CHAPTER IV

OFF-DIAGONAL LONG-RANGE ORDER AND LIQUID ^{4}HE II

In this section we will used the reduced density matrices and ODLRO, and the microscopic quantities for He II, to obtain the thermo-hydrodynamics equation. The system of He II is considered as an interacting Bose system, but the concept of its "condensation" is more complicated than of the ideal Bose gas. When the condensation has occurred, every atom contributes both to the condensate spreading throughout the volume occupied by the system in the presence of strong interaction between the ^{4}He atoms and to localized region of higher density (localized to within about the average interatomic spacing, 4.5 Å). The condensate density ρ_c of He II at 0 K has been estimated to be between 8% and 25% of the total density ρ. This result have been comfirmed by many experimental works (21). Therefore, the condensate density ρ_c should be distinguished from the superfluid density ρ_s. The superfluid density ρ_s is equal to the total density at o K. Both the condensate density and superfluid density decrease with increase of temperature from 0 K and vanish at the lambda transition temperature, T_λ

4.1 Thermo-Hydrodynamic Equation of He II

A two-fluid model was proposed (22) for He II as the condensate and depletion model: the total density

$$\rho \quad = \quad \rho_c + \rho_d \qquad (4\text{-}1)$$

where ρ_d is called the depletion density. ρ_d is equal to the total density ρ at T_λ. The appearance of the non-factorized part (ODLRO) in Ω_1 for the condensation in He II system is written as

$$\Omega_1(\vec{x}'\,;\,\vec{x}'') \quad = \quad \phi^*(\vec{x}'')\,\phi(\vec{x}') + \Lambda_1(\vec{x}'\,;\,\vec{x}'') \qquad (4\text{-}2)$$

where $\Lambda_1(\vec{x}'\,;\,\vec{x}'') \longrightarrow 0$ when $|\vec{x}' - \vec{x}''| \longrightarrow \infty$

Liquid ^{4}He as well as a "quantum liquid" is differentiated from ordinary liquid at below T_λ by the appearance of macroscopic wave function $\phi(\vec{x})$. The wave function $\phi(\vec{x})$ is the statistical average $< \psi(\vec{x}) >$ over an ensemble such that it has a definite phase and amplitude(23). $\psi(\vec{x})$ is the spinless boson annihilation operator for ^{4}He atoms. $|\phi(\vec{x})|^2$ is defined as the "condensate" density ρ_c below T_λ, and $\phi(\vec{x})$ is refered to as the "condensate (macroscopic) wave function":

$$\phi(\vec{x}) \quad = \quad \{\rho_c(\vec{x})\}^{1/2}\exp\{i\Theta(\vec{x})\} \qquad (4\text{-}3)$$

where $\rho_c(\vec{x}) = |\phi(\vec{x})|^2$. The condensate velocity is defined as

$$\begin{aligned}
\hat{P}\phi(x) &= -i\hbar\nabla\phi(\vec{x}) \\
&= -i\hbar\nabla\{\rho_c(\vec{x})\}^{1/2}\exp\, i\Theta(\vec{x}) \\
&= \hbar\nabla\Theta(\vec{x})\{\rho_c(\vec{x})\}^{1/2}\exp\, i\Theta(\vec{x}) \\
&= \hbar\nabla\Theta(\vec{x})\phi(\vec{x})
\end{aligned}$$

and $\vec{P} = m\vec{v}_c$;

$$m\vec{v}_c \quad = \quad \hbar\nabla\Theta(\vec{x})$$

$$\vec{v}_c \quad = \quad \frac{\hbar}{m}\nabla\Theta(\vec{x}) \qquad (4\text{-}4)$$

and the "depletion" density ρ_d is then given by

$$\rho_d(\vec{x}) = \Lambda_1(\vec{x};\vec{x}) \tag{4-5}$$

where $\Lambda_1(x;x)$ is written in terms of two real function; an even function $\rho_d(\vec{x}\,'; \vec{x}\,'')$ and an odd function $\chi(\vec{x}\,'; \vec{x}\,'')$

$$\Lambda_1(\vec{x}\,'; \vec{x}\,'') = \rho_d(\vec{x}\,'; \vec{x}\,'')\exp\{i\chi(\vec{x}\,'; \vec{x}\,'')\} \tag{4-6}$$

whose phase χ vanishes when $\vec{x}\,' = \vec{x}\,''$. The depletion velocity is defined from the depletion current density as

$$\rho_d\vec{v}_d = \frac{\hbar}{2mi}\lim_{x'\to x''}(\nabla' - \nabla'')\Lambda_1(\vec{x}\,'; \vec{x}\,'') \tag{4-7}$$

where $$\vec{v}_d = \frac{\hbar}{2m}\lim_{x'\to x''}(\nabla' - \nabla'')\chi(\vec{x}\,'; \vec{x}\,'') \tag{4-8}$$

The condensate current density is defined as

$$\rho_c\vec{v}_c = \frac{\hbar}{2mi}\lim_{x'\to x''}(\nabla' - \nabla'')\phi^*(\vec{x}\,'')\phi(\vec{x}\,') \tag{4-9}$$

The total current density of the He II system becomes

$$\vec{J} = \rho\vec{v} = \rho_c\vec{v}_c + \rho_d\vec{v}_d \tag{4-10}$$

In the bulk system, the depletion "bulk" density which is spatially indepen- independent is defined as

$$\Lambda_1(\vec{x}\,'; \vec{x}\,'') = \Lambda_1(|\vec{x}\,' - \vec{x}\,''|) = \rho_d h(|\vec{x}\,' - \vec{x}\,''|)\exp\{i\chi(|\vec{x}\,' - \vec{x}\,''|)\} \tag{4-11}$$

where $h(\vec{r})$ is real function and approaches zero at 0 K when r is much greater than the average interatomic spacing

Yang(20) has also shown that the factorization of $\Omega_1(\vec{x}'\,;\,\vec{x}'')$ in the limit $|\vec{x}' - \vec{x}''| \longrightarrow \infty$ implies the factorization of the second order reduced density matrix $\Omega_2(\vec{x}_1',\vec{x}_2';\ \vec{x}_1'',\vec{x}_2'')$ of the Bose system. A very plausible form(12,24) of Ω_2 which shows the presence of ODLRO may be written as

$$\begin{aligned}
\Omega_2(\vec{x}_1',\vec{x}_2';\ \vec{x}_1'',\vec{x}_2'') = {} & \$_1(|\vec{x}_1' - \vec{x}_2'|)\$_1(|\vec{x}_1'' - \vec{x}_2''|)\phi^*(\vec{x}_1'')\phi^*(\vec{x}_2'')\phi(\vec{x}_1')\phi(\vec{x}_2') \\
& +\$_2(|\vec{x}_1' - \vec{x}_2'|)\$_2(|\vec{x}_1'' - \vec{x}_2''|)\ \phi^*(\vec{x}_1'')\phi(\vec{x}_1')\Lambda_1(\vec{x}_2';\vec{x}_2'') \\
& +\phi^*(\vec{x}_1'')\phi(\vec{x}_2')\Lambda_1(\vec{x}_1';\vec{x}_2'') + \phi^*(\vec{x}_2'')\phi(\vec{x}_1')\Lambda_1(\vec{x}_2';\vec{x}_1'') \\
& +\phi^*(\vec{x}_2'')\phi(\vec{x}_2')\Lambda_2(\vec{x}_1'';\vec{x}_1') + \Lambda_2(\vec{x}_1',\vec{x}_2';\ \vec{x}_1'',\vec{x}_2'')
\end{aligned} \qquad (4\text{-}12)$$

where $\$_1(\vec{r})$ and $\$_2(\vec{r})$ are the screening factors for the "core" condition. The function $h(\vec{r})$ and the screening factors $\$(\vec{r})$ can be obtained from the expression of McMillan(2). The function Λ_2 is required to satisfy all condition analogous to the condition(12) of Ω_2

We will now obtain the thermo-hydrodynamic equations of motion for the bulk system of ^4He II. Inserting equation (4-2) into equation (3-26) and taking the limit as $|\vec{x}' - \vec{x}''| \longrightarrow \infty$ gives an exact equation of motion for the condensate macroscopic wave function ϕ_1, separable in $\vec{x}'$ and $\vec{x}''$ is obtained, ie.,

$$i\hbar\frac{\partial\phi(\vec{x}')}{\partial t} = \frac{-\hbar^2}{2m}\nabla^2_{\vec{x}'}\phi(\vec{x}') + \lim_{|\vec{x}'-\vec{x}''|\to\infty}\frac{\int V(\vec{x}' - \vec{y})\Omega_2(\vec{x}',\vec{y};\ \vec{x}'',\vec{y})\,d\vec{y}}{\phi^*(\vec{x}'')} \qquad (4\text{-}13)$$

and
$$\phi(\vec{x}') = \rho(\vec{x})^{1/2}e^{i\Theta(\vec{x}')}$$

$$i\hbar\frac{\partial\{\rho_c^{1/2}(\vec{x}')e^{i\Theta(\vec{x}')}\}}{\partial t} = \frac{-\hbar^2}{2m}\nabla'^2\rho_c^{1/2}(\vec{x}')e^{i\Theta(\vec{x}')} + \lim_{|\vec{x}'-\vec{x}''|\to\infty}\frac{\int V(\vec{x}'-\vec{y})\Omega_2(\vec{x}',\vec{y};\vec{x}'',\vec{y})d\vec{y}}{\phi^*(\vec{x}'')} \qquad (4\text{-}14)$$

thus

$$\frac{i\hbar}{m}\frac{\partial\{\rho_c^{1/2}(\vec{x}')e^{i\Theta(\vec{x}')}\}}{\partial t} = \frac{i\hbar}{m}\left(\frac{1}{2}\frac{e^{i\Theta(\vec{x}')}}{\rho_c^{1/2}(\vec{x}')}\frac{\partial\rho_c(\vec{x}')}{\partial t} + i\rho_c^{1/2}(\vec{x}')e^{i\Theta(\vec{x}')}\frac{\partial\Theta(\vec{x}')}{\partial t}\right)$$

$$= \frac{i\hbar}{2m\rho_c^{1/2}(\vec{x}')}e^{i\Theta(\vec{x}')}\frac{\partial\rho_c(\vec{x}')}{\partial t} - \frac{\hbar}{m}\rho_c^{1/2}(\vec{x}')e^{i\Theta(\vec{x}')}\frac{\partial\Theta(\vec{x}')}{\partial t} \qquad (4\text{-}15)$$

and

$$\frac{-\hbar^2}{2m}\nabla'^2\rho_c^{1/2}(\vec{x}')e^{i\Theta(\vec{x}')} = \frac{-\hbar^2}{2m}\nabla'\left(\frac{e^{i(\vec{x}')}}{2\rho_c^{1/2}(\vec{x}')}\nabla'\rho_c(\vec{x}') + i\rho_c^{1/2}(\vec{x}')e^{i\Theta(\vec{x}')}\nabla'\Theta(x')\right)$$

$$= \frac{-\hbar^2}{2m}\left(\frac{-1}{4}\frac{e^{i\Theta(\vec{x}')}}{\rho_c^{3/2}(x')}(\nabla'\rho_c(\vec{x}'))^2 + \frac{ie^{i\Theta}}{\rho_c^{1/2}(\vec{x}')}\nabla'\Theta(\vec{x}')\nabla'\rho_c(\vec{x}')\right.$$

$$+ \frac{e^{i\Theta(\vec{x}')}}{2\rho_c^{1/2}(\vec{x}')}\nabla'^2\rho_c(\vec{x}') + ie^{i\Theta(\vec{x}')}\nabla'\rho(\vec{x}')\nabla\Theta(\vec{x}')$$

$$\left. - \rho_c^{1/2}(\vec{x}')e^{i\Theta(\vec{x}')}(\nabla'\Theta(\vec{x}'))^2 + i\rho_c^{1/2}(\vec{x}')e^{i\Theta(\vec{x}')}\nabla'^2\Theta(\vec{x}')\right)$$

$$= \left(\frac{\hbar^2}{8m}\frac{e^{i\Theta(\vec{x}')}}{\rho_c^{3/2}(\vec{x}')}(\nabla'\rho_c(\vec{x}'))^2 - \frac{\hbar^2}{4m}\frac{e^{i\Theta(\vec{x}')}}{\rho_c^{1/2}(\vec{x}')}\nabla'^2\rho_c(\vec{x}')\right.$$

$$\left. - \frac{\hbar^2}{2m}\rho_c^{1\,2}(\vec{x}').(\nabla'\Theta(\vec{x}'))\right) - i\left(\frac{\hbar^2}{4m}\frac{e^{i\Theta(\vec{x}')}}{\rho_c^{1/2}(\vec{x}')}\nabla'\Theta(\vec{x}').\right.$$

$$\left.\nabla'\rho_c(\vec{x}') + \frac{\hbar^2}{2m^2}\rho_c^{1/2}(\vec{x}')e^{i\Theta(\vec{x}')}\nabla'^2\Theta(\vec{x}')\right) \qquad (4\text{-}16)$$

Inserting equations(4-15) and (4-16) into equation(4-14), and separating the real part and imaginary part, for real part, we get

$$\frac{-\hbar\rho_c^{1/2}}{m}(\vec{x}')e^{i\Theta(\vec{x}')}\frac{\partial\Theta(\vec{x}')}{\partial t} = \frac{\hbar^2}{8m^2\rho_c^{3/2}(\vec{x}')}e^{i\Theta(\vec{x}')}(\nabla'\rho(\vec{x}'))^2 - \frac{\hbar^2}{4m^2\rho_c^{1/2}(\vec{x}')}e^{i\Theta(\vec{x}')}\nabla'\rho_c(\vec{x}')$$

$$\frac{-\hbar^2}{2m^2}\rho_c^{1/2}(\vec{x}')e^{i\Theta(\vec{x}')}\ \nabla'\Theta(\vec{x}'))^2 + \mathrm{Re}\lim_{|\vec{x}'-\vec{x}''|\to\infty}\frac{\int V(\vec{x}',\vec{y})\Omega_2(\vec{x}',\vec{y};\vec{x}'',\vec{y})d\vec{y}}{m\phi^*(\vec{x}'')}$$

$$= \frac{\hbar^2(\nabla'\rho_c(\vec{x}'))^2}{8m^2\rho_c(\vec{x}')} - \frac{\hbar^2}{4m^2}\nabla'^2\rho_c(\vec{x}') - \frac{v_c^2}{2}$$

$$+ \mathrm{Re}\lim_{|\vec{x}'-\vec{x}''|\to\infty}\frac{\int V(\vec{x}',\vec{y})\Omega_2(\vec{x}',\vec{y};\vec{x}'',\vec{y})\ d\vec{y}}{m\phi^*(\vec{x}'')\phi(\vec{x}')}$$

$$= -\frac{\hbar^2}{2m^2}\left[\frac{\nabla'^2\rho_c(\vec{x}')}{2\rho_c(\vec{x}')} - \frac{(\nabla'\rho_c(\vec{x}'))^2}{4\rho_c^2(\vec{x}')}\right] - \frac{v_c^2}{2} + \mathrm{Re}\ F$$

$$= -\frac{\hbar^2}{2m^2}\frac{\nabla'^2|\phi(\vec{x}')|}{|\phi(\vec{x}')|} - \frac{v_c^2}{2} + \mathrm{Re}\ F \qquad (4\text{-}17)$$

where $F = \lim_{|\vec{x}'-\vec{x}''|\to\infty}\frac{\int V(\vec{x}',\vec{y})\Omega_2(\vec{x}',\vec{y};\vec{x}'',\vec{y})\ d\vec{y}}{m\phi^*(\vec{x}'')\phi(\vec{x}')}$

Putting $(-\nabla')$ into equation (4-17), thus

$$\frac{\hbar}{m}\frac{\partial\nabla'\Theta(\vec{x}')}{\partial t} = -\nabla'\left[\frac{-\hbar^2}{2m}\frac{\nabla'^2|\phi(\vec{x}')|}{|\phi(\vec{x}')|} + \mathrm{Re}\ F\right] - \frac{\nabla v_c^2}{2} \qquad (4\text{-}18)$$

where $\vec{v}_c = \frac{\hbar}{m}\nabla'\Theta(\vec{x}')$

We thus obtain, from equation (4-18)

$$\frac{\partial\vec{v}_c}{\partial t} + \frac{\nabla' v_c^2}{2} = -\nabla'\left[\frac{-\hbar^2}{2m^2}\frac{\nabla'^2|\phi(\vec{x}')|}{|\phi(\vec{x}')|} + \mathrm{Re}\ F\right] \qquad (4\text{-}19)$$

since $\nabla(\vec{a}.\vec{b}) = (\vec{a}.\nabla)\vec{b} + (\vec{b}.\nabla)\vec{a} + \vec{a}\times(\nabla\times\vec{b}) + (\vec{b}\times(\nabla\times\vec{a})$

If $\vec{a} = \vec{b} = \vec{v}_c$

we get $\nabla v_c^2 = 2(\vec{v}_c.\nabla)\vec{v}_c + 2\vec{v}_c\times(\nabla\times\vec{v}_c)$

$$\frac{\nabla v_c^2}{2} = (\vec{v}_c.\nabla)\vec{v}_c$$

from equation (4-19), we have

$$\frac{\partial \vec{v}_c}{\partial t} + (\vec{v}_c \cdot \nabla')\vec{v}_c = -\nabla\left[\frac{-\hbar^2}{2m^2}\frac{\nabla^2|\phi(x\ )|}{|\phi(x\ )|} + \text{Re } F\right]. \qquad (4\text{-}20)$$

For imaginary, we have

$$\frac{\hbar}{2m}\frac{e^{i\Theta(\vec{x}')}}{\rho_c^{1/2}(\vec{x}')}\frac{\partial\rho_c(x')}{\partial t} = \frac{-\hbar^2}{4m^2}\frac{e^{i\Theta(\vec{x}')}}{\rho_c^{1/2}(\vec{x}')}\nabla'\Theta(\vec{x}')\nabla'\rho_c(\vec{x}') - \frac{\hbar^2}{4m^2}\frac{e^{i\Theta(\vec{x}')}}{\rho_c^{1/2}(\vec{x}')}\nabla'\Theta(\vec{x}')\nabla'\rho_c(\vec{x}')$$

$$\frac{-\hbar^2}{2m^2}\rho_c^{1/2}(\vec{x}')e^{i\Theta(\vec{x}')}\nabla'^2\Theta(\vec{x}')$$

$$+ \text{Im}_{|\vec{x}'-\vec{x}''|\to\infty}\frac{\int V(\vec{x}',\vec{y})\Omega_2(\vec{x}',\vec{y};\vec{x}'',\vec{y})\ d\vec{y}}{m\phi^*(\vec{x}'')}$$

$$\frac{\hbar}{2m\rho_c(\vec{x}')}\frac{\partial\rho_c(\vec{x}')}{\partial t} = \frac{-\hbar}{4m\rho_c(\vec{x}')}\vec{v}_c\cdot\nabla'\rho_c(\vec{x}') - \frac{\hbar}{4m\rho_c(\vec{x}')}\vec{v}_c\cdot\nabla'\rho_c(\vec{x}') - \frac{\hbar^2}{2m^2}\nabla'^2\Theta + \text{Im } F$$

$$= \frac{-\hbar}{2m\rho_c(\vec{x}')}\vec{v}_c\cdot\nabla'\rho_c(\vec{x}') - \frac{\hbar^2}{2m^2}\nabla'^2\Theta(\vec{x}') + \text{Im } F$$

$$\frac{\partial\rho_c(\vec{x}')}{\partial t} = -\vec{v}_c\nabla'\rho_c(\vec{x}') - \frac{\hbar\rho_c}{m}(\vec{x}')\nabla'^2\Theta(\vec{x}') + \frac{2m\rho_c(\vec{x}')}{\hbar}.\text{Im } F$$

$$= -\vec{v}_c\nabla'\rho_c(\vec{x}') - \rho_c\nabla'\cdot\vec{v}_c + \frac{2m\rho_c(\vec{x}')}{\hbar}\ \text{Im } F$$

$$= -\nabla'\cdot(\rho_c(\vec{x}')\vec{v}_c) + \frac{2m\rho_c(\vec{x}')}{\hbar}\ \text{Im } F$$

$$\frac{\partial\rho_c(\vec{x}\ )}{\partial t} + \nabla\ \cdot(\rho_c(\vec{x}\ )\vec{v}_c) = \frac{2m\rho_c(\vec{x}\ )}{\hbar}\ \text{Im } F \qquad (4\text{-}21)$$

where

$$F = \lim_{|\vec{x}'-\vec{x}''|\to\infty}\frac{\int V(\vec{x}',\vec{y})\Omega_2(\vec{x}',\vec{y};\vec{x}'',\vec{y})\ d\vec{y}}{m\phi^*(\vec{x}'')\phi(\vec{x}')}$$

Inserting equation (4-12) into above equation, we obtain

$$F = \lim_{|\vec{x}'-\vec{x}''|\to\infty} \frac{\int V(\vec{x}'-\vec{y})\,\$_1(|\vec{x}'-\vec{y}|)\,\$_1(|\vec{x}''-\vec{y}|)\,\phi^*(\vec{x}'')\,\phi^*(\vec{y})\,\phi(\vec{x}')\,\phi(\vec{y})\;d\vec{y}}{m\phi^*(\vec{x}'')\,\phi(\vec{x}')}$$

$$+ \lim_{|\vec{x}'-\vec{x}''|\to\infty} \frac{\int V(\vec{x}'-\vec{y})\,\$_2(|\vec{x}'-\vec{y}|)\,\$_2(|\vec{x}''-\vec{y}|)\,\phi^*(\vec{x}'')\,\phi(\vec{x}')\,\Lambda_1(\vec{y};\vec{y})\;d\vec{y}}{m\phi^*(\vec{x}'')\,\phi(\vec{x}')}$$

$$+ \lim_{|\vec{x}'-\vec{x}''|\to\infty} \frac{\int V(\vec{x}'-\vec{y})\,\$_2(|\vec{x}'-\vec{y}|)\,\$_2(|\vec{x}''-\vec{y}|)\,\phi^*(\vec{x}'')\,\phi(\vec{y})\,\Lambda_1(\vec{x}';\vec{y})\;d\vec{y}}{m\phi^*(\vec{x}'')\,\phi(\vec{x}')}$$

$$+ \lim_{|\vec{x}'-\vec{x}''|\to\infty} \frac{\int V(\vec{x}'-\vec{y})\,\$_2(|\vec{x}'-\vec{y}|)\,\$_2(|\vec{x}''-\vec{y}|)\,\phi^*(\vec{y})\,\phi(\vec{x}')\,\Lambda_1(\vec{y};\vec{x}')\;d\vec{y}}{m\phi^*(\vec{x}'')\,\phi(\vec{x}')}$$

$$+ \lim_{|\vec{x}'-\vec{x}''|\to\infty} \frac{\int V(\vec{x}'-\vec{y})\,\$_2(|\vec{x}'-\vec{y}|)\,\$_2(|\vec{x}''-\vec{y}|)\,\phi^*(\vec{y})\,\phi(\vec{y})\,\Lambda_1(\vec{x}';\vec{y})\;d\vec{y}}{m\phi^*(\vec{x}'')\,\phi(\vec{x}')}$$

$$+ \lim_{|x'-x''|\to\infty} \frac{\int V(\vec{x}'-\vec{y})\,\Lambda_2(\vec{x}',\vec{y};\vec{x}'',\vec{y})\;d\vec{y}}{m\phi^*(\vec{x}'')\,\phi(\vec{x}')}$$

$$= \frac{1}{m}\int V(\vec{x}'-\vec{y})\,\$_1(|\vec{x}'-\vec{y}|)\,\phi^*(\vec{y})\,\phi(\vec{y})\;d\vec{y}$$

$$+\frac{1}{m}\int V(\vec{x}'-\vec{y})\,\$_2(|\vec{x}'-\vec{y}|)\,\Lambda_1(\vec{y};\vec{y})\;d\vec{y}$$

$$+\frac{1}{m}\int \frac{V(\vec{x}'-\vec{y})\,\$_2(|\vec{x}'-\vec{y}|)\,\phi(\vec{y})\,\Lambda_1(\vec{x}';\vec{y})}{\phi(\vec{x}')}\;d\vec{y}$$

$$= \frac{\rho_c}{m}\int V(\vec{r})\,\$_1(r)\;d^3r + \frac{\rho_d}{m}\int V(\vec{r})\,\$_2(r)\;d^3r$$

$$+ \frac{1}{m}\int \frac{V(\vec{r})\,\$_2(r)\,\phi(\vec{y})\,\rho_d\,h(\vec{r})\,e^{i\chi(\vec{x}',\vec{y})}}{\phi(\vec{x}')}\;d^3r$$

From equations(4-5) and (4-7) in the paper of Fröhlich(17), we get

$$e^{i\chi(\vec{x}',\vec{y})} = 1 - \frac{im}{\hbar} r_\kappa v_{d\kappa}(\vec{x}') - \frac{im}{2\hbar} r_\kappa r_\ell \partial_\ell v_{d\kappa} - \frac{m^2}{2\hbar^2} r_\kappa r_\ell v_{d\kappa}(\vec{x}')\,v_{d\ell}(\vec{x}') + \cdots$$

$$\text{and} \qquad \phi(\vec{y}) = \phi(\vec{x}\,') + r_\kappa \partial_\kappa \phi(\vec{x}\,') + \frac{1}{2} r_\kappa r_\ell \partial_\kappa \partial_\ell \phi(\vec{x}\,') + \text{-----}$$

$$\text{Now} \qquad r_\kappa \partial_\kappa \phi(\vec{x}\,') = r_\kappa \partial_\kappa \rho_c^{1/2}(\vec{x}\,') e^{i\Theta(\vec{x}\,')}$$

$$= r_\kappa \left[i\rho_c^{1/2}(\vec{x}\,') e^{i\Theta(\vec{x}\,')} \partial_\kappa \Theta(\vec{x}\,') + \frac{1\rho_c^{1/2}(\vec{x}\,')}{2\;\rho_c(\vec{x}\,')} e^{i\Theta(\vec{x}\,')} \partial_\kappa \rho_c(\vec{x}\,') \right]$$

$$= r_\kappa \phi(\vec{x}\,') \left[i\partial_\kappa \Theta(\vec{x}\,') + \frac{\partial_\kappa \rho_c(\vec{x}\,')}{2\rho_c(\vec{x}\,')} \right]$$

$$\text{and} \qquad \frac{1}{2} r_\kappa r_\ell \partial_\kappa \partial_\ell \phi(\vec{x}\,') = \frac{1}{2} r_\kappa r_\ell \partial_\ell \left[i\rho_c^{1/2}(\vec{x}\,') e^{i\Theta(\vec{x}\,')} \partial_\kappa \Theta(\vec{x}\,') \right.$$

$$\left. + \frac{1\rho_c^{1/2}(\vec{x}\,')}{2\;\rho_c(x')} e^{i\Theta(\vec{x}\,')} \partial_\kappa \rho_c(x') \right]$$

$$= \frac{1}{2} r_\kappa r_\ell \left[i\rho_c^{1/2}(\vec{x}\,') e^{i\Theta(\vec{x}\,')} \partial_\ell \partial_\kappa \Theta(\vec{x}\,') \right.$$

$$- \rho_c^{1/2}(\vec{x}\,') e^{i\Theta(\vec{x}\,')} \partial_\ell \Theta(\vec{x}\,') \partial_\kappa \Theta(\vec{x}\,')$$

$$+ \frac{i\rho_c^{1/2}}{2\rho}(\vec{x}\,') e^{i\Theta(\vec{x}\,')} \partial_\ell \rho_c(\vec{x}\,') \partial_\kappa \Theta(\vec{x}\,')$$

$$+ \frac{1}{2} \left\{ \rho_c^{1/2}(\vec{x}\,') \left[e^{i\Theta(\vec{x}\,')} \frac{\partial_\ell \partial_\kappa \rho_c(\vec{x}\,')}{\rho_c(\vec{x}\,')} \right. \right.$$

$$\left. + \frac{ie^{i\Theta(\vec{x}\,')} \partial_\ell \Theta(\vec{x}\,') \partial_\kappa \rho_c(\vec{x}\,')}{\rho_c(\vec{x}\,')} \right]$$

$$\left. \left. - \frac{1}{2} \frac{e^{i\Theta(\vec{x}\,')} \rho_c^{1/2}(\vec{x}\,') \partial_\ell \rho_c(\vec{x}\,') \partial_\kappa \rho_c(\vec{x}\,')}{\rho_c^2(\vec{x}\,')} \right\} \right]$$

$$= \frac{1}{2} r_\kappa r_\ell \phi(\vec{x}\,') \left[i\partial_\ell \partial_\kappa \Theta(\vec{x}\,') - \partial_\ell \Theta(\vec{x}\,') \partial_\kappa \Theta(\vec{x}\,') \right.$$

$$+ \frac{i\partial_\kappa \Theta(\vec{x}\,') \partial_\ell \rho_c(\vec{x}\,')}{2\rho_c(\vec{x}\,')} + \frac{\partial_\ell \partial_\kappa \rho_c(\vec{x}\,')}{2\rho_c(\vec{x}\,')}$$

$$\left. + \frac{i\partial_\ell \Theta(\vec{x}\,') \partial_\kappa \rho_c(\vec{x}\,')}{2\rho_c(\vec{x}\,')} - \frac{\partial_\ell \rho_c(\vec{x}\,') \partial_\kappa \rho_c(\vec{x}\,')}{4\rho^2(\vec{x}\,')} \right]$$

Therefore

$$\phi(\vec{y}) = \phi(\vec{x}') + r_\kappa \phi(\vec{x}')\{i\partial_\kappa \Theta(\vec{x}') + \frac{\partial\rho_C(\vec{x}')}{2\rho_C(\vec{x}')}\}$$

$$+ \frac{1}{2} r_\kappa r_\ell \phi(\vec{x}')\{i\partial_\ell \partial_\kappa \Theta(\vec{x}') - \partial_\ell \Theta(\vec{x}')\partial_\kappa \Theta(\vec{x}')$$

$$+ \frac{i\partial_\kappa \Theta(\vec{x}')\partial\rho_C(\vec{x}')}{2\rho_C(\vec{x}')} + \frac{\partial_\ell \partial_\kappa \rho_C(\vec{x}')}{2\rho_C(\vec{x}')} + \frac{i\partial_\ell \Theta(\vec{x}')\partial_\kappa \rho_C(\vec{x}')}{2\rho_C(\vec{x}')}$$

$$- \frac{\partial_\ell \rho_C(\vec{x}')\partial_\kappa \rho_C(\vec{x}')}{4\rho_C^2(\vec{x}')}\}$$

$$\frac{\phi(\vec{y})}{\phi(\vec{x}')} = 1 + ir_\kappa \partial_\kappa \Theta(\vec{x}') + \frac{r_\kappa \partial\rho_C(\vec{x}')}{2\rho_C(x')} + \frac{ir_\kappa r_\ell \partial_\ell \partial_\kappa \Theta(\vec{x}')}{2}$$

$$- \frac{1}{2} r_\kappa r_\ell \partial_\ell \Theta(\vec{x}')\partial_\kappa \Theta(\vec{x}') + i\frac{r_\kappa r_\ell \partial_\kappa \Theta(\vec{x}')\partial\rho_C(\vec{x}')}{4\rho_C(\vec{x}')}$$

$$+ \frac{r_\kappa r_\ell \partial_\ell \partial_\kappa \rho_C(\vec{x}')}{4\rho_C(\vec{x}')} + i\frac{r_\kappa r_\ell \partial_\ell \Theta(\vec{x}')\partial_\kappa \rho_C(\vec{x}')}{4\rho_C(\vec{x}')}$$

$$- \frac{r_\kappa r_\ell \partial_\ell \rho_C(\vec{x}')\partial_\kappa \rho_C(\vec{x}')}{8\rho_C^2(\vec{x}')}$$

Furthermore

$$\frac{\phi(\vec{y})e^{i\chi(\vec{x}',\vec{y})}}{\phi(\vec{x}')} = 1 + ir_\kappa \partial_\kappa \Theta(\vec{x}') + \frac{r_\kappa \partial\rho_C(\vec{x}')}{2\rho_C(\vec{x}')} + i\frac{r_\kappa r_\ell \partial_\ell \partial_\kappa \Theta(\vec{x}')}{2}$$

$$- \frac{1}{2} r_\kappa r_\ell \partial_\ell \Theta(\vec{x}')\partial_\kappa \Theta(\vec{x}') + i\frac{r_\kappa r_\ell \partial_\kappa \Theta(\vec{x}')\partial_\ell \rho_C(\vec{x}')}{4\rho_C(\vec{x}')}$$

$$+ \frac{1}{4}\frac{r_\kappa r_\ell \partial_\ell \partial_\kappa \rho_C(\vec{x}')}{\rho_C(\vec{x}')} + i\frac{r_\kappa r_\ell \partial_\ell \Theta(\vec{x}')\partial_\kappa \rho_C(\vec{x}')}{4\rho_C(\vec{x}')}$$

$$- \frac{1}{8}\frac{r_\kappa r_\ell \partial_\ell \rho_C(\vec{x}')\partial_\kappa \rho_C(\vec{x}')}{\rho_C^2(\vec{x}')} - \frac{im}{\hbar} r_\kappa v_{d\kappa}(\vec{x}')$$

$$+ \frac{m}{\hbar} r_\kappa^2 v_{d\kappa}(\vec{x}')\partial_\kappa \Theta(\vec{x}') - \frac{im}{2\hbar} r_\kappa^2 v_{d\kappa}(\vec{x}')\frac{\partial\rho_C(\vec{x}')}{\rho_C(\vec{x}')}$$

$$+ \frac{m}{2\hbar} r_\kappa^2 r_\ell v_{d\kappa}(\vec{x}')\partial_\ell \partial_\kappa \Theta(\vec{x}') + \frac{im}{2\hbar} r_\kappa^2 r_\ell v_{d\kappa}(\vec{x}')\partial_\ell \Theta(\vec{x}')\partial_\kappa \Theta(\vec{x}')$$

$$+ \frac{m}{4\hbar\rho_c(\vec{x}')} r_\kappa^2 r_\ell v_{d\kappa}(\vec{x}')\partial_\kappa\Theta(\vec{x}')\partial_\kappa\rho_c(\vec{x}') - \frac{im}{4\hbar\rho_c(\vec{x}')} r_\kappa^2 r_\ell v_{d\kappa}(\vec{x}')\partial_\ell\partial_\kappa\rho_c(\vec{x}')$$

$$+ \frac{m}{4\hbar\rho_c(\vec{x}')} r_\kappa^2 r_\ell v_{d\kappa}(\vec{x}')\partial_\ell\Theta(\vec{x}')\partial_\kappa\rho_c(\vec{x}') + \frac{im}{8\hbar\rho_c^2} r_\kappa^2 r_\ell \partial_\ell\rho_c(\vec{x}')\partial_\kappa\rho_c(\vec{x}')$$

$$- \frac{im}{2\hbar} r_\kappa r_\ell \partial_\ell v_{d\kappa}(\vec{x}') + \frac{m}{2\hbar} r_\kappa^2 r_\ell \partial_\ell v_{d\kappa}\partial_\kappa\Theta(\vec{x}')$$

$$- \frac{im}{2\hbar} r_\kappa^2 r_\ell \partial_\kappa v_{d\ell}(\vec{x}')\frac{\partial\rho_c(\vec{x}')}{\rho_c(\vec{x}')} + \frac{m}{4\hbar} r_\kappa^2 r_\ell \partial_\ell v_{d\kappa}(\vec{x}')\partial_\ell\partial_\kappa\Theta(\vec{x}')$$

$$+ \frac{im}{4\hbar} r_\kappa^2 r_\ell^2 \partial_\ell v_{d\kappa}(\vec{x}')\partial_\ell\Theta(\vec{x}')\partial_\kappa\Theta(\vec{x}')$$

$$- \frac{m}{8\hbar} r_\kappa^2 r_\ell^2 \partial_\kappa v_{d\kappa}(\vec{x}')\partial_\kappa\Theta(\vec{x}')\partial\rho_c(\vec{x}') - \frac{im}{8\hbar} r_\kappa^2 r_\ell^2 \partial_\ell v_{d\kappa}(\vec{x}')\partial_\ell\partial_\kappa\rho_c(\vec{x}')$$

$$+ \frac{m}{8\hbar\rho_c(\vec{x}')} r_\kappa^2 r_\ell^2 \partial_\ell v_{d\kappa}(\vec{x}')\partial_\ell\Theta(\vec{x}')\partial_\kappa\rho_c(\vec{x}')$$

$$+ \frac{im}{16\hbar\rho_c^2(\vec{x}')} r_\kappa^2 r_\ell^2 \partial_\kappa v_{d\kappa}(\vec{x}')\partial_\ell\rho_c(\vec{x}')\partial_\kappa\rho_c(\vec{x}') - \frac{m^2}{2\hbar^2} r_\kappa r_\ell v_{d\kappa}(\vec{x}')v_{d\ell}(\vec{x}')$$

$$- \frac{im^2}{2\hbar^2} r_\kappa^2 r_\ell v_{d\kappa}(\vec{x}')v_{d\ell}(\vec{x}')\partial_\kappa\Theta(\vec{x}') - \frac{m^2}{4\hbar^2} r_\kappa^2 r_\ell v_{d\kappa}(\vec{x}')v_{d\ell}(\vec{x}')\frac{\partial\rho_c(\vec{x}')}{\rho_c(\vec{x}')}$$

$$- \frac{im^2}{4\hbar^2} r_\kappa^2 r_\ell v_{d\kappa}(\vec{x}')v_{d\ell}(\vec{x}')\partial_\ell\partial_\kappa\Theta(\vec{x}')$$

$$+ \frac{m^2}{4\hbar^2} r_\kappa^2 r_\ell^2 v_{d\kappa}(\vec{x}')v_{d\ell}(\vec{x}')\partial_\ell\Theta(\vec{x}')\partial_\kappa\Theta(\vec{x}')$$

$$- \frac{im^2}{8\hbar^2} r_\kappa^2 r_\ell^2 v_{d\kappa}(\vec{x}')v_{d\ell}(\vec{x}')\partial_\kappa\Theta(\vec{x}')\partial_\kappa\rho_c(\vec{x}')$$

$$- \frac{m^2}{8\hbar^2} r_\kappa^2 r_\ell^2 v_{d\kappa}(\vec{x}')v_{d\ell}(\vec{x}')\partial_\ell\partial_\kappa\rho_c(\vec{x}')$$

$$- \frac{im^2}{8\hbar^2} r_\kappa^2 r_\ell^2 v_{d\ell}(\vec{x}')v_{d\ell}(\vec{x}')\partial_\ell\Theta(\vec{x}')\partial_\kappa\rho_c(\vec{x}')$$

$$+ \frac{m^2}{16\hbar^2\rho_c(\vec{x}')} r_\kappa^2 r_\ell^2 v_{d\kappa}(\vec{x}') v_{d\ell}(\vec{x}') \partial_\ell \rho_c(\vec{x}') \partial_\kappa \rho_c(\vec{x}')$$

$$\frac{\phi(\vec{y}) e^{i\chi(\vec{x}',\vec{y})}}{\phi(\vec{x}')} = 1 + \frac{im}{\hbar} r_\kappa v_{\kappa c}(\vec{x}') + \frac{r_\kappa \partial_\kappa \rho_c(x')}{2\rho_c(\vec{x}')} + \frac{im}{2\hbar} r_\kappa r_\ell \partial_\ell v_{\kappa c} - \frac{m^2}{2\hbar^2} r_\kappa r_\ell v_c^2$$

$$+ \frac{im}{4\hbar\rho_c(\vec{x}')} r_\kappa r_\ell v_{\kappa c}(\vec{x}') \partial_\ell \rho_c(\vec{x}') + \frac{r_\kappa r_\ell \partial_\ell \partial_\kappa \rho_c(\vec{x}')}{4\rho_c(\vec{x}')}$$

$$+ \frac{im}{4\hbar\rho_c(\vec{x}')} r_\kappa r_\ell v_{\ell c}(\vec{x}') \partial_\kappa \rho_c(\vec{x}') - \frac{r_\kappa r_\ell \partial_\ell \rho_c(\vec{x}') \partial_\kappa \rho_c(\vec{x}')}{8\rho_c^2(\vec{x}')}$$

$$- \frac{im}{\hbar} r_\kappa v_{\kappa d}(\vec{x}') + \frac{m^2}{\hbar^2} r_\kappa^2 v_{\kappa d}(\vec{x}') v_{\kappa c}(\vec{x}') - \frac{im}{2\hbar} r_\kappa^2 v_{\kappa d}(\vec{x}') \frac{\partial_\kappa \rho_c(\vec{x}')}{\rho_c(\vec{x}')}$$

$$- \frac{im}{2\hbar} r_\kappa r_\ell \partial_\ell v_{\kappa d}(\vec{x}') - \frac{m^2}{2\hbar^2} r_\kappa r_\ell v_d^2(\vec{x}')$$

$$= \{1 + \frac{r_\kappa \partial_\kappa \rho_c(\vec{x}')}{2\rho_c(\vec{x}')} - \frac{m^2}{2\hbar^2} r_\kappa r_\ell v_c^2(\vec{x}') + \frac{r_\kappa r_\ell \partial_\ell \partial_\kappa \rho_c(\vec{x}')}{4\rho_c(\vec{x}')}$$

$$- \frac{r_\kappa r_\ell \partial_\ell \rho_c(\vec{x}') \partial_\kappa \rho_c(\vec{x}')}{8\rho_c^2(\vec{x}')} + \frac{m^2}{\hbar^2} r_\kappa r_\kappa \vec{v}_{d\ell}(\vec{x}') \vec{v}_c(\vec{x}')$$

$$- \frac{m^2}{2\hbar} r_\kappa r_\ell v_d^2(\vec{x}')\} + i\{\frac{m}{\hbar} r_\kappa v_{\kappa c} + \frac{m}{2\hbar} r_\kappa r_\ell \partial_\ell v_{\kappa c}(\vec{x}')$$

$$+ \frac{m}{4\hbar\rho_c(\vec{x}')} r_\kappa r_\ell v_{\ell\kappa c}(\vec{x}') \partial_\kappa \rho_c(\vec{x}') + \frac{m}{4\hbar\rho_c(\vec{x}')} r_\kappa r_\ell v_{\ell\kappa c}(\vec{x}') \partial_\kappa \rho_c(\vec{x}')$$

$$- \frac{m}{\hbar} r_\kappa v_{d\kappa}(\vec{x}') - \frac{m}{2\hbar\rho_c(\vec{x}')} r_\kappa r_\ell v_{\ell d\kappa}(\vec{x}') \partial_\kappa \rho_c(\vec{x}') - \frac{m}{2\hbar} r_\kappa r_\ell \partial_\ell v_{d\kappa}(\vec{x}')\}$$

$$= \{1 - \frac{m^2}{2\hbar^2} r_\kappa r_\ell (v_c^2 - 2v_d v_c + v_d^2) + \frac{r_\kappa \partial\kappa \rho_c(\vec{x}')}{2\rho_c(\vec{x}')}$$

$$+ \frac{r_\kappa r_\ell \partial_\ell \partial_\kappa \rho_c(\vec{x}')}{4\rho_c(\vec{x}')} - \frac{r_\kappa r_\ell \partial_\ell \rho_c(\vec{x}') \partial_\kappa \rho_c(\vec{x}')}{8\rho_c^2(\vec{x}')}\}$$

$$- i\{ \frac{m}{2\hbar} r_\kappa r_\ell \partial_\ell (v_{\kappa d} - v_{\kappa c}) + \frac{m}{\hbar} r_\kappa (v_{\kappa d} - v_{\kappa c})$$

$$+ \frac{m}{2\hbar} \frac{r_\ell r_\kappa v_{\kappa c}}{\rho_c(\vec{x}')} \partial_\kappa \rho_c(\vec{x}') - \frac{m}{2\hbar \rho_c(\vec{x}')} r_\ell r_\kappa v_{\kappa d} \partial_\kappa \rho_c(\vec{x}')\}.$$

Use has been made of the relations ($f(r)$ is a function of $|r|$)

$$\int r_\kappa f(r)\, d^3r = 0 , \quad \int r_\kappa r_\ell r_\upsilon f(r)\, d^3r = 0$$

$$\int r_\kappa r_\ell f(r)\, d^3r = \delta_{\kappa\ell} \frac{1}{3} \int r^2 f(r)\, d^3r$$

Thus $\frac{1}{m}\int V(\vec{r})\, \$_2(\vec{r}) h(\vec{r}) \rho_d \frac{\phi(\vec{y})}{\phi(\vec{x}')} e^{i\chi(\vec{x}',\vec{y})}\, d^3r$

$$= \frac{1}{m}\int V(\vec{r})\, \$_2(\vec{r}) h(\vec{r}) \rho_d \{1 - \frac{m^2}{2\hbar^2} r_\kappa r_\ell (\vec{v}_d - \vec{v}_c)^2$$

$$+ \frac{r_\kappa \partial_\kappa \rho_c(\vec{x}')}{2\rho_c(\vec{x}')} + \frac{r_\kappa r_\ell \partial_\ell \partial_\kappa \rho_c(\vec{x}')}{4\rho_c(\vec{x}')}$$

$$- \frac{r_\kappa r_\ell \partial_\kappa \rho_c(\vec{x}') \partial_\kappa \rho_c(\vec{x}')}{8\rho_c^2(\vec{x}')}\} - i\{ \frac{m}{2\hbar} r_\kappa r_\ell \partial_\ell (v_{\kappa d} - v_{\kappa c})$$

$$+ \frac{m}{\hbar} r_\kappa (v_{\kappa d} - v_{\kappa c}) + \frac{m}{2\hbar} \frac{r_\ell r_\kappa v_{\kappa c} \partial_\kappa \rho_c(\vec{x}')}{\rho_c(\vec{x}')}$$

$$- \frac{m}{2\hbar \rho_c(\vec{x}')} r_\ell r_\kappa v_{\kappa d} \partial_\kappa \rho_c(\vec{x}')\}$$

$$= \frac{\rho_d}{m}\int V(\vec{r})\, \$_2(\vec{r}) h(\vec{r})\, d^3r - \frac{m}{2\hbar} \frac{\rho_d}{3} (\vec{v}_d - \vec{v}_c)^2 \int V(\vec{r})$$

$$\$_2(\vec{r}) h(\vec{r}) r^2 d^3r - \frac{\rho_d}{m} \frac{\nabla^2 \rho_c}{12\rho_c} \int V(\vec{r})\, \$_2(\vec{r}) h(\vec{r}) r^2 d^3r$$

$$- \frac{\rho_d}{m} \frac{(\nabla\rho_c)^2}{24\, \rho_c^2} \int V(\vec{r})\, \$_2(\vec{r}) h(\vec{r}) r^2 d^3r$$

$$- i\{ \frac{\rho_d}{2\hbar} \frac{\nabla.}{3}(\vec{v}_d - \vec{v}_c) \int V(\vec{r}) \$_2(\vec{r}) h(\vec{r}) d^3r$$

$$- \frac{\rho_d}{2\hbar} (\vec{v}_d - \vec{v}_c) . \frac{\nabla \rho_c}{3\rho_c} \int V(\vec{r}) \$_2(\vec{r}) h(\vec{r}) r^2 d^3r\} \qquad (4\text{-}22)$$

The real part of F is

$$\text{Re } F = \frac{\rho_c}{m} \int V(\vec{r}) \$_1(\vec{r}) d^3r + \frac{\rho_d}{m} \int V(\vec{r}) \$_2(\vec{r}) d^3r$$

$$+ \frac{\rho_d}{m} \int V(\vec{r}) \$_2(\vec{r}) h(\vec{r}) d^3r - \frac{m\rho_d}{6\hbar^2} \int V(\vec{r}) \$_2(\vec{r})$$

$$h(\vec{r}) r^2 d^3r \ (\vec{v}_d - \vec{v}_c)^2 + \frac{\rho_d}{m} \frac{\nabla^2 \rho_c}{12\rho_c} \int V(\vec{r}) \$_2(\vec{r})$$

$$h(\vec{r}) r^2 d^3r - \frac{\rho_d}{m} \frac{(\nabla \rho_c)^2}{24 \rho_c^2} \int V(\vec{r}) \$_2(\vec{r}) h(\vec{r}) r^2 d^3r.$$

Letting

$$\tilde{\mu}(\rho_c, \rho_d) = \frac{\rho_c}{m} \int V(\vec{r}) \$_1(\vec{r}) d^3r + \frac{\rho_d}{m} \int V(\vec{r}) \$_2(\vec{r}) d^3r + \frac{\rho_d}{m} \int V(\vec{r}) \$_2(\vec{r}) h(\vec{r}) d^3r \qquad (4\text{-}23)$$

which will be identified as the ground state energy later on,

and

$$\alpha = \frac{m\rho}{3\hbar^2} \int V(\vec{r}) \$_2(\vec{r}) h(\vec{r}) r^2 d^3r \qquad (4\text{-}24)$$

$$\tilde{\mu}^0(\rho_c, \rho_d) = \frac{\rho_d}{12m\rho_c} (\nabla^2 \rho_c) \int V(\vec{r}) \$_2(\vec{r}) h(\vec{r}) r^2 d^3r$$

$$- \frac{\rho_d}{24m\rho_c^2} (\nabla \rho_c)^2 \int V(\vec{r}) \$_2(\vec{r}) h(\vec{r}) r^2 d^3r \qquad (4\text{-}25)$$

and

$$\text{Re } F = \tilde{\mu}(\rho_c, \rho_d) - \frac{\alpha \rho_d}{2} (\vec{v}_d - \vec{v}_c)^2 + \tilde{\mu}^0(\rho_c, \rho_d) \qquad (4\text{-}25a)$$

we thus obtain, the equation of motion for $\vec{v}_c$, ρ_c and ρ_d,

$$\frac{\partial v_c}{\partial t} + (\vec{v}_c.\nabla)\vec{v}_c = -\nabla\{ \frac{-\hbar^2}{8m^2}\frac{(\nabla\rho_c)^2}{\rho_c^2} + \frac{\hbar^2}{4m^2\rho_c}\nabla^2\rho_c + \tilde{\mu}(\rho_c,\rho_d) - \frac{\alpha\rho_d}{2\rho}(\vec{v}_d - \vec{v}_c)^2 + \tilde{\mu}^0(\rho_c,\rho_d)\}$$

(4-26)

For the imaginary part of equation (4-22) is

$$\frac{2m\rho_c \mathrm{Im}\, F}{\hbar} = - \frac{m}{6\hbar^2}\int V(\vec{r})\$_2(\vec{r})h(\vec{r})r^2 d^3r\ \nabla.\{\rho_c\rho_d(\vec{v}_d - \vec{v}_c)$$

$$- \frac{m\rho_d}{3\hbar^2}(\vec{v}_d - \vec{v}_c).\nabla\rho_c\int V(\vec{r})\$_2(\vec{r})h(\vec{r})r^2 d^3r$$

$$= - \frac{\alpha}{2\rho}\nabla.\{\rho_c\rho_d(\vec{v}_d - \vec{v}_c)\} + \Phi(\rho_c,\rho_d,\vec{v}_c,\vec{v}_d)$$

where $\Phi(\rho_c,\rho_d,\vec{v}_c,\vec{v}_d) = - \frac{m\rho_d}{3\hbar^2}(\vec{v}_d - \vec{v}_c).\nabla\rho_c\int V(\vec{r})\$_2(\vec{r})h(\vec{r})r^2 d^3r$

We thus obtain, from equation (4-21)

$$\frac{\partial\rho_c}{\partial t} + \nabla.(\rho_c\vec{v}_c) = - \frac{\alpha}{2\rho}\nabla.\{\rho_c\rho_d(\vec{v}_d - \vec{v}_c)\} + \Phi(\rho_c,\rho_d,\vec{v}_c,\vec{v}_d) \quad (4\text{-}27)$$

From the mass conservation law, we have

$$\frac{\partial\rho}{\partial t} + \nabla.\vec{J} = 0$$

$$\frac{\partial(\rho_c + \rho_d)}{\partial t} + \nabla.(\rho_c\vec{v}_c + \rho_d\vec{v}_d) = 0$$

$$\frac{\partial\rho_c}{\partial t} + \nabla.(\rho_c\vec{v}_c) + \frac{\partial\rho_d}{\partial t} + \nabla.\rho_d\vec{v}_d = 0 \quad (4\text{-}28)$$

Inserting equation (4-27) into equation (4-28), we obtain

$$\frac{\partial \rho_d}{\partial t} + \nabla\cdot(\rho_d \vec{v}_d) = \frac{\alpha}{2\rho}\{\nabla\cdot\rho_c\rho_d(\vec{v}_d - \vec{v}_c)\} - \Phi(\rho_c,\rho_d,\vec{v}_c,\vec{v}_d) \qquad (4\text{-}29)$$

Since the hydrodynamic equation of the two-fluid model must be based on the hydrodynamic equation of a single fluid the Navier-Stoke equation (3-28) as derived microscopically by Fröhlich(17) can be written for He II as

$$\rho_c\frac{\partial \vec{v}_c}{\partial t} + \rho_c(\vec{v}_c\cdot\nabla)\vec{v}_c + \rho_d\frac{\partial \vec{v}_d}{\partial t} + \rho_d(\vec{v}_d\cdot\nabla)\vec{v}_d = -\nabla P + \eta\nabla^2\vec{v}_d \qquad (4\text{-}30)$$

in the bulk system.

Multiplying equation (4-26) by ρ_c yield

$$\rho_c\frac{\partial \vec{v}_c}{\partial t} + \rho_c(\vec{v}_c\cdot\nabla)\vec{v}_c = -\rho_c\nabla\{\frac{-\hbar^2}{8m^2}\frac{(\nabla\rho_c)^2}{\rho_c^2} + \frac{\hbar^2}{4m^2\rho_c}\nabla^2\rho_c + \tilde{\mu}(\rho_c,\rho_d)$$

$$- \frac{\alpha\rho_d}{2\rho}(\vec{v}_d - \vec{v}_c)^2 \qquad (4\text{-}31)$$

Substracting equation (4-31) with equation (4-30), we obtain

$$\rho_d\frac{\partial \vec{v}_d}{\partial t} + \rho_d(\vec{v}_d\cdot\nabla)\vec{v}_d = \rho_c\nabla\{\frac{-\hbar^2}{8m^2}\frac{(\nabla\rho_c)^2}{\rho_c^2} + \frac{\hbar^2\nabla^2\rho_c}{4m^2\rho_c} + \tilde{\mu}(\rho_c,\rho_d) - \frac{\alpha\rho_d}{2\rho}(\vec{v}_d - \vec{v}_c)^2\}$$

$$-\nabla P + \eta\nabla^2\vec{v}_d \qquad (4\text{-}32)$$

$$\frac{\partial \vec{v}_d}{\partial t} + (\vec{v}_d\cdot\nabla)\vec{v}_d = \frac{\rho_c}{\rho_d}\nabla\{\frac{-\hbar^2}{8m^2}\frac{(\nabla\rho_c)^2}{\rho_c^2} + \frac{\hbar^2\nabla^2\rho_c}{4m^2\rho_c} + \tilde{\mu}(\rho_c,\rho_d) - \frac{\alpha\rho_d}{2\rho}(\vec{v}_d - \vec{v}_c)^2\}$$

$$- \frac{\nabla P}{\rho_d} + \frac{\eta\nabla^2\vec{v}_c}{\rho_d} \qquad (4\text{-}33)$$

One can also obtain the equation of motion for the momentum density $\vec{J}$, from

$$\vec{J} = \rho_c\vec{v}_c + \rho_d\vec{v}_d$$

Multiplying equation (4-27) by v_c, we have

$$\vec{v}_c\frac{\partial\rho_c}{\partial t} + \{\nabla.(\rho_c\vec{v}_c)\}\vec{v}_c = \{-\frac{\alpha}{2\rho}\nabla.\rho_c\rho_d(\vec{v}_d - \vec{v}_c) + \Phi\}\vec{v}_c \qquad (4\text{-}34)$$

Multiplying equation (4-29) by v_d, we have

$$\vec{v}_d\frac{\partial\rho_d}{\partial t} + \{\nabla.(\rho_d\vec{v}_d)\}\vec{v}_d = \{\frac{\alpha}{2\rho}\nabla.\rho_c\rho_d(\vec{v}_d - \vec{v}_c) - \Phi\}\vec{v}_d \qquad (4\text{-}35)$$

From equations(4-31), (4-32), (4-34) and (4-35), in the bulk system, we obtain

$$\frac{\partial\{\rho_c\vec{v}_c + \rho_d\vec{v}_d\}}{\partial t} + \{\rho_c(\vec{v}_c.\nabla)\vec{v}_c + \rho_d(\vec{v}_d.\nabla)\vec{v}_d + (\nabla.(\rho_c\vec{v}_c))\vec{v}_c + (\nabla.(\rho_d\vec{v}_d))\vec{v}_d\}$$

$$= \{\frac{\alpha}{2\rho}\nabla.(\rho_c\rho_d(\vec{v}_d - \vec{v}_c)) - \Phi\}(\vec{v}_d - \vec{v}_c) - \nabla P + n\nabla^2\vec{v}_d$$

$$\frac{\partial\vec{J}}{\partial t} + \nabla.\vec{P} = \{\frac{\alpha}{2\rho}\nabla.(\rho_c\rho_d(\vec{v}_d - \vec{v}_c)) - \Phi\}(\vec{v}_d - \vec{v}_c) + n\nabla^2\vec{v}_d \qquad (4\text{-}36)$$

where

$$\nabla.\vec{P} = \{\rho_c(\vec{v}_c.\nabla)\vec{v}_c + \rho_d(\vec{v}_d.\nabla)\vec{v}_d + (\nabla.(\rho_c\vec{v}_c))\vec{v}_c + (\nabla.(\rho_d\vec{v}_d))\vec{v}_d\} + \nabla P$$

To obtain the equation for the energy conservation law, one first obtains the energy density $\mathcal{E}$ per unit mass, from the Hamitonian H of the N-body system, equation (3-3). The energy density $\mathcal{E}(\vec{x})$, which is defined via $< H(\vec{x}) > = \int\mathcal{E}(\vec{x})\, d\vec{x}$, can be written as,

$$\mathcal{E}(\vec{R}) = -\frac{\hbar^2}{2m}\lim_{\vec{r}\to 0}\nabla^2_{\vec{r}}\Omega_1(\vec{r},\vec{R}) + \frac{1}{2}\int V(\vec{x}-\vec{y})\Omega_2(\vec{x},\vec{y};\vec{x},\vec{y})\, d\vec{y} \qquad (4\text{-}38)$$

which $\vec{r} = \vec{x}' - \vec{x}''$ and $\vec{R} = \dfrac{\vec{x}' + \vec{x}''}{2}$, and $2\nabla_{\vec{r}} = \nabla' - \nabla''$ (we will show this in appendix A)

$$\mathcal{E}(\vec{x}) = \frac{-\hbar^2}{2m} \lim_{\vec{x}' \to \vec{x}''} \frac{(\nabla' - \nabla'')(\nabla' - \nabla'')}{4} \Omega_1(\vec{x}';\vec{x}'') + \frac{1}{2}\int V(\vec{x}-\vec{y})\Omega_2(\vec{x}',\vec{y};\vec{x}'',\vec{y})\, d\vec{y}$$

$$= \frac{-\hbar^2}{8m} \lim_{\vec{x}' \to \vec{x}''} (\nabla' - \nabla'')(\nabla' - \nabla'')\phi^*(\vec{x}'')\phi(\vec{x}') + \Lambda_1(\vec{x}';\vec{x}'')$$

$$+ \frac{1}{2}\int V(\vec{x}-\vec{y})\Omega_2(\vec{x}',\vec{y};\vec{x}'',\vec{y})\, d\vec{y}$$

$$= \frac{-\hbar^2}{8m} \lim_{\vec{x}' \to \vec{x}''} (\nabla' - \nabla'')(\nabla' - \nabla'')\phi^*(\vec{x}'')\phi(\vec{x}')$$

$$\frac{-\hbar^2}{8m} \lim_{\vec{x}' \to \vec{x}''} (\nabla' - \nabla'')(\nabla' - \nabla'')\Lambda_1(\vec{x}';\vec{x}'') + \frac{1}{2}\int V(\vec{x}-\vec{y})\Omega_2(\vec{x}',\vec{y};\vec{x}'',\vec{y})\, d\vec{y} \qquad (4\text{-}39)$$

Since

$$\nabla'\phi^*(\vec{x}'')\phi(\vec{x}') = \phi^*(\vec{x}'')\phi(\vec{x}')\left(i\nabla'\Theta(\vec{x}') + \frac{\nabla'\rho}{2\rho}\right)$$

and
$$\nabla''\phi^*(\vec{x}'')\phi(\vec{x}') = \phi^*(\vec{x}'')\phi(\vec{x}')\left(-i\nabla''\Theta(\vec{x}'') + \frac{\nabla''\rho}{2\rho}\right)$$

we get

$$(\nabla' - \nabla'')\phi^*(\vec{x}'')\phi(\vec{x}') = \phi^*(\vec{x}'')\phi(\vec{x}')\left(i\nabla'\Theta(\vec{x}') + i\nabla''\Theta(\vec{x}'') + \frac{\nabla'\rho}{2\rho} - \frac{\nabla''\rho}{2\rho}\right)$$

$$= \phi^*(\vec{x}'')\phi(\vec{x}')\left(\frac{im\vec{v}_C(\vec{x}')}{\hbar} + \frac{im\vec{v}_C(\vec{x}'')}{\hbar} + \frac{\nabla'\rho}{2\rho} - \frac{\nabla''\rho}{2\rho}\right)$$

Proceeding further, we have

$$\frac{-\hbar^2}{8m} \lim_{\vec{x}' \to \vec{x}''} (\nabla' - \nabla'')(\nabla' - \nabla'')\phi^*(\vec{x}'')\phi(\vec{x}')$$

$$= \frac{-\hbar^2}{8m} \lim_{\vec{x}' \to \vec{x}''} (\nabla' - \nabla'')\phi^*(\vec{x}'')\phi(\vec{x}')\Big\{\frac{im\vec{v}_C(\vec{x}')}{\hbar} + \frac{im\vec{v}_C(\vec{x}'')}{\hbar}$$

$$+ \frac{\nabla'\rho}{2\rho} - \frac{\nabla''\rho}{2\rho}\Big\}$$

$$= \frac{m}{4}\frac{\hbar}{2mi}\lim_{\vec{x}'\to\vec{x}''}(\nabla' - \nabla'')\phi^*(\vec{x}'')\phi(\vec{x}')$$

$$= \frac{m}{2}\rho_c\vec{v}_c\cdot\vec{v}_c$$

$$= \frac{m}{2}\rho_c v_c^2 \qquad (4-40)$$

and

$$\frac{-\hbar^2}{8m}\lim_{\vec{x}'\to\vec{x}''}(\nabla' - \nabla'')(\nabla' - \nabla'')\Lambda_1(\vec{x}';\vec{x}'')$$

$$= \frac{-\hbar^2}{8m}\lim_{\vec{x}'\to\vec{x}''}(\nabla' - \nabla'')(\nabla' - \nabla'')\rho_d(\vec{x}';\vec{x}'')e^{i\chi(\vec{x}';\vec{x}'')}$$

$$\nabla'\rho_d(\vec{x}';\vec{x}'')e^{i\chi(\vec{x}';\vec{x}'')} = \rho_d(\vec{x}';\vec{x}'')e^{i\chi(\vec{x}';\vec{x}'')}\{i\nabla'\chi(\vec{x}';\vec{x}'') + \frac{\nabla'\rho_d}{\rho_d}\}$$

$$\nabla''\rho_d(\vec{x}';\vec{x}'')e^{i\chi(\vec{x}';\vec{x}'')} = \rho_d(\vec{x}';\vec{x}'')e^{i\chi(\vec{x}';\vec{x}'')}\{-i\nabla''\chi(\vec{x}';\vec{x}'') + \frac{\nabla''\rho_d}{\rho_d}\}$$

$$(\nabla' - \nabla'')\rho_d(\vec{x}';\vec{x}'')e^{i\chi(\vec{x}';\vec{x}'')} = \rho_d(\vec{x}';\vec{x}'')e^{i\chi(\vec{x}';\vec{x}'')}\{i(\nabla' - \nabla'')\chi(\vec{x}';\vec{x}'') + \frac{\nabla'\rho_d}{\rho_d} - \frac{\nabla''\rho_d}{\rho_d}$$

$$= \Lambda_1(\vec{x}';\vec{x}'')\{i(\nabla' - \nabla'')\chi(\vec{x}';\vec{x}'') + \frac{\nabla'\rho_d}{\rho_d} - \frac{\nabla''\rho_d}{\rho_d}\}$$

$$\frac{-\hbar^2}{8m}\lim_{\vec{x}'\to\vec{x}''}(\nabla' - \nabla'')(\nabla' - \nabla'')\Lambda_1(\vec{x}';\vec{x}'')$$

$$= \frac{-\hbar^2}{8m}\lim_{\vec{x}'\to\vec{x}''}(\nabla' - \nabla'')\Lambda_1(\vec{x}';\vec{x}'')\{i(\nabla' - \nabla'')\chi(\vec{x}';\vec{x}'')$$

$$+ \frac{\nabla'\rho_d}{\rho_d} - \frac{\nabla''\rho_d}{\rho_d}\}$$

$$= \frac{m}{2}\frac{\hbar}{2mi}\lim_{\vec{x}'\to\vec{x}''}(\nabla' - \nabla'')\ {}_1(\vec{x}';\vec{x}'')\frac{\hbar}{2m}\lim_{\vec{x}'\to\vec{x}''}(\nabla' - \nabla'')\chi(\vec{x}';$$

$$\frac{-\hbar^2}{2m}\lim_{\vec{x}'\to\vec{x}''}\frac{(\nabla' - \nabla'')\rho_d\, e^{i\chi(\vec{x}';\vec{x}'')}}{4}\cdot\frac{(\nabla' - \nabla'')\rho_d}{\rho_d}$$

$$= \frac{m}{2}\rho_d\vec{v}_d\cdot\vec{v}_d - \frac{\hbar^2}{2m}\lim_{\vec{x}'\to\vec{x}''}(\nabla' - \nabla'')(\nabla' - \nabla'')\rho_d$$

$$= \frac{m}{2}\rho_d v_d^2 - \frac{\hbar^2}{2m}\lim_{\vec{x}'\to\vec{x}''}\nabla_{\vec{r}}^2|\Omega_1(\vec{r},\vec{R})| \qquad (4\text{-}41)$$

which $|\Omega_1(\vec{r},\vec{R})| = \rho_c + \rho_d = \rho$, in the bulk system

and

$$\frac{1}{2}\int V(\vec{x}-\vec{y})\Omega_2(\vec{x},\vec{y};\vec{x},\vec{y})\,d\vec{y} = \frac{1}{2}\int V(\vec{x}-\vec{y})\Big[\, \$_1^2(|\vec{x}-\vec{y}|)\phi^*(\vec{x})\phi(\vec{x})\phi^*(\vec{y})\phi(\vec{y})$$

$$+ \$_2^2(|\vec{x}-\vec{y}|)\{\phi^*(\vec{x})\phi(\vec{x})\Lambda_1(\vec{y};\vec{y}) + \phi^*(\vec{x})\phi(\vec{y})\Lambda_1(\vec{x};\vec{y})$$

$$+ \phi^*(\vec{y})\phi(\vec{x})\Lambda_1(\vec{y};\vec{x}) + \phi^*(\vec{y})\phi(\vec{y})\Lambda_1(\vec{x};\vec{x})$$

$$+ \Lambda_2(\vec{x},\vec{y};\vec{x},\vec{y})\}\Big]\,d\vec{y}$$

$$= \frac{1}{2}\int V(\vec{r})\Big[\$_1^2(\vec{r})\rho_c^2 + \$_2^2(\vec{r})\{2\rho_c\rho_d$$

$$+ 2\phi^*(\vec{x})\phi(\vec{y})\Lambda_1(\vec{x};\vec{y})\}\Big]\,d^3r$$

$$= \frac{\rho_c^2}{2}\int V(\vec{r})\$_1^2(\vec{r})\,d^3r + \rho_c\rho_d\int V(\vec{r})\$_2^2(\vec{r})\,d^3r$$

$$+ \int V(\vec{r})\$_2^2(\vec{r})\phi^*(\vec{x})\phi(\vec{y})\rho_d h(\vec{r})e^{i\chi(\vec{x};\vec{y})}\,d^3r$$

$$= \frac{\rho_c^2}{2}\int V(\vec{r})\$_1^2(\vec{r})\,d^3r + \rho_c\rho_d\int V(\vec{r})\$_1^2(\vec{r})\,d^3r$$

$$+ \rho_c\rho_d\int V(\vec{r})\$_2^2(\vec{r})h(\vec{r})e^{-i\Theta(\vec{x})}\rho_c^{1/2}e^{i\Theta(\vec{y})}e^{i\chi(\vec{x},\vec{y})}\,d^3r$$

$$= \frac{\rho_c^2}{2}\int V(\vec{r})\,\$_1^2(\vec{r})\; d^3r + \rho_c\rho_d\int V(\vec{r})\,\$_1^2(\vec{r})\; d^3r$$

$$+ \rho_c\rho_d\int V(\vec{r})\,\$_2^2(\vec{r})\,h(\vec{r})\; \frac{\phi(\vec{y})\,e^{i\chi(\vec{x};\vec{y})}}{\phi(\vec{x})}\, d^3r$$

where

$$\frac{\phi(y)\,e^{i\chi(\vec{x};\vec{y})}}{\phi(x)} = 1 - \frac{m^2}{2\hbar^2} r_\kappa r_\ell (\vec{v}_d - \vec{v}_c)^2 + \frac{r_\kappa r_\kappa \rho_c(\vec{x}')}{2\rho_c(x')} + \frac{r_\kappa r_\ell \partial_\ell \partial_\kappa \rho_c}{4\rho_c(\vec{x}')} - \frac{r_\kappa r_\ell \partial_\kappa \rho_c(\vec{x}')\,\partial_\kappa \rho_c(\vec{x}')}{8\rho_c^2(\vec{x}')}$$

We obtain

$$\frac{1}{2}\int V(\vec{x}-\vec{y})\,\Omega_2(\vec{x},\vec{y};\vec{x},y)\,d^3r = \frac{\rho_c^2}{2}\int V(\vec{r})\,\$_1^2(\vec{r})\; d^3r + \rho_c\rho_d\int V(\vec{r})\,\$_1^2(\vec{r})\; d^3r$$

$$+ \rho_c\rho_d\int V(\vec{r})\,\$_2^2(\vec{r})\,h(\vec{r})\Big\{1 - \frac{m^2}{2\hbar^2} r_\kappa r_\ell (\vec{v}_d - \vec{v}_c)^2$$

$$+ \frac{r_\kappa r_\kappa \rho_c(\vec{x}')}{2\rho_c(\vec{x}')} + \frac{r_\kappa r_\ell \partial_\kappa \partial_\ell \rho_c}{4\rho_c(\vec{x}')} - \frac{r_\kappa r_\ell \partial_\kappa \rho_c(\vec{x}')\,\partial_\kappa \rho_c(\vec{x}')}{8\rho_c^2(\vec{x}')}\Big\}\, d^3r$$

$$= \frac{\rho_c^2}{2}\int V(\vec{r})\,\$_1^2(\vec{r})\; d^3r + \rho_c\rho_d\int V(\vec{r})\,\$_1^2(\vec{r})\; d^3r$$

$$+ \rho_c\rho_d\int V(\vec{r})\,\$_2^2(\vec{r})\,h(\vec{r})\; d^3r$$

$$- \rho_c\rho_d(\vec{v}_d - \vec{v}_c)^2 \frac{m}{2}\frac{m}{3\hbar^2}\int V(\vec{r})\,\$_2^2(\vec{r})\,h(\vec{r})\,r^2 d^3r$$

$$+ \rho_c\rho_d \frac{\nabla^2\rho_c}{12\rho_c}\int V(\vec{r})\,\$_2(\vec{r})\,h(\vec{r})\,r^2 d^3r$$

$$- \rho_c\rho_d \frac{(\nabla\rho_c)^2}{24\rho_c^2}\int V(\vec{r})\,\$_2^2(\vec{r})\,h(\vec{r})\,r^2 d^3r \qquad (4\text{-}42)$$

Inserting equations(4-40),(4-41) and (4-42) into equation (4-39), we obtain the energy density

$$\mathcal{E}(\vec{x}) = \frac{m}{2}\rho_c v_c^2 + \frac{m}{2}\rho_d v_d^2 - \frac{\hbar^2}{2m}\lim_{\vec{r}\to 0}\nabla^2_{\vec{r}}|\Omega_1(\vec{r},\vec{R})| + \frac{\rho_c^2}{2}\int V(\vec{r})\$_1^2(\vec{r})\,d^3r$$

$$+ \rho_c\rho_d\int V(\vec{r})\$_2^2(\vec{r})\{1 + h(\vec{r})\}d^3r - \rho_c\rho_d(\vec{v}_d - \vec{v}_c)\frac{m\beta}{2\rho}$$

$$+ \rho_c\rho_d\frac{\nabla^2\rho_c}{12\rho_c}\int V(\vec{r})\$_2^2(\vec{r})h(\vec{r})r^2d^3r$$

$$- \rho_c\rho_d\frac{(\nabla\rho_c)^2}{24\rho_c^2}\int V(\vec{r})\$_2^2(\vec{r})h(\vec{r})r^2d^3r \qquad (4\text{-}43)$$

where

$$\beta = \frac{m\rho}{3\hbar^2}\int V(\vec{r})\$_2^2(\vec{r})h(\vec{r})r^2d^3r \qquad (4\text{-}44)$$

Energy density per unit mass is given by

$$\tilde{\mathcal{E}} = \frac{\mathcal{E}}{m}$$

and so we have

$$\tilde{\mathcal{E}}(\vec{x}) = \frac{1}{2}\rho_c v_c^2 + \frac{1}{2}\rho_d v_d^2 - \frac{\hbar^2}{2m^2}\lim_{\vec{r}\to 0}\nabla^2_r|\Omega_1(\vec{r},\vec{R})| + \frac{\rho_c^2}{2m}\int V(\vec{r})\$_1^2(\vec{r})\,d^3r$$

$$+ \frac{\rho_c\rho_d}{m}\int V(\vec{r})\$_2^2(\vec{r})\{1 + h(\vec{r})\}d^3r - \beta\frac{\rho_c\rho_d}{2\rho}(\vec{v}_d - \vec{v}_c)^2$$

$$+ \frac{\rho_c\rho_d}{m}\frac{\nabla^2\rho_c}{12\rho_c}\int V(\vec{r})\$_2^2(\vec{r})h(\vec{r})r^2d^3r$$

$$- \frac{\rho_c\rho_d}{m}\frac{(\nabla\rho_c)^2}{24\rho_c^2}\int V(\vec{r})\$_2^2(\vec{r})h(\vec{r})r^2d^3r$$

$$= \frac{1}{2}\rho_c v_c^2 + \frac{1}{2}\rho_d v_d^2 - \beta\frac{\rho_c\rho_d}{2\rho}(\vec{v}_d - \vec{v}_c)^2 + \tilde{\xi}_0(\rho_c,\rho_d)$$

$$+ \frac{\rho_c\rho_d}{m}\frac{\nabla^2\rho_c}{12\rho_c}\int V(\vec{r})\$_2^2(\vec{r})h(\vec{r})r^2d^3r$$

$$- \frac{\rho_c \rho_d}{m} \frac{(\nabla\rho_c)^2}{24\rho_c^2} \int V(\vec{r}) \$_2^2(\vec{r}) h(\vec{r}) r^2 d^3r \tag{4-45}$$

which
$$\mathcal{E}_0(\rho_c,\rho_d) = \frac{\rho^2}{2m}\int V(\vec{r}) \$_1^2(\vec{r}) d^3r + \frac{\rho_c \rho_d}{m}\int V(\vec{r}) \$_1^2(\vec{r}) \{1+h(\vec{r})\} d^3r$$

$$- \frac{\hbar^2}{2m} \lim_{\vec{r}\to 0} \nabla_{\vec{r}}^2 |\Omega_1(\vec{r},\vec{R})| \tag{4-46}$$

The last term of equation (4-46) is the "internal" kinetic energy density which is independent of the macroscopic flow velocities.

We will now use the second quantization method to determine the chemical potential at 0 K (ground state energy). The reduced density matrices are defined in the second quantization representation as

$$\Omega_n(\vec{x}_1',\ldots,\vec{x}_n';\vec{x}_1'',\ldots,\vec{x}_n'') = \mathrm{Tr}\ \Omega\Psi^\dagger(\vec{x}_1'')\ldots\Psi^\dagger(\vec{x}_n'')\Psi(\vec{x}_1')\ldots\Psi(\vec{x}_n') \tag{4-47}$$

where Ω is the von Neumann density operator of the system satisfying

$$i\hbar\frac{\partial\Omega}{\partial t} = \left[H,\Omega \right]$$

H is the Hamiltonian operator of the system. $\Psi^\dagger(\Psi)$ is the creation(annihilation) operator of the spinless boson which satisfies the commutation rules. We now consider the first order reduced density matrix Ω_1 at any temperature below T_λ. We write Ω_1 in the form of expression (4-47) as

$$\Omega_1(\vec{x},t;\vec{x}',t) = \mathrm{Tr}\ \Omega\Psi^\dagger(\vec{x}',t)\Psi(\vec{x},t)$$

$$= \langle\Psi^\dagger(\vec{x}',t)\Psi(\vec{x},t)\rangle \tag{4-48}$$

in the Heisenberg representation. In the ground state, we have

$$\Omega_1(\vec{x},t;\vec{x}',t) = \langle E_0,N|\Psi^\dagger(\vec{x}',t)\Psi(\vec{x},t)|E_0,N\rangle$$

$$= \sum_i \langle E_0,N|\Psi^\dagger(\vec{x}',t)|E_i,N-1\rangle\langle E_i,N-1|\Psi(\vec{x},t)|E_0,N\rangle$$

$$= \langle E_0,N|\Psi^\dagger(\vec{x}',t)|E_0,N-1\rangle\langle E_0,N-1|\Psi(\vec{x},t)|E_0,N\rangle$$

$$+\sum_{i\neq 0}\langle E_0,N|\Psi^\dagger(\vec{x}',t)|E_i,N-1\rangle\langle E_i,N-1|\Psi(\vec{x},t)|E_0.N\rangle$$

(4-49)

where $|E_0,N-1\rangle$ is the same (to order 1/N) ground state as $|E_0,N\rangle$.

Comparison between equation (4-49) and the form Ω_1 given by equation (4-2), for the ODLRO in the ground state, gives the condensate macroscopic wave function(23) of the system in the ground state as

$$\phi_0(\vec{x},t) = \langle E_0,N-1|\Psi(\vec{x},t)|E_0 N\rangle$$

$$= \langle E_0,N-1|e^{iHt/\hbar}\psi(x)e^{-iHt/\hbar}|E_0,N\rangle$$

$$= (\rho_c^0)^{1/2}\ e^{-im\mu_0 t/\hbar} \qquad (4\text{-}50)$$

hich is independent of space in the bulk system, and

$$m\mu_0 = E_0(N) - E_0(N-1) \qquad (4\text{-}51)$$

where $m\mu_0$ is the chemical potential or the ground state energy per particle in the ground state. From equation (4-13) we get

$$i\hbar\frac{\partial\phi(\vec{x}')}{\partial t} = \frac{-\hbar^2}{2m}\nabla^2_{\vec{x}}\phi(\vec{x}') + \lim_{|\vec{x}'-\vec{x}''|\to\infty}\frac{\int V(\vec{x}'-\vec{y})\,{}_2(\vec{x}',\vec{y};\vec{x}'',\vec{y})\,d\vec{y}}{\phi^*(\vec{x}')}$$

which is the exact equation of motion for the condensate macroscopic wave function ϕ.

Using equation (4-50) and form of Ω_2 given by (4-12) in the exact equation of motion for ϕ_0 (equation(4-13)), in the ground state, we have

$$m\mu_0 = \mathrm{Re} \lim_{|\vec{x}-\vec{x}'|\to\infty} \frac{\int V(\vec{x}-\vec{y})\,\Omega_2(\vec{x},\vec{y};\vec{x}',\vec{y})\,d\vec{y}}{\phi_0^*(\vec{x}')\,\phi_0(\vec{x})}$$

$$= \rho_c^0 \int V(\vec{r})\, S_1(\vec{r})\, d^3r + \rho_d^0 \int V(\vec{r})\, S_2(\vec{r})\, d^3r$$

$$+ \rho_d^0 \int V(\vec{r})\, S_2(\vec{r})\, h(\vec{r})\, d^3r \qquad (4\text{-}52)$$

where the real function $h(\vec{r})$ comes from the definition of the depletion "bulk" density ρ_d as

$$|\Lambda_1(\vec{x}';\vec{x}'')| = |\Lambda_1(|\vec{x}'-\vec{x}''|)| = \rho_d h(|\vec{x}'-\vec{x}''|)$$

and ρ_d^0 denotes the depletion density of liquid ^{4}He II in the ground state

Comparison equation (4-52) and equation (4-25a) we have $\vec{v}_d = \vec{v}_c$ at 0 K when ρ_d is not equal to zero. Since the total current density $\vec{J} = \rho\vec{v} = \rho_c\vec{v}_c + \rho_d\vec{v}_d$, it can be seen that

$$\vec{v}_c = \vec{v}_d = \vec{v} \quad , \quad \text{in the ground state.}$$

The equation (4-23) is the chemical potential at 0 K or the ground state energy.

4.2 Condensate Fraction in ^{4}He II

It is widely believed that the ratio of the (bulk) condensate to the total density = ρ_c/ρ is in the range of 0.05 at 0 K for ^{4}He II based on theoretical considerations. An experimental confirmation of this fact would be describable. We will review the method for determining the condensate fraction at all temperature from knowledge of the pair distribution function, which was proposed by Cummings, we will then extend it.

We must first understood that ODLRO in Ω_1 is a statement that each helium atom must be regarded as being partly in the condensate, and partly localized within a distance given by the range of $\Lambda_1(\vec{x}' - \vec{x}'')$. The situation may be pictures as follows; (above the lambda transition, T_λ = 2.17 K) the helium atoms are localized to within an angstrom or so. As the temperature is lowered in this region, the diminishing thermal motion, and increasing localization, is reflected experimentally in a heightening of the maxima of $g(\vec{r},T>T_\lambda)$. As the temperature is lower below T_λ, each helium atom begin to contribute to the uniform condensate density. Scattering of X-rays or neutrons will take place only at the localized "lump", the depletion part of the density associated with Λ_1. As the temperature is lowered in the region below T_λ, the "melting" of the "lump" into the condensate will be reflected as a diminishing scattering intensity. This will result in a lowering of the maxima of $g(\vec{r},T)$ will decreasing temperature. A reversal in trend of maxima height of $g(\vec{r},T)$ is predicted as T is lowered through T_λ.

of $g(\vec{r},T)$ is predicted as T is lowered through T_λ.

It may further be expected that the relative localization, or structure, of the lump will remain relatively unchanged as T decreas below T_λ. Above T_λ both average kinetic and potential energy can be lowered by decreasing the thermal motion, since the average potential energy per particle is

$$< V > \quad = \quad \frac{1}{2}\rho \int V(\vec{r})\, g(\vec{r})\, d\vec{r} \tag{4-53}$$

and the largest maxima of $g(\vec{r})$ multiplies the negative part of $V(\vec{r})$. The lamda transition is presumably the point below which the system cannot decrease its energy in this manner any longer, since the increasing localization with decreasing temperature would lead to an increasing kinetic energy via the uncertainty principle. Below this point, $T = T_\lambda$, energy can be lowered further by having each helium particle go partly into the condensate. This would results in a lowering of both kinetic and potential energy.

Since the condensation(ODLRO) must also appear in the second reduced density matrix Ω_2, we can write it as

$$\Omega_2(\vec{x},\vec{y};\vec{x},\vec{y}) \quad = \quad \Omega_2(|\vec{x}-\vec{y}|) \quad = \quad \rho^2 g(\vec{r},T)$$

$$= \quad \$_1^2(\vec{r})\rho_c^2 + 2\$_2^2(\vec{r})\rho_c\rho_d + 2\$_2^2(\vec{r})\rho_c|\Lambda_1(\vec{r})| + \Lambda_2(\vec{r}) \tag{4-54}$$

The function $\Lambda_1(0) = \rho_d$ and $\Lambda_1(\vec{r}) = 0$ for $r > r_1$, where r_1 is about 4.5 Å the point where the first reduced density matrix $\Omega_1(r)$ becomes equal to the condensate density ρ_c (fig 4.1), and where $g(\vec{r},T)$ assume the value 1 for the second time. The function Λ_2 must satisfy all condition analogous to the condition required of Ω_2. Thus $\Lambda_2(0) = 0$, and $\Lambda_2(\vec{r}) = \rho_d^2$ for $r>r_2$, where r_2 is several times larger than r_1. We define

$$|\Lambda_2(\vec{r})| = \rho_d^2 g(\vec{r},T) \tag{4-55}$$

The bulk "depletion" density ρ_d is defined via

$$|\Lambda_1(\vec{r})| = \rho_d h(\vec{r}) \tag{4-56}$$

where $h(\vec{r})$ approaches zero when $r>r_1$. The screening factors, $\$_1(\vec{r})$ and $\$_2(\vec{r})$, have been introduced in the expression (4-54) because of the "core" condition required of Ω_2

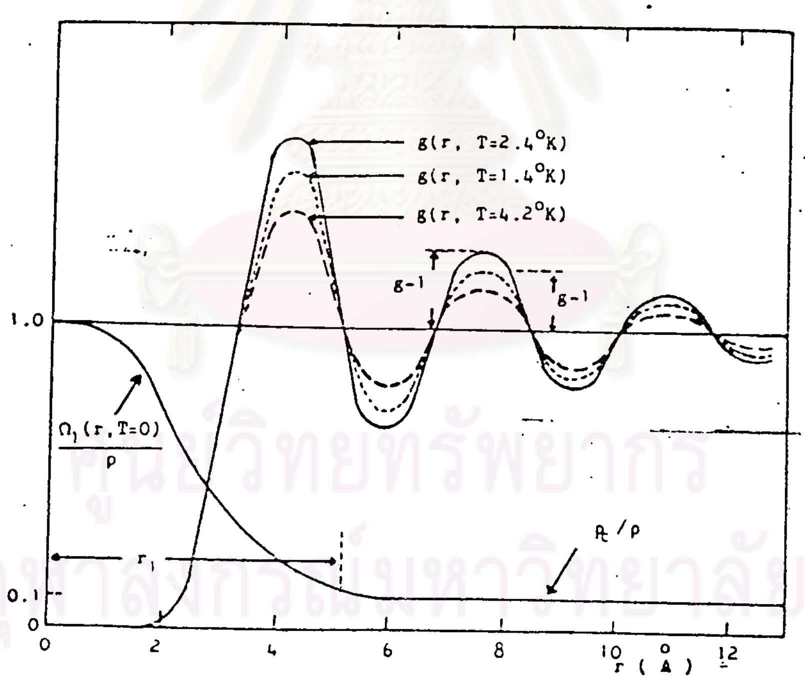

Fig 4.1 The r-dependence of $g(r,T)$, and Ω_1/ρ : observe that $\Omega_1(r)$ attains its asymptotic value of $\tilde{\rho}$ at $r \approx 4.5$ Å (cf. ref.5).

$\tilde{g}(\vec{r},T>T_\lambda) = g(\vec{r},T)$, and $\rho_c = 0$ above T_λ; $g(\vec{r},T<T_\lambda) - g(\vec{r},T>T_\lambda)$ whenever $g(\vec{r},T>T_\lambda) = 1$(which is the "crossing points" observed by Gordon et al, see Fig 4.1)

In the expression (4-54) in the region $r_1>r>r_2$, the screening factor, $\$_1(\vec{r})$ and $\$_2(\vec{r})$, approach unity,and the function $h(\vec{r})$ approachs zero

$$\Omega_2(\vec{r}) = \rho^2 g(\vec{r},T) = \rho_c^2 + 2\rho_c\rho_d + \rho_d^2\tilde{g}(\vec{r},T) \qquad (4\text{-}57)$$

and

$$\rho = \rho_c + \rho_d \qquad (4\text{-}58)$$

$$\rho^2 g(\vec{r},T) = (\rho - \rho_d)^2 + 2(\rho - \rho_d)\rho_d + \rho_d^2\tilde{g}(\vec{r},T)$$

$$= \rho^2 - 2\rho\rho_d + \rho_d^2 + 2\rho\rho_d - 2\rho_d^2 + \rho_d^2\tilde{g}(\vec{r},T)$$

$$= \rho^2 - \rho_d^2 + \rho_d^2\tilde{g}(\vec{r},T)$$

$$\rho^2 g(\vec{r},T) - \rho^2 = \rho_d^2\tilde{g}(\vec{r},T) - \rho_d^2$$

$$\rho^2\{g(\vec{r},T) - 1\} = \rho_d^2\{\tilde{g}(\vec{r},T) - 1\}$$

$$\frac{\rho_d}{\rho} = \left[\frac{g(\vec{r},T) - 1}{\tilde{g}(\vec{r},T) - 1}\right]^{1/2} \qquad (4\text{-}59)$$

This equation gives ρ_d/ρ as a function of the measurable $g(\vec{r},T) - 1$ and $\tilde{g}(\vec{r},T) - 1$ represents the structure of the depletion "lumps", and $g(\vec{r},T)-1$ may be expected to be equal to $g(\vec{r},T \simeq T_\lambda)-1$. Since $\rho = \rho_c + \rho_d$, equation

(4-59) gives

$$\frac{\rho-\rho_c}{\rho} = \left[\frac{g(\vec{r},T) - 1}{g(\vec{r},T\ T\ ) - 1}\right]^{1/2}$$

$$\frac{\rho_c}{\rho} = 1 - \left[\frac{g(\vec{r},T) - 1}{g(\vec{r},T\ T\ ) - 1}\right]^{1/2}$$

which gives the dependence of ρ_c/ρ on the measurable $g(\vec{r},T) - 1$

Therefore, we must not think that one ^{4}He atom is in the condensate and another is not, when the condensation has occured. Instead each and very atom must be regarded as being in the condensate and partly localized within a distance about the average interatomic spacing (ie, the range of $\Lambda_1(|\vec{x}'-\vec{x}''|)$ is about 4.5 Å). The diffraction pattern from the measurement of the liquid structure factor $S(\vec{k})$ by X-rays of neutron diffraction will be due to the "lumps" of higher density (the depletion part of the density associated with Λ_1) and not from the uniform background of condensate (if $\rho = \rho_c$ at T = 0 K, there is not structure the case of the ideal Bose gas). The measurement of the relative intensity of the diffraction patterns, at different temperature below T_λ; will show the diminishing of scattering intensity from the depleting part since the "lumps" will have melted into the condensate as the temperature is lowered. The total density remains nearly constant.

On the other hand when the temperature is above T_λ the helium atoms are partly localized to within an angstrom or so. When the temperature is lowered below T_λ, each helium atom will begin to contribute to a (structureless) uniform condensate density, spreading throughout the volume occupied by the system. Above T_λ, the lowering temperature causes diminishing thermal motion and increasing localization which have experimentally causes a heightening of the maxima of the pair distribution function $g(\vec{r},T>T_\lambda)$. The pair distribution function $g(\vec{r},T)$ is relate to the liquid structure factor $S(\vec{k})$ by a Fourier transform. The largest experimental error in $S(\vec{k})$ occurs for very small momentum transfer $\vec{k}$. We expect the maxima of $g(\vec{r})$ to be increasingly in error as $\vec{r}$ is increased . Thus the most reliable value for ρ_c/ρ is expected to come from the second maximum or minimum of $g(r)$. For $r_1<r<r_2$, equation (4-60) should be independent of $\vec{r}$ ideally.

For the numerical value of the condensate fraction ρ_c/ρ, we will use the connection between Landau two-fluid model and Bose-Einstein condensation, which Visoottiviseth(29) has proposed. Visoottiviseth(5) should that the expression for the energy density obtained from the microscopic point of view is

$$\mathcal{E} = \frac{1}{2}\rho v^2 + \frac{\rho_c\rho_d}{2\rho}(1-\beta)(\vec{v}_d-\vec{v}_c)^2 + \tilde{\mathcal{E}}_0(\rho_c,\rho_d) \qquad (4\text{-}61)$$

and from the phenomenological two-fluid view point is

$$\mathcal{E} = \frac{1}{2}\rho v^2 + \frac{\rho_s\rho_n}{2\rho}(\vec{v}_n-\vec{v}_s)^2 + \mathcal{E}_0(\rho,S,\rho_n) \qquad (4\text{-}62)$$

The terms which depend on the macroscopic flow velocities are then equated in two expressions,

$$\rho_c\rho_d(1-\beta)(\vec{v}_d-\vec{v}_c)^2 = \rho_s\rho_n(\vec{v}_n-\vec{v}_s)^2 \qquad (4\text{-}63)$$

He has also shown that the expression for the chemical potential obtained from both the microscopic view point is

$$\mu \quad = \quad \tilde{\mu}(\rho_c,\rho_d) - \frac{\alpha\rho_d}{2\rho}(\vec{v}_n - \vec{v}_s)^2 \qquad (4-64)$$

and from phenomenological point of view is

$$\mu(P,T,\vec{v}_n-\vec{v}_s) \quad \simeq \quad \mu(P,T) - \frac{\rho_n}{2\rho}(\vec{v}_n - \vec{v}_s)^2 \qquad (4-65)$$

Again equating the terms depending on the macroscopic flow velocities,

$$\alpha\rho_d(\vec{v}_d - \vec{v}_c)^2 \quad = \quad \rho_n(\vec{v}_n - \vec{v}_s)^2 \qquad (4-66)$$

Finally, he should that the comparison between the relations obtained from equation (4-63) and equation (4-66) yields the connection

$$\rho_c \quad = \quad \left[\frac{\alpha}{(1 - \beta)}\right]\rho_s \qquad (4-67)$$

where α and β are the equations (4-24) and (4-44) respectively. We will use the expression of α and β calculating the numerical value of condensate fraction ρ_c/ρ at 0 K.