CHAPTER III

SOLUTION

In the following section we are going to solve for the solution to the governing equation derived in the previous chapter. For convenience the governing equation and the boundary conditions shall be summarized as follow.

Governing Equation

$$\frac{d^2W}{dz^2} - \frac{2iW\,\Omega\,\sin\varphi}{\nu} = \frac{2\omega a^2k^3\cosh 2k(z+h)}{\sinh^2 kh} \qquad 3.1$$

Boundary Conditions

$$\frac{du}{dz} = \frac{\tau}{\rho\nu} + 4\omega a^2k^2\coth kh \quad , \text{ at } z = 0 \qquad 3.2$$

$$W = \frac{5\omega a^2 k}{4\sinh^2 kh} \quad , \text{ at } z=-h \qquad 3.3$$

Eq.(3.3) is a non-homogeneous differential equation, the complete solution is the sum of the complementary solution, which is the solution of Eq.(3.1) with zero on the right hand side instead of a function, and the particular solution.

3.1 Complementary Solution

The complementary solution is the solution to the homogeneous equation

$$\frac{d^2W}{dz^2} - \frac{2iW\,\Omega\,\sin\varphi}{\nu} = 0 \qquad 3.4$$

The solution of this equation is in the from

$$W = e^{mz} \qquad 3.5$$

Substituting Eq.(3.5) into Eq.(3.4) yields the auxiliary equation

$$m^2 - \frac{2i\,\Omega \sin\varphi}{\nu} = 0$$

Solving for m, we get

$$m = \pm\sqrt{\frac{2i\,\Omega \sin\varphi}{\nu}}$$

or

$$m = \pm\frac{(1+i)}{\delta} \qquad 3.6$$

where δ is Ekman's depth which is defined as

$$\delta = \sqrt{\frac{\nu}{\Omega \sin\varphi}}$$

Sustituting the values of m given by Eq.(3.6) into Eq.(3.5) yields basic solutions. $W = e^{\frac{(1+i)z}{\delta}}$ and $W = e^{\frac{-(1+i)z}{\delta}}$. The complementary solution is the linear combination of these two basic solutions, e.g.,

$$W_c = Me^{(1+i)\frac{z}{\delta}} + Ne^{-(1+i)\frac{z}{\delta}} \qquad 3.7$$

where M and N are constants to be determined by the boundary conditions.

3.2 Particular Solution

Assuming that the particular solution is in the form

$$Wp = (C + iD)e^{2kz} + (E + iF)e^{-2kz} \qquad 3.8$$

Sustituting into the governing equation, Eq.(3.1) yields

$$\frac{-2i(E+iF)e^{-2kz}}{\delta^2} + 4k^2(C+iD)e^{2kz} + 4k^2(E+iF)e^{-2kz} - \frac{2\,(C+iD)e^{2kz}}{\delta^2}$$

$$= \frac{2\omega a^2k^3\cosh 2k(z+h)}{\sinh^2 kh}$$

Solving for C, D, E and F, we get

$$C = \frac{\omega a^2 k^5 e^{2kh}}{(4k^4+\delta^{-4})\sinh^2 kh}, \qquad D = \frac{\omega \delta^{-2} a^2 k^3 e^{2kh}}{2(4k^4+\delta^{-4})\sinh^2 kh}$$

$$E = \frac{\omega a^2 k^5 e^{-2kh}}{(4k^4+\delta^{-4})\sinh^2 kh}, \qquad F = \frac{\omega \delta^{-2} a^2 k^3 e^{-2kh}}{2(4k^4+\delta^{-4})\sinh^2 kh}$$

Substituting C, D, E and F back into Eq.(3.8) yield the particular solution,

$$Wp = \frac{\omega a^2 k(1+\frac{1}{2}\, i\, k^{-2}\delta^{-2})\cosh 2k(z+h)}{2(1+\frac{1}{4}\, k^{-4}\delta^{-4})\sinh^2 kh} \qquad 3.9$$

3.3 Complete Solution

By combining the complementary solution with particular solution we get

$$W = Me^{(1+i)\frac{z}{\delta}} + Ne^{-(1+i)\frac{z}{\delta}} + \frac{\omega a^2 k(1+\frac{1}{2}\, i\, k^{-2}\delta^{-2})\cosh 2k(z+h)}{2(1+\frac{1}{4}\, k^{-4}\delta^{-4})\sinh^2 kh} \qquad 3.10$$

Substituting Eq.(3.10) in to Eq.(3.2) and Eq.(3.3) respectively yields

$$Me^{-(1+i)\frac{h}{\delta}} + Ne^{(1+i)\frac{h}{\delta}} + \frac{\omega a^2 k(1+\frac{1}{2}\, i\, k^{-2}\delta^{-2})}{2(1+\frac{1}{4}\, k^{-4}\delta^{-4})\sinh^2 kh} = \frac{5\omega a^2 k}{4\sinh^2 kh} \qquad 3.11$$

$$\frac{(1+i)}{\delta}(M-N) + \frac{\omega a^2 k^2(1+\frac{1}{2}\, i\, k^{-2}\delta^{-2})\sinh 2kh}{(1+\frac{1}{4}\, k^{-4}\delta^{-4})\sinh^2 kh} = 4\omega a^2 k^2 \coth kh + \frac{\tau}{\rho\nu} \qquad 3.12$$

Solving for M and N we obtain

$$M = \Big[\frac{5\omega a^2 k}{4\sinh^2 kh} - \frac{\omega a^2 k(1+\frac{1}{2}\, i\, k^{-2}\delta^{-2})\{\delta k(1-i)e^{(1+i)\frac{h}{\delta}}\sinh 2kh + 1\}}{2(1+\frac{1}{4}\, k^{-4}\delta^{-4})\sinh^2 kh}$$

$$+ \frac{\delta}{2}(1-i)e^{(1+i)\frac{h}{\delta}}\left(4\omega a^2 k^2 \coth kh + \frac{\tau}{\rho\nu}\right)\Big] \Big/ \left(e^{(1+i)\frac{h}{\delta}} + e^{-(1+i)\frac{h}{\delta}}\right)$$

3.13

$$N = \left[\frac{5\omega a^2 k}{4\sinh^2 kh} + \frac{\omega a^2 k(1+\frac{1}{2}\, i\, k^{-2}\delta^{-2})\{\delta k(1-i)e^{-(1+i)\frac{h}{\delta}}\sinh 2kh - 1\}}{2(1+\frac{1}{4}\, k^{-4}\delta^{-4})\sinh^2 kh}\right.$$

$$\left. - \frac{\delta}{2}(1-i)e^{-(1+i)\frac{h}{\delta}}(4\omega a^2 k^2 \coth kh + \frac{\tau}{\rho\surd})\right] \Big/ (e^{(1+i)\frac{h}{\delta}} + e^{-(1+i)\frac{h}{\delta}}) \qquad 3.14$$

Substituting Eq.(3.13) and Eq.(3.14) into Eq.(3.10) yields

$$W = \frac{5\omega a^2 k\cosh(1+i)\frac{z}{\delta}}{4\sinh^2 kh\,\cosh(1+i)\frac{h}{\delta}} + \frac{\delta(1-i)(4\omega a^2k^2\coth kh + \frac{\tau}{\rho\surd})\sinh(1+i)\frac{(z+h)}{\delta}}{2\cosh(1+i)\frac{h}{\delta}}$$

$$+ \omega a^2 k(1+\frac{1}{2}\, ik^{-2}\delta^{-2})[e^{-(1+i)\frac{z}{\delta}}\{\delta k(1-i)e^{-(1+i)\frac{h}{\delta}}\sinh 2kh - 1\}$$

$$- e^{(1+i)\frac{z}{\delta}}\{\delta k(1-i)e^{(1+i)\frac{h}{\delta}}\sinh 2kh + 1\}]$$

$$/\ 4(1 + \frac{1}{4}k^{-4}\delta^{-4})\cosh(1+i)\frac{h}{\delta}\,\sinh^2 kh$$

$$+ \frac{\omega a^2 k(1 + \frac{1}{2}ik^{-2}\delta^{-2})\cosh 2k(z+h)}{2(1 + \frac{1}{4}\, k^{-4}\delta^{-4})\sinh^2 kh} \qquad 3.15$$

Eq.(3.15) gives the total mass transport velocity vector. By separating the right hand side into the real part and imaginary part we get the x-components of mass tranport velocity as follow.

$$
\begin{aligned}
u = {} & [\ \delta\tau\ [\ \{ \cos\frac{(z+h)}{\delta}\sinh\frac{(z+h)}{\delta} + \sin\frac{(z+h)}{\delta}\cosh\frac{(z+h)}{\delta} \} \cos\frac{h}{\delta}\cosh\frac{h}{\delta} \\
& + \{ \sin\frac{(z+h)}{\delta}\cosh\frac{(z+h)}{\delta} - \cos\frac{(z+h)}{\delta}\sinh\frac{(z+h)}{\delta} \} \sin\frac{h}{\delta}\sinh\frac{h}{\delta}\ ]] \\
& \Big/ 2\rho\sqrt{(\cos^2\frac{h}{\delta}\cosh^2\frac{h}{\delta} + \sin^2\frac{h}{\delta}\sinh^2\frac{h}{\delta})} \\
& + \omega a^2 k [\cos\frac{h}{\delta}\cosh\frac{h}{\delta} [2\cos\frac{z}{\delta}\cosh\frac{z}{\delta}\ (\frac{5}{4\sinh^2 kh} - \frac{1}{2(1+\frac{1}{4}k^{-4}\delta^{-4})\sinh^2 kh}) \\
& + \frac{\sin\frac{z}{\delta}\sinh\frac{z}{\delta}}{2\delta^2k^2(1+\frac{1}{4}k^{-4}\delta^{-4})\sinh^2 kh} \\
& + \delta\cos\frac{(z+h)}{\delta}\sinh\frac{(z+h)}{\delta}\{4k\coth kh - \frac{(2\delta^2k^2+1)\sinh 2kh}{2\delta^2k(1+\frac{1}{4}k^{-4}\delta^{-4})\sinh^2 kh}\} \\
& + \delta\sin\frac{(z+h)}{\delta}\cosh\frac{(z+h)}{\delta}\{4k\coth kh - \frac{(2\delta^2k^2-1)\sinh 2kh}{2\delta^2k(1+\frac{1}{4}k^{-4}\delta^{-4})\sinh^2 kh}\}] \\
& + \sin\frac{h}{\delta}\sinh\frac{h}{\delta}\ [2\sin\frac{z}{\delta}\sinh\frac{z}{\delta}\ \{\frac{5}{4\sinh^2 kh} - \frac{1}{2(1+\frac{1}{4}k^{-4}\delta^{-4})\sinh^2 kh}\} \\
& - \frac{\cos\frac{z}{\delta}\cosh\frac{z}{\delta}}{2\delta^2k^2(1+\frac{1}{4}k^{-4}\delta^{-4})\sinh^2 kh} \\
& + \delta\sin\frac{(z+h)}{\delta}\cosh\frac{(z+h)}{\delta}\{4k\coth kh - \frac{(2\delta^2k^2+1)\sinh 2kh}{2\delta^2k(1+\frac{1}{4}k^{-4}\delta^{-4})\sinh^2 kh}\} \\
& - \delta\cos\frac{(z+h)}{\delta}\sinh\frac{(z+h)}{\delta}\{4k\coth kh - \frac{(2\delta^2k^2-1)\sinh 2kh}{2\delta^2k(1+\frac{1}{4}k^{-4}\delta^{-4})\sinh^2 kh}\}]] \\
& \Big/ 2(\cos^2\frac{h}{\delta}\cosh^2\frac{h}{\delta} + \sin^2\frac{h}{\delta}\sinh^2\frac{h}{\delta}) + \frac{\omega a^2 k\cosh 2k(z+h)}{2(1+\frac{1}{4}k^{-4}\delta^{-4})\sinh^2 kh}
\end{aligned}
\qquad \ldots\ldots\ldots\ldots\ldots(3.16)
$$

$$
\begin{aligned}
v = \; & [\; \delta\tau\, [\, \{\sin\frac{(z+h)}{\delta}\cosh\frac{(z+h)}{\delta} - \cos\frac{(z+h)}{\delta}\sinh\frac{(z+h)}{\delta}\}\cos\frac{h}{\delta}\cosh\frac{h}{\delta} \\
& - \{\cos\frac{(z+h)}{\delta}\sinh\frac{(z+h)}{\delta} + \sin\frac{(z+h)}{\delta}\cosh\frac{(z+h)}{\delta}\}\sin\frac{h}{\delta}\sinh\frac{h}{\delta}\;]] \\
& \Big/ 2\rho\surd\,(\cos^2\frac{h}{\delta}\cosh^2\frac{h}{\delta} + \sin^2\frac{h}{\delta}\sinh^2\frac{h}{\delta}) \\
& + \omega a^2 k\,[\cos\frac{h}{\delta}\cosh\frac{h}{\delta}\,[2\sin\frac{z}{\delta}\sinh\frac{z}{\delta}\,\{\frac{5}{4\sinh^2 kh} - \frac{1}{2(1+\frac{1}{4}k^{-4}\delta^{-4})\sinh^2 kh}\} \\
& - \frac{\cos\frac{z}{\delta}\cosh\frac{z}{\delta}}{2\delta^2k^2(1+\frac{1}{4}k^{-4}\delta^{-4})\sinh^2 kh} \\
& + \delta\sin\frac{(z+h)}{\delta}\cosh\frac{(z+h)}{\delta}\{4k\coth kh - \frac{(2\delta^2k^2+1)\sinh 2kh}{2\delta^2k(1+\frac{1}{4}k^{-4}\delta^{-4})\sinh^2 kh}\} \\
& - \delta\cos\frac{(z+h)}{\delta}\sinh\frac{(z+h)}{\delta}\{4k\coth kh - \frac{(2\delta^2k^2-1)\sinh 2kh}{2\delta^2k(1+\frac{1}{4}k^{-4}\delta^{-4})\sinh^2 kh}\}] \\
& - \sin\frac{h}{\delta}\sinh\frac{h}{\delta}\,[2\cos\frac{z}{\delta}\cosh\frac{z}{\delta}\,\{\frac{5}{4\sinh^2 kh} - \frac{1}{2(1+\frac{1}{4}k^{-4}\delta^{-4})\sinh^2 kh}\} \\
& + \frac{\sin\frac{z}{\delta}\sinh\frac{z}{\delta}}{2\delta^2k^2(1+\frac{1}{4}k^{-4}\delta^{-4})\sinh^2 kh} \\
& + \delta\cos\frac{(z+h)}{\delta}\sinh\frac{(z+h)}{\delta}\{4k\coth kh - \frac{(2k^2\delta^2+1)\sinh 2kh}{2\delta^2k(1+\frac{1}{4}k^{-4}\delta^{-4})\sinh^2 kh}\} \\
& + \delta\sin\frac{(z+h)}{\delta}\cosh\frac{(z+h)}{\delta}\{4k\coth kh - \frac{(2k^2\delta^2-1)\sinh 2kh}{2\delta^2k(1+\frac{1}{4}k^{-4}\delta^{-4})\sinh^2 kh}\}]] \\
& \Big/ 2(\cos^2\frac{h}{\delta}\cosh^2\frac{h}{\delta} + \sin^2\frac{h}{\delta}\sinh^2\frac{h}{\delta}) \\
& + \frac{\omega a^2\cosh 2k(z+h)}{4\delta^2k(1+\frac{1}{4}k^{-4}\delta^{-4})\sinh^2 kh} \qquad \text{...........(3.17)}
\end{aligned}
$$