.1, we get that

$$(S \smallsetminus (A \cup B)) \cap (S \smallsetminus (A \cup B) + n) \underset{\text{a.e.}}{=} \varnothing,$$

for all $n \in \mathbb{Z} \setminus \{0\}$. Set $C = \{x \in T_2 \setminus (A \cup B) : \tau_2(x) \in T_2 \setminus (A \cup B)\}$. To show that $\lambda(C) = 0$, we suppose that $\lambda(C) > 0$. Then there exists a nonzero integer m such that $\lambda((T_2 \setminus (A \cup B)) + m \cap (T_2 \setminus (A \cup B))) > 0$. Hence

$$\lambda((S \setminus (A \cup B)) + m \cap (S \setminus (A \cup B))) > 0$$

which is a contradiction. Thus $\lambda(C) = 0$. Hence

$$\tau_2(T_2 \setminus (A \cup B)) \underset{\text{a.e.}}{\subseteq} A \cup B.$$

Therefore, $\tau_2(T_2 \setminus (A \cup B)) \underset{\text{a.e.}}{=} A \cup B$. From Proposition 3.3.8, we obtain that $\delta_2(D_2 \setminus (A \cup B)) \underset{\text{a.e.}}{=} A \cup B$. Thus $T_2 \setminus (A \cup B) \underset{\text{a.e.}}{\overset{\tau}{\sim}} (A \cup B)$ and $D_2 \setminus (A \cup B) \underset{\text{a.e.}}{\overset{\delta}{\sim}} (A \cup B)$. Then $S \setminus (A \cup B)$ is a wavelet set which as a consequence of Theorem 3.1.5. □

Example 3.4.4. *Let* $S = \left[-1, -\frac{1}{2}\right) \cup \left[\frac{1}{2}, 1\right) \cup [3, 4)$. *Then* $T_1 = \varnothing, T_2 = S$ *and* $D_1 = \left[-1, -\frac{1}{2}\right) \cup \left[\frac{1}{2}, \frac{3}{4}\right), D_2 = \left[\frac{3}{4}, 1\right) \cup [3, 4)$. *Thus we can define*

$$\tau_2 : \left[-1, -\frac{1}{2}\right) \underset{-4}{\overset{+4}{\rightleftarrows}} \left[3, \frac{7}{2}\right); \left[\frac{1}{2}, 1\right) \underset{-3}{\overset{+3}{\rightleftarrows}} \left[\frac{7}{2}, 4\right) \quad \textit{and} \quad \delta_2 : \left[\frac{3}{4}, 1\right) \underset{\times \frac{1}{4}}{\overset{\times 4}{\rightleftarrows}} [3, 4).$$

Hence $A = [3, 4)$ *and* $B = \varnothing$, *and so* $S \setminus (A \cup B) = \left[-1, -\frac{1}{2}\right) \cup \left[\frac{1}{2}, 1\right)$. *Then* $\lambda(S \setminus (A \cup B)) = 1$, *which implies that* $S \setminus (A \cup B)$ *is a wavelet set. In fact, this Shannon wavelet set is the only wavelet subset of* S.

Next, we define another measure μ by

$$\mu(E) := \int_E \frac{1}{|x|} \mathrm{d}\lambda = \int_E \frac{1}{|x|} \mathrm{d}x$$

where E is Lebesgue measurable. It is easy to see that μ is absolutely continuous with respect to λ. Indeed, we shall show that λ is also absolutely continuous with respect to μ in the next lemma.

Lemma 3.4.5. *λ is absolutely continuous with respect to μ.*

Proof. Let E be Lebesgue measurable such that $\lambda(E) > 0$. Then there exist an integer n such that $\lambda(E \cap (n, n+1)) > 0$, say $E_n := E \cap (n, n+1)$. If $n \geq 0$, then $\frac{1}{n+1} < \frac{1}{x} = \frac{1}{|x|}$ for all $x \in E_n$, and so

$$0 < \frac{1}{n+1}\lambda(E_n) = \int_{E_n} \frac{1}{n+1} \mathrm{d}\lambda \leq \int_{E_n} \frac{1}{|x|} \mathrm{d}\lambda \leq \int_E \frac{1}{|x|} \mathrm{d}\lambda = \mu(E).$$

If $n < 0$, then $0 < \frac{1}{|n|} = -\frac{1}{n} < -\frac{1}{x} = \frac{1}{|x|}$ for all $x \in E_n$, and so

$$0 < \frac{1}{|n|}\lambda(E_n) = \int_{E_n} \frac{1}{|n|} \mathrm{d}\lambda \leq \int_{E_n} \frac{1}{|x|} \mathrm{d}\lambda \leq \int_E \frac{1}{|x|} \mathrm{d}\lambda = \mu(E).$$

Therefore, $\mu(E) > 0$.

Consequently, If E is Lebesgue measurable such that $\mu(E) = 0$, then $\lambda(E) = 0$. Thus λ is absolutely continuous with respect to μ. □

We then have a dilation counterpart of Lemma 3.4.1 whose proof is similar.

Lemma 3.4.6. *Suppose Q is a measurable subset of $\mathbb{R}$ such that $\bigcup_{k\in\mathbb{Z}} 2^k Q = \mathbb{R}$ (μ-a.e.). Then the followings are equivalent.*

(a) $Q \cap 2^k Q = \varnothing$ (μ-a.e.) whenever k is a nonzero element in $\mathbb{Z}$,

(b) $\mu(Q \cap \mathbb{R}^+) = \ln 2$ and $\mu(Q \cap \mathbb{R}^-) = \ln 2$.

Proof. Assume that the statement (a) holds. Set $f(x) = \sum_{j\in\mathbb{Z}} \mathbb{1}_Q(2^j x)$. Thus $f(x) = 1$ a.e. on $\mathbb{R}$. Then

$$\begin{aligned} \mu(Q \cap \mathbb{R}^+) &= \int \mathbb{1}_{Q\cap\mathbb{R}^+} \frac{\mathrm{d}x}{x} \\ &= \sum_{j\in\mathbb{Z}} \int_{2^j}^{2^{j+1}} \mathbb{1}_Q(x) \frac{\mathrm{d}x}{x} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j\in\mathbb{Z}} \int_1^2 \mathbb{1}_Q(2^j y)\frac{\mathrm{d}y}{y} \\
&= \int_1^2 \sum_{j\in\mathbb{Z}} \mathbb{1}_Q(2^j y)\frac{\mathrm{d}y}{y} \\
&= \int_1^2 f(y)\frac{\mathrm{d}y}{y} \\
&= \int_1^2 \frac{\mathrm{d}y}{y} \\
&= \ln 2.
\end{aligned}
$$

Similarly, we obtain that $\mu(Q\cap\mathbb{R}^-) = \ln 2$.

Conversely, Assume that the statement (b) holds. By assumption, we obtain that $f(x) \geq 1$ a.e. on $\mathbb{R}$. It is easy to verify that

$$\int_1^2 f(x)\frac{\mathrm{d}x}{x} = \int_{\mathbb{R}} \mathbb{1}_{Q\cap\mathbb{R}^+}(x)\frac{\mathrm{d}x}{|x|} = \mu(Q\cap\mathbb{R}^+) = \ln 2$$

and

$$\int_{-2}^{-1} f(x)\frac{\mathrm{d}x}{x} = \int_{\mathbb{R}} \mathbb{1}_{Q\cap\mathbb{R}^-}(x)\frac{\mathrm{d}x}{|x|} = \mu(Q\cap\mathbb{R}^-) = \ln 2.$$

It implies that $f(x) = 1$ a.e. on $[-2,-1]\cup[1,2]$, and so $f(x) = 1$ a.e. on $\mathbb{R}$. Thus $Q\cap 2^kQ \underset{\text{a.e.}}{=} \varnothing$ whenever k is a nonzero element in $\mathbb{Z}$. □

Proposition 3.4.7. *Let S be a 2-basic set with the sets A and B defined in Definition 3.3.1. If $A\cap B \underset{\text{a.e.}}{\subseteq} T_2\cap D_2$, and $\mu((S\setminus(A\cup B))\cap\mathbb{R}^+) = \mu((S\setminus(A\cup B))\cap\mathbb{R}^-) = \ln 2$, then $S\setminus(A\cup B)$ is a wavelet set.*

Proof. Similar to that of Proposition 3.4.3. □

Clearly, any subset of a 2-basic set covering $\mathbb{R}$ by translation and dilation is 2-basic. Another special property of the sets A and B constructed from a 2-basic set S is that if $A\cup B \subseteq T_2\cap D_2$, then the sets $\tilde{T}_2$ and $\tilde{D}_2$ of the good part

$\tilde{S} = S\backslash(A \cup B)$ are equal. Here, $\tilde{T_2}$ is the set of all points in $\tilde{S}$ whose integral translates belong to $\tilde{S}$ exactly twice and $\tilde{D}_2$ is defined analogously.

Lemma 3.4.8. *Let S be a 2-basic set and $\tilde{S} = S \smallsetminus (A \cup B)$. If $A \cup B \underset{\text{a.e.}}{\subseteq} T_2 \cap D_2$, then $\tilde{S}$ is a 2-basic set with $\tilde{T}_2 = \tilde{D}_2$. Moreover,*

$$\tilde{T}_1 = T_1 \cup \tau_2(A \cup B), \qquad \tilde{D}_1 = D_1 \cup \delta_2(A \cup B),$$

$$\tilde{T}_2 = T_2 \smallsetminus (A \cup B \cup \tau_2(A \cup B)) \quad \textit{and} \quad \tilde{D}_2 = D_2 \smallsetminus (A \cup B \cup \delta_2(A \cup B)).$$

Proof. Assume that $A \cup B \underset{\text{a.e.}}{\subseteq} T_2 \cap D_2$. From Proposition 3.3.8, we obtain that

$$(A \cup B) \cap \tau_2(T_2 \cap (A \cup B)) \underset{\text{a.e.}}{=} \varnothing \text{ and } (A \cup B) \cap \delta_2(D_2 \cap (A \cup B)) \underset{\text{a.e.}}{=} \varnothing.$$

It implies that

$$(A \cup B) \underset{\text{a.e.}}{\subseteq} \tau_2(T_2 \smallsetminus (A \cup B)) \text{ and } (A \cup B) \underset{\text{a.e.}}{\subseteq} \delta_2(D_2 \smallsetminus (A \cup B)).$$

Then $A \cup B \underset{\text{a.e.}}{\subseteq} \bigcup_{n \in \mathbb{Z}} (T_2 \smallsetminus (A \cup B)) + n$ and $A \cup B \underset{\text{a.e.}}{\subseteq} \bigcup_{2^j \in \mathbb{Z}} 2^j (D_2 \smallsetminus (A \cup B))$, which implies that $S \underset{\text{a.e.}}{\subseteq} \bigcup_{n \in \mathbb{Z}} \tilde{S} + n$ and $S \underset{\text{a.e.}}{\subseteq} \bigcup_{j \in \mathbb{Z}} 2^j \tilde{S}$. Thus $\mathbb{R} \underset{\text{a.e.}}{\subseteq} \bigcup_{n \in \mathbb{Z}} \tilde{S} + n$ and $\mathbb{R} \underset{\text{a.e.}}{\subseteq} \bigcup_{j \in \mathbb{Z}} 2^j \tilde{S}$, and so $\tau_{\tilde{S}} \geq 1$ a.e. and $\delta_{\tilde{S}} \geq 1$ a.e. Since $\tilde{S} \subseteq S$, we get that $\tau_{\tilde{S}} \leq \tau_S \leq 2$ and $\delta_{\tilde{S}} \leq \delta_S \leq 2$. Thus $\tilde{S}$ is a 2-basic set. It is easily seen that

$$\tau_2(T_2 \cap (A \cup B)) = (T_2 \smallsetminus D_2) \cup (T_2 \cap \delta_2(D_2 \cap A)) \cup \tau_2(T_2 \cap B)$$

and

$$\delta_2(D_2 \cap (A \cup B)) = \delta_2(D_2 \cap A) \cup (D_2 \smallsetminus T_2) \cup (D_2 \cap \tau_2(T_2 \cap B)).$$

It then follows that

$$\begin{aligned} &T_2 \smallsetminus [A \cup B \cup \tau_2(T_2 \cap (A \cup B))] \\ &\quad = T_2 \smallsetminus [A \cup B \cup (T_2 \smallsetminus D_2) \cup (T_2 \cap \delta_2(D_2 \cap A)) \cup \tau_2(T_2 \cap B)] \end{aligned}$$

$$= T_2 \setminus [A \cup B \cup (T_2 \setminus D_2) \cup \delta_2(D_2 \cap A) \cup \tau_2(T_2 \cap B)]$$

$$= [T_2 \setminus (T_2 \setminus D_2)] \setminus [A \cup B \cup \delta_2(D_2 \cap A) \cup \tau_2(T_2 \cap B)]$$

$$= [T_2 \cap D_2] \setminus [A \cup B \cup \delta_2(D_2 \cap A) \cup \tau_2(T_2 \cap B)]$$

$$= [D_2 \setminus (D_2 \setminus T_2)] \setminus [A \cup B \cup \delta_2(D_2 \cap A) \cup \tau_2(T_2 \cap B)]$$

$$= D_2 \setminus [A \cup B \cup \delta_2(D_2 \cap A) \cup (D_2 \setminus T_2) \cup \tau_2(T_2 \cap B)]$$

$$= D_2 \setminus [A \cup B \cup \delta_2(D_2 \cap A) \cup (D_2 \setminus T_2) \cup (D_2 \cap \tau_2(T_2 \cap B))]$$

$$= D_2 \setminus [A \cup B \cup \delta_2(D_2 \cap (A \cup B))].$$

Because $\tilde{T}_2$ must be subset of T_2, we obtain that $\tilde{T}_2 = T_2 \setminus [A \cup B \cup \tau_2(T_2 \cap (A \cup B))]$. Similarly, we get that $\tilde{D}_2 = D_2 \setminus [A \cup B \cup \delta_2(D_2 \cap (A \cup B))]$. Then $\tilde{T}_2 = \tilde{D}_2$. Hence $\tilde{T}_1 = T_1 \cup \tau_2(T_2 \cap (A \cup B))$ and $\tilde{D}_1 = D_1 \cup \delta_2(D_2 \cap (A \cup B))$. Furthermore, $\tilde{T}_1 = \tilde{D}_1$. □

Example 3.4.9. *Let* $S = [-1, -\frac{1}{3}) \cup [\frac{1}{2}, \frac{3}{2})$. *Then we have*

$$\begin{aligned} T_1 &= \left[\frac{2}{3}, 1\right), \\ T_2 &= \left[-1, -\frac{1}{3}\right) \cup \left[\frac{1}{2}, \frac{2}{3}\right) \cup \left[1, \frac{3}{2}\right), \\ D_1 &= \left[-\frac{2}{3}, -\frac{1}{2}\right) \cup \left[\frac{3}{4}, 1\right) \quad \textit{and} \\ D_2 &= \left[-1, -\frac{2}{3}\right) \cup \left[-\frac{1}{2}, -\frac{1}{3}\right) \cup \left[\frac{1}{2}, \frac{3}{4}\right) \cup \left[1, \frac{3}{2}\right). \end{aligned}$$

Thus we can define

$$\tau_2 : \left[-1, -\frac{1}{2}\right) \underset{-2}{\overset{+2}{\rightleftarrows}} \left[1, \frac{3}{2}\right); \left[-\frac{1}{2}, -\frac{1}{3}\right) \underset{-1}{\overset{+1}{\rightleftarrows}} \left[\frac{1}{2}, \frac{2}{3}\right) \quad \textit{and}$$

$$\delta_2 : \left[-1, -\frac{2}{3}\right) \underset{\times 2}{\overset{\times \frac{1}{2}}{\rightleftarrows}} \left[-\frac{1}{2}, -\frac{1}{3}\right); \left[\frac{1}{2}, \frac{3}{4}\right) \underset{\times \frac{1}{2}}{\overset{\times 2}{\rightleftarrows}} \left[1, \frac{3}{2}\right).$$

From $T_2 \setminus D_2 = \left[-\frac{2}{3}, -\frac{1}{2}\right)$ *and* $D_2 \setminus T_2 = \left[\frac{2}{3}, \frac{3}{4}\right)$, *we obtain that* $A_0 = \left[\frac{4}{3}, \frac{3}{2}\right)$ *and* $B_0 = \left[\frac{4}{3}, \frac{3}{2}\right)$. *This implies that*

$$\delta_2(D_2 \cap A_0) \cap T_2 = \left[\frac{2}{3}, \frac{3}{4}\right) \cap T_2 = \varnothing$$

and

$$\tau_2(T_2 \cap B_0) \cap D_2 = \left[-\frac{2}{3}, -\frac{1}{2}\right) \cap D_2 = \varnothing.$$

Thus $A_n = \varnothing$ *and* $B_n = \varnothing$ *for all* $n \in \mathbb{N}$. *It implies that* $A \cup B = \left[\frac{4}{3}, \frac{3}{2}\right)$, *and so* $\tau_2(A \cup B) = \left[-\frac{2}{3}, -\frac{1}{2}\right)$, $\delta_2(A \cup B) = \left[\frac{2}{3}, \frac{3}{4}\right)$. *Therefore,* $\tilde{S} = S \setminus (A \cup B) = \left[-1, -\frac{1}{3}\right) \cup \left[\frac{1}{2}, \frac{3}{4}\right)$. *Moreover,*

$$\begin{aligned} \tilde{T}_1 = \tilde{D}_1 &= \left[-\frac{2}{3}, -\frac{1}{2}\right) \cup \left[\frac{2}{3}, 1\right) \quad \textit{and} \\ \tilde{T}_2 = \tilde{D}_2 &= \left[-1, -\frac{2}{3}\right) \cup \left[-\frac{1}{2}, -\frac{1}{3}\right) \cup \left[\frac{1}{2}, \frac{2}{3}\right) \cup \left[1, \frac{4}{3}\right). \end{aligned}$$

3.5 Two-basic sets satisfying $T_2 = D_2$

It follows from Theorem 3.3.3 that if a 2-basic set S is to contain a wavelet set W, then W cannot intersect $A \cup B$. So our next goal is to find wavelet subsets of $\tilde{S} = S \setminus (A \cup B)$. In light of Lemma 3.4.8, we shall only consider 2-basic sets satisfying $T_2 = D_2$. Let us begin with a proposition and a corollary.

Proposition 3.5.1. *Let* S *be a measurable subset of* $\mathbb{R}$ *such that* $1 \le \tau_S \le 2$ a.e. *Set* $T_1 = S \cap \tau_S^{-1}(1)$ *and* $T_2 = S \cap \tau_S^{-1}(2)$. *Then* $\lambda(T_1) = 1$ *if and only if* $\lambda(T_2) = 0$.

Proof. Recall $T_1 \cap (T_1 + k) = \varnothing$ for all $k \in \mathbb{Z} \setminus \{0\}$. Assume that $\lambda(T_1) = 1$. By Lemma 3.4.2, we get that $\bigcup_{k\in\mathbb{Z}}(T_1 + k) \underset{\text{a.e.}}{=} \mathbb{R}$. Thus $\bigcup_{k\in\mathbb{Z}}[(T_1+k) \cap T_2] \underset{\text{a.e.}}{=} T_2$. Since $T_1 + k \subseteq \tau_S^{-1}(1)$ for all $k \in \mathbb{Z}$, we get that $(T_1 + k) \cap T_2 = \varnothing$ for all $k \in \mathbb{Z}$. Hence

$\lambda(T_2) = 0$. Conversely, we assume that $\lambda(T_2) = 0$. Thus $S \underset{\text{a.e.}}{=} T_1$, and so $1 \leq \tau_{T_1}$ a.e. Hence $\bigcup_{k\in\mathbb{Z}}(T_1 + k) \underset{\text{a.e.}}{=} \mathbb{R}$. By Lemma 3.4.1, we obtain that $\lambda(T_1) = 1$. □

Corollary 3.5.2. *Let S be a 2-basic set such that $T_2 = D_2$. If $\lambda(T_1) = 1$, then S is a wavelet set.*

Proof. Assume that $\lambda(T_1) = 1$. Then $T_2 \underset{\text{a.e.}}{=} \varnothing$, and so $D_2 \underset{\text{a.e.}}{=} \varnothing$. Hence $S \underset{\text{a.e.}}{=} T_1$ and $S \underset{\text{a.e.}}{=} D_1$. Thus $\tau_S = \tau_{T_1}$ a.e. and $\delta_S = \delta_{D_1}$ a.e. This implies that $\tau_S = 1$ a.e. and $\delta_S = 1$ a.e., that is, S is a wavelet set. □

Definition 3.5.3. *Let S be a 2-basic set with $G := T_2 = D_2$ and $\lambda(G) > 0$. Then S is called a $\tau\delta$-***separable set** *if for each measurable subset X of G with $\lambda(X) > 0$ and $\tau_2(X) \underset{\text{a.e.}}{=} X \underset{\text{a.e.}}{=} \delta_2(X)$, there exists a measurable subset Y of X such that $\lambda(Y) > 0$, $\tau_2(Y) \underset{\text{a.e.}}{=} \delta_2(Y)$, and $\delta_2(Y) \cap Y \underset{\text{a.e.}}{=} \varnothing$.*

Remark. *Observe that the disjoint union $Y \cup \delta_2(Y)$ may not cover X. Hence the existence of such a set Y is considerably weaker than being able to partition X into $\tau_2(Y) = \delta_2(Y)$ and Y. So we conjecture that for any non-null set X with $\tau_2(X) = X = \delta_2(X)$ there exists a non-null subset Y of X for which $\tau_2(Y) = \delta_2(Y)$, and $\delta_2(Y) \cap Y = \varnothing$.*

Next, We will provide that $\tau\delta$-separability is a necessary and sufficient condition for a 2-basic set S, with $G := T_2 = D_2$, to contain a wavelet set.

Lemma 3.5.4. *Let S be a 2-basic set with $G := T_2 = D_2$ and $\lambda(G) > 0$. If S contains a wavelet set W_1, then $W_2 = T_1 \cup R_1$ is a wavelet subset of S where $R_1 = S \setminus W_1$. Furthermore, $S = W_1 \cup W_2$ and $\lambda(W_1 \triangle W_2) > 0$.*

Proof. Suppose that S contains a wavelet set W_1. Without loss of generality we assume that $T_1 \subseteq W_1$. Set $R_1 = S \setminus W_1$ and $W_2 = T_1 \cup R_1$. Because $T_2 = D_2$, we have $W_2 = D_1 \cup R_1$. Since $T_1 \subseteq W_1$ and $T_2 = D_2$, which implies that $R_1 \subseteq T_2 \cap D_2$. By Theorem 3.1.5., we have $T_2 \setminus R_1 \underset{\text{a.e.}}{\overset{\tau}{\sim}} R_1$ and $D_2 \setminus R_1 \underset{\text{a.e.}}{\overset{\delta}{\sim}} R_1$. Put $R_2 = G \setminus R_1$. Thus $R_2 \subseteq T_2 \cap D_2$, $T_2 \setminus R_2 \underset{\text{a.e.}}{\overset{\tau}{\sim}} R_2$ and $D_2 \setminus R_2 \underset{\text{a.e.}}{\overset{\delta}{\sim}} R_2$. Then $W_2 = T_1 \cup R_1 = S \setminus R_2$ is a wavelet set which follows from By Theorem 3.1.5. The proof is complete. □

Lemma 3.5.5. *Let S be a 2-basic set with $G := T_2 = D_2$ and $\lambda(G) > 0$. If S contains a wavelet set, then S is a $\tau\delta$-separable set.*

Proof. Assume that S contains a wavelet set. Then S contains two wavelet sets W_1 and W_2 satisfying the properties in Lemma 3.5.4. It is easy to check that $\tau_2(T_2 \cap W_1) \underset{\text{a.e.}}{=} T_2 \cap W_2$ and $\delta_2(D_2 \cap W_1) \underset{\text{a.e.}}{=} D_2 \cap W_2$. To show that S is a $\tau\delta$-separable set, let X be a measurable subset of G such that $\lambda(X) > 0$ and $\tau_2(X) \underset{\text{a.e.}}{=} X \underset{\text{a.e.}}{=} \delta_2(X)$. Choose $Y = X \cap W_1$. By the properties of X, it implies that $\lambda(Y) > 0$ and $\delta_2(Y) \underset{\text{a.e.}}{=} X \cap W_2 \underset{\text{a.e.}}{=} \tau_2(Y)$. Moreover, $\delta_2(Y) \cap Y \underset{\text{a.e.}}{=} \varnothing$. Hence S is a $\tau\delta$-separable set. □

Lemma 3.5.6. *Let G be a Lebesgue measurable set with finite measure and let I be an index set. Suppose $\{E_\alpha : \alpha \in I\}$ is a family of measurable subset of G such that for all $\alpha, \beta \in I$, either $E_\alpha \underset{\text{a.e.}}{\subseteq} E_\beta$ or $E_\beta \underset{\text{a.e.}}{\subseteq} E_\alpha$. Then there exists a sequence $\{E_{\alpha_n}\}_{n=1}^{\infty}$ in the family such that $E_\alpha \underset{\text{a.e.}}{\subseteq} \bigcup_{n\in\mathbb{N}} E_{\alpha_n}$ for all $\alpha \in I$.*

Proof. Put $L = \sup_{\alpha\in I} \lambda(E_\alpha)$. Obviously, $\lambda(L) \leq \lambda(G) < \infty$. If there exists $\alpha_0 \in I$ with $\lambda(E_{\alpha_0}) = L$, then we set $E_{\alpha_n} := E_{\alpha_0}$ for all $n \in \mathbb{N}$, and so $E_\alpha \underset{\text{a.e.}}{\subseteq} E_{\alpha_0}$ for all $\alpha \in I$. Assume that $\lambda(E_\alpha) < L$ for all $\alpha \in I$. For each $n \in \mathbb{N}$, there exits $\alpha_n \in I$ such that $L < \lambda(E_{\alpha_n}) + \frac{1}{n}$. Set $C = \bigcup_{n\in\mathbb{N}} E_{\alpha_n}$. We will show that $E_\alpha \underset{\text{a.e.}}{\subseteq} C$ for all $\alpha \in I$. Let $\alpha \in I$. Then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < L - \lambda(E_\alpha)$. Then $\lambda(E_\alpha) < L - \frac{1}{n} < \lambda(E_{\alpha_n})$. This implies that $E_\alpha \underset{\text{a.e.}}{\subseteq} E_{\alpha_n} \underset{\text{a.e.}}{\subseteq} C$. □

Lemma 3.5.7. *Let S be a 2-basic set with $G := T_2 = D_2$ and $\lambda(G) > 0$. If S is a $\tau\delta$-separable set, then there exists a measurable subset M of G such that $\tau_2(M) \underset{\text{a.e.}}{=} G \setminus M$ and $\delta_2(M) \underset{\text{a.e.}}{=} G \setminus M$. Moreover, $S \setminus M$ is a wavelet set.*

Proof. Let $\mathcal{M}_G = \{E \in \mathcal{M} : E \subseteq G\}$ where $\mathcal{M}$ is the collection of Lebesgue measurable sets. Let a relation $\underset{\text{a.e.}}{=}$ among the members of $\mathcal{M}_G$ be defined by setting $E \underset{\text{a.e.}}{=} F$ when $E \underset{\text{a.e.}}{\subseteq} F$ and $F \underset{\text{a.e.}}{\subseteq} E$. Then $\underset{\text{a.e.}}{=}$ is an equivalence relation on $\mathcal{M}_G$. Let $[\mathcal{M}_G]$ be the collection of all the equivalence classes with respect to the equivalence relation $\underset{\text{a.e.}}{=}$. By the $\tau\delta$-separable property of S and $\lambda(G) > 0$, $\tau_2(G) = G = \delta_2(G)$ and hence there exists a measurable subset Z of G such that

$$\lambda(Z) > 0,\ \tau_2(Z) \underset{\text{a.e.}}{=} \delta_2(Z),\ \text{and } \delta_2(Z) \cap Z \underset{\text{a.e.}}{=} \varnothing.$$

Let $\mathcal{P} = \{E \in [\mathcal{M}_G] : Z \underset{\text{a.e.}}{\subseteq} E,\ \tau_2(E) \underset{\text{a.e.}}{=} \delta_2(E) \text{ and } \delta_2(E) \cap E \underset{\text{a.e.}}{=} \varnothing\}$. Clearly, $\mathcal{P} \neq \varnothing$ because $Z \in \mathcal{P}$. Moreover, $(\mathcal{P}, \underset{\text{a.e.}}{\subseteq})$ is a partially ordered set. Let $\mathcal{L} = \{E_\alpha : \alpha \in I\}$ be a linearly ordered subset of $\mathcal{P}$, that is, if $E, F \in \mathcal{L}$, then $E \underset{\text{a.e.}}{\subseteq} F$ or $F \underset{\text{a.e.}}{\subseteq} E$, where I is an index set. By Lemma 3.5.6, there exists a sequence $\{E_{\alpha_n}\}_{n=1}^{\infty}$ in $\mathcal{L}$ such that $E_\alpha \underset{\text{a.e.}}{\subseteq} C = \bigcup_{n\in\mathbb{N}} E_{\alpha_n}$ for all $\alpha \in I$. Clearly, $Z \underset{\text{a.e.}}{\subseteq} C$ and $\tau_2(C) \underset{\text{a.e.}}{=} \delta_2(C)$. It is easy to verify that

$$\delta_2(C) \cap C = \left[\bigcup_{n\in\mathbb{N}} \delta_2(E_{\alpha_n})\right] \cap \bigcup_{m\in\mathbb{N}} E_{\alpha_m} = \bigcup_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \delta_2(E_{\alpha_n}) \cap E_{\alpha_m}.$$

Fix $n, m \in \mathbb{N}$. If $E_{\alpha_n} \underset{\text{a.e.}}{\subseteq} E_{\alpha_m}$, then $\delta_2(E_{\alpha_n}) \cap E_{\alpha_m} \underset{\text{a.e.}}{\subseteq} \delta_2(E_{\alpha_m}) \cap E_{\alpha_m} \underset{\text{a.e.}}{=} \varnothing$. If $E_{\alpha_m} \underset{\text{a.e.}}{\subseteq} E_{\alpha_n}$, then $\delta_2(E_{\alpha_n}) \cap E_{\alpha_m} \underset{\text{a.e.}}{\subseteq} \delta_2(E_{\alpha_n}) \cap E_{\alpha_n} \underset{\text{a.e.}}{=} \varnothing$. Hence $\delta_2(C) \cap C \underset{\text{a.e.}}{=} \varnothing$, and so $C \in \mathcal{P}$. By Zorn's Lemma [10], $\mathcal{P}$ has a maximal element, say M. Then $Z \underset{\text{a.e.}}{\subseteq} M \subseteq G$, $\tau_2(M) \underset{\text{a.e.}}{=} \delta_2(M)$ and $\delta_2(M) \cap M \underset{\text{a.e.}}{=} \varnothing$, and so $\lambda(M) > 0$. We shall prove that $G \underset{\text{a.e.}}{=} M \cup \delta_2(M)$. Suppose that $\lambda(G \setminus [M \cup \delta_2(M)]) > 0$. Set $H := G\backslash(M \cup \delta_2(M))$. Clearly, $\lambda(H) > 0$ and $\delta_2(H) \underset{\text{a.e.}}{=} H \underset{\text{a.e.}}{=} \tau_2(H)$. By the above argument with G replaced by H, there exists a measurable subset N of H such that $\lambda(N) > 0$, $\tau_2(N) \underset{\text{a.e.}}{=} \delta_2(N)$ and $\delta_2(N) \cap N \underset{\text{a.e.}}{=} \varnothing$. Then $M \subsetneq M \cup N$ which contradicts the fact that M is a maximal element of $\mathcal{P}$. Hence $G \underset{\text{a.e.}}{=} M \cup \delta_2(M)$. Since $\delta_2(M) \cap M \underset{\text{a.e.}}{=} \varnothing$, we obtain that $G \setminus M \underset{\text{a.e.}}{=} \delta_2(M)$, and so $G \setminus M \underset{\text{a.e.}}{=} \tau_2(M)$. Thus $G \setminus M \underset{\text{a.e.}}{\overset{\delta}{\sim}} M$ and $G \setminus M \underset{\text{a.e.}}{\overset{\tau}{\sim}} M$. Consequently, by Theorem 3.1.5, $S \setminus M$ is a wavelet set. □

Theorem 3.5.8. *Let S be a 2-basic set with $G := T_2 = D_2$ and $\lambda(G) > 0$. Then S contains a wavelet set if and only if S is a $\tau\delta$-separable set.*

Proof. This result is obvious from Lemma 3.5.5 and Lemma 3.5.7. □

Next, we shall investigate when S is a $\tau\delta$-separable set. Consider a measurable subset X of G with $\lambda(X) > 0$ and $\tau_2(X) \underset{\text{a.e.}}{=} X \underset{\text{a.e.}}{=} \delta_2(X)$, we can always find partitions of X by τ_2 and δ_2. For instance,

$$\{x \in X : \tau_2(x) - x > 0\} \text{ and } \{x \in X : \tau_2(x) - x < 0\}$$

are partitions of X under τ_2 and

$$\{x \in X : 0 < \delta_2(x)/x < 1\} \text{ and } \{x \in X : \delta_2(x)/x > 1\}$$

are partitions of X under δ_2. Let U_1, U_2, V_1, and V_2 be measurable subsets of X such that

$$U_1 \cap U_2 \underset{\text{a.e.}}{=} \varnothing \underset{\text{a.e.}}{=} V_1 \cap V_2,\ U_1 \cup U_2 \underset{\text{a.e.}}{=} X \underset{\text{a.e.}}{=} V_1 \cup V_2, \text{ and } \tau_2(U_1) \underset{\text{a.e.}}{=} U_2,\ \delta_2(V_1) \underset{\text{a.e.}}{=} V_2.$$

Note that $U_i \cap V_j \underset{\text{a.e.}}{=} \varnothing$ for at most two pairs of (i, j), for $i, j \in \{1, 2\}$.

Lemma 3.5.9. *If $U_i \cap V_j$ has measure zero for at least one pair of $i, j \in \{1, 2\}$, then there exists a measurable subset Y of X such that $\lambda(Y) > 0$, $\tau_2(Y) \underset{\text{a.e.}}{=} \delta_2(Y)$, and $\delta_2(Y) \cap Y \underset{\text{a.e.}}{=} \varnothing$.*

Proof. Assume that $U_i \cap V_j$ has measure zero for at least one pair of $i, j \in \{1, 2\}$. Without loss of generality we assume that $U_2 \cap V_2 \underset{\text{a.e.}}{=} \varnothing$. If $U_i \cap V_j$ has measure zero for another pair of $i, j \in \{1, 2\}$, then it implies that $U_1 \cap V_1 \underset{\text{a.e.}}{=} \varnothing$, and hence $U_1 \underset{\text{a.e.}}{=} V_2$ and we choose $Y = U_1$.

Assume that $U_i \cap V_j$ has measure zero for only one pair of $i, j \in \{1, 2\}$. Then $U_2 \underset{\text{a.e.}}{\subseteq} V_1$, $V_2 \underset{\text{a.e.}}{\subseteq} U_1$ and $\lambda(U_1 \cap V_1) > 0$. For each $n \in \mathbb{Z}$, let

$$Y_n = (\tau_2 \circ \delta_2)^n (U_1 \cap V_1)$$

where $(\tau_2 \circ \delta_2)^0 = id_G$ and $(\tau_2 \circ \delta_2)^{-n} = (\delta_2 \circ \tau_2)^n$ for $n \in \mathbb{N}$. Set $Y = \bigcup_{n \in \mathbb{Z}} Y_n$. It is clear that $\lambda(Y) > 0$ and $\delta_2(Y) = \tau_2(Y)$. Clearly,

$$Y \cap \delta_2(Y) = \bigcup_{n \in \mathbb{Z}} Y_n \cap \bigcup_{m \in \mathbb{Z}} \delta_2(Y_m) = \bigcup_{n \in \mathbb{Z}} \bigcup_{m \in \mathbb{Z}} [Y_n \cap \delta_2(Y_m)].$$

We must show that $Y \cap \delta_2(Y) \underset{\text{a.e.}}{=} \varnothing$. It suffices to show that $Y_n \cap \delta_2(Y_m) \underset{\text{a.e.}}{=} \varnothing$ for all $n, m \in \mathbb{Z}$. First, we show that $Y_n \underset{\text{a.e.}}{\subseteq} U_2$ for all $n \in \mathbb{N}$ by induction. When $n = 1$, from $\delta_2(Y_0) \underset{\text{a.e.}}{\subseteq} V_2 \underset{\text{a.e.}}{\subseteq} U_1$, we get that $Y_1 = (\tau_2 \circ \delta_2)(Y_0) \underset{\text{a.e.}}{\subseteq} U_2$. Suppose $Y_n \underset{\text{a.e.}}{\subseteq} U_2$ for $n \geq 1$. Since $U_2 \underset{\text{a.e.}}{\subseteq} V_1$, it implies that $Y_n \underset{\text{a.e.}}{\subseteq} V_1$. Thus $\delta_2(Y_n) \underset{\text{a.e.}}{\subseteq} V_2 \underset{\text{a.e.}}{\subseteq} U_1$, and so $Y_{n+1} = (\tau_2 \circ \delta_2)(Y_n) \underset{\text{a.e.}}{\subseteq} U_2$. This prove our claim.

Next, we shall prove that $Y_{-n} \underset{\text{a.e.}}{\subseteq} V_2$ for all $n \in \mathbb{N}$. Since $\tau_2(Y_0) \underset{\text{a.e.}}{\subseteq} U_2 \underset{\text{a.e.}}{\subseteq} V_1$, we obtain that $Y_{-1} = (\tau_2 \circ \delta_2)^{-1}(Y_0) = (\delta_2 \circ \tau_2)(Y_0) \underset{\text{a.e.}}{\subseteq} V_2$. Assume $Y_{-n} \underset{\text{a.e.}}{\subseteq} V_2$ for $n \geq 1$. From $V_2 \underset{\text{a.e.}}{\subseteq} U_1$, we get that $Y_{-n} \underset{\text{a.e.}}{\subseteq} U_1$, which implies $\tau_2(Y_{-n}) \underset{\text{a.e.}}{\subseteq} U_2 \underset{\text{a.e.}}{\subseteq} V_1$. Thus $Y_{-(n+1)} = (\tau_2 \circ \delta_2)^{-1}(Y_{-n}) = (\delta_2 \circ \tau_2)(Y_{-n}) \underset{\text{a.e.}}{\subseteq} V_2$. By induction, we have $Y_{-n} \underset{\text{a.e.}}{\subseteq} V_2$ for all $n \in \mathbb{N}$.

Now, we will prove that $Y_n \cap \delta_2(Y_m) \underset{\text{a.e.}}{=} \varnothing$ for all $n, m \in \mathbb{Z}$. Fix $n, m \in \mathbb{Z}$.

If $n, m \geq 1$, then $Y_n \cap \delta_2(Y_m) \underset{\text{a.e.}}{\subseteq} V_1 \cap V_2 \underset{\text{a.e.}}{=} \varnothing$.

If $n, m \leq -1$, then $Y_n \cap \delta_2(Y_m) \underset{\text{a.e.}}{\subseteq} V_2 \cap V_1 \underset{\text{a.e.}}{=} \varnothing$.

If $n = 0$ and $m \geq 0$, then $Y_0 \cap \delta_2(Y_m) \underset{\text{a.e.}}{\subseteq} V_1 \cap V_2 \underset{\text{a.e.}}{=} \varnothing$.

If $n = 0$ and $m < 0$, then, from $m + 1 \leq 0$,

$$Y_0 \cap \delta_2(Y_m) = Y_0 \cap \tau_2((\tau_2 \circ \delta_2)(Y_m)) = Y_0 \cap \tau_2(Y_{m+1}) \underset{\text{a.e.}}{\subseteq} U_1 \cap U_2 \underset{\text{a.e.}}{=} \varnothing.$$

If $n \geq 0$ and $m = 0$, then $Y_n \cap \delta_2(Y_0) \underset{\text{a.e.}}{\subseteq} V_1 \cap V_2 \underset{\text{a.e.}}{=} \varnothing$.

If $n < 0$ and $m = 0$, then $Y_n \cap \delta_2(Y_0) = \delta_2(\delta_2(Y_n) \cap Y_0) \underset{\text{a.e.}}{=} \delta_2(\varnothing) = \varnothing$.

If $n > 0$ and $m < 0$, then $m = -k$ for some $k \in \mathbb{N}$. And if $n \geq k$, that is, $n - k \geq 0$, we get that

$$\begin{aligned} Y_n \cap \delta_2(Y_m) &= (\tau_2 \circ \delta_2)^n(Y_0) \cap \delta_2((\delta_2 \circ \tau_2)^k(Y_0)) \\ &= (\tau_2 \circ \delta_2)^n(Y_0) \cap (\tau_2 \circ \delta_2)^{k-1}(\tau_2(Y_0)) \\ &= (\tau_2 \circ \delta_2)^{k-1}((\tau_2 \circ \delta_2)^{n-k+1}(Y_0) \cap \tau_2(Y_0)) \\ &= (\tau_2 \circ \delta_2)^{k-1}(\tau_2[\delta_2((\tau_2 \circ \delta_2)^{n-k}(Y_0)) \cap Y_0]) \\ &= (\tau_2 \circ \delta_2)^{k-1}(\tau_2[\delta_2(Y_{n-k}) \cap Y_0]) \\ &\underset{\text{a.e.}}{=} \varnothing. \end{aligned}$$

But if $n < k$, that is, $m + 1 + n = -k + 1 + n \leq 0$, we obtain that

$$\begin{aligned} Y_n \cap \delta_2(Y_m) &= (\tau_2 \circ \delta_2)^n(Y_0) \cap \delta_2((\delta_2 \circ \tau_2)^k(Y_0)) \\ &= (\tau_2 \circ \delta_2)^n(Y_0) \cap (\tau_2 \circ \delta_2)^{k-1}(\tau_2(Y_0)) \\ &= (\tau_2 \circ \delta_2)^n(Y_0 \cap (\tau_2 \circ \delta_2)^{k-1-n}(\tau_2(Y_0))) \\ &= (\tau_2 \circ \delta_2)^n(Y_0 \cap (\tau_2((\delta_2 \circ \tau_2)^{k-1-n}(Y_0)))) \\ &= (\tau_2 \circ \delta_2)^n(Y_0 \cap (\tau_2((\delta_2 \circ \tau_2)^{-(m+1+n)}(Y_0)))) \\ &= (\tau_2 \circ \delta_2)^n(Y_0 \cap (\tau_2((\tau_2 \circ \delta_2)^{m+1+n}(Y_0)))) \\ &= (\tau_2 \circ \delta_2)^n(Y_0 \cap (\tau_2(Y_{m+1+n}))) \end{aligned}$$

$$\underset{\text{a.e.}}{=} (\tau_2 \circ \delta_2)^n(\varnothing)$$
$$= \varnothing.$$

Finally, if $n < 0$ and $m > 0$, then $Y_n \cap \delta_2(Y_m) = \delta_2(\delta_2(Y_n) \cap Y_m) \underset{\text{a.e.}}{=} \delta_2(\varnothing) = \varnothing$. Then $Y \cap \delta_2(Y) \underset{\text{a.e.}}{=} \varnothing$. □

Theorem 3.5.10. *Let S be a 2-basic with $G := T_2 = D_2$ and $\lambda(G) > 0$. Let U_1, U_2, V_1, and V_2 be measurable subsets of G such that $U_1 \cap U_2 \underset{\text{a.e.}}{=} \varnothing \underset{\text{a.e.}}{=} V_1 \cap V_2$, $U_1 \cup U_2 \underset{\text{a.e.}}{=} G \underset{\text{a.e.}}{=} V_1 \cup V_2$, and $\tau_2(U_1) \underset{\text{a.e.}}{=} U_2$, $\delta_2(V_1) \underset{\text{a.e.}}{=} V_2$. If $U_i \cap V_j$ has measure zero for at least one pair of $i, j \in \{1, 2\}$, then S is a $\tau\delta$-separable set, and hence S contains a wavelet set.*

Proof. Assume that $U_i \cap V_j$ has measure zero for at least one pair of $i, j \in \{1, 2\}$. Without loss of generality we assume that $U_2 \cap V_2 \underset{\text{a.e.}}{=} \varnothing$. Suppose X is a measurable subset of G with $\lambda(X) > 0$ and $\tau_2(X) \underset{\text{a.e.}}{=} X \underset{\text{a.e.}}{=} \delta_2(X)$. It is easy to check that $\tau_2(X \cap U_1) \underset{\text{a.e.}}{=} X \cap U_2$ and $\delta_2(X \cap V_1) \underset{\text{a.e.}}{=} X \cap V_2$. Then we have $\lambda(X \cap U_i) > 0$ and $\lambda(X \cap V_i) > 0$ for al $i = 1, 2$. Moreover,

$$(X \cap U_1) \cup (X \cap U_2) \underset{\text{a.e.}}{=} X \underset{\text{a.e.}}{=} (X \cap V_1) \cup (X \cap V_2)$$

and

$$(X \cap U_1) \cap (X \cap U_2) \underset{\text{a.e.}}{=} \varnothing \underset{\text{a.e.}}{=} (X \cap V_1) \cap (X \cap V_2).$$

From $U_2 \cap V_2 \underset{\text{a.e.}}{=} \varnothing$, we get that $(X \cap U_2) \cap (X \cap V_2) \underset{\text{a.e.}}{=} \varnothing$. If $(X \cap U_1) \cap (X \cap V_2) \underset{\text{a.e.}}{=} \varnothing$, then

$$\begin{aligned} X \cap V_2 &= (X \cap V_2) \cap X \\ &\underset{\text{a.e.}}{=} (X \cap V_2) \cap [(X \cap U_1) \cup (X \cap U_2)] \\ &= [(X \cap V_2) \cap (X \cap U_1)] \cup [(X \cap V_2) \cap (X \cap U_2)] \end{aligned}$$

$$= \varnothing$$

which contradicts the fact that $\lambda(X \cap V_2) > 0$. Then $\lambda((X \cap U_1) \cap (X \cap V_2)) > 0$. Similarly, we can prove that $\lambda((X \cap U_2) \cap (X \cap V_1)) > 0$. By Lemma 3.5.9, the proof is complete. □

Corollary 3.5.11. *Let S be a 2-basic with $G := T_2 = D_2$ and $\lambda(G) > 0$. If there exists a measurable subset V of G such that $V \cup \delta_2(V) \underset{\text{a.e.}}{=} G$, $V \cap \delta_2(V) \underset{\text{a.e.}}{=} \varnothing$, and $V \cap \tau_2(V) \underset{\text{a.e.}}{=} \varnothing$, then S contains a wavelet set.*

Proof. Assume that there exists a measurable subset V of G such that $V \cup \delta_2(V) \underset{\text{a.e.}}{=} G$, $V \cap \delta_2(V) \underset{\text{a.e.}}{=} \varnothing$ and $V \cap \tau_2(V) \underset{\text{a.e.}}{=} \varnothing$. Thus $\tau_2(G \setminus [V \cup \tau_2(V)]) = G \setminus [V \cup \tau_2(V)]$. Then we can find a measurable subset Z of $G \setminus [V \cup \tau_2(V)]$ such that $Z \cap \tau_2(Z) = \varnothing$ and $Z \cup \tau_2(Z) = G \setminus [V \cup \tau_2(V)]$. Set $U_1 = Z \cup V$, $U_2 = \tau_2(Z \cup V)$, $V_1 = \delta_2(V)$, and $V_2 = V$. Hence $U_1 \cap U_2 \underset{\text{a.e.}}{=} \varnothing \underset{\text{a.e.}}{=} V_1 \cap V_2$, $U_1 \cup U_2 \underset{\text{a.e.}}{=} G = V_1 \cup V_2$, and $\tau_2(U_1) \underset{\text{a.e.}}{=} U_2$, $\delta_2(V_1) \underset{\text{a.e.}}{=} V_2$. Moreover, $U_2 \cap V_2 \underset{\text{a.e.}}{=} \varnothing$. The proof is complete once we apply Theorem 3.5.10. □

Remark. *It is still unknown whether if S is 2-basic with $G := T_2 = D_2$ and $\lambda(G) > 0$, then S always contains a wavelet set, as we cannot find an example of a 2-basic set, with $G := T_2 = D_2$ and $\lambda(G) > 0$ but there is no wavelet subset. It is the same problem as in the remark on Theorem 3.3.3.*

CHAPTER IV

CONSTRUCTION OF WAVELET SETS FROM TWO-BASIC SETS

In this chapter, we provide examples of wavelet sets which are constructed from a 2-basic set S satisfying $A \cup B \subseteq T_2 \cap D_2$. First, we construct wavelet sets from a 2-basic set S such that $S \setminus (A \cup B)$ has measure one. After that we consider a 2-basic set S such that $S \setminus (A \cup B)$ has measure greater than one.

4.1 $S \setminus (A \cup B)$ has measure one

Example 4.1.1. *Let* $S = \left[-\frac{1}{2^k}, -\frac{1}{2^{k+1}}\right) \cup \left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right) \cup [1, 2)$ *where* $k \geq 2$. *Obviously,* S *is 2-basic. Since* $k \geq 2$, *we have* $\frac{1}{2^k} < 1 - \frac{1}{2^k}$. *Then we obtain that*

$$\begin{aligned} D_1 &= \left[-\frac{1}{2^k}, -\frac{1}{2^{k+1}}\right), \\ D_2 &= \left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right) \cup [1, 2), \\ T_1 &= \left[1, 1 + \frac{1}{2^{k+1}}\right) \cup \left[1 + \frac{1}{2^k}, 2 - \frac{1}{2^k}\right) \cup \left[2 - \frac{1}{2^{k+1}}, 2\right) \text{ and} \\ T_2 &= \left[-\frac{1}{2^k}, -\frac{1}{2^{k+1}}\right) \cup \left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right) \cup \left[1 + \frac{1}{2^{k+1}}, 1 + \frac{1}{2^k}\right) \\ &\quad \cup \left[2 - \frac{1}{2^k}, 2 - \frac{1}{2^{k+1}}\right). \end{aligned}$$

Thus we can define

$$\tau_2 \ : \ \left[-\frac{1}{2^k}, -\frac{1}{2^{k+1}}\right) \underset{-2}{\overset{+2}{\rightleftarrows}} \left[2 - \frac{1}{2^k}, 2 - \frac{1}{2^{k+1}}\right);$$

$$\left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right) \underset{-1}{\overset{+1}{\rightleftarrows}} \left[1+\frac{1}{2^{k+1}}, 1+\frac{1}{2^k}\right) \text{ and}$$
$$\delta_2 \ : \ \left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right) \underset{\times\frac{1}{2^{k+1}}}{\overset{\times 2^{k+1}}{\rightleftarrows}} [1,2).$$

We see that $\left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right) \xrightarrow{+1} \left[1+\frac{1}{2^{k+1}}, 1+\frac{1}{2^k}\right) \subseteq [1,2)$ *and* $[1,2) \xrightarrow{\times\frac{1}{2^{k+1}}} \left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right)$. *From* $D_2 \smallsetminus T_2 = D_2 \cap T_1 = T_1 \subseteq [1,2)$, *we have* $B_0 \subseteq \left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right)$. *It implies that* $B_n \subseteq \left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right)$ *for all* $n \in \mathbb{N}$. *Then* $B \subseteq T_2$. *Since* $T_2 \smallsetminus D_2 \subseteq T_2 \cap D_1 = D_1$, *we get that* $A_0 \subseteq [1,2)$, *which implies that* $A_n \subseteq \left[1+\frac{1}{2^{k+1}}, 1+\frac{1}{2^k}\right)$. *Thus* $A \subseteq D_2$. *Hence* $A \cup B \subseteq T_2 \cap D_2$. *Moreover,* $A \cap B = \varnothing$. *We can find that*

$$\begin{aligned} B_0 &= \left[\frac{1}{2^{k+1}}, \frac{1}{2^{k+1}} + (\frac{1}{2^{k+1}})^2\right) \cup \left[\frac{1}{2^{k+1}} + \frac{1}{2^k}\cdot\frac{1}{2^{k+1}}, \frac{1}{2^k} - \frac{1}{2^k}\cdot\frac{1}{2^{k+1}}\right) \\ &\quad \cup \left[\frac{1}{2^k} - (\frac{1}{2^{k+1}})^2, \frac{1}{2^k}\right) \text{ and} \\ B_n &= \frac{1}{2^{k+1}} \cdot \left[\frac{1-(\frac{1}{2^{k+1}})^{n+1}}{1-\frac{1}{2^{k+1}}}, \frac{1-(\frac{1}{2^{k+1}})^{n+2}}{1-\frac{1}{2^{k+1}}}\right) \cup [p_n, q_n) \cup [s_n, t_n) \end{aligned}$$

where

$$\begin{aligned} p_n &= \frac{1}{2^{k+1}} \cdot \frac{1-(\frac{1}{2^{k+1}})^{n+1}}{1-\frac{1}{2^{k+1}}} + (\frac{1}{2^{k+1}})^{n+1} + \frac{1}{2^k}\cdot(\frac{1}{2^{k+1}})^{n+1} \\ q_n &= \frac{1}{2^{k+1}} \cdot \frac{1-(\frac{1}{2^{k+1}})^{n+1}}{1-\frac{1}{2^{k+1}}} + \frac{1}{2^k}\cdot(\frac{1}{2^{k+1}})^{n} - \frac{1}{2^k}\cdot(\frac{1}{2^{k+1}})^{n+1} \\ s_n &= \frac{1}{2^{k+1}} \cdot \frac{1-(\frac{1}{2^{k+1}})^{n}}{1-\frac{1}{2^{k+1}}} + \frac{1}{2^k}\cdot(\frac{1}{2^{k+1}})^{n} - (\frac{1}{2^{k+1}})^{n+2} \\ t_n &= \frac{1}{2^{k+1}} \cdot \frac{1-(\frac{1}{2^{k+1}})^{n}}{1-\frac{1}{2^{k+1}}} + \frac{1}{2^k}\cdot(\frac{1}{2^{k+1}})^{n} \end{aligned}$$

for all $n \in \mathbb{N}$. *Similarly, we have*

$$\begin{aligned} A_0 &= \left[2-\frac{1}{2^k}, 2-\frac{1}{2^{k+1}}\right), \\ A_1 &= \left[1+\frac{1}{2^k} - \frac{1}{2^k}\cdot\frac{1}{2^{k+1}}, 1+\frac{1}{2^k} - (\frac{1}{2^{k+1}})^2\right) \text{ and} \\ A_n &= [u_n, v_n) \end{aligned}$$

where

$$\begin{aligned} u_n &= \frac{1-(\frac{1}{2^{k+1}})^n}{1-\frac{1}{2^{k+1}}} + \frac{1}{2^k}\cdot(\frac{1}{2^{k+1}})^{n-1} - \frac{1}{2^k}\cdot(\frac{1}{2^{k+1}})^n \text{ and} \\ v_n &= \frac{1-(\frac{1}{2^{k+1}})^n}{1-\frac{1}{2^{k+1}}} + \frac{1}{2^k}\cdot(\frac{1}{2^{k+1}})^{n-1} - (\frac{1}{2^{k+1}})^{n+1} \end{aligned}$$

for all $n \geq 2$. It is easy check that

$$\begin{aligned} \lambda(B_n) &= (\frac{1}{2^{k+1}})^{n+2} + (\frac{1}{2^{k+1}})^{n+1}\cdot(1-\frac{1}{2^{k-1}}) + (\frac{1}{2^{k+1}})^{n+2} \text{ and} \\ \lambda(A_n) &= (\frac{1}{2^{k+1}})^{n+1} \end{aligned}$$

for all $n \in \mathbb{N}_0$. By Proposition 3.3.2, that is $A_n \cap A_m = \varnothing$ and $B_n \cap B_m = \varnothing$ if $n \neq m$, we have $\lambda(A \cup B) = \frac{1}{2^k}$. Then $\lambda(S \setminus (A \cup B)) = 1$. Consequently, by Theorem 3.4.3, we obtain that

$$\begin{aligned} S \setminus (A \cup B) &= \tau_2(A \cup B) \cup T_1 \\ &= \tau_2(A) \cup \tau_2(B) \cup T_1 \\ &= (T_2 \setminus D_2) \cup \bigcup_{n\in\mathbb{N}} \tau_2(A_n) \cup \bigcup_{n\in\mathbb{N}} \delta_2(B_n) \cup T_1 \\ &= \left[-\frac{1}{2^k}, -\frac{1}{2^{k+1}}\right) \cup \bigcup_{n\in\mathbb{N}} (A_n - 1) \cup \bigcup_{n\in\mathbb{N}} 2^{k+1}\cdot B_n \\ &\quad \cup \left[1, 1+\frac{1}{2^{k+1}}\right) \cup \left[1+\frac{1}{2^k}, 2-\frac{1}{2^k}\right) \cup \left[2-\frac{1}{2^{k+1}}, 2\right) \end{aligned}$$

is a wavelet set.

Similarly, we believe that

(a) $\left[-\frac{1}{2^k}, -\frac{1}{2^{k+1}}\right) \cup \left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right) \cup [m, m+1)$ where $k \geq 2$ and $m \in \mathbb{N}$

(b) $[-2x, -x) \cup [x, 2x) \cup [1, 2)$ where $0 < x < \frac{1}{4}$

are 2-basic sets that contain a wavelet set.

4.2 $S \setminus (A \cup B)$ has measure greater than one

In this section, we consider a 2-basic set S for which $S \setminus (A \cup B)$ has measure greater than one. By Lemma 3.4.8, we have that $\tilde{S} = S \setminus (A \cup B)$ satisfy $\tilde{T_2} = \tilde{D_2}$. We do not have an explicit construction of a wavelet set from a 2-basic set $\tilde{S}$ satisfying $\tilde{T_2} = \tilde{D_2} := G$. However, if we can find partitions of G as in Theorem 3.5.10, then we have a method to construct a wavelet set as in the proof of Lemma 3.5.9. We shall illustrate the method to construct a wavelet set in the next example.

Example 4.2.1. *Let* $S = [-2,-1) \cup \left[-\frac{1}{2},-\frac{1}{3}\right) \cup \left[\frac{1}{4},\frac{1}{2}\right) \cup \left[1,\frac{5}{4}\right) \cup \left[\frac{5}{3},2\right)$. *Then we obtain that*

$$\begin{aligned} D_1 &= \left[-\frac{4}{3},-1\right) \cup \left[\frac{5}{16},\frac{5}{12}\right), \\ D_2 &= \left[-2,-\frac{4}{3}\right) \cup \left[-\frac{1}{2},-\frac{1}{3}\right) \cup \left[\frac{1}{4},\frac{5}{16}\right) \cup \left[\frac{5}{12},\frac{1}{2}\right) \cup \left[1,\frac{5}{4}\right) \cup \left[\frac{5}{3},2\right), \\ T_1 &= \varnothing \text{ and} \\ T_2 &= [-2,-1) \cup \left[-\frac{1}{2},-\frac{1}{3}\right) \cup \left[\frac{1}{4},\frac{1}{2}\right) \cup \left[1,\frac{5}{4}\right) \cup \left[\frac{5}{3},2\right). \end{aligned}$$

Thus we can define

$$\begin{aligned} \tau_2 \ : \ & \left[-2,-\frac{7}{4}\right) \underset{-3}{\overset{+3}{\rightleftarrows}} \left[1,\frac{5}{4}\right); \left[-\frac{7}{4},-\frac{3}{2}\right) \underset{-2}{\overset{+2}{\rightleftarrows}} \left[\frac{1}{4},\frac{1}{2}\right); \\ & \left[-\frac{3}{2},-\frac{4}{3}\right) \underset{-1}{\overset{+1}{\rightleftarrows}} \left[-\frac{1}{2},-\frac{1}{3}\right); \left[-\frac{4}{3},-1\right) \underset{-3}{\overset{+3}{\rightleftarrows}} \left[\frac{5}{3},2\right) \text{ and} \\ \delta_2 \ : \ & \left[-2,-\frac{4}{3}\right) \underset{\times 4}{\overset{\times\frac{1}{4}}{\rightleftarrows}} \left[-\frac{1}{2},-\frac{1}{3}\right); \left[\frac{1}{4},\frac{5}{16}\right) \underset{\times\frac{1}{4}}{\overset{\times 4}{\rightleftarrows}} \left[1,\frac{5}{4}\right); \\ & \left[\frac{5}{12},\frac{1}{2}\right) \underset{\times\frac{1}{4}}{\overset{\times 4}{\rightleftarrows}} \left[\frac{5}{3},2\right). \end{aligned}$$

We see that $\left[-2,-\frac{4}{3}\right) \xrightarrow{\times\frac{1}{4}} \left[-\frac{1}{2},-\frac{1}{3}\right)$ *and* $\left[-\frac{1}{2},-\frac{1}{3}\right) \xrightarrow{-1} \left[-\frac{3}{2},-\frac{4}{3}\right) \subseteq \left[-2,-\frac{4}{3}\right)$. *It is clear that* $B = \varnothing$. *From* $T_2 \setminus D_2 = D_1 = \left[-\frac{4}{3},-1\right) \cup \left[\frac{5}{16},\frac{5}{12}\right)$, *we have*

$A_0 = \left[\frac{5}{3}, 2\right) \cup \left[-\frac{27}{16}, -\frac{19}{12}\right)$. *Thus* $A_1 = \left[-\frac{19}{12}, -\frac{3}{2}\right) \cup \left[-\frac{91}{64}, -\frac{67}{48}\right) \subseteq \left[-2, -\frac{4}{3}\right)$. *Then we obtain that, for each* $n \geq 2$, $A_n = [a_n, b_n) \cup [c_n, d_n)$ *where*

$$\begin{aligned}
a_n &= (\tau_2 \circ \delta_2)^{n-1}(-\frac{19}{12}) = (\frac{1}{4})^{n-1}(-\frac{19}{12}) - \frac{4}{3}(1 - (\frac{1}{4})^{n-1}) \\
b_n &= (\tau_2 \circ \delta_2)^{n-1}(-\frac{3}{2}) = (\frac{1}{4})^{n-1}(-\frac{3}{2}) - \frac{4}{3}(1 - (\frac{1}{4})^{n-1}) \\
c_n &= (\tau_2 \circ \delta_2)^{n-1}(-\frac{91}{64}) = (\frac{1}{4})^{n-1}(-\frac{91}{64}) - \frac{4}{3}(1 - (\frac{1}{4})^{n-1}) \\
d_n &= (\tau_2 \circ \delta_2)^{n-1}(-\frac{67}{48}) = (\frac{1}{4})^{n-1}(-\frac{67}{48}) - \frac{4}{3}(1 - (\frac{1}{4})^{n-1}).
\end{aligned}$$

It is easy to check that $d_n = a_{n+1}$ *for all* $n \geq 2$. *Hence*

$$\begin{aligned}
\bigcup_{n\geq 2} A_n &= \bigcup_{n\geq 2} [a_n, b_n) \cup [c_n, d_n) \\
&= [a_2, b_2) \cup \bigcup_{n\geq 2} [c_n, d_n) \cup [a_{n+1}, b_{n+1}) \\
&= \left[-\frac{67}{48}, -\frac{11}{8}\right) \cup \bigcup_{n\geq 2} [c_n, b_{n+1}).
\end{aligned}$$

Then $A = \left[-\frac{27}{16}, -\frac{3}{2}\right) \cup \left[-\frac{91}{64}, -\frac{11}{8}\right) \cup \bigcup_{n\geq 2} [c_n, b_{n+1}) \cup \left[\frac{5}{3}, 2\right)$ *and* $A \subseteq D_2$. *Moreover,* $\delta_2(A) = \left[-\frac{27}{64}, -\frac{3}{8}\right) \cup \left[-\frac{91}{256}, -\frac{11}{32}\right) \cup \bigcup_{n\geq 2} \frac{1}{4} [c_n, b_{n+1}) \cup \left[\frac{5}{12}, \frac{1}{2}\right)$. *Set* $\tilde{S} := S \setminus (A \cup B)$. *Thus*

$$\begin{aligned}
\tilde{S} &= \left[-2, -\frac{27}{16}\right) \cup \left[-\frac{3}{2}, -\frac{91}{64}\right) \cup \left[-\frac{11}{8}, -\frac{347}{256}\right) \cup \bigcup_{n\geq 2} [b_{n+1} c_{n+1},) \\
&\quad \cup \left[-\frac{4}{3}, -1\right) \cup \left[-\frac{1}{2}, -\frac{1}{3}\right) \cup \left[\frac{1}{4}, \frac{1}{2}\right) \cup \left[1, \frac{5}{4}\right).
\end{aligned}$$

Then, by Lemma 3.4.8, we have

$$\begin{aligned}
\tilde{T}_2 = \tilde{D}_2 &= D_2 \setminus (A \cup B \cup \delta_2(A \cup B)) \\
&= \left[-2, -\frac{27}{16}\right) \cup \left[-\frac{3}{2}, -\frac{91}{64}\right) \cup \left[-\frac{11}{8}, -\frac{347}{256}\right) \cup \bigcup_{n\geq 2} [b_{n+1}, c_{n+1})
\end{aligned}$$

$$\cup\left[-\frac{1}{2},-\frac{27}{64}\right)\cup\left[-\frac{3}{8},-\frac{91}{256}\right)\cup\left[-\frac{11}{32},-\frac{347}{1024}\right)\cup\bigcup_{n\geq 2}\frac{1}{4}\left[b_{n+1},c_{n+1}\right)$$

$$\cup\left[\frac{1}{4},\frac{5}{16}\right)\cup\left[1,\frac{5}{4}\right) \text{ and}$$

$$\begin{aligned}\tilde{T}_1=\tilde{D}_1 &= D_1\cup\delta_2(A\cup B)\\ &= \left[-\frac{4}{3},-1\right)\cup\left[-\frac{27}{64},-\frac{3}{8}\right)\cup\left[-\frac{91}{256},-\frac{11}{32}\right)\cup\bigcup_{n\geq 2}\frac{1}{4}\left[c_n,b_{n+1}\right)\\ &\quad\cup\left[\frac{5}{16},\frac{5}{12}\right)\cup\left[\frac{5}{12},\frac{1}{2}\right).\end{aligned}$$

Set $G:=\tilde{T}_2=\tilde{D}_2$ *and*

$$\begin{aligned}\tilde{U}_1 &:= \{x\in G:\tau_2(x)-x<0\}\\ &= \cup\left[-\frac{1}{2},-\frac{27}{64}\right)\cup\left[-\frac{3}{8},-\frac{91}{256}\right)\cup\left[-\frac{11}{32},-\frac{347}{1024}\right)\cup\bigcup_{n\geq 2}\frac{1}{4}\left[b_{n+1}c_{n+1},\right)\\ &\quad\cup\left[\frac{1}{4},\frac{5}{16}\right)\cup\left[1,\frac{5}{4}\right),\\ \tilde{U}_2 &:= \{x\in G:\tau_2(x)-x>0\}\\ &= \left[-2,-\frac{27}{16}\right)\cup\left[-\frac{3}{2},-\frac{91}{64}\right)\cup\left[-\frac{11}{8},-\frac{347}{256}\right)\cup\bigcup_{n\geq 2}\left[b_{n+1}c_{n+1},\right),\end{aligned}$$

$$\begin{aligned}\tilde{V}_1 &:= \{x\in G:0<\frac{\delta_2(x)}{x}<1\}\\ &= \left[-2,-\frac{27}{16}\right)\cup\left[-\frac{3}{2},-\frac{91}{64}\right)\cup\left[-\frac{11}{8},-\frac{347}{256}\right)\cup\bigcup_{n\geq 2}\left[b_{n+1}c_{n+1},\right)\cup\left[1,\frac{5}{4}\right),\\ \tilde{V}_2 &:= \{x\in G:\frac{\delta_2(x)}{x}>1\}\\ &= \cup\left[-\frac{1}{2},-\frac{27}{64}\right)\cup\left[-\frac{3}{8},-\frac{91}{128}\right)\cup\left[-\frac{11}{64},-\frac{347}{1024}\right)\cup\bigcup_{n\geq 2}\frac{1}{4}\left[b_{n+1}c_{n+1},\right)\\ &\quad\cup\left[\frac{1}{4},\frac{5}{16}\right).\end{aligned}$$

Then $\tilde{U}_2\cap\tilde{V}_2=\varnothing$ *and* $\tilde{U}_1\cap\tilde{V}_1=\left[1,\frac{5}{4}\right)$. *By Theorem 3.5.10,* $\tilde{S}$ *contains a wavelet set. Next, we will find a wavelet subset of* $\tilde{S}$ *by using the method in Lemma 3.5.9.*

For each $n \in \mathbb{Z}$, let

$$Y_n = (\tau_2 \circ \delta_2)^n(\tilde{U}_1 \cap \tilde{V}_1)$$

where $(\tau_2 \circ \delta_2)^0 = id_G$ *and* $(\tau_2 \circ \delta_2)^{-n} = (\delta_2 \circ \tau_2)^n$ *on* G*, for* $n \in \mathbb{N}$*. Then* $Y_1 = \left[-\frac{7}{4}, -\frac{27}{16}\right) \subseteq \left[-2, -\frac{4}{3}\right)$ *and* $Y_{-1} = \left[-\frac{1}{2}, -\frac{7}{16}\right) \subseteq \left[-\frac{1}{2}, -\frac{1}{3}\right)$*. From the fact that* $\left[-2, -\frac{4}{3}\right) \xrightarrow{\times \frac{1}{4}} \left[-\frac{1}{2}, -\frac{1}{3}\right)$ *and* $\left[-\frac{1}{2}, -\frac{1}{3}\right) \xrightarrow{-1} \left[-\frac{3}{2}, -\frac{4}{3}\right) \subseteq \left[-2, -\frac{4}{3}\right)$*, we get that, for each* $n \geq 2$*,* $Y_n = [s_n, t_n)$ *and* $Y_{-n} = [x_n, z_n)$ *where*

$$\begin{aligned}
s_n &= (\tau_2 \circ \delta_2)^{n-1}(-\frac{7}{4}) = (\frac{1}{4})^{n-1}(-\frac{7}{4}) - \frac{4}{3}(1 - (\frac{1}{4})^{n-1}) \\
t_n &= (\tau_2 \circ \delta_2)^{n-1}(-\frac{27}{16}) = (\frac{1}{4})^{n-1}(-\frac{27}{16}) - \frac{4}{3}(1 - (\frac{1}{4})^{n-1}) \\
x_n &= (\delta_2 \circ \tau_2)^{n-1}(-\frac{1}{2}) = (\frac{1}{4})^{n-1}(-\frac{1}{2}) - \frac{1}{3}(1 - (\frac{1}{4})^{n-1}) \\
z_n &= (\delta_2 \circ \tau_2)^{n-1}(-\frac{7}{16}) = (\frac{1}{4})^{n-1}(-\frac{7}{16}) - \frac{1}{3}(1 - (\frac{1}{4})^{n-1}).
\end{aligned}$$

Clearly, $Y_n \subseteq \left[-2, -\frac{4}{3}\right)$ *and* $Y_{-n} \subseteq \left[-\frac{1}{2}, -\frac{1}{3}\right)$ *for all* $n \in \mathbb{N}$*. Moreover,* $Y_n \cap Y_m = \varnothing$ *if* $n \neq m$*. It is clear that* $b_n = \delta_2(x_{n+1}) = 4x_{n+1}$*,* $c_n = t_{n+1}$ *and* $s_n = \delta_2(z_n) = 4z_n$ *for all* $n \geq 2$*. Then*

$$\begin{aligned}
\bigcup_{n \geq 2} [b_{n+1}, c_{n+1}) &= \bigcup_{n \geq 2} [4x_{n+2}, 4z_{n+2}) \cup [s_{n+2}, t_{n+2}) \\
&= \bigcup_{n \geq 2} \delta_2([x_{n+2}, z_{n+2})) \cup \bigcup_{n \geq 2} [s_{n+2}, t_{n+2}) \\
&= \bigcup_{n \geq 2} Y_{-(n+2)} \cup \bigcup_{n \geq 2} Y_{n+2},
\end{aligned}$$

and so $\bigcup_{n \geq 2} \frac{1}{4} [b_{n+1}, c_{n+1}) = \delta_2(\bigcup_{n \geq 2} [b_{n+1}, c_{n+1})) = \delta_2(\bigcup_{n \geq 2} Y_{-(n+2)} \cup \bigcup_{n \geq 2} Y_{n+2})$*. Furthermore,*

$$\begin{aligned}
\bigcup_{-3 \leq n \leq 3} Y_n \cup \bigcup_{-3 \leq n \leq 3} \delta_2(Y_n) &= \left[-2, -\frac{27}{16}\right) \cup \left[-\frac{3}{2}, -\frac{91}{64}\right) \cup \left[-\frac{11}{8}, -\frac{347}{256}\right) \\
&\quad \cup \left[-\frac{1}{2}, -\frac{27}{64}\right) \cup \left[-\frac{3}{8}, -\frac{91}{128}\right) \cup \left[-\frac{11}{64}, -\frac{347}{1024}\right)
\end{aligned}$$

$$\cup \left[\frac{1}{4}, \frac{5}{16}\right) \cup \left[1, \frac{5}{4}\right).$$

Now, set $Y = \bigcup_{n\in\mathbb{Z}} Y_n$. *Then* $Y \cup \delta_2(Y) = G$ *and* $\delta_2(Y) = \tau_2(Y)$. *From the fact the* $\tilde{U}_2 \cap \tilde{V}_2 = \varnothing$,*see in Lemma 3.5.9, it implies that* $Y \cap \delta_2(Y) = \varnothing$. *Then* $W_1 = \tilde{D}_1 \cup Y$ *and* $W_2 = \tilde{D}_1 \cup \delta_2(Y)$ *are wavelet subsets of* $\tilde{S}$.

Next, we will give another 2-basic set containing a wavelet set. We first thought that the 2-basic set in the next example satisfies $A \cup B \subseteq T_2 \cap D_2$ but does not contain a wavelet set. Indeed, it contains a wavelet set.

Example 4.2.2. *Let* $S = [-\frac{1}{4}, -\frac{1}{8}) \cup [-\frac{1}{16}, -\frac{1}{32}) \cup [\frac{1}{16}, \frac{1}{8}) \cup [1, 2)$. *Then* S *is a 2-basic set. We obtain that*

$$D_1 = \varnothing$$

$$D_2 = S$$

$$T_1 = \left[1, \frac{17}{16}\right) \cup \left[\frac{9}{8}, \frac{7}{4}\right) \cup \left[\frac{15}{8}, \frac{31}{16}\right) \cup \left[\frac{63}{32}, 2\right)$$

$$T_2 = \left[-\frac{1}{4}, -\frac{1}{8}\right) \cup \left[-\frac{1}{16}, -\frac{1}{32}\right) \cup \left[\frac{1}{16}, \frac{1}{8}\right) \cup \left[\frac{17}{16}, \frac{9}{8}\right) \cup \left[\frac{7}{4}, \frac{15}{8}\right) \cup \left[\frac{31}{16}, \frac{63}{32}\right).$$

Thus we can define

$$\tau_2 : \left[-\frac{1}{4}, -\frac{1}{8}\right) \underset{-2}{\overset{+2}{\rightleftarrows}} \left[\frac{7}{4}, \frac{15}{8}\right); \left[-\frac{1}{16}, -\frac{1}{32}\right) \underset{-2}{\overset{+2}{\rightleftarrows}} \left[\frac{31}{16}, \frac{63}{32}\right);$$

$$\left[\frac{1}{16}, \frac{1}{8}\right) \underset{-1}{\overset{+1}{\rightleftarrows}} \left[\frac{17}{16}, \frac{9}{8}\right), \text{ and}$$

$$\delta_2 : \left[-\frac{1}{4}, -\frac{1}{8}\right) \underset{\times 4}{\overset{\times \frac{1}{4}}{\rightleftarrows}} \left[-\frac{1}{16}, -\frac{1}{32}\right); \left[\frac{1}{16}, \frac{1}{8}\right) \underset{\times \frac{1}{16}}{\overset{\times 16}{\rightleftarrows}} [1, 2).$$

We see that $[\frac{1}{16}, \frac{1}{8}) \xrightarrow{+1} [\frac{17}{16}, \frac{9}{8}) \subseteq [1, 2)$ *and* $[1, 2) \xrightarrow{\times \frac{1}{16}} [\frac{1}{16}, \frac{1}{8})$. *It is clear that* $A = \varnothing$. *From* $D_2 \setminus T_2 = T_1 \subseteq [1, 2)$, *we have* $B_0 \subseteq [\frac{1}{16}, \frac{1}{8})$ *and* $B_n \subseteq [\frac{1}{16}, \frac{1}{8})$ *for all* $n \in \mathbb{N}$. *Then* $B \subseteq T_2$, *and so* $A \cup B \subseteq T_2 \cap D_2$. *By the definitions of* δ_2 *and* τ_2, *we have*

$$B = \bigcup_{n \in \mathbb{N}_0} [x_n, y_n) \cup [z_n, p_n) \cup [q_n, r_n) \cup [s_n, t_n)$$

where

$$\begin{aligned}
x_n &= (\delta_2 \circ \tau_2)^n \circ \delta_2(1) &&= \frac{1}{15}(1 - \frac{1}{16^{n+1}}) \\
y_n &= (\delta_2 \circ \tau_2)^n \circ \delta_2(\frac{17}{16}) &&= \frac{1}{15}(1 - \frac{1}{16} \cdot \frac{1}{16^{n+1}}) \\
z_n &= (\delta_2 \circ \tau_2)^n \circ \delta_2(\frac{9}{8}) &&= \frac{1}{15}(1 + \frac{7}{8} \cdot \frac{1}{16^{n+1}}) \\
p_n &= (\delta_2 \circ \tau_2)^n \circ \delta_2(\frac{7}{4}) &&= \frac{1}{15}(1 + \frac{41}{4} \cdot \frac{1}{16^{n+1}}) \\
q_n &= (\delta_2 \circ \tau_2)^n \circ \delta_2(\frac{15}{8}) &&= \frac{1}{15}(1 + \frac{97}{8} \cdot \frac{1}{16^{n+1}}) \\
r_n &= (\delta_2 \circ \tau_2)^n \circ \delta_2(\frac{31}{16}) &&= \frac{1}{15}(1 + \frac{209}{16} \cdot \frac{1}{16^{n+1}}) \\
s_n &= (\delta_2 \circ \tau_2)^n \circ \delta_2(\frac{63}{32}) &&= \frac{1}{15}(1 + \frac{433}{32} \cdot \frac{1}{16^{n+1}}) \\
t_n &= (\delta_2 \circ \tau_2)^n \circ \delta_2(2) &&= \frac{1}{15}(1 + 14 \cdot \frac{1}{16^{n+1}})
\end{aligned}$$

for all $n \in \mathbb{N}_0$*. It is easy to check that* $z_n \le p_n \le q_n \le r_n \le s_n \le t_n$, $x_{n+1} = y_n$ *and* $t_{n+1} = z_n$ *for all* $n \in \mathbb{N}_0$*. It implies that* $\bigcup_{n=0}^{\infty}[x_n, y_n) = \left[\frac{1}{16}, \frac{1}{15}\right)$*. Since*

$$\bigcup_{n=0}^{\infty} [s_n, t_n) = \bigcup_{n=1}^{\infty} [s_n, t_n) \cup \left[\frac{63}{512}, \frac{1}{8}\right) = \bigcup_{n=0}^{\infty} [s_{n+1}, t_{n+1}) \cup \left[\frac{63}{512}, \frac{1}{8}\right),$$

we obtain that

$$\bigcup_{n=0}^{\infty} [z_n, p_n) \cup \bigcup_{n=0}^{\infty} [s_n, t_n) = \bigcup_{n=0}^{\infty} [s_{n+1}, p_n) \cup \left[\frac{63}{512}, \frac{1}{8}\right).$$

Then

$$B = \left[\frac{1}{16}, \frac{1}{15}\right) \cup \bigcup_{n=0}^{\infty} [s_{n+1}, p_n) \cup \bigcup_{n=0}^{\infty} [q_n, r_n) \cup \left[\frac{63}{512}, \frac{1}{8}\right),$$

and so

$$\delta_2(B) = \left[1, \frac{16}{15}\right) \cup \bigcup_{n=0}^{\infty} 16 \cdot [s_{n+1}, p_n) \cup \bigcup_{n=0}^{\infty} 16 \cdot [q_n, r_n) \cup \left[\frac{63}{32}, 2\right).$$

Hence

$$\left[\frac{1}{16},\frac{1}{8}\right)\setminus B=\bigcup_{n=0}^{\infty}[r_{n+1},s_{n+1})\cup\bigcup_{n=0}^{\infty}[p_n,q_n)\cup\left[\frac{31}{256},\frac{63}{512}\right),$$

which implies that

$$[1,2)\setminus\delta_2(B)=\bigcup_{n=0}^{\infty}16\cdot[r_{n+1},s_{n+1})\cup\bigcup_{n=0}^{\infty}16\cdot[p_n,q_n)\cup\left[\frac{31}{16},\frac{63}{32}\right).$$

Set $\tilde{S}:=S\setminus(A\cup B)$. *By Lemma 3.4.8, we have*

$$\begin{aligned}\tilde{T}_1=\tilde{D}_1 &= \left[1,\frac{16}{15}\right)\cup\bigcup_{n=0}^{\infty}16\cdot[s_{n+1},p_n)\cup\bigcup_{n=0}^{\infty}16\cdot[q_n,r_n)\cup\left[\frac{63}{32},2\right) \text{ and}\\ \tilde{T}_2=\tilde{D}_2 &= \left[-\frac{1}{4},-\frac{1}{8}\right)\cup\left[-\frac{1}{16},-\frac{1}{32}\right)\\ &\quad\cup\bigcup_{n=0}^{\infty}[r_{n+1},s_{n+1})\cup\bigcup_{n=0}^{\infty}[p_n,q_n)\cup\left[\frac{31}{256},\frac{63}{512}\right)\\ &\quad\cup\bigcup_{n=0}^{\infty}16\cdot[r_{n+1},s_{n+1})\cup\bigcup_{n=0}^{\infty}16\cdot[p_n,q_n)\cup\left[\frac{31}{16},\frac{63}{32}\right).\end{aligned}$$

Next, we shall find a subset Y *of* $\tilde{D}_2$ *satisfying* $Y\cap\delta_2(Y)=\varnothing$ *and* $\delta_2(Y)=\tau_2(Y)$. *Choose* $Y=\left[-\frac{1}{4},-\frac{1}{8}\right)\cup\bigcup_{n=0}^{\infty}[p_n,q_n)\cup\bigcup_{n=0}^{\infty}16\cdot[r_{n+1},s_{n+1})\cup\left[\frac{31}{16},\frac{63}{32}\right)$. *Then* $\delta_2(Y)=\left[-\frac{1}{16},-\frac{1}{32}\right)\cup\bigcup_{n=0}^{\infty}[r_{n+1},s_{n+1})\cup\left[\frac{31}{256},\frac{63}{512}\right)\cup\bigcup_{n=0}^{\infty}16\cdot[p_n,q_n)=\tilde{D}_2\setminus Y$, *and so* $Y\cap\delta_2(Y)=\varnothing$. *Moreover,* $Y\cup\delta_2(Y)=\tilde{D}_2$. *Since, for each* $n\in\mathbb{N}_0$,

$$p_n+1=\frac{1}{15}\left(1+\frac{41}{4}\cdot\frac{1}{16^{n+1}}\right)+1=\frac{16}{15}\left(1+\frac{41}{4}\cdot\frac{1}{16^{n+2}}\right)=16\cdot p_{n+1}$$

and

$$q_n+1=\frac{1}{15}\left(1+\frac{97}{8}\cdot\frac{1}{16^{n+1}}\right)+1=\frac{16}{15}\left(1+\frac{97}{8}\cdot\frac{1}{16^{n+2}}\right)=16\cdot q_{n+1},$$

we obtain that $\tau_2\left(\bigcup_{n=0}^{\infty}[p_n,q_n)\right)=\bigcup_{n=0}^{\infty}[p_n,q_n)+1=\bigcup_{n=1}^{\infty}16\cdot[p_n,q_n)$. *From*

$$r_n+1=\frac{1}{15}\left(1+\frac{209}{16}\cdot\frac{1}{16^{n+1}}\right)+1=\frac{16}{15}\left(1+\frac{209}{16}\cdot\frac{1}{16^{n+2}}\right)=16\cdot r_{n+1}$$

and

$$s_n+1=\frac{1}{15}\left(1+\frac{433}{32}\cdot\frac{1}{16^{n+1}}\right)+1=\frac{16}{15}\left(1+\frac{433}{32}\cdot\frac{1}{16^{n+2}}\right)=16\cdot s_{n+1}$$

for each $n \in \mathbb{N}_0$, *it implies that* $\tau_2(\bigcup_{n=0}^{\infty}[r_{n+1}, s_{n+1})) = \bigcup_{n=1}^{\infty} 16 \cdot [r_{n+1}, s_{n+1})$, *that is,* $\tau_2(\bigcup_{n=1}^{\infty} 16 \cdot [r_{n+1}, s_{n+1})) = \bigcup_{n=0}^{\infty}[r_{n+1}, s_{n+1})$. *It is easy to check that* $\tau_2([-\frac{1}{4}, -\frac{1}{8})) = 16 \cdot [p_0, q_0)$, $\tau_2(16 \cdot [r_1, s_1)) = [\frac{31}{256}, \frac{63}{512})$, $\tau_2([\frac{31}{16}, \frac{63}{32})) = [-\frac{1}{16}, -\frac{1}{32})$. *Therefore,* $\tau_2(Y) = \delta_2(Y) = \tilde{D}_2 \setminus Y$, *which implies that* $\tilde{S}$ *contains a wavelet set. Indeed,* $W_1 = \tilde{D}_1 \cup Y$ *and* $W_2 = \tilde{D}_1 \cup \delta_2(Y)$ *are wavelet subsets of* $\tilde{S}$ *as a consequence of Theorem 3.1.5.*

CHAPTER V

CONCLUSION

This thesis has been concerned with the geometry of 2-basic sets containing a wavelet set. First, we presented a few properties of a supersets of wavelet sets. Next, we introduced the concept of a 2-basic set and provided an easy and natural characterization of a 2-basic set containing a wavelet set. That is, a 2-basic set S contains a wavelet set if and only if there exists a measurable subset R of S such that $R \underset{\text{a.e.}}{\subseteq} T_2 \cap D_2$, $T_2 \setminus R \underset{\text{a.e.}}{\overset{\tau}{\sim}} R$, and $D_2 \setminus R \underset{\text{a.e.}}{\overset{\delta}{\sim}} R$. In addition, we provided a procedure to construct two sequences of subsets of a 2-basic set S whose union, denoted by $A \cup B$, never intersects with wavelet subsets of S. Hence, $A \cup B$ is called the "bad part" of a 2-basic set. Indeed, we showed that $A \cup B \underset{\text{a.e.}}{\subsetneq} T_2 \cap D_2$ is a necessary condition for a 2-basic set to contain a wavelet set. Conversely, we obtained that $A \cup B \underset{\text{a.e.}}{\subsetneq} T_2 \cap D_2$ and $\lambda(S \setminus (A \cup B)) = 1$ is a sufficient condition for a 2-basic set to contain a wavelet set. In this case, $S \setminus (A \cup B)$ is the only wavelet subset of S. Furthermore, we proved that $\tilde{S} = S \setminus (A \cup B)$ is 2-basic with $\tilde{T}_2 = \tilde{D}_2$ and $\tilde{T}_1 = \tilde{D}_1$ if $A \cup B \underset{\text{a.e.}}{\subsetneq} T_2 \cap D_2$. Therefore, we provided a characterization of a 2-basic set S, with $T_2 = D_2$, to contain a wavelet set. The property used in the characterization is called $\tau\delta$-separable which is believed to be so weak that every measurable set is $\tau\delta$-separable. Moreover, we gave a sufficient condition for a 2-basic set S, with $G := T_2 = D_2$, to contain a wavelet set. That is, if there exists a measurable subset V of G such that $V \cup \delta_2(V) \underset{\text{a.e.}}{=} G$, $V \cap \delta_2(V) \underset{\text{a.e.}}{=} \varnothing$, and $V \cap \tau_2(V) \underset{\text{a.e.}}{=} \varnothing$, then S contains a wavelet set. Finally, we presented a few examples of construction of wavelet sets from some 2-basic sets.

Although, we obtained some necessary conditions and sufficient conditions for a 2-basic set to contain a wavelet set, we still do not know whether every 2-basic set with $T_2 = D_2$ must contain a wavelet set. This is equivalent to the problem whether there exists an example of a 2-basic set for which $A \underset{\text{a.e.}}{\subseteq} D_2$ and $B \underset{\text{a.e.}}{\subseteq} T_2$ but there is no wavelet subset. Another interesting problem is whether one can find another easy sufficient condition for $A \underset{\text{a.e.}}{\subseteq} D_2$ and $B \underset{\text{a.e.}}{\subseteq} T_2$. We enumerate here a few open problems worth investigating further.

(i) Is there a sufficient condition for $A \underset{\text{a.e.}}{\subseteq} D_2$ and $B \underset{\text{a.e.}}{\subseteq} T_2$ which is easy to verify?

(ii) Is $A \cup B \underset{\text{a.e.}}{\subseteq} T_2 \cap D_2$ a sufficient condition for a 2-basic set S to contain a wavelet set?

(iii) Does a 2-basic set S satisfying the property that $T_2 = D_2$ always contain a wavelet set?

(iv) Is a 2-basic set S with $T_2 = D_2$ $\tau\delta$-separable?

(v) Can these results be extended to n-basic with $n > 2$?

(vi) Do these results somehow have connections with the open problem posed by Larson?

The problems (ii), (iii) and (iv) are equivalent.

REFERENCES

[1] Benedetto, J.J., Leon, M.: The construction of single wavelets in D-dimensions, *J. Geom. Anal.* **11**(1), 1-15 (2001).

[2] Benedetto, J.J., Sumetkijakan, S.: A fractal set constructed from a class of wavelet sets, *Contemp. Math.* **313**, 19-35 (2002).

[3] Benedetto, J.J., Sumetkijakan, S.: Tight frames and geometric properties of wavelet sets, *Adv. Comp. Math.* **24**(1-4), 35-56 (2006).

[4] Cabrelli, C., Molter, U.: Geometric methods in wavelet theory, *Revista de la Unión Matemática Argentina* **46**(2), 121-132 (2005).

[5] Calogero, A.: A characterization of wavelets on general lattices, *J. Geom. Anal.* **10**(4), 597-622 (2000).

[6] Dai, X., Diao, Y., Gu, Q.: Frame wavelets with frame set support in the frequency domain, *Ill. J. Math.* **48**(2), 539-558 (2004).

[7] Dai, X., Larson, D.R.: Wandering vectors for unitary systems and orthogonal wavelets, *Mem. Am. Math. Soc.* **134**(640), (1998).

[8] Dai, X., Larson, D.R., Speegle, D.M.: Wavelet sets in $\mathbb{R}^n$, *J. Fourier Anal. Appl.* **3**(4), 451-456 (1997).

[9] Dai, X., Larson, D.R., Speegle, D.M.: Wavelet sets in $\mathbb{R}^n$ II, *Contemp. Math.* **216**, 15-40 (1998).

[10] Folland, G. B.: *Real Analysis: Modern Techniques and Their Applications*, 2^{nd} ed., John Wiley and Sons, Inc., New York, 1999.

[11] Gröchenig, K., Madych, W.R.: Multiresolution analysis, Haar bases, and self-similar tilings of $\mathbb{R}^n$, *IEEE Trans. Inf. Theory* **38**(2), 556-568 (1992).

[12] Hernández, E., Weiss, G.: *A First Course on Wavelets*, CRC Press, Boca Raton, 1996.

[13] Hernández, E., Wang, X., Weiss, G.: Smoothing minimally supported frequency wavelets. Part I., *J. Fourier Anal. Appl.* **2**(4), 329-340 (1996).

[14] Ionascu, E.J.: A new construction of wavelet sets, *Real Anal. Exch.* **28**(2), 593-609 (2003).

[15] Ionascu, E.J., Larson, D.R., Pearcy, C.M.: On wavelet sets, *J. Fourier Anal. Appl.* **4**(6), 711-721 (1998).

[16] Ionascu, E.J., Wang, Y.: Simultaneous translational and multiplicative tiling and wavelet sets in $\mathbb{R}^2$, *Indiana Univ. Math. J.* **55**(6), 1935-1949 (2006).

[17] Larson, D.R.: Unitary systems and wavelet sets, In *Wavelet Analysis and Applications*, Appl. Numer. Harmon. Anal., Birkhäuser, Basel, 2007.

[18] Lemarié, P.G., Meyer, Y.: Ondelettes et bases hilbertiennes (Wavelets and Hilbert bases), *Rev. Mat. Iberoam.* **2**(12), 1-18 (1987).

[19] Ólafsson, G., Speegle, D.: Wavelets, wavelet sets, and linear actions on $\mathbb{R}^n$, *Contemp. Math.* **345**, 253-281 (2004).

[20] Rzeszotnik, ́Z., Speegle, D.: On wavelets interpolated from a pair of wavelet sets, *Proc. Am. Math. Soc.* **130**(10), 2921-2930 (2002).

[21] Soardi, P.M, Weiland, D.: Single wavelets in n-dimensions, *J. Fourier Anal. Appl.* **4**(3), 299-315 (1998).

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