Chapter 2
Theoretical Background and Transport Equation

Scientists do not exactly understand the acceleration of particles near the Sun, or their "injection function," the number of particles released into the interplanetary medium as a function of time. One way to study this is to use an appropriate transport equation for solar energetic particles in the interplanetary medium. This helps us to understand observational information relevant to the acceleration of particles near the Sun. The main objective of this work is to fit data in order to determine the injection of particles from the Sun as a function of time and energy. Therefore, it is necessary to consider the theoretical background relevant to the transport equation as described in this Chapter.

2.1 Irregular Magnetic Field from the Sun

There are 3 layers in the Sun's atmosphere: the photosphere, chromosphere, and corona. The outermost portion of the Sun, the corona, has a temperature of about 10^6 K, so its pressure is higher than that of the surrounding interplanetary medium. The pressure in the interplanetary medium is close to zero because the density of plasma is very low. Because of the pressure difference, plasma flows (non-uniformly) from the corona into the interplanetary medium with a speed of ≈ 400 km/s. This flow is called the "solar wind."

The energetic particles released from the Sun move at relativistic speeds in the interplanetary medium, which comprises the solar wind plasma continuously

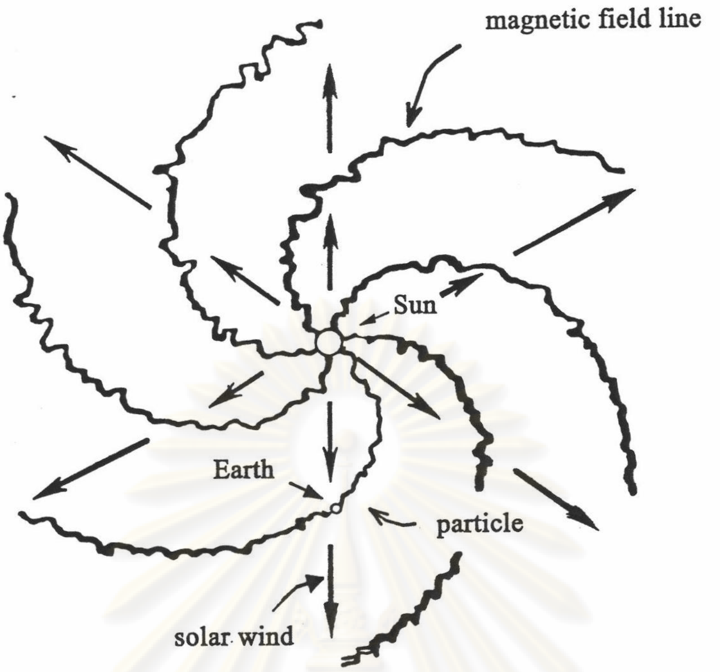

Figure 2.1: Solar wind and the interplanetary magnetic field.

moving at a slower speed. The solar wind drags magnetic field lines out from the Sun while the Sun is rotating around its axis. The magnetic field lines are dragged along with the plasma because the plasma is nearly conducting (Roelof 1969). The magnetic field lines are therefore curved as in Figure 2.1. Since the solar wind is highly turbulent, the interplanetary magnetic field which is dragged out is very irregular, too.

To understand the dragging of magnetic field lines, consider the time derivative of the magnetic flux, $d\Phi/dt$, through any surface, s, bounded by a closed contour, L, and moving with the plasma. After a small time, Δt, L propagates with a velocity, $\vec{u}$, to a new contour, L', as in Figure 2.2. We define

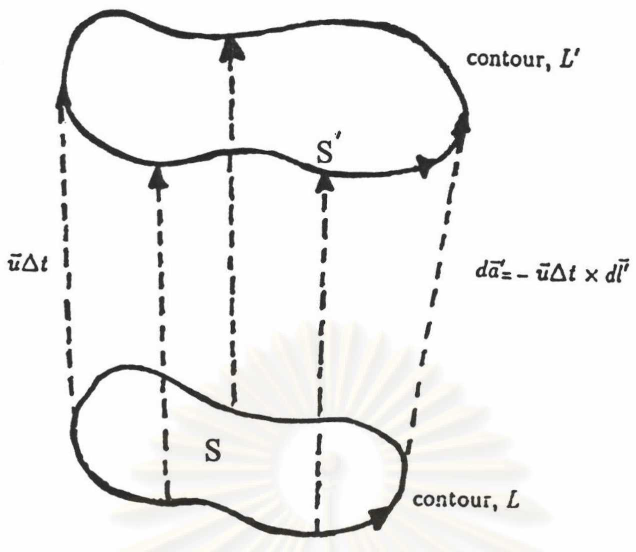

Figure 2.2: The propagation of magnetic flux through a closed contour, L.

s' as the surface bounded by L', giving

$$\frac{d\Phi}{dt} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(\int_{s'} \vec{B}(t + \Delta t) \cdot d\vec{a}' - \int_{s} \vec{B}(t) \cdot d\vec{a} \right). \tag{2.1}$$

From Green's theorem, Stokes's theorem, and Ampere's Law,

$$\frac{d\Phi}{dt} = - \lim_{\Delta t \to 0} c \oint_{L'} \left(\vec{E} + \frac{1}{c} \vec{u} \times \vec{B} \right) \cdot d\vec{l}, \tag{2.2}$$

where $\vec{u}$ is the plasma velocity. Now $\vec{E}_r = \gamma \left(\vec{E} + \vec{u} \times \vec{B}/c \right)$, where $\vec{E}_r$ is the electric field in the rest frame of the plasma and $\gamma = (1 - u^2/c^2)^{-1/2}$. If the plasma is completely conducting, $\vec{E}_r \equiv 0$ and $d\Phi/dt \equiv 0$. This shows that magnetic field lines are dragged along with the plasma (Roelof 1969).

Assuming that the plasma flows radially out from the Sun, the magnetic field lines lie along the Archimedean spirals $\phi = \phi_0 - \Omega r \sin\theta/u$, where θ is measured from the North Ecliptic Pole. We have

$$\Delta r = u\Delta t, \;\; \Delta\phi = -\Omega\Delta t, \;\; \frac{dr}{d\phi} = -\frac{u}{\Omega}, \tag{2.3}$$

where Ω is the angular rate of rotation of the Sun. We define $\phi = 0$ when $r = 0$, so $d\phi/dt = 0$, and

$$\begin{aligned} \vec{B}(r,\theta) &= a(r,\theta)\left(\vec{e}_r + \frac{r\sin\theta d\phi}{dr}\vec{e}_\phi\right), \\ &= a(r,\theta)\left(\vec{e}_r - \frac{\Omega r\sin\theta}{u}\vec{e}_\phi\right). \end{aligned} \tag{2.4}$$

We must have $\vec{\nabla}\cdot\vec{B} = 0$, so

$$\begin{aligned} 0 &= \frac{1}{r^2}\frac{\partial}{\partial r}r^2 B_r + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta B_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}B_\phi, \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}r^2 a, \\ a &= \frac{f(\theta)}{r^2}. \end{aligned} \tag{2.5}$$

If $\vec{B}_0(\theta) \equiv \vec{B}(r_0,\theta)$ then

$$\begin{aligned} f(\theta) &= r_0^2 B_{0r}(\theta) = -\frac{u r_0}{\Omega\sin\theta}B_{0\phi}(\theta) \\ \text{and } B_{0\phi} &= -\frac{\Omega r_0\sin\theta}{u}B_{0r}. \end{aligned} \tag{2.6}$$

Finally, the magnetic field, lying along the Archimedean spiral at a distance r from the Sun, is

$$B(r,\theta) = B_0(\theta)\left(\frac{r_0}{r}\right)^2\left(\vec{e}_r - \frac{\Omega r\sin\theta}{u}\vec{e}_\phi\right), \tag{2.7}$$

where r_0 is the corotation radius of the Sun (~ 0.01 AU), and $\vec{e}_r$ and $\vec{e}_\phi$ are unit vectors in the radial and azimuthal directions, respectively (Roelof 1969).

The orbit of a charged particle in a uniform magnetic field is a helix around the magnetic field line with a polar angle, θ, which is the angle between the velocity of the particle and the magnetic field line. We define

$$\mu \equiv \cos\theta, \tag{2.8}$$

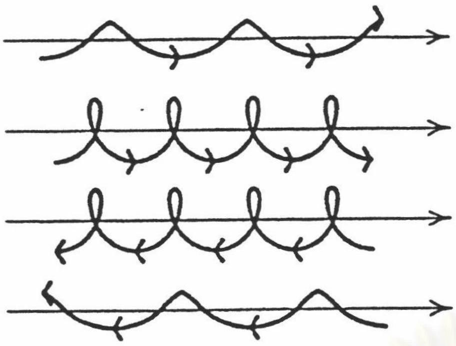

Figure 2.3: Particle orbits in a uniform magnetic field for various pitch angles.

where μ is a constant for a uniform magnetic field, and can randomly vary in the irregular interplanetary magnetic field. The sign of μ will show the direction of motion: when $\theta < 90°$, $\mu > 0$, and then the particles move outward from the Sun, and when $\theta > 90°$, $\mu < 0$, and then the particles move toward the Sun. If the magnetic field is smooth, then the form of this orbit will be constant, but when the interplanetary magnetic field is irregular, the value of μ changes randomly. The process is called "pitch angle scattering," and represents a diffusive process, i.e., a random walk, in μ. Here we use the coefficient of pitch angle scattering as

$$\varphi(\mu) = A|\mu|^{q-1}(1-\mu^2), \tag{2.9}$$

where q is the spectral index of the power law for interplanetary field fluctuations at wave number k within an interval dk, Q_{xx} is the spectral power at a reference wave number k_0, and

$$A = 2\pi \frac{v}{R^2} Q_{xx}(k_0 r_L)^q, \tag{2.10}$$

where R is the rigidity of particle, and r_L is the Larmor radius. Thus Q_{xx}, q, and k_0 are parameters of the spectrum of the irregular field (Jokipii 1966; Jokipii 1971; Earl 1973).

2.2 Transport of Particles: Fokker-Planck Equation

A major goal of this research is to fit data in order to determine the injection of particles from the Sun as a function of time and energy. Therefore, it is necessary to consider the influences that affect the transport. We use the theory of focused transport to explain the motion of solar energetic particles in the interplanetary space. In this work we consider the following transport processes: systematic changes in the position along the magnetic field, z, systematic changes in the pitch angle cosine, μ, systematic changes in the momentum, p, and random changes in pitch angle scattering (diffusion in μ) due to the magnetic field irregularities. The distribution function, $F(t,\mu,z,p)$, is defined as the density of particles in terms of p, μ, z, and the time, t:

$$F(t,\mu,z,p) = \frac{d^3N}{d\mu dz dp}. \tag{2.11}$$

Then the evolution of F is governed by a Fokker-Planck equation,

$$\frac{\partial F(t,\mu,z,p)}{\partial t} = -\frac{\partial}{\partial z}\left(\frac{\Delta z}{\Delta t}F\right) - \frac{\partial}{\partial \mu}\left(\frac{\Delta \mu}{\Delta t}F\right) - \frac{\partial}{\partial p}\left(\frac{\Delta p}{\Delta t}F\right) + \textit{second order term} \tag{2.12}$$

where first-order terms correspond to systematic processes, and second-order terms correspond to random processes.

Jokipii (1966) used a Fokker-Planck equation to explain the diffusive transport of particles in interplanetary medium, including the effects of the streaming of particles and the distribution of scattering:

$$\frac{\partial f(t,\mu,z)}{\partial t} = -\mu v\frac{\partial f(t,\mu,z)}{\partial z} + \frac{\partial}{\partial \mu}\frac{\varphi(\mu)}{2}\frac{\partial f(t,\mu,z)}{\partial \mu}, \tag{2.13}$$

where t is the time since the solar flare occurrence, z is the arclength along the magnetic field, v is the particle speed, μ is the cosine of the pitch angle, or v_z/v, f is the particle distribution function and φ is the coefficient of pitch angle scattering (Jokipii 1971; Earl 1973). This equation explains the cosmic-ray distribution as a function of the distance along the magnetic and the cosine of the pitch angle.

Earl (1976a) developed the transport equation further by including the effects of adiabatic focusing:

$$\frac{\partial f(t,\mu,z)}{\partial t} + \mu v \frac{\partial f(t,\mu,z)}{\partial z} = -\frac{v}{2L(z)}(1-\mu^2)\frac{\partial f(t,\mu,z)}{\partial \mu} + \frac{\partial}{\partial \mu}\frac{\varphi(\mu)}{2}\frac{\partial f(t,\mu,z)}{\partial \mu}, \tag{2.14}$$

where the first term on the right-hand side shows the effect of adiabatic focusing (Roelof 1969), and L is the scale length for spatial variations of the guiding field,

$$\frac{1}{L(z)} = -\frac{1}{B}\frac{\partial B}{\partial z}. \tag{2.15}$$

Particles are considered to undergo pitch-angle scattering, which is the effect of small-scale irregularities in the interplanetary magnetic field, and focusing, which is an effect of the large-scale divergence of the field at increasing distance from the Sun.

Later, Ruffolo (1995) considered these effects in two reference frames, which are the fixed frame and solar wind frame (Figure 2.4). In the fixed frame, the particle velocity is defined as $\vec{v}$. The large-scale structure of the magnetic field is constant, and the focusing conserves the magnitude of the velocity, $v = |\vec{v}|$. In the solar wind frame, the small-scale irregularities are frozen in the solar-wind frame, so the scattering conserves the magnitude of the solar wind frame velocity, $v' = \left|\vec{v'}\right|$. We consider the effects of focusing and scattering in the solar

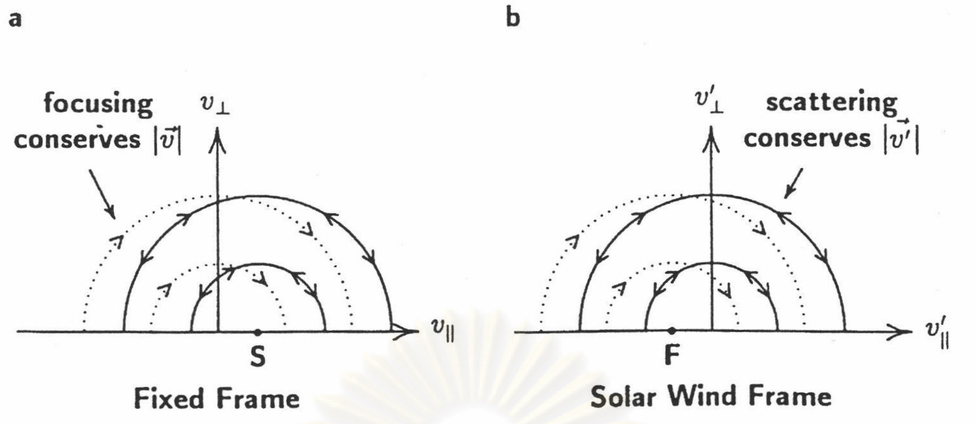

Figure 2.4: The adiabatic focusing and pitch-angle scattering in the fixed frame (a) and the solar wind frame (b).

wind frame, because it is computationally easier to consider scattering in the frame in which it conserves the particle speed.

Focusing always preserves the magnitude of velocity in the fixed frame (v), but does not preserve the magnitude of velocity in the solar wind frame (v'). The process of focusing always makes the velocity of the particle in the solar wind frame closer to the origin.

We find the (non-relativistic) rate of deceleration as

$$\dot{v}' = \left.\frac{dv'}{d\mu}\right|_{v} \dot{\mu}, \tag{2.16}$$

and the focusing in the fixed frame gives the rate of change of μ (Earl 1976a) as

$$\dot{\mu} = \frac{v}{2L(z)}(1-\mu^2). \tag{2.17}$$

When we substitute $\dot{\mu}$ from equation (2.17) into equation (2.16), we find that

$$\dot{v}' = \frac{-v_{sw}v'}{2L(z)}(1-\mu'^2). \tag{2.18}$$

For relativistic particle speeds, the appropriate formula for the rate of change of the momentum is

$$\dot{p}' = -\frac{v_{sw}p'}{2L(z)}(1-\mu'^2), \tag{2.19}$$

where p' is the momentum of the particle in the solar wind frame and v_{sw} is the solar wind velocity. From these processes, adiabatic deceleration is a decrease in momentum from the transformation of the adiabatic focusing from the fixed frame to the solar wind frame (Ruffolo 1995).

For the Archimedean spiral magnetic field in a frame that is corotating with the Sun, the solar wind velocity, v^c_{sw}, is parallel to the magnetic field at each point:

$$v^c_{sw} = v_{sw}\sec[\psi(z)], \tag{2.20}$$

where $\psi(z)$ is the angle between the solar wind direction and the average magnetic field as shown in Figure 2.5.

The change in the momentum of a particle in the solar wind frame is due to two effects. One effect is associated with adiabatic focusing. From equations (2.19) and (2.20), we get

$$\dot{p}' = \frac{-v^c_{sw}p'}{2L(z)}(1-\mu^2). \tag{2.21}$$

Another effect is due to the position dependence of v^c_{sw}. In other words, at different locations, z, the local solar wind frame and v^c_{sw} are different [moving with $v^c_{sw}(z)$ with respect to the fixed frame], leading to "differential convection."

In the solar wind frame, we consider the streaming of a particle moving from one point to another point along the same field line. The transformation between the fixed frame and local solar wind frame gives $p_{||} = p'_{||} + (E/c^2)v^c_{sw}$ or $p'_{||} = p_{||} - (E/c^2)v^c_{sw}$. The momentum difference at different points is

$$\Delta p'_{||} = -\frac{E}{c^2}v_{sw}(\Delta\sec\psi). \tag{2.22}$$

Figure 2.5: Illustration of the Archimedean spiral field and $\psi(z)$.

In terms of the distance along the field line between the two different points,

$$\Delta p'_{||} = -\frac{E}{c^2} v_{sw} \left(\frac{d}{dz} \sec \psi \right) \Delta z. \tag{2.23}$$

Substituting $\Delta z = v_{||} \Delta t$ into equation (2.23), we get the rate of deceleration of the momentum component along the field line:

$$\dot{p}'_{||} = -p_{||} v_{sw} \left(\cos \psi \frac{d}{dr} \sec \psi \right), \tag{2.24}$$

where $\frac{d}{dz} \sec \psi = \cos \psi \frac{d}{dr} \sec \psi$. From the relation $\dot{p}' = (p'_{||}/p') \dot{p}_{||}{}'$, we get the rate of deceleration due to changes in v^c_{sw}:

$$\dot{p}' = -p' v_{sw} \left(\cos \psi \frac{d}{dr} \sec \psi \right) \mu'^2, \tag{2.25}$$

Summing the two effects, the total deceleration rate is

$$\dot{p}' = -p' v_{sw} \left[\frac{\sec \psi}{2L(z)} (1 - \mu'^2) + \cos \psi \frac{d}{dr} \sec \psi \ \mu'^2 \right]. \tag{2.26}$$

There are effects of the solar wind on the z-transport solar energetic particles, in particular because of solar wind convection. Since we consider the effects of the solar wind on z in the corotating frame and the focusing preserves v in that frame, we find the velocity along the magnetic field line, $v_{||}$, from

$$\begin{aligned} p_{||} &= \mu' p' + \frac{E}{c^2} v_{sw} \sec\psi, \\ &= \mu' p' + \frac{E}{c^2} v_{sw} \sec\psi, \\ v_{||} &= \frac{E'}{E} \mu' v' + v_{sw} \sec\psi. \end{aligned} \tag{2.27}$$

From these equations we get the rate of streaming and convection with the effects of the solar wind in the corotating frame:

$$\dot{z} = v_z = v_{||} \tag{2.28}$$

$$= \mu' v' + \left(1 - \mu'^2 \frac{v'^2}{c^2}\right) v_{sw} \sec\psi, \tag{2.29}$$

where the factor $E'/E = 1 - \mu v_{sw} \sec\psi / c^2$ is the ratio of the particle energy in the solar wind frame to that in the fixed frame.

There are also effects of the solar wind speed on the pitch angle. The rate of change of μ, the pitch angle cosine, is due to the focusing in the fixed frame as in equation (2.17) and the differential convection from the position-dependent solar wind velocity. For the effect of focusing, we get

$$\dot{\mu}' = \left.\frac{d\mu'}{d\mu}\right|_p \dot{\mu} = \frac{v'}{2L(z)} \left(1 + \mu' \frac{v_{sw}}{v'} \sec\psi - \mu' v_{sw} \frac{v'}{c^2} \sec\psi\right) (1 - \mu'^2) \tag{2.30}$$

(Ruffolo 1995). The differential convection only affects the parallel component of the momentum, so we have

$$d(p_\perp'^2) = 0 = d[p'^2(1 - \mu'^2)] \tag{2.31}$$

$$= 2(1 - \mu'^2) p' dp' - 2\mu' p'^2 d\mu' \tag{2.32}$$

$$\dot{\mu}' = \frac{1 - \mu'^2}{\mu'} \frac{\dot{p}'}{p'}. \tag{2.33}$$

From equation (2.33) and equation (2.25), the effect of differential convection is

$$\dot{\mu}' = -v_{sw}\left(\cos\psi \frac{d}{dr}\sec\psi\right)\mu'(1-\mu'^2). \tag{2.34}$$

Thus the total rate of change of μ due to effects of the solar wind is

$$\begin{aligned}\dot{\mu}' &= \frac{v'}{2L(z)}\left[1+\mu'\frac{v_{sw}}{v'}\sec\psi - \mu'\frac{v_{sw}v'}{c^2}\sec\psi\right](1-\mu'^2) \\ &\quad - v_{sw}\left(\cos\psi\frac{d}{dr}\sec\psi\right)\mu'(1-\mu'^2).\end{aligned} \tag{2.35}$$

2.3 Transport Equation for Solar Energetic Particles in Interplanetary Space

As previous section, we have the equations of the total rates of change of z, μ, and p due to effects of the solar wind as in equations (2.29), (2.35), and (2.26). We use these formulae in a transport equation for solar energetic particles. The Fokker-Planck equation (2.12) for solar energetic particles in the interplanetary medium can be improved by writing this equation in terms of changes of p and μ in the local solar wind frame and z and t in the fixed (or slowly corotating) frame:

$$\begin{aligned}\frac{\partial F(t,\mu,z,p)}{\partial t} &= -\frac{\partial}{\partial z}\left(\frac{\Delta z}{\Delta t}F\right) - \frac{\partial}{\partial \mu}\left(\frac{\Delta \mu}{\Delta t}F\right) \\ &\quad + \frac{\partial}{\partial \mu}\left[\frac{\varphi(\mu)}{2}\frac{\partial}{\partial \mu}\left(\frac{E'}{E}F\right)\right] - \frac{\partial}{\partial p}\left(\frac{\Delta p}{\Delta t}F\right),\end{aligned} \tag{2.36}$$

where $E'/E = 1 - \mu v_{sw}\sec\psi/c^2$ is the ratio of the particle energy in the solar wind frame to that in the fixed frame, a factor in the pitch-angle scattering term. We know $\Delta z/\Delta t$, $\Delta\mu/\Delta t$ and $\Delta p/\Delta t$ from equations (2.29), (2.35), and (2.26), respectively, so we can get the appropriate transport equation (Ruffolo 1995):

$$
\begin{aligned}
\frac{\partial F(t,\mu,z,p)}{\partial t} =& -\frac{\partial}{\partial z}\mu v F(t,\mu,z,p) \\
& - \frac{\partial}{\partial z}\left(1-\mu^2\frac{v^2}{c^2}\right) v_{sw}\sec\psi F(t,\mu,z,p) \\
& - \frac{\partial}{\partial \mu}\frac{v}{2L(z)}\left[1+\mu\frac{v_{sw}}{v}\sec\psi - \mu\frac{v_{sw}v}{c^2}\sec\psi\right](1-\mu^2)F(t,\mu,z,p) \\
& + \frac{\partial}{\partial \mu} v_{sw}(\cos\psi\frac{d}{dr}\sec\psi)\mu(1-\mu^2)F(t,\mu,z,p) \\
& + \frac{\partial}{\partial \mu}\frac{\varphi(\mu)}{2}\frac{\partial}{\partial \mu}F(t,\mu,z,p) \\
& + \frac{\partial}{\partial p} p v_{sw}\left(\frac{\sec\psi}{2L(z)}(1-\mu^2)+\cos\psi\frac{d}{dr}\sec\psi\mu^2\right)F(t,\mu,z,p),
\end{aligned}
\tag{2.37}
$$

where $\psi(z)$ is the angle between the field line and the radial direction (Figure 2.5), $L(z)$ is the focusing length, $\varphi(\mu)$ is the pitch-angle scattering coefficient, r is the radial distance from the Sun, v is the particle speed, v_{sw} is the solar wind speed, μ is the cosine of the pitch angle, $F(t,\mu,z,p)$ is the density of particles in terms of p, μ, and z, the position along the magnetic field, p is the momentum of the particle, and c is the speed of light.

This is the equation we use for simulating the energetic particle propagation from the Sun. We find the spatial mean free path of the solar energetic particles, λ, from

$$\lambda \equiv \frac{3D}{v}, \tag{2.38}$$

where D is a spatial diffusion coefficient (Jokipii 1966, 1968; Hasselmann & Wibberenz, 1968), and

$$D = \frac{v^2}{4}\int_{-1}^{1}\frac{(1-\mu^2)^2}{\varphi(\mu)}d\mu. \tag{2.39}$$