CHAPTER II

MATHEMATICAN FORMULATION

2.1 Derivation of Governing Equation

The Lagrangian equation governing the wave-induced mass transport, according to Unluata and Mei (1970), is given by

$$\surd \frac{d^2u}{dz^2} = \frac{\bar{p}}{\rho} + \frac{2\surd \omega a^2 k^3 \cosh 2k(z+h)}{\sinh^2 kh} \qquad 2.1$$

where $\bar{p}$ = pressure gradient in the direction of wave propagation

u = second order steady Lagrangian velocity in the direction of wave propagation (x-direction)

$\surd$ = kinematic eddy viscosity

Since we are seeking for a model in the case of an ocean of infinite lateral extent the pressure gradient term shall be neglected, and taking Coriolis effect into consideration, Eq.(2.1) could be generalized as

$$\surd \frac{d^2u}{dz^2} = -2v\,\Omega \sin\varphi + \frac{2\surd \omega a^2 k^3 \cosh 2k(z+h)}{\sinh^2 kh} \qquad 2.2$$

$$\surd \frac{d^2v}{dz^2} = 2u\,\Omega \sin\varphi \qquad 2.3$$

where v = second order steady Lagrangian velocity in the y - direction

Ω = angular velocity of earth rotation

φ = latitude

Let define the complex variable

$$W = u + iv \qquad 2.4$$

Sine "u" and "v" are velocity in x- and y-directions respectively, the complex variabe W shall therefore repesent the total velocity. Differentiating Eq.(2.4) with respect to z yields

$$\frac{dW}{dz} = \frac{du}{dz} + i\,\frac{dv}{dz} \qquad 2.5$$

Since the wind is blowing in the x-direction, the velocity gradient in the y-direction at the surface must be vanish. Hence,

$$\frac{dv}{dz} = 0 \qquad \text{at } z = 0 \qquad 2.6$$

Differentiate Eq.(2.5) with respect to z yields

$$\frac{d^2W}{dz^2} = \frac{d^2u}{dz^2} + i\,\frac{d^2v}{dz^2}$$

or

$$\frac{d^2u}{dz^2} = \frac{d^2W}{dz^2} - i\,\frac{d^2v}{dz^2} \qquad 2.7$$

Substituting Eq.(2.2) and Eq.(2.3) into Eq.(2.7) we get.

$$\frac{d^2W}{dz^2} - \frac{2(ui-v)\,\Omega \sin\varphi}{\surd} = \frac{2\omega a^2k^3\cosh 2k(z+h)}{\sinh^2 kh} \qquad 2.8$$

Multiplying Eq.(2.4) by i yields iw = iu - v. Substituting this into Eq.(2.8) leads to the governing equation of our problem

$$\frac{d^2W}{dz^2} - \frac{2iW\,\Omega \sin\varphi}{\surd} = \frac{2\omega a^2k^3\cosh 2k(z+h)}{\sinh^2 kh} \qquad 2.9$$

2.2 Boundary Conditions

It will be assumed, as Madsen did, that the velocity gradient at the surface is the sum of the velocity gradient due to the nonzero shear stress by the wind and the wave-induced velocity gradient as predicted by Longuet-Higgins' model (see Eq.(1.3) in Chapter I), e.g.

$$\frac{du}{dz} = \frac{\tau}{\rho\nu} + 4\omega a^2 k^2 \coth kh \qquad \text{at } z = 0 \, . \qquad 2.10$$

where τ = surface wind shear stress

The velocity at the bottom in the x-direction shall be assuned to be the velocity near the bottom as given by Eq.(1.2), e.g.

$$u = \frac{5\omega a^2 k}{4\sinh^2 kh} \, , \qquad \text{at } z = -h \qquad 2.11$$

The y-component of the bottom velocity shall be assumed to be equal to zero,

$$v = 0 \, , \qquad \text{at } z = -h \qquad 2.12$$

Substituting Eq.(2.11) and Eq.(2.12) in to Eq.(2.4) yields the bottom boundary condition

$$W = \frac{5\omega a^2 k}{4\sinh^2 kh} \, , \qquad \text{at } z = -h \qquad 2.13$$