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$\mathcal{B}Q$-ELEMENTS OF SOME SEMIGROUPS

Miss Ngarmcherd Danpattanamongkon

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| By | Miss Ngarmcherd Danpattanamongkon |
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Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Doctoral Degree

........................................Dean of the Faculty of Science
(Professor Supot Hannongbua, Dr.rer.nat.)

THESIS COMMITTEE

........................................Chairman
(Associate Professor Patanee Udomkavanich, Ph.D.)

........................................Thesis Advisor
(Professor Yupaporn Kemprasit, Ph.D.)

........................................Examiner
(Associate Professor Amorn Wasanawichit, Ph.D.)

........................................Examiner
(Assistant Professor Sajee Pianskool, Ph.D.)

........................................External Examiner
(Associate Professor Somporn Sutinuntopas, Ph.D.)

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เราเรียกสมาชิก $x$ ของกึ่งกรุป $S$ ว่า *สมาชิกบีคิว* ถ้าไบไอดีลและควอซีไอดีลของ $S$ ก่อกำเนิดโดย $x$ เป็นสิ่งเดียวกัน นั่นคือ $\{x\} \cup xS^1x = S^1x \cap xS^1$ สมาชิกบีคิวเป็นกรณีทั่วไปอันหนึ่งของสมาชิกปรกติในกึ่งกรุป ในงานวิจัยนี้เราให้ลักษณะเฉพาะของสมาชิกบีคิวในกึ่งกรุป $(\mathbb{Z}_n, \cdot)$ เราให้ลักษณะเฉพาะของสมาชิกบีคิวในกึ่งกรุปของการแปลงของเซตและกึ่งกรุปของการแปลงเชิงเส้นบางชนิดด้วย ยิ่งไปกว่านั้นเราบอกสมาชิกบีคิวของกึ่งกรุปต่อไปนี้ $(\mathbb{Z}^+, +)$, $(\mathbb{Z}^+, \cdot)$, $(\mathbb{Z}, \cdot)$, กึ่งกรุปช่วงบน $\mathbb{R}$ ภายใต้การบวก และกึ่งกรุปช่วงบน $\mathbb{R}$ ภายใต้การคูณ

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต............................................

ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก..................................

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A $\mathcal{B}Q$*-element* of a semigroup $S$ is an element $x$ such that the bi-ideal and the quasi-ideal of $S$ generated by $x$ coincide, i.e., $\{x\} \cup xS^1x = S^1x \cap xS^1$. $\mathcal{B}Q$-elements are a generalization of regular elements in semigroups. In this research, the $\mathcal{B}Q$-elements of the semigroup $(\mathbb{Z}_n, \cdot)$ are characterized. We also characterize the $\mathcal{B}Q$-elements of some semigroups of transformations of sets and linear transformations. In addition, the $\mathcal{B}Q$-elements of the following semigroups are determined : $(\mathbb{Z}^+, +), (\mathbb{Z}^+, \cdot), (\mathbb{Z}, \cdot)$, the additive interval semigroups and the multiplicative interval semigroups on $\mathbb{R}$.

Department : Mathematics and Computer Science ........ Student's Signature ........

Advisor's Signature ........

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# CONTENTS

# CHAPTER I

# INTRODUCTION

The notions of bi-ideal and quasi-ideal for semigroups were introduced by Good and Hughes [5] in 1952 and Steinfeld [17] in 1956, respectively. Bi-ideals and quasi-ideals are generalizations of one-sided ideals and every quasi-ideal is a bi-ideal. The notation $\mathcal{B}Q$ was given by Kapp [8] to denote the class of all semigroups whose sets of bi-ideals and quasi-ideals coincide, i.e., $\mathcal{B}Q$ is the class of all semigroups whose bi-ideals are quasi-ideals. Mielke [12] called the semigroups in the class $\mathcal{B}Q$ the $\mathcal{B}Q$-semigroups. In 1961, Lajos [11] showed that every regular semigroup is a $\mathcal{B}Q$-semigroup.

Regularity is a crucial notion in semigroup theory. The following standard semigroups are regular : the full transformation semigroup, the partial transformation semigroup and the 1-1 partial transformation semigroup (the symmetric inverse semigroup) on a nonempty set, the semigroup of all linear transformations from a vector space over a filed into itself under composition and the full $n \times n$ matrix multiplicative semigroup over a division ring. It is well-known that every semigroup is embedded in the full transformation semigroup on a nonempty set.

We know that for an element $x$ in $\mathrm{Reg}(S)$, the set of all regular elements of a semigroup $S$, the bi-ideal and the quasi-ideal generated by $x$ coincide. However, the converse is not true in general. To generalize a regular element, a $\mathcal{B}Q$-element is defined. By a $\mathcal{B}Q$-*element* of a semigroup $S$ we mean an element $x$ in $S$ such that the bi-ideal and the quasi-ideal of $S$ generated by $x$ coincide. The set of all $\mathcal{B}Q$-elements of a semigroup $S$ is denoted by $\mathcal{B}Q(S)$. Then $\mathrm{Reg}(S) \subseteq \mathcal{B}Q(S)$. It is not necessarily true that $S$ is a $\mathcal{B}Q$-semigroup if every element of $S$ is a $\mathcal{B}Q$-element. In fact, Calais [2] showed that $S$ is a $\mathcal{B}Q$-semigroup if and only if the bi-ideal and the quasi-ideal of $S$ generated by any two elements coincide.

This research is organized as follows :

Chapter II contains the basic definitions, notations and quoted results which will be used in this research. Some examples are also provided.

In Chapter III, the following additive and multiplicative semigroups of integers are studied : $(\mathbb{Z}^+, +), (\mathbb{Z}^+, \cdot), (\mathbb{Z}, \cdot)$ and some of their subsemigroups. The $\mathcal{B}Q$-elements of these semigroups are characterized.

In Chapter IV, the $\mathcal{B}Q$-elements of all the additive and multiplicative interval semigroups on $\mathbb{R}$ are determined. It indicates that the regular elements and the $\mathcal{B}Q$-elements of these semigroups are the same.

Ehrlich [4] proved that the multiplicative semigroup $\mathbb{Z}_n$ is regular if and only if $n$ is square-free. Recently, Alkam and Osba [1] characterized its regular elements in terms of Euler's phi-function. In Chapter V, the regular elements of the multiplicative semigroup $\mathbb{Z}_n$ are characterized differently and its $\mathcal{B}Q$-elements are determined. Our characterizations show that the regular elements and the $\mathcal{B}Q$-elements are almost the same. It is shown that $\mathcal{B}Q(\mathbb{Z}_n, \cdot) = \operatorname{Reg}(\mathbb{Z}_n, \cdot)$ if $4 \nmid n$ and $\mathcal{B}Q(\mathbb{Z}_n, \cdot) = \operatorname{Reg}(\mathbb{Z}_n, \cdot) \cup \left\{ \overline{\left(\frac{n}{2}\right)} \right\}$ if $4 \mid n$. Including in this chapter, we provide some sufficient conditions for the semigroup $(k\mathbb{Z}_n, \cdot)$ to have the property that $\mathcal{B}Q(k\mathbb{Z}_n, \cdot) = k\mathbb{Z}_n$.

Chapter VI deals with the following subsemigroups of $T(X)$ and $L_F(V)$ where $T(X)$ is the full transformations semigroup on a nonempty set $X$ and $L_F(V)$ is the semigroup under composition of all linear transformations from a vector space $V$ over a field $F$ into itself :

$$\begin{aligned}
M(X) &= \{\alpha \in T(X) \mid \alpha \text{ is 1-1}\}, \\
UE(X) &= \{\alpha \in T(X) \mid \alpha \text{ is onto and } |a\alpha^{-1}| = |b\alpha^{-1}| \text{ for all } a, b \in X\}, \\
M_F(V) &= \{\alpha \in L_F(V) \mid \alpha \text{ is 1-1}\} \\
&\quad (= \{\alpha \in L_F(V) \mid \ker \alpha = \{0\}\}), \\
E_F(V) &= \{\alpha \in L_F(V) \mid \alpha \text{ is onto}\} \\
&\quad (= \{\alpha \in L_F(V) \mid V\alpha = V\}).
\end{aligned}$$

Notice that for $\alpha \in E_F(V)$ and $v \in V, v\alpha^{-1} = w + \ker \alpha$ where $w\alpha = v$, so $|v\alpha^{-1}| = |\ker \alpha|$. We show in this chapter that the $\mathcal{B}Q$-elements of each semigroup must be regular. It is shown that $\mathcal{B}Q(M(X)) = G(X) = \mathcal{B}Q(UE(X))$ and $\mathcal{B}Q(M_F(V)) = G_F(V) = \mathcal{B}Q(E_F(V))$ where $G(X)$ is the symmetric group on $X$ (the group under composition of all bijections on $X$) and $G_F(V)$ the group under composition of all isomorphisms from $V$ onto itself.

# CHAPTER II

# PRELIMINARIES

The cardinality of a set $X$ is denoted by $|X|$.

The value of a mapping $\alpha$ at $x$ in the domain of $\alpha$ shall be written as $x\alpha$. For convenience, we use a bracket notation to represent a mapping. For instance,

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ stands for the mapping $\alpha$ with $\operatorname{dom}\alpha = \{a, b\}$, $\operatorname{ran}\alpha = \{c, d\}$,

$a\alpha = c$ and $b\alpha = d$,

$\begin{pmatrix} A & x \\ a & x' \end{pmatrix}$ stands for the mapping $\beta$ with $\operatorname{dom}\beta = X$,

$$\operatorname{ran}\beta = \{a\} \cup \{x' \mid x \in X \setminus A\} \text{ and } x\beta = \begin{cases} a & \text{if } x \in A, \\ x' & \text{if } x \in X \setminus A. \end{cases}$$

By the above notation, a mapping $\alpha$ can be written as $\alpha = \begin{pmatrix} x\alpha^{-1} \\ x \end{pmatrix}_{x \in \operatorname{ran}\alpha}$.

Here, $\operatorname{ran}\alpha$ denotes the range (image) of $\alpha$. The notation $\dot{\cup}$ stands for a disjoint union.

Let $S$ be a semigroup and let 1 be a symbol not representing any element of $S$. Extend the binary operation on $S$ to $S \cup \{1\}$ by

$$1\,1 = 1 \qquad \text{and} \qquad 1\,a = a = a\,1 \qquad \text{for every } a \in S.$$

Then $S \cup \{1\}$ is a semigroup with identity 1. Let

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity}, \\ S \cup \{1\} & \text{if } S \text{ has no identity}. \end{cases}$$

An element $x$ of a semigroup $S$ is called a *regular element* if $x = xyx$ for some $y \in S$ and $S$ is called a *regular semigroup* if every element of $S$ is regular. The set of all regular elements of a semigroup $S$ will be denoted by $\mathrm{Reg}(S)$, i.e.,

$$\mathrm{Reg}(S) = \{x \in S \mid x = xyx \text{ for some } y \in S\}.$$

Note that a subsemigroup of a regular semigroup need not be regular. However, an ideal of a regular semigroup is a regular semigroup. If $I$ is an ideal of a semigroup $S$ and $x \in I$ is such that $x = xyx$ for some $y \in S$, then $x = x(yxy)x$ and $yxy \in I$, so $x$ is regular in $I$. Hence

$$I \cap \mathrm{Reg}(S) = \mathrm{Reg}(I).$$

A subsemigroup $Q$ of a semigroup $S$ is called a *quasi-ideal* of $S$ if $SQ \cap QS \subseteq Q$ and by a *bi-ideal* of $S$ we mean a subsemigroup $B$ of $S$ such that $BSB \subseteq B$. Clearly, every one-sided ideal of $S$ is a quasi-ideal of $S$ and every quasi-ideal of $S$ is a bi-ideal of $S$. Notice that if $S$ is commutative, then the quasi-ideals and the ideals of $S$ are identical.

**Example 2.1.** Let $F$ be a field and $M_n(F)$ the multiplicative semigroup of $n \times n$ matrices over $F$ where $n > 1$. Let

$$Q = \left\{ \begin{bmatrix} x & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \mid x \in F \right\}.$$

Then $Q$ is a subsemigroup of $M_n(F)$,

$$M_n(F)Q = \left\{ \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ x_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_n & 0 & \cdots & 0 \end{bmatrix} \mid x_1, \ldots, x_n \in F \right\},$$

and

$$QM_n(F) = \left\{ \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \mid x_1, \ldots, x_n \in F \right\},$$

so $M_n(F)Q \cap QM_n(F) = Q$. Thus $Q$ is a quasi-ideal but not a one-sided ideal of $M_n(F)$.

**Example 2.2.** Let $n \geq 4$, $SU_n(F)$ the multiplicative semigroup of strictly upper triangular $n \times n$ matrices over $F$ and

$$B = \left\{ \begin{bmatrix} 0 & \cdots & 0 & x & 0 \\ 0 & \cdots & 0 & 0 & y \\ 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \mid x, y \in F \right\}.$$

Then $B^2 = \{\mathbf{0}\}$ where $\mathbf{0}$ is the zero $n \times n$ matrix over $F$. Since

$$SU_n(F)B = \left\{ \begin{bmatrix} 0 & \cdots & 0 & x \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix} \mid x \in F \right\} = BSU_n(F)$$

and

$$BSU_n(F)B = \{\mathbf{0}\} \subseteq B,$$

it follows that $B$ is a bi-ideal but not a quasi-ideal of $SU_n(F)$.

Example 2.1 and Example 2.2 show that quasi-ideals generalize one-sided ideals and bi-ideals generalize quasi-ideals, respectively.

The class of all semigroups whose sets of bi-ideals and quasi-ideals coincide is denoted by $\mathcal{B}Q$ (Kapp [8]) and a semigroup in $\mathcal{B}Q$ is called a *$\mathcal{B}Q$-semigroup* (Mielke [12]).

For a nonempty subset $X$ of $S$, let $(X)_q$ and $(X)_b$ denote respectively the quasi-ideal and the bi-ideal of $S$ generated by $X$, i.e., $(X)_q$ is the intersection of all quasi-ideals of $S$ containing $X$ and $(X)_b$ is the intersection of all bi-ideals of $S$ containing $X$ ([18] p. 10 and p. 12). If $X = \{x_1, x_2, ..., x_n\}$, we may write $(x_1, x_2, ..., x_n)_q$ and $(x_1, x_2, ..., x_n)_b$ for $(\{x_1, x_2, ..., x_n\})_q$ and $(\{x_1, x_2, ..., x_n\})_b$, respectively. Observe that for every nonempty subset $X$ of $S$, $(X)_b \subseteq (X)_q$ since every quasi-ideal of $S$ is a bi-ideal of $S$. It is easily seen that $S$ is a $\mathcal{BQ}$-semigroup if and only if $(X)_b = (X)_q$ for every nonempty subset $X$ of $S$. The following facts are well-known.

**Theorem 2.3** ([3], p. 84-85)**.** *For a nonempty subset $X$ of a semigroup $S$,*

$$(X)_q = S^1X \cap XS^1 \ (= X \cup (SX \cap XS))$$

*and*

$$(X)_b = X \cup XS^1X \ (= X \cup X^2 \cup XSX).$$

*Hence if $S$ has an identity, then $(X)_q = SX \cap XS$ and $(X)_b = X \cup XSX$.*

An important $\mathcal{BQ}$-semigroup was introduced by Lajos [11] as follows :

**Theorem 2.4** ([11])**.** *Every regular semigroup is a $\mathcal{BQ}$-semigroup.*

Theorem 2.4 is a special case of the following fact given by Kapp [9].

**Theorem 2.5** ([9])**.** *If $B$ is a bi-ideal of a semigroup $S$ such that $B \subseteq \mathrm{Reg}(S)$, then $B$ is a quasi-ideal of $S$.*

In fact, we can prove the next theorem which generalizes Theorem 2.5.

**Theorem 2.6.** *Let $X$ be a nonempty subset of a semigroup $S$. If $X \subseteq \mathrm{Reg}(S)$, then $(X)_b = (X)_q$.*

*Proof.* Assume that $X \subseteq \mathrm{Reg}(S)$, i.e., for every $x \in X, x = xyx$ for some $y \in S$. We know that $(X)_b \subseteq (X)_q$. To show that $(X)_q \subseteq (X)_b$, let $x \in (X)_q$. Since $(X)_q = S^1X \cap XS^1, x = s_1y = zs_2$ for some $s_1, s_2 \in S^1$ and $y, z \in X$. Let $w \in S$ be such that $y = ywy$. Then

$$x = s_1y = s_1ywy = (s_1y)wy = (zs_2)wy = z(s_2w)y \in XSX \subseteq (X)_b.$$

This shows that $(X)_q \subseteq (X)_b$, and hence the result follows. $\square$

Calais [2] characterized any $\mathcal{B}Q$-semigroup as follows :

**Theorem 2.7** ([2]). *A semigroup $S$ is a $\mathcal{B}Q$-semigroup if and only if $(x,y)_b = (x,y)_q$ for all $x, y \in S$.*

It follows from Theorem 2.7 that if $S$ is a $\mathcal{B}Q$-semigroup, then $(x)_b = (x)_q$ for all $x \in S$. It can be seen from the proof of Theorem 2.7 given in [18] p. 76 that the following theorem holds.

**Theorem 2.8.** *A commutative semigroup $S$ is a $\mathcal{B}Q$-semigroup if and only if $(x)_b = (x)_q$ for all $x \in S$.*

The following example shows that the converse of Theorem 2.8 need not be true if $S$ is noncommutative.

**Example 2.9.** Let $S = \{0,1,2,3,4\}$ and define the operation $\cdot$ on $S$ by

| $\cdot$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 1 |
| 2 | 0 | 0 | 0 | 0 | 1 |
| 3 | 0 | 0 | 0 | 0 | 3 |
| 4 | 0 | 1 | 2 | 1 | 4 |

Then $(S,\cdot)$ is a semigroup (Selfridge [16]). Notice that $(S,\cdot)$ is noncommutative. First, we show that $S$ is not a $\mathcal{B}Q$-semigroup. Let $A = \{0,2,3\}$. Since $xy = 0$ for all $x, y \in A$, $A$ is a subsemigroup of $S$. From the given table, we have

$$SA = \{0,1,2\}, \quad AS = \{0,1,3\}$$

and

$$ASA = \{0\}.$$

Then $A$ is a bi-ideal of $S$ but $SA \cap AS = \{0,1,2\} \cap \{0,1,3\} = \{0,1\} \not\subseteq A$, so $A$ is not a quasi-ideal of $S$. Hence $S$ is not a $\mathcal{B}Q$-semigroup.

Next, we show that $(x)_b = (x)_q$ for all $x \in S$. Since $0 = 0^3$ and $4 = 4^3$, by Theorem 2.6, $(0)_b = (0)_q$ and $(4)_b = (4)_q$. Also, we have that

$$
\begin{aligned}
(1)_q &= \{1\} \cup (S1 \cap 1S) \\
&= \{1\} \cup (\{0,1\} \cap \{0,1\}) \\
&= \{0,1\}, \\
(1)_b &= \{1, 1^2\} \cup 1S1 \\
&= \{1,0\} \cup \{0,1\}1 \\
&= \{1,0\} \cup \{0\} \\
&= \{0,1\}, \\
(2)_q &= \{2\} \cup (S2 \cap 2S) \\
&= \{2\} \cup (\{0,2\} \cap \{0,1\}) \\
&= \{0,2\}, \\
(2)_b &= \{2, 2^2\} \cup 2S2 \\
&= \{2,0\} \cup \{0,1\}2 \\
&= \{2,0\} \cup \{0\} \\
&= \{0,2\}, \\
(3)_q &= \{3\} \cup (S3 \cap 3S) \\
&= \{3\} \cup (\{0,1\} \cap \{0,3\}) \\
&= \{0,3\}, \\
(3)_b &= \{3, 3^2\} \cup 3S3 \\
&= \{3,0\} \cup \{0,3\}3 \\
&= \{3,0\} \cup \{0\} \\
&= \{0,3\}.
\end{aligned}
$$

It follows that for all $x \in S, (x)_b = (x)_q$.

As mentioned previously, $\mathcal{B}Q$-semigroups have been defined. It is reasonable to define $\mathcal{B}Q$-elements of semigroups accordingly as follows : by a $\mathcal{B}Q$-*element* of a semigroup $S$ we mean an element $x$ of $S$ such that $(x)_b = (x)_q$. It follows from Theorem 2.6 that every regular element of $S$ is a $\mathcal{B}Q$-element. For convenience, let $\mathcal{B}Q(S)$ be the set of all $\mathcal{B}Q$-elements of $S$. Then $\text{Reg}(S) \subseteq \mathcal{B}Q(S)$. This inclusion can be proper. This implies that in a semigroup, $\mathcal{B}Q$-elements are a generalization of regular elements.

**Example 2.10.** Let $S$ be a zero semigroup with zero 0, i.e., $xy = 0$ for all $x, y \in S$ and assume that $|S| > 1$. It is clearly seen that $\mathrm{Reg}(S) = \{0\}$ and for all $x \in S$, $(x)_q = \{0, x\} = (x)_b$. It follows that $\mathcal{B}Q(S) = S \supsetneq \mathrm{Reg}(S)$.

**Example 2.11.** From Example 2.9, we can directly check that $\mathrm{Reg}(S) = \{0, 4\}$. It was shown that $\mathcal{B}Q(S) = S = \{0, 1, 2, 3, 4\}$.

**Remark 2.12.** By Theorem 2.7, if $S \in \mathcal{B}Q$, then $\mathcal{B}Q(S) = S$. However, Example 2.9 shows that the converse is not generally true. By Theorem 2.8, these statements are equivalent if $S$ is commutative.

Let $\mathbb{Z}$ and $\mathbb{R}$ denote respectively the set of all integers and the set of all real numbers and let $\mathbb{Z}^+ = \{x \in \mathbb{Z} \mid x > 0\}$. For $a, b \in \mathbb{Z}$ and $a \neq 0$, let $a \mid b$ means that $b$ is divisible by $a$.

Pearson [13] introduced without proof all the multiplicative interval semigroups on $\mathbb{R}$. There are 15 types as follows :

(i) $\mathbb{R}$,

(ii) $\{0\}$,

(iii) $\{1\}$,

(iv) $(0, \infty)$,

(v) $[0, \infty)$,

(vi) $(a, \infty)$ where $a \geq 1$,

(vii) $[a, \infty)$ where $a \geq 1$,

(viii) $(0, b)$ where $0 < b \leq 1$,

(ix) $(0, b]$ where $0 < b \leq 1$,

(x) $[0, b)$ where $0 < b \leq 1$,

(xi) $[0, b]$ where $0 < b \leq 1$,

(xii) $(a, b)$ where $-1 \leq a < 0 < a^2 \leq b \leq 1$,

(xiii) $(a, b]$ where $-1 \leq a < 0 < a^2 \leq b \leq 1$,

(xiv) $[a, b)$ where $-1 \leq a < 0 < a^2 < b \leq 1$,

(xv) $[a, b]$ where $-1 \leq a < 0 < a^2 \leq b \leq 1$.

Ritkeao [15] gave a proof in detail for these facts in his master thesis by making use of the supremum and the infimum of subsets of $\mathbb{R}$. It can be shown similarly that all the additive interval semigroups on $\mathbb{R}$ have 6 types as follows :

(i) $\mathbb{R}$,

(ii) $\{0\}$,

(iii) $[a, \infty)$ where $a \geq 0$,

(iv) $(a, \infty)$ where $a \geq 0$,

(v) $(-\infty, b]$ where $b \leq 0$,

(vi) $(-\infty, b)$ where $b \leq 0$.

For $n \in \mathbb{Z}^+, \mathbb{Z}_n$ denotes the set of all integers modulo $n$. Then $\mathbb{Z}_n$ contains $n$ elements and

$$\mathbb{Z}_n = \{\overline{0}, \overline{1}, \dots, \overline{n-1}\} = \{\overline{x} \mid x \in \mathbb{Z}\}$$

where $\overline{x}$ is the equivalence class of $x$ modulo $n$. We have that for $k \in \mathbb{Z}$,

$$\begin{aligned} k\mathbb{Z}_n = (k, n)\mathbb{Z}_n &= \{\overline{0}, \overline{(k, n)}, 2\overline{(k, n)}, \dots, \left(\frac{n}{(k, n)} - 1\right)\overline{(k, n)}\} \\ &= \{(k, n)\overline{x} \mid x \in \mathbb{Z}\}, \\ |k\mathbb{Z}_n| &= \frac{n}{(k, n)} \end{aligned}$$

where for $a, b \in \mathbb{Z}$ not both 0, $(a, b)$ denotes the g.c.d. of $a$ and $b$. We know that $(a, b) = as + bt$ for some $s, t \in \mathbb{Z}$ and $(a, b) = 1$, i.e., $a$ and $b$ are relatively prime if and only if $ax + by = 1$ for some $x, y \in \mathbb{Z}$. Notice that $k\mathbb{Z}_n$ is a subsemigroup of $(\mathbb{Z}_n, \cdot)$. Ehrlich [4] gave the following result.

**Theorem 2.13** ([4]). *The semigroup $(\mathbb{Z}_n, \cdot)$ is regular if and only if $n$ is square-free.*

Recall that $n$ is *square-free* if there is no integer $a > 1$ such that $a^2 \mid n$. Then $n$ is square-free if and only if either $n = 1$ or $n$ is a product of distinct primes. Notice that if $n$ is square-free, then $(x, n) = (x^k, n)$ for all $x \in \mathbb{Z}$ and $k \in \mathbb{Z}^+$. By making use of Theorem 2.4 and Theorem 2.13, the following theorem was given in [10].

**Theorem 2.14** ([10]). *The semigroup $(\mathbb{Z}_n, \cdot)$ is a $\mathcal{BQ}$-semigroup if and only if either $n = 4$ or $n$ is square-free.*

Let $X$ be a nonempty set, $1_X$ the identity mapping on $X$ and $T(X)$ the full transformation semigroup on $X$ (the semigroup under composition of all mappings $\alpha : X \to X$). It is well-known that $T(X)$ is a regular semigroup ([6], p. 4). Let $M(X), E(X)$ and $G(X)$ be the subsemigroups of $T(X)$ defined by

$$\begin{aligned}
M(X) &= \{\alpha \in T(X) \mid \alpha \text{ is 1-1}\},\\
E(X) &= \{\alpha \in T(X) \mid \alpha \text{ is onto}\},\\
G(X) &= \text{the symmetric group on } X\\
&\quad (= \{\alpha \in T(X) \mid \alpha \text{ is 1-1 and onto}\}).
\end{aligned}$$

Then $M(X) = G(X)\,[E(X) = G(X)]$ if and only if $X$ is finite. It is evident that $G(X) \subseteq \text{Reg}(M(X))$ and $G(X) \subseteq \text{Reg}(E(X))$. Notice that by right cancellation and left cancellation of $M(X)$ and $E(X)$, respectively, we have that $\text{Reg}(M(X)) \subseteq G(X)$ and $\text{Reg}(E(X)) \subseteq G(X)$. Hence $\text{Reg}(M(X)) = G(X)$ and $\text{Reg}(E(X)) = G(X)$.

Let $V$ be a vector space over a field $F$ and $L_F(V)$ the semigroup under composition of all linear transformations $\alpha : V \to V$. It is known that $L_F(V)$ is a regular semigroup ([7], p. 63). Define the subsemigroups $M_F(V), E_F(V)$ and $G_F(V)$ respectively as follows :

$$\begin{aligned}
M_F(V) &= \{\alpha \in L_F(V) \mid \alpha \text{ is 1-1}\}\\
&\quad (= \{\alpha \in L_F(V) \mid \ker \alpha = \{0\}\}),
\end{aligned}$$

$$E_F(V) = \{\alpha \in L_F(V) \mid \alpha \text{ is onto}\}$$
$$( = \{\alpha \in L_F(V) \mid V\alpha = V\}),$$
$$G_F(V) = \{\alpha \in L_F(V) \mid \alpha \text{ is an isomorphism}\}.$$

Then $M_F(V) = G_F(V)\,[E_F(V) = G_F(V)]$ if and only if $V$ is finite-dimensional. We have similarly that $\mathrm{Reg}(M_F(V)) = G_F(V) = \mathrm{Reg}(E_F(V))$.

If $\alpha \in L_F(V)$ and $v \in \operatorname{ran}\alpha$, then we have

$$v\alpha^{-1} = w + \ker\alpha$$

where $w\alpha = v$. But $|w + \ker\alpha| = |\ker\alpha|$ for all $w \in V$, so we have that for all $v_1, v_2 \in \operatorname{ran}\alpha$, $|v_1\alpha^{-1}| = |v_2\alpha^{-1}|$. Hence

$$\text{for all } \alpha \in E_F(V) \text{ and } v_1, v_2 \in V,\ \ |v_1\alpha^{-1}| = |v_2\alpha^{-1}|.$$

This is not true for $E(X)$ where $X$ is infinite. Let $a \in X$. Then $|X \smallsetminus \{a\}| = |X|$. Let $\varphi : X \smallsetminus \{a\} \to X$ be a bijection and let $\alpha \in E(X)$ be defined by

$$\alpha = \begin{pmatrix} x & a \\ x\varphi & a \end{pmatrix}_{x \in X \smallsetminus \{a\}}.$$

It follows that $|x\alpha^{-1}| = 1$ if $x \in X \smallsetminus \{a\}$ and $|a\alpha^{-1}| = 2$. The following example shows that $|a\alpha^{-1}| \neq |b\alpha^{-1}|$ for all distinct $a, b \in \mathbb{Z}^+$. Define $\alpha \in E(\mathbb{Z}^+)$ by

$$1\alpha = 1,$$
$$2\alpha = 3\alpha = 2,$$
$$4\alpha = 5\alpha = 6\alpha = 3,$$
$$7\alpha = 8\alpha = 9\alpha = 10\alpha = 4, ...$$

i.e., $1\alpha = 1$ and $\left(\left\{1, 2, ..., \sum_{i=1}^{n} i\right\} \smallsetminus \left\{1, 2, ..., \sum_{i=1}^{n-1} i\right\}\right)\alpha = \{n\}$ for all $n \in \mathbb{Z}^+ \smallsetminus \{1\}$.

Then $|n\alpha^{-1}| = n$ for all $n \in \mathbb{Z}^+$.

Next, we define the subset $UE(X)$ of $E(X)$ in order that $UE(X)$ has the same property as $E_F(V)$ mentioned above. Let

$$UE(X) = \{\alpha \in E(X) \mid |a\alpha^{-1}| = |b\alpha^{-1}| \text{ for all } a, b \in X\}.$$

To show that $UE(X)$ is a subsemigroup of $E(X)$, let $\alpha, \beta \in UE(X)$ and $a, b \in X$. Then $|a\beta^{-1}| = |b\beta^{-1}|$ and $|x\alpha^{-1}| = |y\alpha^{-1}|$ for all $x, y \in X$. Let $\varphi : b\beta^{-1} \to a\beta^{-1}$ be a bijection. Then

$$\begin{aligned}
|a(\alpha\beta)^{-1}| &= |a\beta^{-1}\alpha^{-1}| \\
&= \left| \dot{\bigcup_{y \in a\beta^{-1}}} y\alpha^{-1} \right| \\
&= \left| \dot{\bigcup_{x \in b\beta^{-1}}} (x\varphi)\alpha^{-1} \right| \qquad \text{(since } a\beta^{-1} = (b\beta^{-1})\varphi \text{ and } \varphi \text{ is 1-1)} \\
&= \left| \dot{\bigcup_{x \in b\beta^{-1}}} x\alpha^{-1} \right| \qquad \text{(by [14], p. 144)} \\
&= |b\beta^{-1}\alpha^{-1}| \\
&= |b(\alpha\beta)^{-1}|.
\end{aligned}$$

This proves that $UE(X)$ is a subsemigroup of $E(X)$, as desired. Note that $UE(X)$ contains $G(X)$ as a subgroup. Then $UE(X) = G(X)$ if $X$ is finite. We also have that $G(X) \subseteq \operatorname{Reg}(UE(X)) \subseteq \operatorname{Reg}(E(X)) = G(X)$. It follows that $\operatorname{Reg}(UE(X)) = G(X)$.

# CHAPTER III

# ADDITIVE AND MULTIPLICATIVE SEMIGROUPS OF INTEGERS

In this chapter, we characterize the $\mathcal{B}Q$-elements of certain additive and multiplicative semigroups of integers.

Notice that the semigroups in this chapter are commutative. Therefore if $S$ is an additive semigroup of integers, then for any $x \in S$

$$(x)_q = \{x\} \cup (x + S) \qquad \text{and} \qquad (x)_b = \{x, 2x\} \cup (2x + S).$$

In particular, if $0 \in S$, then $(x)_q = x + S$ and $(x)_b = \{x\} \cup (2x + S)$. If $S$ is a multiplicative semigroup of integers, then for any $x \in S$,

$$(x)_q = \{x\} \cup xS \qquad \text{and} \qquad (x)_b = \{x, x^2\} \cup x^2 S.$$

In particular, if $1 \in S$, then $(x)_q = xS$ and $(x)_b = \{x\} \cup x^2 S$.

It can be easily seen that $\operatorname{Reg}(\mathbb{Z}^+, +) = \varnothing$. The first theorem shows that $\mathcal{B}Q(\mathbb{Z}^+, +) \neq \varnothing$ and 1 is the only $\mathcal{B}Q$-element of $(\mathbb{Z}^+, +)$.

Note that $(\mathbb{Z}^+, +)$ is the infinite cyclic semigroup generated by 1.

**Theorem 3.1.** $\mathcal{B}Q(\mathbb{Z}^+, +) = \{1\}$.

*Proof.* We have that

$$(1)_q = \{1\} \cup (1 + \mathbb{Z}^+) = \mathbb{Z}^+$$

and

$$(1)_b = \{1, 2\} \cup (2 + \mathbb{Z}^+) = \mathbb{Z}^+$$

which imply that $1 \in \mathcal{B}Q(\mathbb{Z}^+, +)$.

To show the reverse inclusion, let $x \in \mathcal{B}Q(\mathbb{Z}^+, +)$. Then

$$\begin{aligned}\{x, x+1, x+2, x+3, ...\} &= \{x\} \cup \{x+1, x+2, x+3, ...\} \\ &= \{x\} \cup (x + \mathbb{Z}^+) \\ &= (x)_q \\ &= (x)_b \\ &= \{x, 2x\} \cup (2x + \mathbb{Z}^+) \\ &= \{x, 2x\} \cup \{2x+1, 2x+2, 2x+3, ...\} \\ &= \{x, 2x, 2x+1, 2x+2, 2x+3, ...\}.\end{aligned}$$

Since $x < x+1 < x+2 < ...$ and $x < 2x < 2x+1 < 2x+2 < ...$, it follows that $x + 1 = 2x$ which implies that $x = 1$.

Hence the theorem is proved. $\square$

We can see easily that $\mathrm{Reg}(\mathbb{Z}^+, \cdot) = \{1\}$. The next theorem shows that 1 is also the only $\mathcal{B}Q$-element of the semigroup $(\mathbb{Z}^+, \cdot)$. Notice that 1 is the identity of $(\mathbb{Z}^+, \cdot)$.

**Theorem 3.2.** $\mathcal{B}Q(\mathbb{Z}^+, \cdot) = \{1\}$.

*Proof.* Since

$$(1)_q = 1\mathbb{Z}^+ = \mathbb{Z}^+ \qquad \text{and} \qquad (1)_b = \{1\} \cup 1^2\mathbb{Z}^+ = \mathbb{Z}^+,$$

we have that 1 is a $\mathcal{B}Q$-element of $(\mathbb{Z}^+, \cdot)$.

Next, let $x \in \mathcal{B}Q(\mathbb{Z}^+, \cdot)$. Then

$$\begin{aligned}\{x, 2x, 3x, ...\} &= x\mathbb{Z}^+ \\ &= (x)_q \\ &= (x)_b \\ &= \{x\} \cup x^2\mathbb{Z}^+ \\ &= \{x, x^2, 2x^2, 3x^2, ...\}.\end{aligned}$$

But $x < 2x < 3x < ...$ and $x \leq x^2 < 2x^2 < ...$, it follows that $2x = x^2$ or $2x = 2x^2$. If $2x = x^2$, then $3x = 2x^2$, so $x = \dfrac{3}{2}$, a contradiction. Thus $2x = 2x^2$ which implies that $x = 1$. Hence the result is obtained. $\square$

We can see that for $n \in \mathbb{Z}^+, n\mathbb{Z}^+$ is a subsemigroup of both semigroups $(\mathbb{Z}^+, +)$ and $(\mathbb{Z}^+, \cdot)$. But $(n\mathbb{Z}^+, +)$ is the infinite cyclic semigroup generated by $n$, so $(n\mathbb{Z}^+, +) \cong (\mathbb{Z}^+, +)$. By Theorem 3.1, we have

**Theorem 3.3.** $\mathcal{B}Q(n\mathbb{Z}^+, +) = \{n\}$.

It is easily seen that $\mathrm{Reg}(n\mathbb{Z}^+, \cdot) = \varnothing$ if $n > 1$. We shall show that $\mathcal{B}Q(n\mathbb{Z}^+, \cdot)$ is also empty if $n > 1$.

**Theorem 3.4.** $\mathcal{B}Q(n\mathbb{Z}^+, \cdot) = \varnothing$ *for all* $n > 1$.

*Proof.* Let $n \in \mathbb{Z}^+ \smallsetminus \{1\}$ and suppose on the contrary that $\mathcal{B}Q(n\mathbb{Z}^+, \cdot) \neq \varnothing$. Let $x \in \mathcal{B}Q(n\mathbb{Z}^+, \cdot)$. Then

$$\begin{aligned}\{x, xn, 2xn, 3xn, ...\} &= \{x\} \cup xn\mathbb{Z}^+ \\ &= (x)_q \\ &= (x)_b \\ &= \{x, x^2\} \cup x^2n\mathbb{Z}^+ \\ &= \{x, x^2, x^2n, 2x^2n, ...\}.\end{aligned}$$

Since $x < xn < 2xn < 3xn < ...$ and $x < x^2 < x^2n < 2x^2n < ...$, we have that $3xn = 2x^2n$, so $x = \dfrac{3}{2}$, a contradiction. Hence the result follows. $\square$

For each $n \in \mathbb{Z}^+$, let

$$A_n = \{n, n+1, n+2, ...\}.$$

Then $A_n$ is a subsemigroup of $(\mathbb{Z}^+, +)$ and $A_1 = \mathbb{Z}^+$. It is evident that $\mathrm{Reg}(A_n, +) = \varnothing$ for all $n \in \mathbb{Z}^+$. We show in the next theorem that if $n > 1$, then $\mathcal{B}Q(A_n, +) = \varnothing$.

**Theorem 3.5.** $\mathcal{B}Q(A_n, +) = \varnothing$ *for all* $n > 1$.

*Proof.* Let $n \in \mathbb{Z}^+ \smallsetminus \{1\}$. To show that $\mathcal{B}Q(A_n, +) = \varnothing$, suppose on the contrary that $\mathcal{B}Q(A_n, +) \neq \varnothing$. Let $x \in \mathcal{B}Q(A_n, +)$. Then

$$\begin{aligned}\{x, x+n, x+n+1, x+n+2, ...\} &= \{x\} \cup (x + \{n, n+1, n+2, ...\}) \\ &= \{x\} \cup (x + A_n) \\ &= (x)_q\end{aligned}$$

$$\begin{aligned}
&= (x)_b \\
&= \{x, 2x\} \cup (2x + A_n) \\
&= \{x, 2x\} \cup (2x + \{n, n+1, n+2, ...\}) \\
&= \{x, 2x, 2x + n, 2x + n + 1, 2x + n + 2, ...\}.
\end{aligned}$$

Since $x < x+n < x+n+1 < x+n+2 < ...$ and $x < 2x < 2x+n < 2x+n+1 < ...$, it follows that $x+n+1 = 2x+n$. This implies that $x = 1$ which is a contradiction because $n > 1$. Therefore the desired result follows. $\square$

If $x, y \in \mathbb{Z}$ are such that $x = xyx$, then

$$x(1 - yx) = 0$$

which implies that $x = 0, 1$ or $-1$. It is clear that $-1, 0, 1 \in \mathrm{Reg}(\mathbb{Z}, \cdot)$. Hence $\mathrm{Reg}(\mathbb{Z}, \cdot) = \{-1, 0, 1\}$. We show in the next theorem that $\mathcal{B}Q(\mathbb{Z}, \cdot) = \mathrm{Reg}(\mathbb{Z}, \cdot)$.

**Theorem 3.6.** $\mathcal{B}Q(\mathbb{Z}, \cdot) = \{-1, 0, 1\}$.

*Proof.* Since $\mathrm{Reg}(\mathbb{Z}, \cdot) \subseteq \mathcal{B}Q(\mathbb{Z}, \cdot)$, we have that $\{-1, 0, 1\} \subseteq \mathcal{B}Q(\mathbb{Z}, \cdot)$. For the reverse inclusion, let $x \in \mathcal{B}Q(\mathbb{Z}, \cdot)$. Then

$$\begin{aligned}
\{0, \pm x, \pm 2x, ...\} &= x\mathbb{Z} \\
&= (x)_q \\
&= (x)_b \\
&= \{x\} \cup x^2\mathbb{Z} \\
&= \{0, x, \pm x^2, \pm 2x^2, ...\}.
\end{aligned}$$

**Case 1 :** $-x \in \{0, x\}$. Then $x = 0$.

**Case 2 :** $-x \notin \{0, x\}$. Then $-x = x^2k$ for some $k \in \mathbb{Z} \smallsetminus \{0\}$. Thus $-1 = xk$ which implies that $x = 1$ or $-1$.

Therefore the theorem is proved. $\square$

If $n \in \mathbb{Z}$, then $n\mathbb{Z}$ is a subsemigroup of $(\mathbb{Z}, \cdot)$. Notice that $n\mathbb{Z} = \mathbb{Z}$ if $n = 1$ or $-1$. Let $n \notin \{-1, 1\}$ and $x, y \in \mathbb{Z}$ be such that $nx = (nx)(ny)(nx)$. Then

$nx(1-n^2xy)=0$ which implies that $nx=0$ or $n^2xy=1$. But $n\neq 1$ and $n\neq -1$, so $n^2xy=1$ cannot occur. Hence $nx=0$. Therefore $\mathrm{Reg}(n\mathbb{Z},\cdot)=\{0\}$. We show in the next theorem that $\mathcal{B}Q(n\mathbb{Z},\cdot)$ also contains only 0.

**Theorem 3.7.** *If $n\notin\{-1,1\}$, then $\mathcal{B}Q(n\mathbb{Z},\cdot)=\{0\}$.*

*Proof.* It is evident if $n=0$. Assume that $n\neq 0$. Then $n\notin\{-1,0,1\}$, so $n\mathbb{Z}$ does not contain 1 and $-1$. Since $0\in\mathrm{Reg}(n\mathbb{Z},\cdot)$ and $\mathrm{Reg}(n\mathbb{Z},\cdot)\subseteq\mathcal{B}Q(n\mathbb{Z},\cdot)$, we have that $0\in\mathcal{B}Q(n\mathbb{Z},\cdot)$. In fact, it is clear that $(0)_q=\{0\}=(0)_b$. Let $x\in\mathcal{B}Q(n\mathbb{Z},\cdot)$ and suppose that $x\neq 0$. Then

$$\begin{aligned}\{0,x,\pm nx,\pm 2nx,...\}&=\{x\}\cup x(n\mathbb{Z})\\&=(x)_q\\&=(x)_b\\&=\{x,x^2\}\cup x^2(n\mathbb{Z})\\&=\{0,x,x^2,\pm nx^2,\pm 2nx^2,...\},\end{aligned}$$

so $nx=0,x,x^2$ or $knx^2$ for some $k\in\mathbb{Z}\setminus\{0\}$. Since $n\neq 0, n\neq 1$ and $x\neq 0$, it follows that $nx\neq 0$ and $nx\neq x$. If $nx=x^2$, then $-nx=-x^2\notin\{0,x,x^2,\pm nx^2,\pm 2nx^2,...\}$, a contradiction. If $nx=knx^2$, then $1=kx$, so $x=\pm 1\notin n\mathbb{Z}$ which is a contradiction. Hence the result follows. $\square$

Next, for $n\in\mathbb{Z}^+$, let

$$\mathbb{Z}(n)=\{x\in\mathbb{Z}\mid |x|\geq n\}.$$

Then

$$\mathbb{Z}(n)=\{\pm n,\pm(n+1),\pm(n+2),...\}$$

which is a subsemigroup of $(\mathbb{Z},\cdot)$ not containing 0. Since

$$\mathbb{Z}(1)=\mathbb{Z}\setminus\{0\},$$

we clearly have that $\mathrm{Reg}(\mathbb{Z}(1),\cdot)=\{-1,1\}$. If $n>1$, then $-1,1\notin\mathbb{Z}(n)$ and we clearly obtain that $\mathrm{Reg}(\mathbb{Z}(n),\cdot)=\varnothing$. We show in the next theorem that the $\mathcal{B}Q$-elements of $(\mathbb{Z}(n),\cdot)$ are also regular.

**Theorem 3.8.**

$$\mathcal{B}Q(\mathbb{Z}(n),\cdot) = \begin{cases} \{-1,1\} & \textit{if } n = 1, \\ \varnothing & \textit{if } n > 1. \end{cases}$$

*Proof.* First, we consider $(\mathbb{Z}(1),\cdot)$. We have that 1 is the identity of $(\mathbb{Z}(1),\cdot)$. Since $\mathrm{Reg}(\mathbb{Z}(1),\cdot) \subseteq \mathcal{B}Q(\mathbb{Z}(1),\cdot)$, it follows that $-1, 1 \in \mathcal{B}Q(\mathbb{Z}(1),\cdot)$. Let $x \in \mathcal{B}Q(\mathbb{Z}(1),\cdot)$. Then

$$\begin{aligned} \{\pm x, \pm 2x, \pm 3x, ...\} &= x(\mathbb{Z}(1)) \\ &= (x)_q \\ &= (x)_b \\ &= \{x\} \cup x^2(\mathbb{Z}(1)) \\ &= \{x, \pm x^2, \pm 2x^2, \pm 3x^2, ...\}. \end{aligned}$$

Since $0 \notin \mathbb{Z}(1), -x \neq x$. Then $-x = kx^2$ for some $k \in \mathbb{Z} \smallsetminus \{0\}$, so $-1 = kx$ and hence $x \in \{-1, 1\}$. Therefore $\mathcal{B}Q(\mathbb{Z}(1),\cdot) = \{-1, 1\}$.

Next, assume that $n > 1$. Then $(\mathbb{Z}(n),\cdot)$ has no identity. To show that $\mathcal{B}Q(\mathbb{Z}(n),\cdot) = \varnothing$, suppose not. Let $x \in \mathcal{B}Q(\mathbb{Z}(n),\cdot)$. Then

$$\begin{aligned} \{x, \pm xn, \pm x(n+1), \pm x(n+2), ...\} &= \{x\} \cup x(\mathbb{Z}(n)) \\ &= (x)_q \\ &= (x)_b \\ &= \{x, x^2\} \cup x^2(\mathbb{Z}(n)) \\ &= \{x, x^2, \pm x^2 n, \pm x^2(n+1), \pm x^2(n+2), ...\}, \end{aligned}$$

so $xn = x^2$ or $x^2 k$ for some $k \in \mathbb{Z}(n)$. If $xn = x^2$, then $-xn = -x^2 \notin \{x, x^2, \pm x^2 n, \pm x^2(n+1), \pm x^2(n+2), ...\}$, a contradiction. If $xn = x^2 k$, then $n = xk$, so $1 < n \leq |k| = \left|\frac{n}{x}\right| \leq 1$ since $|x| \geq n$, which is a contradiction.

Hence the theorem is proved. $\square$

# CHAPTER IV

# ADDITIVE AND MULTIPLICATIVE INTERVAL SEMIGROUPS ON $\mathbb{R}$

In this chapter, we characterize the $\mathcal{B}Q$-elements of all interval semigroups on $\mathbb{R}$ under addition and multiplication. It is shown that the $\mathcal{B}Q$-elements of these semigroups are regular.

All the 15 types of the multiplicative interval semigroups on $\mathbb{R}$ are recalled as follows :

(i) $\mathbb{R}$,

(ii) $\{0\}$,

(iii) $\{1\}$,

(iv) $(0, \infty)$,

(v) $[0, \infty)$,

(vi) $(a, \infty)$ where $a \geq 1$,

(vii) $[a, \infty)$ where $a \geq 1$,

(viii) $(0, b)$ where $0 < b \leq 1$,

(ix) $(0, b]$ where $0 < b \leq 1$,

(x) $[0, b)$ where $0 < b \leq 1$,

(xi) $[0, b]$ where $0 < b \leq 1$,

(xii) $(a, b)$ where $-1 \leq a < 0 < a^2 \leq b \leq 1$,

(xiii) $(a, b]$ where $-1 \leq a < 0 < a^2 \leq b \leq 1$,

(xiv) $[a, b)$ where $-1 \leq a < 0 < a^2 < b \leq 1$,

(xv) $[a, b]$ where $-1 \leq a < 0 < a^2 \leq b \leq 1$.

All the additive interval semigroups on $\mathbb{R}$ are also recalled as follows :

(i) $\mathbb{R}$,

(ii) $\{0\}$,

(iii) $[a, \infty)$ where $a \geq 0$,

(iv) $(a, \infty)$ where $a \geq 0$,

(v) $(-\infty, b]$ where $b \leq 0$,

(vi) $(-\infty, b)$ where $b \leq 0$.

First, we determine the $\mathcal{B}Q$-elements of all the multiplicative interval semigroups on $\mathbb{R}$. It is clearly seen that the multiplicative interval semigroups on $\mathbb{R}$ of type (i) - (v) are regular semigroups. Then the following theorem is directly obtained.

**Theorem 4.1.** *If $I$ is a multiplicative interval semigroup on $\mathbb{R}$ of type $(i) - (v)$, then $\mathcal{B}Q(I) = I$.*

It is not difficult to see that if $I$ is a multiplicative interval semigroup on $\mathbb{R}$ of type (vi) - (xv), then

$$\mathrm{Reg}(I) = I \cap \{-1, 0, 1\}.$$

The following theorem shows that the $\mathcal{B}Q$-elements of these semigroups are regular.

**Theorem 4.2.** *If $I$ is a multiplicative interval semigroup on $\mathbb{R}$ of type $(vi) - (xv)$, then*

$$\mathcal{B}Q(I) = I \cap \{-1, 0, 1\}.$$

*Proof.* Since $\operatorname{Reg}(I) \subseteq \mathcal{B}Q(I)$, we have that $I \cap \{-1, 0, 1\} \subseteq \mathcal{B}Q(I)$.

**Case 1 :** $I$ is of type (vi) or (vii). Then $I = (a, \infty)$ or $I = [a, \infty)$ for some $a \geq 1$. To show that $\mathcal{B}Q(I) = I \cap \{1\}$, it suffices to show that for $x \in I$ with $x > 1$, we have $(x)_q \smallsetminus (x)_b \neq \varnothing$. Let $x \in I$ be such that $x > 1$. Then

$$(x)_q = \{x\} \cup xI = \begin{cases} \{x\} \cup (xa, \infty) & \text{if } I = (a, \infty), \\ \{x\} \cup [xa, \infty) & \text{if } I = [a, \infty) \end{cases}$$

and

$$(x)_b = \{x, x^2\} \cup x^2 I = \begin{cases} \{x, x^2\} \cup (x^2 a, \infty) & \text{if } I = (a, \infty), \\ \{x, x^2\} \cup [x^2 a, \infty) & \text{if } I = [a, \infty). \end{cases}$$

**Subcase 1.1 :** $a = 1$. Then

$$(x)_q = [x, \infty) \qquad \text{and} \qquad (x)_b = \{x\} \cup [x^2, \infty).$$

Since $x < \dfrac{x + x^2}{2} < x^2$, so $\dfrac{x + x^2}{2} \in (x)_q \smallsetminus (x)_b$.

**Subcase 1.2 :** $a > 1$. Then

$$x < xa \leq x^2 < \frac{x^2 + x^2 a}{2} < x^2 a,$$

so $\dfrac{x^2 + x^2 a}{2} \in (x)_q \smallsetminus (x)_b$.

**Case 2 :** $I$ is of type (viii)-(xi). Then $I = (0, b), (0, b], [0, b)$ or $[0, b]$ for some $0 < b \leq 1$. Therefore $I \cap \{-1, 0, 1\} = I \cap \{0, 1\}$. To show that $\mathcal{B}Q(I) = I \cap \{0, 1\}$, it suffices to show that for $x \in I$ with $0 < x < 1$, we have $(x)_q \smallsetminus (x)_b \neq \varnothing$. Let $x \in I$ be such that $0 < x < 1$. Then

$$(x)_q = \{x\} \cup xI = \begin{cases} \{x\} \cup (0, xb) & \text{if } I = (0, b), \\ \{x\} \cup (0, xb] & \text{if } I = (0, b], \\ \{x\} \cup [0, xb) & \text{if } I = [0, b), \\ \{x\} \cup [0, xb] & \text{if } I = [0, b], \end{cases}$$

and

$$(x)_b = \{x, x^2\} \cup x^2 I = \begin{cases} \{x, x^2\} \cup (0, x^2 b) & \text{if } I = (0, b), \\ \{x, x^2\} \cup (0, x^2 b] & \text{if } I = (0, b], \\ \{x, x^2\} \cup [0, x^2 b) & \text{if } I = [0, b), \\ \{x, x^2\} \cup [0, x^2 b] & \text{if } I = [0, b]. \end{cases}$$

**Subcase 2.1 :** $b = 1$. Then

$$(x)_q = \begin{cases} (0, x] & \text{if } I = (0, b) \text{ or } (0, b], \\ [0, x] & \text{if } I = [0, b) \text{ or } [0, b] \end{cases}$$

and

$$(x)_b = \begin{cases} \{x\} \cup (0, x^2] & \text{if } I = (0, b) \text{ or } (0, b], \\ \{x\} \cup [0, x^2] & \text{if } I = [0, b) \text{ or } [0, b]. \end{cases}$$

Since $x^2 < \dfrac{x^2 + x}{2} < x$, we have that $\dfrac{x^2 + x}{2} \in (x)_q \smallsetminus (x)_b$.

**Subcase 2.2 :** $b < 1$. Then

$$0 < x^2 b < \frac{x^2 b + x^2}{2} < x^2 \leq xb < x,$$

so $\dfrac{x^2 b + x^2}{2} \in (x)_q \smallsetminus (x)_b$.

**Case 3 :** $I$ is of type (xii)-(xv). Then $I = (a, b), (a, b]$ or $[a, b]$ for some $a, b \in \mathbb{R}$ with $-1 \leq a < 0 < a^2 \leq b \leq 1$ or $I = [a, b)$ for some $a, b \in \mathbb{R}$ with $-1 \leq a < 0 < a^2 < b \leq 1$. To show that $\mathcal{B}Q(I) = I \cap \{-1, 0, 1\}$, let $x \in I$ be such that $x \notin \{-1, 0, 1\}$. Then

$$(x)_b = \{x, x^2\} \cup x^2 I = \begin{cases} \{x, x^2\} \cup (x^2 a, x^2 b) & \text{if } I = (a, b), \\ \{x, x^2\} \cup (x^2 a, x^2 b] & \text{if } I = (a, b], \\ \{x, x^2\} \cup [x^2 a, x^2 b] & \text{if } I = [a, b], \\ \{x, x^2\} \cup [x^2 a, x^2 b) & \text{if } I = [a, b). \end{cases}$$

and

$$(x)_q = \{x\} \cup xI = \begin{cases} \{x\} \cup (xa, xb) & \text{if } I = (a, b) \text{ and } x > 0, \\ \{x\} \cup (xb, xa) & \text{if } I = (a, b) \text{ and } x < 0, \\ \{x\} \cup (xa, xb] & \text{if } I = (a, b] \text{ and } x > 0, \\ \{x\} \cup [xb, xa) & \text{if } I = (a, b] \text{ and } x < 0, \\ \{x\} \cup [xa, xb] & \text{if } I = [a, b] \text{ and } x > 0, \\ \{x\} \cup [xb, xa] & \text{if } I = [a, b] \text{ and } x < 0, \\ \{x\} \cup [xa, xb) & \text{if } I = [a, b) \text{ and } x > 0, \\ \{x\} \cup (xb, xa] & \text{if } I = [a, b) \text{ and } x < 0. \end{cases}$$

**Subcase 3.1 :** $b = 1$ and $x > 0$. Then $x^2 < x$, so

$$xa < 0 < x^2 b = x^2 < \frac{x^2 + x}{2} < x = xb.$$

Hence $\frac{x^2+x}{2} \in (x)_q \smallsetminus (x)_b$.

**Subcase 3.2 :** $b = 1, x < 0$ and $a = -1$. Then $x^2 < -x$, so

$$xb = x < 0 < x^2 b = x^2 < \frac{x^2 - x}{2} < -x = xa$$

which implies that $\frac{x^2-x}{2} \in (x)_q \smallsetminus (x)_b$.

**Subcase 3.3 :** $b = 1, x < 0$ and $a > -1$. Then $x^2 a < 0 < x^2 < -x$. Since $xa < 1$ and $x < 0$, we have that $x^2 a > x$. Thus

$$xb = x < \frac{x + x^2 a}{2} < x^2 a < 0 < x^2 \leq xa.$$

It follows that $\frac{x+x^2a}{2} \in (x)_q \smallsetminus (x)_b$.

**Subcase 3.4 :** $b < 1$ and $x > 0$. Then $0 < x^2 b < x^2$, so

$$xa < 0 < x^2 b < \frac{x^2 b + x^2}{2} < x^2 \leq xb < x.$$

Thus $\dfrac{x^2b+x^2}{2} \in (x)_q \smallsetminus (x)_b$.

**Subcase 3.5 :** $b < 1$ and $x < 0$. Then $0 < x^2b < x^2$, so

$$x < xb < 0 < x^2b < \frac{x^2b+x^2}{2} < x^2 \leq xa,$$

and hence $\dfrac{x^2b+x^2}{2} \in (x)_q \smallsetminus (x)_b$.

The proof is thereby complete. $\square$

Next, we characterize the $\mathcal{B}Q$-elements of all the additive interval semigroups on $\mathbb{R}$. Since the additive interval semigroups of type (i) and type (ii) are regular, it follows that their $\mathcal{B}Q$-elements are regular. Therefore the following theorem is obtained.

**Theorem 4.3.** *If $I$ is an additive interval semigroup on $\mathbb{R}$ of type (i) or type (ii), then $\mathcal{B}Q(I) = I$.*

It is clear that if $I$ is an additive interval semigroup on $\mathbb{R}$ of type (iii)-(vi), then

$$\mathrm{Reg}(I) = I \cap \{0\},$$

i.e.,

$$\mathrm{Reg}(I) = \begin{cases} \{0\} & \text{if } 0 \in I, \\ \varnothing & \text{if } 0 \notin I. \end{cases}$$

The next theorem shows that the $\mathcal{B}Q$-elements of the semigroup of these types are regular.

**Theorem 4.4.** *If $I$ is an additive interval semigroup on $\mathbb{R}$ of type (iii) $-$ (vi), then*

$$\mathcal{B}Q(I) = I \cap \{0\}.$$

*Proof.* Since $\mathrm{Reg}(I) \subseteq \mathcal{B}Q(I)$, we have that $I \cap \{0\} \subseteq \mathcal{B}Q(I)$.

**Case 1 :** $I$ is of type (iii) or type (iv). Then $I = [a, \infty)$ or $(a, \infty)$ for some $a \geq 0$. To show that $\mathcal{B}Q(I) = I \cap \{0\}$, it suffices to show that for $x \in I$ with $x > 0$, we have $(x)_q \smallsetminus (x)_b \neq \varnothing$. Let $x \in I$ be such that $x > 0$.

**Subcase 1.1 :** $a = 0$. Then

$$(x)_q = \{x\} \cup (x + I) = [x, \infty)$$

and

$$(x)_b = \{x, 2x\} \cup (2x + I) = \{x\} \cup [2x, \infty).$$

Since $x > 0$, $x < \dfrac{3x}{2} < 2x$. Thus $\dfrac{3x}{2} \in (x)_q \smallsetminus (x)_b$.

**Subcase 1.2 :** $a > 0$. Then

$$(x)_q = \{x\} \cup (x + I) = \begin{cases} \{x\} \cup [x + a, \infty) & \text{if } I = [a, \infty), \\ \{x\} \cup (x + a, \infty) & \text{if } I = (a, \infty) \end{cases}$$

and

$$(x)_b = \{x, 2x\} \cup (2x + I) = \begin{cases} \{x, 2x\} \cup [2x + a, \infty) & \text{if } I = [a, \infty), \\ \{x, 2x\} \cup (2x + a, \infty) & \text{if } I = (a, \infty). \end{cases}$$

Since

$$x < x + a \leq 2x < 2x + \frac{a}{2} < 2x + a,$$

it follows that $2x + \dfrac{a}{2} \in (x)_q \smallsetminus (x)_b$.

**Case 2 :** $I$ is of type (v) or type (vi). Then $I = (-\infty, b]$ or $(-\infty, b)$ for some $b \leq 0$. We can show similarly to Case 1 that $\mathcal{B}Q(I) = I \cap \{0\}$. $\square$

# CHAPTER V

# THE MULTIPLICATIVE SEMIGROUP $\mathbb{Z}_n$

The purpose of this chapter is to characterize the $\mathcal{B}Q$-elements of the multiplicative semigroup $\mathbb{Z}_n$ and give some sufficient conditions for the multiplicative semigroup $k\mathbb{Z}_n$ to have the property that every element is a $\mathcal{B}Q$-element.

Recall that $\mathbb{Z}_n$ contains $n$ elements,

$$\mathbb{Z}_n = \{\overline{0}, \overline{1}, \ldots, \overline{n-1}\} = \{\overline{x} \mid x \in \mathbb{Z}\}$$

where $\overline{x}$ is the equivalence class of $x$ modulo $n$. Also, for $k \in \mathbb{Z}$ we have that $k\mathbb{Z}_n = (-k)\mathbb{Z}_n$,

$$\begin{aligned} k\mathbb{Z}_n = (k,n)\mathbb{Z}_n &= \{\overline{0}, \overline{(k,n)}, 2\overline{(k,n)}, \ldots, \left(\frac{n}{(k,n)} - 1\right)\overline{(k,n)}\} \\ &= \{(k,n)\overline{x} \mid x \in \mathbb{Z}\}, \\ |k\mathbb{Z}_n| &= \frac{n}{(k,n)}. \end{aligned}$$

Notice that $k\mathbb{Z}_n$ is an ideal of $(\mathbb{Z}_n, \cdot)$. We know that $\mathrm{Reg}(\mathbb{Z}_n, \cdot) \subseteq \mathcal{B}Q(\mathbb{Z}_n, \cdot)$. First, we characterize the regular elements of $(\mathbb{Z}_n, \cdot)$ differently from the one given in [1] and then its $\mathcal{B}Q$-elements are determined. As consequences, we have Theorem 2.13 and Theorem 2.14, respectively.

The regular elements of the semigroup $(\mathbb{Z}_n, \cdot)$ are characterized as follows :

**Theorem 5.1.** *For $x \in \mathbb{Z}$, $\overline{x} \in \mathrm{Reg}(\mathbb{Z}_n, \cdot)$ if and only if $x$ and $\frac{n}{(x,n)}$ are relatively prime.*

*Proof.* Assume that $\overline{x} \in \mathrm{Reg}(\mathbb{Z}_n, \cdot)$. Then $\bar{x} = \bar{x}\bar{y}\bar{x}$ for some $y \in \mathbb{Z}$. Then $\overline{x(xy-1)} = \overline{0}$, so $n \mid x(xy-1)$. Hence $\frac{n}{(x,n)} \,\Big|\, \frac{x}{(x,n)}\,(xy-1)$. But since $\frac{n}{(x,n)}$ and $\frac{x}{(x,n)}$ are relatively prime, it follows that $\frac{n}{(x,n)} \,\Big|\, xy-1$. Then $\frac{nk}{(x,n)} = xy-1$

for some $k \in \mathbb{Z}$. Now we have $xy + \left(\frac{n}{(x,n)}\right)(-k) = 1$. This implies that $x$ and $\frac{n}{(x,n)}$ are relatively prime.

For the converse, assume that $x$ and $\frac{n}{(x,n)}$ are relatively prime. Then $xk + \frac{nl}{(x,n)} = 1$ for some $k, l \in \mathbb{Z}$. Thus $x = x^2k + \frac{xnl}{(x,n)} = x^2k + \left(\frac{x}{(x,n)}\right) ln$ and $\frac{x}{(x,n)} \in \mathbb{Z}$. This implies that $\overline{x} = \overline{x}^2\overline{k}$ and thus $\overline{x} \in \operatorname{Reg}(\mathbb{Z}_n, \cdot)$. $\square$

**Corollary 5.2.** *The semigroup $(\mathbb{Z}_n, \cdot)$ is a regular semigroup if and only if $n$ is square-free.*

*Proof.* Assume that $n$ is not square-free. Then there is $a \in \mathbb{Z}$ such that $a > 1$ and $a^2 \mid n$, so $a \,\Big|\, \frac{n}{a}$ and $\frac{n}{(n,a)} = \frac{n}{a}$. Thus $\left(a, \frac{n}{(n,a)}\right) = a > 1$. By Theorem 5.1, $\overline{a} \notin \operatorname{Reg}(\mathbb{Z}_n, \cdot)$. Hence $(\mathbb{Z}_n, \cdot)$ is not a regular semigroup.

Conversely, assume that $n$ is square-free. Then either $n = 1$ or $n$ is a product of distinct primes. It clearly follows that for every $x \in \mathbb{Z}$, $x$ and $\frac{n}{(x,n)}$ are relatively prime. By Theorem 5.1, $\overline{x} \in \operatorname{Reg}(\mathbb{Z}_n, \cdot)$ for all $x \in \mathbb{Z}$. Thus $(\mathbb{Z}_n, \cdot)$ is a regular semigroup. $\square$

Next, we characterize the $\mathcal{B}Q$-elements of $(\mathbb{Z}_n, \cdot)$. Since $(\mathbb{Z}_n, \cdot)$ is a commutative semigroup having $\overline{1}$ as its identity, we have that

$$\begin{aligned} \mathcal{B}Q(\mathbb{Z}_n, \cdot) &= \{\overline{x} \in \mathbb{Z}_n \mid x \in \mathbb{Z} \text{ and } (\overline{x})_q = (\overline{x})_b\} \\ &= \{\overline{x} \in \mathbb{Z}_n \mid x \in \mathbb{Z} \text{ and } \overline{x}\mathbb{Z}_n = \{\overline{x}\} \cup \overline{x}^2\mathbb{Z}_n\}. \end{aligned}$$

**Theorem 5.3.** *For $x \in \mathbb{Z}$, $\overline{x} \in \mathcal{B}Q(\mathbb{Z}_n, \cdot)$ if and only if either*

(i) *$x$ and $\frac{n}{(x,n)}$ are relatively prime or*

(ii) *$n \mid x^2$ and $\frac{n}{(x,n)} = 2$.*

*Proof.* Assume that $\overline{x} \in \mathcal{B}Q(\mathbb{Z}_n, \cdot)$. Then $\overline{x}\mathbb{Z}_n = \{\overline{x}\} \cup \overline{x}^2\mathbb{Z}_n$.

**Case 1 :** $\overline{x} \in \overline{x}^2\mathbb{Z}_n$. Then $\overline{x} \in \operatorname{Reg}(\mathbb{Z}_n, \cdot)$, so by Theorem 5.1, $x$ satisfies (i).

**Case 2 :** $\overline{x} \notin \overline{x}^2\mathbb{Z}_n$. Since $|\overline{x}^2\mathbb{Z}_n| = \dfrac{n}{(x^2,n)}$ and $|\overline{x}\mathbb{Z}_n| = \dfrac{n}{(x,n)}$, it follows that $1+\dfrac{n}{(x^2,n)} = |\{\overline{x}\}\cup\overline{x}^2\mathbb{Z}_n| = |\overline{x}\mathbb{Z}_n| = \dfrac{n}{(x,n)}$. But $(x,n)|(x^2,n)$, thus $\dfrac{n}{(x^2,n)}\,\bigg|\,\dfrac{n}{(x,n)}$. This implies that $\dfrac{n}{(x^2,n)}\,\bigg|\,1$. Then $\dfrac{n}{(x^2,n)} = 1$, so $(x^2,n) = n$. Therefore $n \mid x^2$ and $\dfrac{n}{(x,n)} = 1 + \dfrac{n}{(x^2,n)} = 1+1 = 2$. Hence $x$ satisfies (ii).

Conversely, assume that $x$ satisfies (i) or (ii). If $x$ satisfies (i), then by Theorem 5.1, $\overline{x} \in \mathrm{Reg}(\mathbb{Z}_n,\cdot)$. But $\mathrm{Reg}(\mathbb{Z}_n,\cdot) \subseteq \mathcal{B}Q(\mathbb{Z}_n,\cdot)$, so $\overline{x} \in \mathcal{B}Q(\mathbb{Z}_n,\cdot)$. Next, let $x$ satisfy (ii). Since $n \mid x^2$, we have that $\overline{x}^2\mathbb{Z}_n = \{\overline{0}\}$, so $\{\overline{x}\} \cup \overline{x}^2\mathbb{Z}_n = \{\overline{0},\overline{x}\}$. We also have that $|\overline{x}\mathbb{Z}_n| = \dfrac{n}{(x,n)} = 2$. Thus $\overline{x}\mathbb{Z}_n = \{\overline{0},\overline{x}\}$. Consequently, $\{\overline{x}\} \cup \overline{x}^2\mathbb{Z}_n$, $= \overline{x}\mathbb{Z}_n$. Hence $\overline{x} \in \mathcal{B}Q(\mathbb{Z}_n,\cdot)$.

The theorem is thereby proved. □

**Corollary 5.4.** *The semigroup $(\mathbb{Z}_n,\cdot)$ is a $\mathcal{B}Q$-semigroup if and only if either $n = 4$ or $n$ is square-free.*

*Proof.* Assume that $n \neq 4$ and $n$ is not square-free. Then there is $a \in \mathbb{Z}$ such that $a > 1$ and $a^2 \mid n$. We claim that $a$ does not satisfy (i) and (ii) of Theorem 5.3. Since $a^2 \mid n$, there is $t \in \mathbb{Z}$ such that $n = a^2t$. Then $\dfrac{n}{(a,n)} = \dfrac{a^2t}{a} = at$, so $\left(a, \dfrac{n}{(a,n)}\right) = a > 1$. Hence $a$ does not satisfy (i). To show that $a$ does not satisfy (ii), i.e., $n \nmid a^2$ or $\dfrac{n}{(a,n)} \neq 2$, suppose that $n \mid a^2$. It follows that $n = a^2$. Since $n \neq 4$, we have that $a > 2$. Hence $\dfrac{n}{(a,n)} = \dfrac{a^2}{a} = a > 2$. Thus $a$ does not satisfy (ii). By Theorem 5.3, $\overline{a} \notin \mathcal{B}Q(\mathbb{Z}_n,\cdot)$. Hence from Theorem 2.8, $(\mathbb{Z}_n,\cdot)$ is not a $\mathcal{B}Q$-semigroup.

For the converse, assume that $n = 4$ or $n$ is square-free. If $n$ is square-free, then by Corollary 5.2 (Theorem 2.13), $(\mathbb{Z}_n,\cdot)$ is a regular semigroup, and hence $(\mathbb{Z}_n,\cdot)$ is a $\mathcal{B}Q$-semigroup by Theorem 2.4. Next, assume that $n = 4$. If $x \in \{0,1,3\}$, then $x$ satisfies (i) of Theorem 5.3. If $x = 2$, then $x$ satisfies (ii) of Theorem 5.3. Hence $\overline{0},\overline{1},\overline{2},\overline{3} \in \mathcal{B}Q(\mathbb{Z}_4,\cdot)$, and therefore $(\mathbb{Z}_4,\cdot)$ is a $\mathcal{B}Q$-semigroup by Theorem 2.8.

Therefore the proof is complete. $\square$

**Example 5.5.** From Theorem 5.1 and Theorem 5.3, we have

$$\begin{aligned}
\operatorname{Reg}(\mathbb{Z}_9,\cdot) &= \{\overline{0},\overline{1},\overline{2},\overline{4},\overline{5},\overline{7},\overline{8}\} = \mathcal{B}Q(\mathbb{Z}_9,\cdot),\\
\operatorname{Reg}(\mathbb{Z}_{18},\cdot) &= \{\overline{0},\overline{1},\overline{2},\overline{4},\overline{5},\overline{7},\overline{8},\overline{9},\overline{10},\overline{11},\overline{13},\overline{14},\overline{16},\overline{17}\} = \mathcal{B}Q(\mathbb{Z}_{18},\cdot),\\
\operatorname{Reg}(\mathbb{Z}_8,\cdot) &= \{\overline{0},\overline{1},\overline{3},\overline{5},\overline{7}\},\ \mathcal{B}Q(\mathbb{Z}_8,\cdot) = \operatorname{Reg}(\mathbb{Z}_8,\cdot)\cup\{\overline{4}\},\\
\operatorname{Reg}(\mathbb{Z}_{12},\cdot) &= \{\overline{0},\overline{1},\overline{3},\overline{4},\overline{5},\overline{7},\overline{8},\overline{9},\overline{11}\},\ \mathcal{B}Q(\mathbb{Z}_{12},\cdot) = \operatorname{Reg}(\mathbb{Z}_{12},\cdot)\cup\{\overline{6}\}.
\end{aligned}$$

From Example 5.5, it is natural to ask whether it is true that $\mathcal{B}Q(\mathbb{Z}_n,\cdot) = \operatorname{Reg}(\mathbb{Z}_n,\cdot)$ if $4\nmid n$ and $\mathcal{B}Q(\mathbb{Z}_n,\cdot) = \operatorname{Reg}(\mathbb{Z}_n,\cdot)\cup\left\{\overline{\left(\frac{n}{2}\right)}\right\}$ and $\overline{\left(\frac{n}{2}\right)}\notin\operatorname{Reg}(\mathbb{Z}_n,\cdot)$ if $4\mid n$. This is generally true as the following theorem shows :

**Theorem 5.6.** *The following statements hold.*

(i) *If $4\nmid n$, then $\mathcal{B}Q(\mathbb{Z}_n,\cdot) = \operatorname{Reg}(\mathbb{Z}_n,\cdot)$.*

(ii) *If $4\mid n$, then $\mathcal{B}Q(\mathbb{Z}_n,\cdot) = \operatorname{Reg}(\mathbb{Z}_n,\cdot)\cup\left\{\overline{\left(\frac{n}{2}\right)}\right\}$ and $\overline{\left(\frac{n}{2}\right)}\notin\operatorname{Reg}(\mathbb{Z}_n,\cdot)$.*

*Proof.* (i) Assume that there is $x\in\mathbb{Z}$ such that $n\mid x^2$ and $\dfrac{n}{(x,n)}=2$. Then $\dfrac{n}{(x,n)}\,\Big|\left(\dfrac{x}{(x,n)}\right)x$ and $2\mid n$. Since $\dfrac{n}{(x,n)}$ and $\dfrac{x}{(x,n)}$ are relatively prime, it follows that $\dfrac{n}{(x,n)}\,\Big|\,x$, so $2\mid x$. Hence $2\mid(x,n)$. Since $n=2(x,n)$ and $2\mid(x,n)$, we have that $4\mid n$. This proves that if $4\nmid n$, then there is no $x\in\mathbb{Z}$ satisfying (ii) of Theorem 5.3. From Theorem 5.1 and Theorem 5.3, we have that if $4\nmid n$, then $\mathcal{B}Q(\mathbb{Z}_n,\cdot)=\operatorname{Reg}(\mathbb{Z}_n,\cdot)$.

(ii) Assume that $4\mid n$. Then $\dfrac{n}{(\frac{n}{2},n)}=2$ and $2\,\Big|\,\dfrac{n}{2}$, so by Theorem 5.1, $\overline{\left(\frac{n}{2}\right)}\notin\operatorname{Reg}(\mathbb{Z}_n,\cdot)$. Since $\left(\dfrac{n}{2}\right)^2=n\left(\dfrac{n}{4}\right)$ which is divisible by $n$, by Theorem 5.3, $\overline{\left(\frac{n}{2}\right)}\in\mathcal{B}Q(\mathbb{Z}_n,\cdot)$. It remains to show that $\mathcal{B}Q(\mathbb{Z}_n,\cdot)\smallsetminus\operatorname{Reg}(\mathbb{Z}_n,\cdot)=\left\{\overline{\left(\frac{n}{2}\right)}\right\}$. Let $x\in\mathbb{Z}$ be such that $\overline{x}\in\mathcal{B}Q(\mathbb{Z}_n,\cdot)$ and $\overline{x}\notin\operatorname{Reg}(\mathbb{Z}_n,\cdot)$. By Theorem 5.1 and Theorem 5.3, $x$ and $\dfrac{n}{(x,n)}$ are not relatively prime, $n\mid x^2$ and $\dfrac{n}{(x,n)}=2$. Then $x\neq 0$ and $n\nmid x$. Let $k\in\mathbb{Z}$ be such that $x=kn+r$ where $0\le r<n$. Since $n\nmid x$, we

have that $0 < r < n$. We also have that $(x,n) \mid r$. Thus $\frac{n}{2} \,\Big|\, r$. Consequently, $\frac{n}{2} \leq r < n$, so $r = \frac{n}{2} + i$ for some $i \in \left\{0, 1, ..., \frac{n}{2} - 1\right\}$. Since $\frac{n}{2} \Big| r$, it follows that $i = 0$. Then $r = \frac{n}{2}$ and therefore $\overline{x} = \overline{r} = \overline{\left(\frac{n}{2}\right)}$. Hence (ii) is proved. $\square$

We have that $k\mathbb{Z}_n \cap \mathrm{Reg}(\mathbb{Z}_n, \cdot) = \mathrm{Reg}(k\mathbb{Z}_n, \cdot)$ since $k\mathbb{Z}_n$ is an ideal of $(\mathbb{Z}_n, \cdot)$. From this fact and Theorem 5.1, the following theorem is directly obtained.

**Theorem 5.7.** *For $\overline{x} \in k\mathbb{Z}_n$, $\overline{x} \in \mathrm{Reg}(k\mathbb{Z}_n, \cdot)$ if and only if $x$ and $\frac{n}{(x,n)}$ are relatively prime.*

To give some necessary conditions for $\frac{n}{(k,n)}$ so that $(k\mathbb{Z}_n, \cdot)$ is a $\mathcal{B}Q$-semigroup, the following lemma is needed.

**Lemma 5.8.** *If $k$ and $\frac{n}{(k,n)}$ are relatively prime, then $(k\mathbb{Z}_n, \cdot) \cong (\mathbb{Z}_{\frac{n}{(k,n)}}, \cdot)$.*

*Proof.* Recall that $\overline{x}$ is the equivalence class of $x$ modulo $n$. In this proof, for $x \in \mathbb{Z}$, let $\tilde{x}$ denote the equivalence class of $x$ modulo $\frac{n}{(k,n)}$. Define $\varphi : k\mathbb{Z}_n \to \mathbb{Z}_{\frac{n}{(k,n)}}$ by

$$(k\overline{x})\varphi = \widetilde{kx} \qquad \text{for all } x \in \mathbb{Z}.$$

To show $\varphi$ is well-defined, let $x_1, x_2 \in \mathbb{Z}$ be such that $k\overline{x_1} = k\overline{x_2}$. Then $n|(kx_1 - kx_2)$, so $\frac{n}{(k,n)} \Big| \frac{k}{(k,n)}(x_1 - x_2)$. Since $\frac{n}{(k,n)}$ and $\frac{k}{(k,n)}$ are relatively prime, $\frac{n}{(k,n)} \Big| (x_1 - x_2)$, so $\widetilde{x_1} = \widetilde{x_2}$ and hence $\widetilde{kx_1} = \widetilde{kx_2}$.

To show $\varphi$ is 1-1, let $x_1, x_2 \in \mathbb{Z}$ be such that $\widetilde{kx_1} = \widetilde{kx_2}$. Then $\frac{n}{(k,n)} \Big| k(x_1 - x_2)$. Since $k$ and $\frac{n}{(k,n)}$ are relatively prime, $\frac{n}{(k,n)} \Big| (x_1 - x_2)$. Thus $n \mid (k,n)(x_1 - x_2)$, so $\overline{(k,n)x_1} = \overline{(k,n)x_2}$. Therefore

$$k\overline{x_1} = \frac{k}{(k,n)}\,\overline{(k,n)x_1} = \frac{k}{(k,n)}\,\overline{(k,n)x_2} = k\overline{x_2}.$$

Since $\varphi$ is 1-1 and $|k\mathbb{Z}_n| = \frac{n}{(k,n)} = \left|\mathbb{Z}_{\frac{n}{(k,n)}}\right|$, it follows that $\varphi$ is onto.

It remains to show that $\varphi$ is a homomorphism. Let $x_1, x_2 \in \mathbb{Z}$. Then

$$
\begin{aligned}
((k\overline{x}_1)(k\overline{x}_2))\varphi &= (k(\overline{x_1kx_2}))\varphi \\
&= \widetilde{kx_1kx_2} \\
&= \widetilde{kx_1}\,\widetilde{kx_2} \\
&= (k\overline{x}_1)\varphi(k\overline{x_2})\varphi.
\end{aligned}
$$

This proves that $(k\mathbb{Z}_n, \cdot) \cong (\mathbb{Z}_{\frac{n}{(k,n)}}, \cdot)$, as desired. □

**Theorem 5.9.** *If* $\dfrac{n}{(k,n)} = 4$ *or* $\dfrac{n}{(k,n)}$ *is square-free, then* $\mathcal{BQ}(k\mathbb{Z}_n, \cdot) = k\mathbb{Z}_n$. *Hence if* $\dfrac{n}{(k,n)} = 4$ *or* $\dfrac{n}{(k,n)}$ *is square-free, then* $(k\mathbb{Z}_n, \cdot)$ *is a* $\mathcal{BQ}$*-semigroup (by Theorem 2.8).*

*Proof.* First, assume that $\dfrac{n}{(k,n)}$ is square-free. Then for each $x \in \mathbb{Z}$, $\left(x, \dfrac{n}{(k,n)}\right) = \left(x^2, \dfrac{n}{(k,n)}\right)$, so

$$
\begin{aligned}
(x)_q &= \{\overline{x}\} \cup \overline{x}(k\mathbb{Z}_n) \\
&= \{\overline{x}\} \cup (kx, n)\mathbb{Z}_n \\
&= \{\overline{x}\} \cup (k,n)\left(\frac{kx}{(k,n)}, \frac{n}{(k,n)}\right)\mathbb{Z}_n \\
&= \{\overline{x}\} \cup (k,n)\left(x, \frac{n}{(k,n)}\right)\mathbb{Z}_n \qquad \left(\text{since } \left(\frac{n}{(k,n)}, \frac{k}{(k,n)}\right) = 1\right) \\
&= \{\overline{x}\} \cup (k,n)\left(x^2, \frac{n}{(k,n)}\right)\mathbb{Z}_n \\
&= \{\overline{x}\} \cup (k,n)\left(\frac{kx^2}{(k,n)}, \frac{n}{(k,n)}\right)\mathbb{Z}_n \\
&= \{\overline{x}\} \cup (kx^2, n)\mathbb{Z}_n \\
&= \{\overline{x}\} \cup \overline{x}^2(k\mathbb{Z}_n) \\
&\subseteq \{\overline{x}, \overline{x}^2\} \cup \overline{x}^2(k\mathbb{Z}_n) \\
&= (x)_b \\
&\subseteq (x)_q
\end{aligned}
$$

which implies that $(x)_q = (x)_b$. Hence $\mathcal{BQ}(k\mathbb{Z}_n) = k\mathbb{Z}_n$.

Next, assume that $\dfrac{n}{(k,n)} = 4$. Then $n = 4(k,n)$ and $\left(k, \dfrac{n}{(k,n)}\right) = 1, 2$ or $4$.

**Case 1 :** $\left(k, \dfrac{n}{(k,n)}\right) = 1$. By Lemma 5.8, $(k\mathbb{Z}_n, \cdot) \cong (\mathbb{Z}_{\frac{n}{(k,n)}}, \cdot) = (\mathbb{Z}_4, \cdot)$ which is a $\mathcal{B}Q$-semigroup by Corollary 5.4.

**Case 2 :** $\left(k, \dfrac{n}{(k,n)}\right) = 2$. We have that $|k\mathbb{Z}_n| = \dfrac{n}{(k,n)} = 4$ and

$$k\mathbb{Z}_n = (k,n)\mathbb{Z}_n = \{\overline{0}, \overline{(k,n)}, \overline{2(k,n)}, \overline{3(k,n)}\}$$

and

$$2(k,n)\mathbb{Z}_n = \{\overline{0}, \overline{2(k,n)}\}.$$

Since $\left(k, \dfrac{n}{(k,n)}\right) = (k,4) = 2$, we have that $2 \mid k$ and $4 \nmid k$, so $k = 4l+2$ for some $l \in \mathbb{Z}$. Then

$$\begin{aligned}
2(k,n)k\mathbb{Z}_n &= (2(k,n)k, n)\mathbb{Z}_n \\
&= (k,n)\left(2k, \frac{n}{(k,n)}\right)\mathbb{Z}_n \\
&= (k,n)(2k,4)\mathbb{Z}_n \\
&= 4(k,n)\mathbb{Z}_n \\
&= n\mathbb{Z}_n \\
&= \{\overline{0}\}.
\end{aligned}$$

Let $a, b \in \mathbb{Z}$ be such that $(k,n) = ka + nb$. Hence

$$\begin{aligned}
1 &= \frac{k}{(k,n)}a + \frac{n}{(k,n)}b \\
&= \frac{k}{(k,n)}a + 4b.
\end{aligned}$$

It implies that $a$ is an odd integer. Let $a = 2m+1$ where $m \in \mathbb{Z}$. Then

$$\begin{aligned}
(\overline{(k,n)})^2 &= \overline{(k,n)(k,n)} \\
&= \overline{(ka+nb)(k,n)} \\
&= \overline{ka(k,n) + nb(k,n)} \\
&= \overline{ka(k,n)} \\
&= \overline{(4l+2)a(k,n)} \\
&= \overline{4la(k,n) + 2a(k,n)} \\
&= \overline{nla + 2a(k,n)} \qquad \text{(since } n = 4(k,n))
\end{aligned}$$

$$
\begin{aligned}
&= \overline{2a(k,n)} \\
&= \overline{2(2m+1)(k,n)} \\
&= \overline{4m(k,n)+2(k,n)} \\
&= \overline{mn+2(k,n)} \\
&= \overline{2(k,n)}.
\end{aligned}
$$

Next, we show that $(\overline{x})_b = (\overline{x})_q$ for all $\overline{x} \in k\mathbb{Z}_n$. Since $\overline{0} \in \mathrm{Reg}(k\mathbb{Z}_n)$, $(\overline{0})_b = (\overline{0})_q$. Also, we have that

$$
\begin{aligned}
(\overline{(k,n)})_q &= \{\overline{(k,n)}\} \cup \overline{(k,n)}(k\mathbb{Z}_n) \\
&= \{\overline{(k,n)}\} \cup ((k,n)k,n)\mathbb{Z}_n \\
&= \{\overline{(k,n)}\} \cup (k,n)\left(k, \frac{n}{(k,n)}\right)\mathbb{Z}_n \\
&= \{\overline{(k,n)}\} \cup 2(k,n)\mathbb{Z}_n && \left(\text{since } \left(k, \frac{n}{(k,n)}\right) = 2\right) \\
&= \{\overline{(k,n)}\} \cup \{\overline{0}, \overline{2(k,n)}\}, \\
(\overline{(k,n)})_b &= \{\overline{(k,n)}, (\overline{(k,n)})^2\} \cup (\overline{(k,n)})^2(k\mathbb{Z}_n) \\
&= \{\overline{(k,n)}, \overline{2(k,n)}\} \cup \overline{2(k,n)}(k\mathbb{Z}_n) && (\text{since } (\overline{(k,n)})^2 = \overline{2(k,n)}) \\
&= \{\overline{(k,n)}, \overline{2(k,n)}\} \cup \{\overline{0}\} && (\text{since } 2(k,n)k\mathbb{Z}_n = \{\overline{0}\}), \\
(\overline{2(k,n)})_q &= \{\overline{2(k,n)}\} \cup \overline{2(k,n)}(k\mathbb{Z}_n) \\
&= \{\overline{2(k,n)}\} \cup \{\overline{0}\}, \\
(\overline{2(k,n)})_b &= \{\overline{2(k,n)}, (\overline{2(k,n)})^2\} \cup (\overline{2(k,n)})^2(k\mathbb{Z}_n) \\
&= \{\overline{2(k,n)}, \overline{8(k,n)}\} \cup \overline{8(k,n)}(k\mathbb{Z}_n) \\
&= \{\overline{2(k,n)}, \overline{0}\} \cup \overline{0}(k\mathbb{Z}_n) \\
&= \{\overline{2(k,n)}, \overline{0}\}, \\
(\overline{3(k,n)})_q &= \{\overline{3(k,n)}\} \cup \overline{3(k,n)}(k\mathbb{Z}_n) \\
&= \{\overline{3(k,n)}\} \cup (3(k,n)k,n)\mathbb{Z}_n \\
&= \{\overline{3(k,n)}\} \cup (k,n)\left(3k, \frac{n}{(k,n)}\right)\mathbb{Z}_n \\
&= \{\overline{3(k,n)}\} \cup (k,n)\,(3k,4)\,\mathbb{Z}_n && \left(\text{since } \frac{n}{(k,n)} = 4\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \{\overline{3(k,n)}\} \cup (k,n)\,(k,4)\,\mathbb{Z}_n && \text{(since } (3,4)=1)\\
&= \{\overline{3(k,n)}\} \cup 2(k,n)\mathbb{Z}_n && \text{(since } (k,4)=2)\\
&= \{\overline{3(k,n)}\} \cup \{\overline{0}, \overline{2(k,n)}\},\\
(\overline{3(k,n)})_b &= \{\overline{3(k,n)}, (\overline{3(k,n)})^2\} \cup (\overline{3(k,n)})^2(k\mathbb{Z}_n)\\
&= \{\overline{3(k,n)}, \overline{18(k,n)}\} \cup \overline{18(k,n)}(k\mathbb{Z}_n)\\
&= \{\overline{3(k,n)}, \overline{2(k,n)}\} \cup \overline{2(k,n)}(k\mathbb{Z}_n)\\
&= \{\overline{3(k,n)}, \overline{2(k,n)}\} \cup \{\overline{0}\}.
\end{aligned}
$$

It follows that for all $\overline{x} \in k\mathbb{Z}_n$, $(\overline{x})_b = (\overline{x})_q$. Therefore $(k\mathbb{Z}_n, \cdot)$ is a $\mathcal{B}Q$-semigroup.

**Case 3 :** $\left(k, \dfrac{n}{(k,n)}\right) = 4$. Since $\dfrac{n}{(k,n)} = 4$, $\dfrac{n}{(k,n)} \Big| k$. Then $k = \dfrac{nl}{(k,n)}$ for some $l \in \mathbb{Z}$, so $k(k,n) = nl$. Let $a, b \in \mathbb{Z}$ be such that $(k,n) = ka + nb$. We have that

$$
\begin{aligned}
(\overline{(k,n)})^2 &= \overline{(k,n)(k,n)}\\
&= \overline{(ka+nb)(k,n)}\\
&= \overline{ka(k,n) + nb(k,n)}\\
&= \overline{ka(k,n)}\\
&= \overline{nla}\\
&= \overline{0}.
\end{aligned}
$$

This implies that $(k\mathbb{Z}_n, \cdot)$ is a zero semigroup since $k\mathbb{Z}_n = (k,n)\mathbb{Z}_n$. Thus $(k\mathbb{Z}_n, \cdot)$ is a $\mathcal{B}Q$-semigroup by Example 2.10.

Therefore the theorem is proved. □

The following example shows that the converse of Theorem 5.9 is not generally true.

**Example 5.10.** It is evident that $(8\mathbb{Z}_{64}, \cdot)$ is a zero semigroup. Then it is a $\mathcal{B}Q$-semigroup by Example 2.10. However, $\dfrac{64}{(8,64)}$ is neither 4 nor square-free.

# CHAPTER VI

# SEMIGROUPS OF TRANSFORMATIONS OF SETS AND LINEAR TRANSFORMATIONS

In this chapter, we determine the $\mathcal{B}Q$-elements of the semigroups $M(X), UE(X)$, $M_F(V)$ and $E_F(V)$ where $X$ is a nonempty set and $V$ is a vector space over a field $F$. We show that the set of regular elements and the set of $\mathcal{B}Q$-elements of each semigroup coincide.

The following semigroups under composition of transformations of sets and linear transformations are recalled :

$$
\begin{aligned}
T(X) &= \{\alpha \mid \alpha : X \to X\}, \\
M(X) &= \{\alpha \in T(X) \mid \alpha \text{ is 1-1}\}, \\
E(X) &= \{\alpha \in T(X) \mid \alpha \text{ is onto}\}, \\
UE(X) &= \{\alpha \in E(X) \mid |a\alpha^{-1}| = |b\alpha^{-1}| \text{ for all } a, b \in X\}, \\
G(X) &= \text{ the symmetric group on } X \\
&\quad ( = \{\alpha \in T(X) \mid \alpha \text{ is 1-1 and onto}\}), \\
L_F(V) &= \{\alpha : V \to V \mid \alpha \text{ is linear}\}, \\
M_F(V) &= \{\alpha \in L_F(V) \mid \alpha \text{ is 1-1}\} \\
&\quad ( = \{\alpha \in L_F(V) \mid \ker \alpha = \{0\}\}), \\
E_F(V) &= \{\alpha \in L_F(V) \mid \alpha \text{ is onto}\} \\
&\quad ( = \{\alpha \in L_F(V) \mid V\alpha = V\}), \\
G_F(V) &= \{\alpha \in L_F(V) \mid \alpha \text{ is an isomorphism}\}.
\end{aligned}
$$

We have mentioned in the preliminaries on page 12-14 that $\mathrm{Reg}(M(X)) = \mathrm{Reg}(E(X)) = \mathrm{Reg}(UE(X)) = G(X)$ and $\mathrm{Reg}(M_F(V)) = \mathrm{Reg}(E_F(V)) = G_F(V)$.

The following facts of linear algebra will be used.

(1) If $B$ and $B'$ are bases of $V$ and $\alpha \in L_F(V)$ is such that $\alpha_{|_B} : B \to B'$ is a bijection, then $\alpha \in G_F(V)$.

(2) If $\alpha \in L_F(V), B_1, B_2$ are bases of $\ker \alpha$ and $\operatorname{ran} \alpha$, respectively and for each $u \in B_2$, let $u' \in u\alpha^{-1}$, then $B_1 \cup \{u' \mid u \in B_2\}$ is a basis of $V$.

**Theorem 6.1.** $\mathcal{B}Q(M(X)) = G(X)$.

*Proof.* If $X$ is finite, then $M(X) = G(X)$ which is a group, so we are done. Assume that $X$ is an infinite set. We have that $G(X) = \operatorname{Reg}(M(X)) \subseteq \mathcal{B}Q(M(X))$.

For the reverse inclusion, let $\alpha \in \mathcal{B}Q(M(X))$. Then $M(X)\alpha \cap \alpha M(X) = \{\alpha\} \cup \alpha M(X)\alpha$. Since $X$ is infinite, there is a subset $A = \{x_n \mid n \in \mathbb{Z}\}$ of $X$ where $x_n \neq x_m$ if $n \neq m$. But since $\alpha$ is 1-1, it follows that $x_n\alpha \neq x_m\alpha$ if $n \neq m$ and $(X \smallsetminus A)\alpha \subseteq X \smallsetminus A\alpha$. Define $\beta, \gamma : X \to X$ by

$$\beta = \begin{pmatrix} x_n & y \\ x_{n+1} & y \end{pmatrix}_{\substack{n \in \mathbb{Z} \\ y \in X \smallsetminus A}}$$

and

$$\gamma = \begin{pmatrix} x_n\alpha & y \\ x_{n+1}\alpha & y \end{pmatrix}_{\substack{n \in \mathbb{Z} \\ y \in X \smallsetminus A\alpha}}.$$

It can be seen that $\beta, \gamma \in G(X)$. We also have that

$$\beta\alpha = \begin{pmatrix} x_n & y \\ x_{n+1}\alpha & y\alpha \end{pmatrix}_{\substack{n \in \mathbb{Z} \\ y \in X \smallsetminus A}} = \alpha\gamma.$$

It follows that $\beta\alpha \in M(X)\alpha \cap \alpha M(X) = \{\alpha\} \cup \alpha M(X)\alpha$. Since $x_n\alpha \neq x_{n+1}\alpha$ for all $n \in \mathbb{Z}$, we have that $\beta\alpha \neq \alpha$. Therefore $\beta\alpha \in \alpha M(X)\alpha$. Thus $\beta\alpha = \alpha\lambda\alpha$ for some $\lambda \in M(X)$. This implies that $\beta = \alpha\lambda$ since $\alpha$ is 1-1. Hence $\operatorname{ran} \lambda = X$ since $\operatorname{ran} \beta = X$. Thus $\lambda \in G(X)$ and so $\alpha = \beta\lambda^{-1} \in G(X)$.

Therefore the theorem is proved. $\square$

**Theorem 6.2.** $\mathcal{B}Q(UE(X)) = G(X)$.

*Proof.* If $X$ is finite, then $UE(X) = G(X)$, so we are done. Suppose that $X$ is infinite. We have that $G(X) = \mathrm{Reg}(UE(X)) \subseteq \mathcal{B}Q(UE(X))$.

To show that $\mathcal{B}Q(UE(X)) \subseteq G(X)$, let $\alpha \in \mathcal{B}Q(UE(X))$. Let $a$ and $b$ be distinct elements of $X$. Since $\alpha \in UE(X), |a\alpha^{-1}| = |b\alpha^{-1}|$. Let $\varphi : a\alpha^{-1} \to b\alpha^{-1}$ be a bijection. Notice that $(X \smallsetminus (a\alpha^{-1} \mathbin{\dot\cup} b\alpha^{-1}))\alpha \subseteq X \smallsetminus \{a, b\}$. Define $\beta, \gamma \in G(X)$ by

$$\beta = \begin{pmatrix} x & y & z \\ x\varphi & y\varphi^{-1} & z \end{pmatrix}_{\substack{x \in a\alpha^{-1} \\ y \in b\alpha^{-1} \\ z \in X \smallsetminus (a\alpha^{-1} \mathbin{\dot\cup} b\alpha^{-1})}}$$

and

$$\gamma = \begin{pmatrix} a & b & z \\ b & a & z \end{pmatrix}_{z \in X \smallsetminus \{a,b\}}.$$

Then $\beta, \gamma \in G(X)$ and

$$\beta\alpha = \begin{pmatrix} x & y & z \\ b & a & z\alpha \end{pmatrix}_{\substack{x \in a\alpha^{-1} \\ y \in b\alpha^{-1} \\ z \in X \smallsetminus (a\alpha^{-1} \mathbin{\dot\cup} b\alpha^{-1})}} = \alpha\gamma.$$

Hence $\beta\alpha = \alpha\gamma \in UE(X)\alpha \cap \alpha UE(X) = \{\alpha\} \cup \alpha UE(X)\alpha$ and $\alpha\gamma \neq \alpha$. Then $\alpha\gamma = \alpha\lambda\alpha$ for some $\lambda \in UE(X)$. Thus $\gamma = \lambda\alpha$ since $\alpha$ is onto. But since $\gamma$ is 1-1 it follows that $\lambda$ is 1-1, so $\lambda \in G(X)$ and hence $\alpha = \lambda^{-1}\gamma \in G(X)$.

Therefore the theorem is proved. $\square$

**Theorem 6.3.** $\mathcal{B}Q(M_F(V)) = G_F(V)$.

*Proof.* If $V$ is finite-dimensional, then we are done. Assume that $\dim_F(V)$ is infinite. Let $B$ be a basis of $V$ and $\alpha \in \mathcal{B}Q(M_F(V))$. Since $B$ is infinite, $B$ has $A = \{u_n \mid n \in \mathbb{Z}\}$ as a subset where $u_n \neq u_m$ if $n \neq m$. Then $u_n\alpha \neq u_m\alpha$ for all distinct $n, m \in \mathbb{Z}$. We also have that $B\alpha = A\alpha \mathbin{\dot\cup} (B \smallsetminus A)\alpha$. Since $\alpha \in M_F(V), \alpha : V \to \operatorname{ran}\alpha$ is an isomorphism. This implies that $B\alpha$ is a basis of $\operatorname{ran}\alpha$. Let $B'$ be a basis of $V$ containing $B\alpha$. Define $\beta, \gamma \in L_F(V)$ on $B$ and $B'$ respectively as follows :

$$\beta = \begin{pmatrix} u_n & v \\ u_{n+1} & v \end{pmatrix}_{\substack{n \in \mathbb{Z} \\ v \in B \smallsetminus A}}$$

and

$$\gamma = \begin{pmatrix} u_n\alpha & v\alpha & w \\ u_{n+1}\alpha & v\alpha & w \end{pmatrix}_{\substack{n\in\mathbb{Z} \\ v\in B\smallsetminus A \\ w\in B'\smallsetminus B\alpha}}.$$

Since $\beta_{|_B}$ is a bijection on $B$ and $\gamma_{|_{B'}}$ is a bijection on $B'$, it follows that $\beta,\gamma \in G_F(V)$. Also, we have

$$\beta\alpha = \begin{pmatrix} u_n & v \\ u_{n+1}\alpha & v\alpha \end{pmatrix}_{\substack{n\in\mathbb{Z} \\ v\in B\smallsetminus A}} = \alpha\gamma,$$

so $\beta\alpha \neq \alpha$ and $\beta\alpha \in M_F(V)\alpha \cap \alpha M_F(V) = \{\alpha\} \cup \alpha M_F(V)\alpha$. Thus $\beta\alpha = \alpha\lambda\alpha$ for some $\lambda \in M_F(V)$. Since $\alpha$ is 1-1, $\beta = \alpha\lambda$. Hence $\operatorname{ran}\lambda = V$ since $\operatorname{ran}\beta = V$, so $\lambda \in G_F(V)$. It follows that $\alpha = \beta\lambda^{-1} \in G_F(V)$. This proves that $\mathcal{B}Q(M_F(V)) \subseteq G_F(V)$. But since $G_F(V) = \operatorname{Reg}(M_F(V)) \subseteq \mathcal{B}Q(M_F(V))$, the result follows. $\square$

**Theorem 6.4.** *$\mathcal{B}Q(E_F(V)) = G_F(V)$.*

*Proof.* If $V$ is finite-dimensional, then $E_F(V) = G_F(V)$, so we are done. Assume that $\dim_F(V)$ is infinite. Let $B$ be a basis of $V$ and let $\alpha \in \mathcal{B}Q(E_F(V))$. Let $A = \{u_n \mid n \in \mathbb{Z}\} \subseteq B$ where $u_n \neq u_m$ if $n \neq m$. Since $V\alpha = V$, for each $u \in B$, choose $u' \in u\alpha^{-1}$. Then $u'\alpha = u$ for all $u \in B$ and $u' \neq v'$ for all distinct $u, v \in B$. Let $B_1$ be a basis of $\ker\alpha$. Then $B_1 \cup \{u' \mid u \in B\}$ is a basis of $V$. Let $\beta,\gamma \in L_F(V)$ be defined respectively on $B_1 \cup \{u' \mid u \in B\}$ and $B$ as follows :

$$\beta = \begin{pmatrix} u'_n & v' & w \\ u'_{n+1} & v' & w \end{pmatrix}_{\substack{n\in\mathbb{Z} \\ v\in B\smallsetminus A \\ w\in B_1}}$$

and

$$\gamma = \begin{pmatrix} u_n & v \\ u_{n+1} & v \end{pmatrix}_{\substack{n\in\mathbb{Z} \\ v\in B\smallsetminus A}}.$$

Then $\beta_{|_{B_1\cup\{u'|u\in B\}}}$ is a bijection on $B_1 \cup \{u' \mid u \in B\}$ and $\gamma_{|_B}$ is a bijection on $B$. It follows that $\beta,\gamma \in G_F(V)$. Also, we have that

$$\beta\alpha = \begin{pmatrix} u'_n & v' & w \\ u_{n+1} & v & 0 \end{pmatrix}_{\substack{n\in\mathbb{Z} \\ v\in B\smallsetminus A \\ w\in B_1}} = \alpha\gamma.$$

Then $\alpha \neq \alpha\gamma$ and $\alpha\gamma \in E_F(V)\alpha \cap \alpha E_F(V) = \{\alpha\} \cup \alpha E_F(V)\alpha$. Thus $\alpha\gamma = \alpha\lambda\alpha$ for some $\lambda \in E_F(V)$. Since $\operatorname{ran}\alpha = V$, $\gamma = \lambda\alpha$. We have that $\lambda$ is 1-1 since $\gamma$ is 1-1. Therefore $\lambda \in G_F(V)$ which implies that $\alpha = \lambda^{-1}\gamma \in G_F(V)$. This proves that $\mathcal{B}Q(E_F(V)) \subseteq G_F(V)$. But $G_F(V) = \operatorname{Reg}(E_F(V)) \subseteq \mathcal{B}Q(E_F(V))$, so $\mathcal{B}Q(E_F(V)) = G_F(V)$, as desired. $\square$

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# VITA

| | |
|---|---|
| **Name** | Miss Ngarmcherd Danpattanamongkon |
| **Date of Birth** | 29 January 1982 |
| **Place of Birth** | Phetchabun, Thailand |
| **Education** | B.Sc.(Mathematic), Chulalongkorn University, 2004 |
| | M.Sc.(Mathematic), Chulalongkorn University, 2006 |
| **Scholarship** | The Ministry Development Staff Project Scholarship |
| | for the M.Sc. program (2 years) |
| | The University Development Commission (UDC) |
| | for the Ph.D. program (3 years) |