CHAPTER III

THEORETICAL CONSIDERATION

3.1 Basic Equations and General Solutions of Poroelastic Materials

Consider a poroelastic medium with a conventional cylindrical polar coordinate system (r,θ,z) defined such that the z-axis is perpendicular to the free surface as shown in Figure 1.1. Let u_i and w_i denote the displacement of the solid matrix and the fluid displacement relative to it, in the i direction $(i=r,z)$, respectively. Then, the constitutive relations for a homogeneous poroelastic material (Biot,1941) can be expressed as follows

$$\sigma_{rr} = 2\mu\frac{\partial u_r}{\partial r} + \lambda e - \alpha p \tag{3.1}$$

$$\sigma_{\theta\theta} = \lambda e + 2\mu\frac{u_r}{r} - \alpha p \tag{3.2}$$

$$\sigma_{zz} = 2\mu\frac{\partial u_z}{\partial z} + \lambda e - \alpha p \tag{3.3}$$

$$\sigma_{zr} = \mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \tag{3.4}$$

$$p = -\alpha M e + M\zeta \tag{3.5}$$

where

$$e = \frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} + \frac{u_r}{r} \tag{3.6}$$

$$\zeta = -\left(\frac{\partial w_r}{\partial r} + \frac{\partial w_z}{\partial z} + \frac{w_r}{r}\right) \tag{3.7}$$

In the above equations, σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz} and σ_{zr} denote the total stress components of the bulk material; e the dilatation of the solid matrix; μ the shear modulus and λ a constant of the bulk material, respectively; p the excess pore fluid pressure (suction is considered negative) and ζ the variation of fluid content per unit reference volume. In addition, α and M are Biot's parameters accounting for compressibility of the two-phased material (Biot, 1941). It is noted that $0 \le \alpha \le 1$ and $0 \le M < \infty$ for all poroelastic materials. For a completely dry material $\alpha = 0$ and $M = 0$, whereas for a material with incompressible constituents, $\alpha = 1$ and $M \to \infty$.

The equations of motion for a poroelastic medium undergoing axisymmetric deformations, in the absence of body forces (solid and fluid) and a fluid source, can be expressed according to Biot (1962) as

$$\mu \nabla^2 u_r + \left(\lambda + \alpha^2 M + \mu\right)\frac{\partial e}{\partial r} - \mu \frac{u_r}{r^2} - \alpha M \frac{\partial \zeta}{\partial r} = \rho\, \ddot{u}_r + \rho_f \ddot{w}_r \tag{3.8}$$

$$\mu \nabla^2 u_z + \left(\lambda + \alpha^2 M + \mu\right)\frac{\partial e}{\partial z} - \alpha M \frac{\partial \zeta}{\partial z} = \rho\, \ddot{u}_z + \rho_f \ddot{w}_z \tag{3.9}$$

$$\alpha M \frac{\partial e}{\partial r} - M \frac{\partial \zeta}{\partial r} = \rho_f \ddot{u}_r + m \ddot{w}_r + b \dot{w}_r \tag{3.10}$$

$$\alpha M \frac{\partial e}{\partial z} - M \frac{\partial \zeta}{\partial z} = \rho_f \ddot{u}_z + m \ddot{w}_z + b \dot{w}_z \tag{3.11}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{3.12}$$

In equations (3.8) to (3.11), an over dot denotes the derivative with respect to the time parameter t; ρ and ρ_f are the mass densities of the bulk material and the pore fluid, respectively, and $m = \rho_f / \beta$ (β=porosity), is a density-like parameter. In addition, b is a parameter accounting for the internal friction due to the relative motion between the solid matrix and the pore fluid. The parameter b is defined as the ratio between the fluid viscosity and the intrinsic permeability of the porous medium.

The motion under consideration is assumed to be time-harmonic with the factor of $e^{i\omega t}$ as the time function, where ω is the frequency of the motion and i is the imaginary number. The term $e^{i\omega t}$ is henceforth suppressed from all expressions for brevity. The radius of the circular plate denoted by a is selected to non-dimensionalize all space dimensions including the co-ordinate frame. Stresses and pore pressure are non-dimensionalized with respect to the shear modulus μ of the bulk material. All variables are replaced by the non-dimensional ones, but the previous notations are used for convenience. In addition, the material properties λ, M, ρ_f, m, b, and frequency ω are nondimensionalized as follows :

$$\lambda^* = \frac{\lambda}{\mu}, M^* = \frac{M}{\mu},\ \rho^* = \frac{\rho_f}{\rho},\ m^* = \frac{m}{\rho},\ b^* = \frac{ab}{\sqrt{\rho\mu}}, \text{ and } \delta = \sqrt{\frac{\rho}{\mu}}\omega a \tag{3.13}$$

The governing partial differential equations, equations (3.8) to (3.11), can be solved by introducing the displacement decomposition based on Helmholtz representation for an axisymmetric vector field given by (Philippacopoulos, 1989):

$$u_r(r,z) = \frac{\partial\Phi_1(r,z)}{\partial r} - \frac{\partial\Psi_1(r,z)}{\partial z} \tag{3.14}$$

$$u_z(r,z) = \frac{\partial\Phi_1(r,z)}{\partial z} - \frac{1}{r}\frac{\partial(r\Psi_1(r,z))}{\partial r} \tag{3.15}$$

$$w_r(r,z) = \frac{\partial\Phi_2(r,z)}{\partial r} - \frac{\partial\Psi_2(r,z)}{\partial z} \tag{3.16}$$

$$w_z(r,z) = \frac{\partial\Phi_2(r,z)}{\partial z} - \frac{1}{r}\frac{\partial(r\Psi_2(r,z))}{\partial r} \tag{3.17}$$

where $\Phi_i(i=1,\ 2)$ and $\Psi_i(i=1,\ 2)$ are functions of scalar and vector fields, respectively. Substitution of equation (3.14) to (3.17) into the equations of motion, equation (3.8) to (3.11), yields two sets of partial differential equations for Φ_1, Φ_2 and Ψ_1, Ψ_2 as

$$[(\lambda^* + \alpha^2 M^* + 2)\nabla^2 + \delta^2]\Phi_1 = -(\alpha M^* \nabla^2 + \rho^* \delta^2)\Phi_2 \tag{3.18}$$

$$(\alpha M^* \nabla^2 + \rho^* \delta^2)\Phi_1 = (ib^* \delta - m^* \delta^2 - M^* \nabla^2)\Phi_2 \tag{3.19}$$

$$(\nabla^2 + \delta^2)\Psi_1 = -\rho^* \delta^2 \Psi_2 \tag{3.20}$$

$$\rho^* \delta^2 \Psi_1 = (ib^* \delta - m^* \delta^2)\Psi_2 \tag{3.21}$$

The n^{th}-order Hankel integral transform of a function $f(r,z)$ is defined as (Sneddon, 1970)

$$\bar{f}(\xi,z) = \int_0^\infty r f(r,z) J_n(\xi r) dr \tag{3.22}$$

and the inverse relationship is given by

$$f(r,z) = \int_0^\infty \xi \bar{f}(\xi,z) J_n(\xi r) d\xi \tag{3.23}$$

where J_n is the Bessel function of the first kind of the n^{th}-order (Watson, 1944)and ξ is the Hankel transform parameter.

Application of the zeroth-order Hankel transform to equations (3.18) to (3.21) yield the following ordinary differential equations

$$\left[\left(\lambda^* + \alpha^2 M^* + 2\right)\left(\frac{d^2}{dz^2} - \xi^2\right) + \delta^2\right]\bar{\Phi}_1 = \left[\alpha M^*\left(\xi^2 - \frac{d^2}{dz^2}\right) - \rho^* \delta^2\right]\bar{\Phi}_2 \tag{3.24}$$

$$\left[\alpha M^*\left(\frac{d^2}{dz^2} - \xi^2\right) + \rho^* \delta^2\right]\bar{\Phi}_1 = \left[ib^* \delta - m^* \delta^2 - M^*\left(\xi^2 - \frac{d^2}{dz^2}\right)\right]\bar{\Phi}_2 \tag{3.25}$$

$$\left(\xi^2 - \frac{d^2}{dz^2} - \delta^2\right)\bar{\Psi}_1 = \rho^* \delta^2 \bar{\Psi}_2 \tag{3.26}$$

$$\rho^* \delta^2 \bar{\Psi}_1 = \left(ib^* \delta - m^* \delta^2\right)\bar{\Psi}_2 \tag{3.27}$$

It can be shown that the general solutions for $\bar{\Phi}_i$ and $\bar{\Psi}_i$ $(i=1,2)$ can be expressed as

$$\bar{\Phi}_1 = Ae^{\gamma_1 z} + Be^{-\gamma_1 z} + Ce^{\gamma_2 z} + De^{-\gamma_2 z} \tag{3.28}$$

$$\bar{\Phi}_2 = \chi_1\left(Ae^{\gamma_1 z} + Be^{-\gamma_1 z}\right) + \chi_1\left(Ce^{\gamma_2 z} + De^{-\gamma_2 z}\right) \tag{3.29}$$

$$\bar{\Psi}_1 = Ee^{\gamma_3 z} + Fe^{-\gamma_3 z} \tag{3.30}$$

$$\bar{\Psi}_2 = \chi_3\left(Ee^{\gamma_3 z} + Fe^{-\gamma_3 z}\right) \tag{3.31}$$

where $A(\xi,\delta)$, $B(\xi,\delta)$, ..., $F(\xi,\delta)$ are the arbitrary functions to be determined by imposing appropriate boundary and/or continuity conditions and the superposed bar denotes the Hankel transform of quantities with respect to the r-coordinate. In addition, the parameters γ_i, χ_i etc., appearing in the above equations are defined as

$$\gamma_i = \sqrt{\xi^2 - L_i^2}, \quad i=1,2 \tag{3.32}$$

$$\gamma_3 = \sqrt{\xi^2 - S^2} \tag{3.33}$$

$$\chi_i = \frac{(\lambda^* + \alpha^2 M^* + 2)L_i^2 - \delta^2}{\rho^* \delta^2 - \alpha M^* L_i^2}, \quad i=1,2 \tag{3.34}$$

$$\chi_3 = \frac{\rho^* \delta}{ib^* - m^* \delta} \tag{3.35}$$

and

$$L_1^2 = \frac{w_1 + \sqrt{w_1^2 - 4w_2}}{2} \tag{3.36}$$

$$L_2^2 = \frac{w_1 - \sqrt{w_1^2 - 4w_2}}{2} \tag{3.37}$$

$$S^2 = (\rho^* \chi_3 + 1)\delta^2 \tag{3.38}$$

$$w_1 = \frac{(m^*\delta^2 - ib^*\delta)(\lambda^* + \alpha^2 M^* + 2) + M^*\delta^2 - 2\alpha M^* \rho^* \delta^2}{(\lambda^* + 2)M^*} \tag{3.39}$$

$$w_2 = \frac{(m^*\delta^2 - ib^*\delta)\delta^2 - (\rho^*)^2\delta^4}{(\lambda^* + 2)M^*} \tag{3.40}$$

In view of equation (3.3) to (3.5), (3.14) to (3.17) and (3.28) to (3.31), it can be shown that the general solutions for the zeroth-order Hankel transforms of u_z, w_z, σ_{zz} and p, and the first-order Hankel transform of u_r, w_r and σ_{zr}, respectively, can be expressed as

$$\bar{u}_r = -\xi\left(Ae^{\gamma_1 z} + Be^{-\gamma_1 z} + Ce^{\gamma_2 z} + De^{-\gamma_2 z}\right) - \gamma_3\left(Ee^{\gamma_3 z} - Fe^{-\gamma_3 z}\right) \tag{3.41}$$

$$\bar{u}_z = \gamma_1\left(Ae^{\gamma_1 z} - Be^{-\gamma_1 z}\right) + \gamma_2\left(Ce^{\gamma_2 z} - De^{-\gamma_2 z}\right) + \xi\left(Ee^{\gamma_3 z} + Fe^{-\gamma_3 z}\right) \tag{3.42}$$

$$\bar{w}_r = -\xi\left(\chi_1\left(Ae^{\gamma_1 z} + Be^{-\gamma_1 z}\right) + \chi_2\left(Ce^{\gamma_2 z} + De^{-\gamma_2 z}\right)\right) - \gamma_3\chi_3\left(Ee^{\gamma_3 z} - Fe^{-\gamma_3 z}\right) \tag{3.43}$$

$$\bar{w}_z = \gamma_1\chi_1\left(Ae^{\gamma_1 z} - Be^{-\gamma_1 z}\right) + \gamma_2\chi_2\left(Ce^{\gamma_2 z} - De^{-\gamma_2 z}\right) - \xi\chi_3\left(Ee^{\gamma_3 z} + Fe^{-\gamma_3 z}\right) \tag{3.44}$$

$$\bar{\sigma}_{zr} = -2\xi\left(\gamma_1\left(Ae^{\gamma_1 z} - Be^{-\gamma_1 z}\right) + \gamma_2\left(Ce^{\gamma_2 z} - De^{-\gamma_2 z}\right)\right) - \left(\xi^2 + \gamma_3^2\right)\left(Ee^{\gamma_3 z} + Fe^{-\gamma_3 z}\right) \tag{3.45}$$

$$\bar{\sigma}_{zz} = \beta_1\left(Ae^{\gamma_1 z} + Be^{-\gamma_1 z}\right) + \beta_2\left(Ce^{\gamma_2 z} + De^{-\gamma_2 z}\right) + 2\xi\gamma_3\left(Ee^{\gamma_3 z} - Fe^{-\gamma_3 z}\right) \tag{3.46}$$

$$\bar{p} = \eta_1\left(Ae^{\gamma_1 z} + Be^{-\gamma_1 z}\right) + \eta_2\left(Ce^{\gamma_2 z} + De^{-\gamma_2 z}\right) \tag{3.47}$$

where the variables η_i and β_i are given by

$$\eta_i = (\alpha + \chi_i)M^* L_i^2, \quad i = 1,2 \tag{3.48}$$

$$\beta_i = 2\gamma_i^2 - \lambda^* L_i^2 - \alpha\eta_i, \quad i = 1,2 \tag{3.49}$$

The fluid discharge is defined as the time derivative of the fluid displacement relative to the solid matrix. The fluid discharge, non-dimensionalized by $\sqrt{\mu/\rho}$, can be expressed as

$$q_n = i\delta w_n \qquad ,n = r,z \tag{3.50}$$

3.2 Indirect Boundary Integral Equation Method

The analytical solution of the interaction problem as shown in Figure 1.1 is mathematically intractable. An alternative method is to formulate the inclusion problem by using the indirect boundary integral equation method (Ohsaki, 1973, Luco and Wong, 1986, Wang and Rajapakse, 1990 and Rajapakse and Senjuntichai, 1995)

Consider a poroelastic domain Ω which include a volume V bounded by a surface S with a Cartesian co-ordinate system (r,θ,z) defined as shown in Figure 3.1. It is assumed that a set of admissible boundary conditions are specified on the surface S. In addition, the pore pressure p or the fluid discharge q_n normal to the surface S has to be specified. The surface S is called a fully permeable surface when p is equal to zero on S whereas an impermeable surface corresponds to the case where $q_n = 0$ on S.

The indirect boundary integral equation method presented herein follows the concepts presented by Luco and Wong (1986) for the case of an ideal elastic half-space and Wang and Rajapakse (1990) for the case of a transversely isotropic half-space, respectively. The present scheme is based on the consideration of an equivalent problem defined with respect to an undisturbed poroelastic medium such as the poroelastic domain Ω with a volume V bounded by a surface S as shown in Figure 3.1. It is assumed that a set of unknown forces with magnitude $f_i(\mathbf{r}',\omega)$ with $i = r,z$ and a fluid source $\Gamma(\mathbf{r}',\omega)$ are applied on the auxiliary surface S' defined interior to real surface S as shown in Figure 3.1. In the case of a time-harmonic problem where the motion is prescribed with a time factor of $e^{i\omega t}$, the analysis is performed directly in the frequency domain.

The displacement $u_i(\mathbf{r},\omega)$, traction $T_i(\mathbf{r},\omega)$, pore pressure $p(\mathbf{r},\omega)$ and fluid discharge in the direction of unit normal $\mathbf{n}$ to an arbitrary plane, denoted by $q_n(\mathbf{r},\omega)$, at any point with position vector $\mathbf{r}$ in V can be expressed as

$$u_i(\mathbf{r},\omega) = \int_{S'} G_{ij}(\mathbf{r},\omega;\mathbf{r}')f_j(\mathbf{r}',\omega)\mathrm{d}S' + \int_{S'} G_{iq}(\mathbf{r},\omega;\mathbf{r}')\Gamma(\mathbf{r}',\omega)\mathrm{d}S' \tag{3.51}$$

$$T_i(\mathbf{r},\omega) = \int_{S'} H_{ij}(\mathbf{r},\omega;\mathbf{r}')f_j(\mathbf{r}',\omega)\mathrm{d}S' + \int_{S'} H_{iq}(\mathbf{r},\omega;\mathbf{r}')\Gamma(\mathbf{r}',\omega)\mathrm{d}S' \tag{3.52}$$

$$p(\mathbf{r},\omega) = \int_{S'} H_{pj}(\mathbf{r},\omega;\mathbf{r}')f_j(\mathbf{r}',\omega)\mathrm{d}S' + \int_{S'} H_{pq}(\mathbf{r},\omega;\mathbf{r}')\Gamma(\mathbf{r}',\omega)\mathrm{d}S' \tag{3.53}$$

$$q_n(\mathbf{r},\omega) = \int_{S'} G_{qj}(\mathbf{r},\omega;\mathbf{r}')f_j(\mathbf{r}',\omega)\mathrm{d}S' + \int_{S'} G_{qq}(\mathbf{r},\omega;\mathbf{r}')\Gamma(\mathbf{r}',\omega)\mathrm{d}S' \tag{3.54}$$

In the above equations, $G_{ij}(\mathbf{r},\omega;\mathbf{r}')$ and $G_{iq}(\mathbf{r},\omega;\mathbf{r}')$ denote the displacement in the i-direction $(i=r,z)$ at a point $\mathbf{r}$ due to an impulsive force in the j-direction $(j=r,z)$ and an impulsive fluid source(the injection of fluid) applied at point $\mathbf{r}'$, respectively; $G_{qj}(\mathbf{r},\omega;\mathbf{r}')$ and $G_{qq}(\mathbf{r},\omega;\mathbf{r}')$ denote fluid discharge in the direction of a vector $\mathbf{n}$ at a point $\mathbf{r}$ due to an impulsive force in the j-direction $(j=r,z)$ and an impulsive fluid source applied at point $\mathbf{r}'$,respectively. For example, if $\mathbf{n}=[0 \quad 1]^{\mathrm{T}}$ then $G_{qr}(\mathbf{r},\omega;\mathbf{r}')$ is the fluid discharge in the z-direction at a point $\mathbf{r}$ due to an impulsive force in the r-direction applied at $\mathbf{r}'$. $H_{ij}(\mathbf{r},\omega;\mathbf{r}')$ and $H_{iq}(\mathbf{r},\omega;\mathbf{r}')$ denote tractions in the i-direction $(i=r,z)$ at a point $\mathbf{r}$ due to an impulsive force in the j-direction $(j=r,z)$ and an impulsive fluid source applied at point $\mathbf{r}'$, respectively; $H_{pj}(\mathbf{r},\omega;\mathbf{r}')$ and $H_{pq}(\mathbf{r},\omega;\mathbf{r}')$ denote excess pore pressure at a point $\mathbf{r}$ due to an impulsive force in the j-direction $(j=r,z)$ and an impulsive fluid source applied at point $\mathbf{r}'$,respectively. It is important to note here that the kernel functions G_{ij}, G_{iq}, H_{ij}, etc. in equations (3.51) to (3.54) are non-singular since $\mathbf{r}\neq\mathbf{r}'$ in the present scheme. Note that for an direct boundary integral equation method when the source and observation points are at the same location, i.e. $\mathbf{r}=\mathbf{r}'$, the singularity of the order $1/|\mathbf{r}-\mathbf{r}'|$ exist in the displacement Green's function (Cheng and Detournay, 1998). In addition, in the case of an elastic material, only equations (3.51) and (3.52) exist with the second integrals in those equations being equal to zero.

The unknown quantities, $f_i(\mathbf{r}',\omega)$ and $\Gamma(\mathbf{r}',\omega)$ in equations (3.51) to (3.54) are determined by taking $\mathbf{r} \in S$. Equations (3.51) to (3.54) represent a set of Fredholm integral equations of the first kind for unknown fields f_j and Γ. In view of the complexity of the kernel functions G_{ij}, H_{ij}, G_{qj},etc., equations (3.51) to (3.54) can be solved only by applying numerical techniques. A numerical solution is obtained by considering N and N' node point on S and S', respectively. Let $\mathbf{F}$ denote a vector whose elements correspond to the unknown quantities (i.e. the forces f_j and fluid source Γ) at node points on S' and defined in the following form:

$$\mathbf{F} = \left[\mathbf{f}_1 \quad \mathbf{f}_2 \quad \mathbf{f}_3 \quad \dots \quad \mathbf{f}_{N'}\right]^{\mathrm{T}} \tag{3.55}$$

where

$$\mathbf{f}_i = \left\langle f_r(\mathbf{r}_i,\omega) \quad f_z(\mathbf{r}_i,\omega) \quad \Gamma(\mathbf{r}_i,\omega)\right\rangle, \qquad i = 1,2,\dots,N' \tag{3.56}$$

Then, a discrete version of equations (3.51) to (3.54) with respect to N and N' node points on S and S', respectively, can be expressed as

$$\mathbf{QF} = \mathbf{R} \tag{3.57}$$

where the elements of the vector $\mathbf{R}$ correspond to the specified boundary conditions at node points on S and the elements of the matrix $\mathbf{Q}$ are expressed in terms of Green's functions. To determine the unknown quantities, the sum of squared errors at N node points on S are minimized where $N > N'$.Then, the errors in equation (3.57) are

$$\mathbf{E} = \mathbf{R} - \mathbf{QF} \tag{3.58}$$

and the square of $\mathbf{E}$ is given by

$$\varepsilon^2 = \left(\mathbf{R}^{\mathrm{T}} - \mathbf{F}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}\right)\left(\mathbf{R} - \mathbf{QF}\right)$$

$$= \mathbf{R}^{\mathrm{T}}\mathbf{R} - \mathbf{F}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}\mathbf{R} - \mathbf{R}^{\mathrm{T}}\mathbf{QF} + \mathbf{F}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}\mathbf{QF} \tag{3.59}$$

The extreme value of ε^2 can be obtained by

$$\frac{\partial \varepsilon^2}{\partial \mathbf{F}} = 2\left(-\mathbf{Q}^{\mathrm{T}}\mathbf{R} + \mathbf{Q}^{\mathrm{T}}\mathbf{QF}\right) = 0 \tag{3.60}$$

resulting in

$$\mathbf{F} = \left[\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\right]^{-1}\mathbf{Q}^{\mathrm{T}}\mathbf{R} \tag{3.61}$$

which is a normal equation for unknown forces and fluid source. The condition for unique estimator of the matrix **F** requires that the matrix $\mathbf{Q}^{\mathrm{T}}\mathbf{Q}$ be nonsingular or $\left|\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\right| \neq 0$ (Beck and Arnold, 1977). This means that any one column in **Q** cannot be proportional to any other column or any linear combination of other columns. The condition $\left|\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\right| \neq 0$ also requires that N, the number of observation points, be equal to or greater than the number of source points N'. Once **F** is known, the complete poroelastic field on the boundary S as well as at points in Ω can be computed directly from equations (3.51) to (3.54).

Let, consider the case where displacement $u_i(\mathbf{r},\omega)$, $i = r, z$, and fluid discharge $q_n(\mathbf{r},\omega)$ on the surface S are specified as equal to $\hat{u}_i(\mathbf{r},\omega)$, $i = r, z$, and $\hat{q}_n(\mathbf{r},\omega)$, respectively. Then,

$$\mathbf{R} = \left[\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3 \quad \ldots \quad \mathbf{u}_N\right]^{\mathrm{T}} \tag{3.62}$$

$$\mathbf{Q} = \left[\mathbf{G}(\mathbf{r}_i, \omega; \mathbf{r}'_j)\right]_{3N\times 3N'}, \quad i = 1,2,\ldots,N, \; j = 1,2,\ldots,N' \tag{3.63}$$

where

$$\mathbf{u}_i = \left\langle \hat{u}_r(\mathbf{r}_i,\omega) \quad \hat{u}_z(\mathbf{r}_i,\omega) \quad \hat{q}_n(\mathbf{r}_i,\omega)\right\rangle, \quad i = 1,2,\ldots,N \tag{3.64}$$

$$\mathbf{G}(\mathbf{r}_i,\omega;\mathbf{r}'_j)=\Delta S'_j\begin{bmatrix} G_{rr} & G_{rz} & G_{rq} \\ G_{zr} & G_{zz} & G_{zq} \\ G_{qr} & G_{qz} & G_{qq} \end{bmatrix} \tag{3.65}$$

and $\Delta S'_j$ denotes the tributary area corresponding to the j^{th} node point on S'.

Another example is the case where displacement $u_i(\mathbf{r},\omega)$, $i=r,z$, and pore pressure $p(\mathbf{r},\omega)$ on the surface S are specified as equal to $\hat{u}_i(\mathbf{r},\omega)$, $i=r,z$, and $\hat{p}(\mathbf{r},\omega)$, respectively. Then,

$$\mathbf{R}=\langle \mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3 \quad \ldots \quad \mathbf{u}_N \rangle^{\mathrm{T}} \tag{3.66}$$

$$\mathbf{Q}=\left[\mathbf{G}(\mathbf{r}_i,\omega;\mathbf{r}'_j)\right]_{3N\times 3N'} \quad ,i=1,2,\ldots,N,\ j=1,2,\ldots,N' \tag{3.67}$$

where

$$\mathbf{u}_i=\langle \hat{u}_r(\mathbf{r}_i,\omega) \quad \hat{u}_z(\mathbf{r}_i,\omega) \quad \hat{p}(\mathbf{r}_i,\omega)\rangle, \quad i=1,2,\ldots,N \tag{3.68}$$

$$\mathbf{G}(\mathbf{r}_i,\omega;\mathbf{r}'_j)=\Delta S'_j\begin{bmatrix} G_{rr} & G_{rz} & G_{rq} \\ G_{zr} & G_{zz} & G_{zq} \\ H_{pr} & H_{pz} & H_{pq} \end{bmatrix} \tag{3.69}$$

3.3 Green's Functions

The required Green's functions for the indirect boundary integral equation method outlined in the last section are obtained by solving the boundary-value problems related to the internally loaded poroelastic half-space. The solutions correspond to the three basic loading configurations, i.e., a vertical ring load, a radial ring load, and a ring fluid source. All types of loading are applied over a circular ring of radius r' at a depth $z=z'$ as shown in Figure 3.3. The solutions can be derived by defining a fictitious plane at $z=z'$ and treating the half-space as a two-domain boundary-value problem. The general solutions for each domain are given by equations (3.41) to (3.47) together with arbitrary functions $A_k(\xi,\delta)$ to $F_k(\xi,\delta)$, where k $(k=1,2)$ is used to identify the domain number. The domain

"1" is bounded by $0 \le z \le z'$ and the domain "2" by $z' \le z < \infty$ as shown in Figure 3.3. Note that for the domain "2", arbitrary functions $A_2(\xi,\delta) = C_2(\xi,\delta) = E_2(\xi,\delta) = 0$ in order to satisfy the condition that the solutions vanish as $z \to \infty$. The boundary conditions corresponding to a fully permeable top surface $(z = 0,\ 0 \le r < \infty)$ can be expressed as

$$\sigma_{zi}^{(1)}(r,0) = 0 \qquad ,i = r,z \tag{3.70}$$

$$p^{(1)}(r,0) = 0 \tag{3.71}$$

where a superscript (1) is used to denote the domain number.

The continuity conditions at the fictitious plane $(z = z',\ 0 \le r < \infty)$ corresponding to a homogeneous poroelastic half-space subjected to a buried vertical/radial loads and a fluid source are given by

$$u_i^{(1)}(r,z') - u_i^{(2)}(r,z') = 0 \qquad ,i = r,z \tag{3.72}$$

$$p^{(1)}(r,z') - p^{(2)}(r,z') = 0 \tag{3.73}$$

$$\sigma_{zi}^{(1)}(r,z') - \sigma_{zi}^{(2)}(r,z') = \frac{F_i(r)}{\mu} \qquad ,i = r,z \tag{3.74}$$

$$w_z^{(1)}(r,z') - w_z^{(2)}(r,z') = \sqrt{\frac{\rho}{\mu}}\frac{iQ(r)}{\delta} \tag{3.75}$$

where $F_i(r)$ and $Q(r)$ are defined according to each loading case as

$$F_z(r) = \delta(r - r') \quad \text{and} \quad F_r(r) = Q(r) = 0 \tag{3.76}$$

for a vertical ring load,

$$F_r(r) = \delta(r - r') \quad \text{and} \quad F_z(r) = Q(r) = 0 \tag{3.77}$$

for a radial ring load,

$$Q(r) = \delta(r - r') \quad \text{and} \quad F_r(r) = F_z(r) = 0 \tag{3.78}$$

for a ring fluid source. In addition, $\delta(\ )$ represents Dirac's delta function.

Substitution of general solutions for displacements, stresses and pore pressure given by equations (3.41) to (3.47) in the boundary conditions, equations (3.70) and (3.71), and the appropriate continuity conditions, equations (3.72) to (3.75), results in a set of linear simultaneous equations to determine arbitrary functions corresponding to the two domains. The explicit solutions for non-zero arbitrary functions corresponding to applied vertical/radial loading and fluid source were presented by Zeng and Rajapakse (1999a). These solutions are given in Appendix.

3.4 Vertical Impedances of Rigid Cylinder in Poroelastic Half-Space

Consider the case of a rigid cylindrical body embedded in a homogeneous poroelastic half space as shown in Figure 1.1. The rigid cylinder is subjected to a time-harmonic vertical force $V_0 e^{i\omega t}$ acting along its centroidal axis. The cylinder is assumed to be perfectly bonded with the half-space and the contact surfaces can be either fully permeable or impermeable. Thus boundary-value problem can be expressed as

$$u_r(r,z) = 0, \quad (r,z) \in S \tag{3.79}$$

$$u_z(r,z) = \Delta_z, \quad (r,z) \in S \tag{3.80}$$

$$\sigma_{zz}(r,0) = \sigma_{zr}(r,0) = p(r,0) = 0, \quad 0 \le r < \infty \tag{3.81}$$

$$\int_S T_z(r,z)\,dS = V_0, \quad (r,z) \in S \tag{3.82}$$

where $T_z(\mathbf{r},\omega)$ denotes the contact traction in the z-direction along the contact surface S. In addition, the hydraulic boundary condition for a fully permeable contact surface is given by

$$p(r,z) = 0, \quad (r,z) \in S \tag{3.83}$$

whereas,

$$q_n(r,z) = 0, \quad (r,z) \in S \tag{3.84}$$

for an impermeable contact surface

The relationship between the magnitude of vertical load V_0 and the magnitude of vertical displacement Δ_z of an embedded cylinder can be obtained from the numerical solution of equations (3.61), and computation of (3.52) and (3.82). The non-dimensional vertical impedance can be expressed as

$$K_{\mathrm{V}} = \frac{V_0}{\mu a \Delta_z} = \frac{1}{\mu a}\left[\mathbf{Z}\left(\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\right)^{-1}\mathbf{Q}^{\mathrm{T}}\mathbf{R}\right]^{\mathrm{T}}\mathbf{A} \tag{3.85}$$

where

$$\mathbf{Z} = \left[\mathbf{H}(\mathbf{r}_i, \omega; \mathbf{r}'_j)\right]_{N \times 3N'}, i = 1,2,\ldots,N, j = 1,2,\ldots,N' \tag{3.86}$$

$$\mathbf{H}(\mathbf{r}_i, \omega; \mathbf{r}'_j) = \Delta S'_j \left[H_{zr} \quad H_{zz} \quad H_{zq}\right] \tag{3.87}$$

$$\mathbf{A} = \langle \Delta S_1 \quad \Delta S_2 \quad \Delta S_3 \quad \ldots \quad \Delta S_N \rangle^{\mathrm{T}} \tag{3.88}$$

and ΔS_i denote the tributary area corresponding to the i^{th} node point on S.

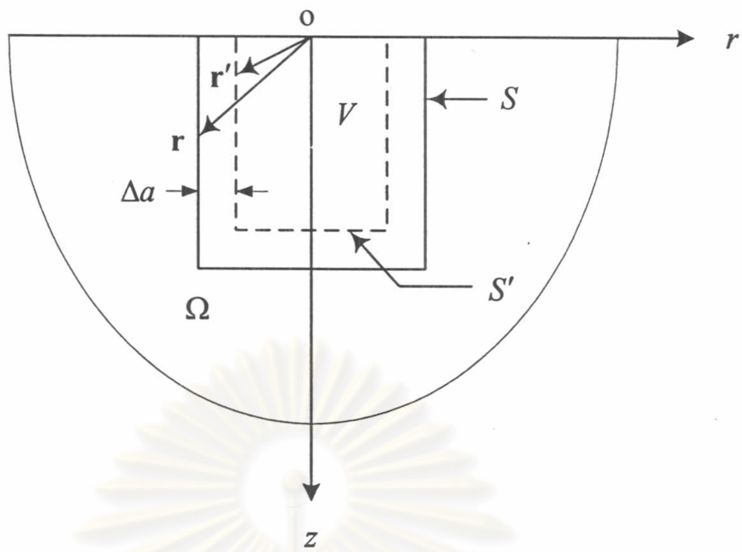

Figure 3.1. System considered in the indirect boundary integral equation

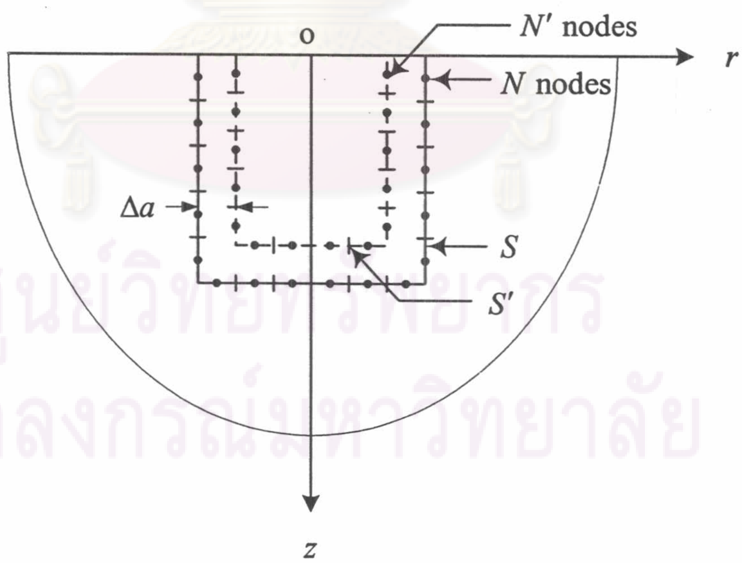

Figure 3.2. Discretization of real surface and auxiliary surface

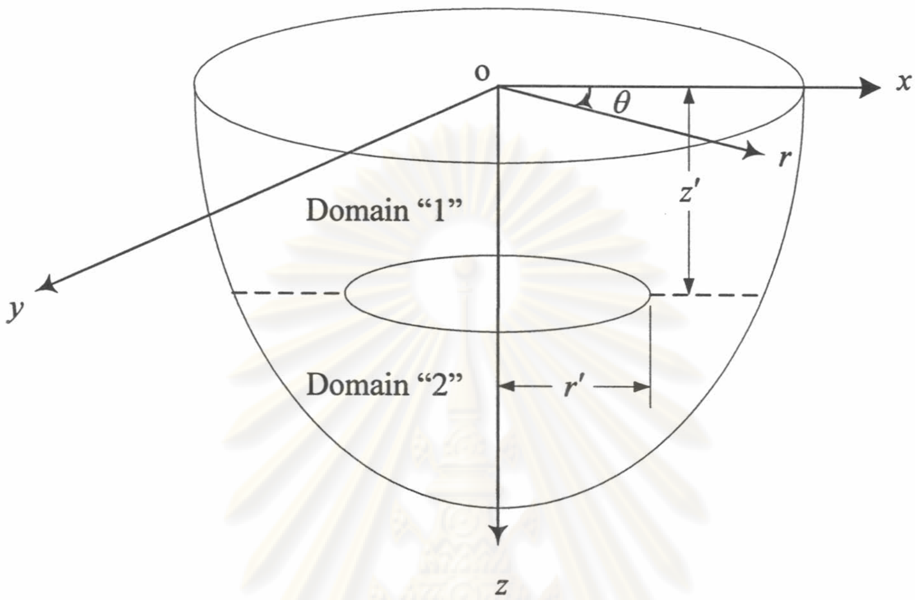

Figure 3.3 Two domain boundary value problem