CHAPTER IV

MATRIX SEMIGROUPS

In this chapter, we are concerned with matrix semigroups. First, we recall the following notation : For any field F and for any positive integer n,

$M_n(F)$ = the matrix semigroup of all n × n matrices over F,

$G_n(F)$ = the matrix group of all n × n nonsingular matrices over F,

$U_n(F)[L_n(F)]$ = the matrix semigroup of all n × n upper [lower] triangular matrices over F

and $D_n(F)$ = the matrix semigroup of all n × n diagonal matrices over F.

There are two main purposes in this chapter. The first one is to show that if F has a proper dense subsemigroup under its multiplication, then so each of the above matrix semigroups does. The second one is to show that if F is a finite field, then none of the above matrix semigroups has a proper dense subsemigroup.

The following lemma give a general property of matrices.

<u>Lemma 4.1.</u> Let A and B be n × n nonsingular matrices over a field F. If det A = det B, then there exists an n × n matrix C over F such that A = BC and det C = 1.

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Proof : Assume that $\det A = \det B$. Then $\det(B^{-1}A) = 1$. Since $B^{-1}A$ is nonsingular, there is an $n \times n$ matrix D over F such that $DB^{-1}A = I_n$ where I_n is the $n \times n$ identity matrix over F. Then $A = BD^{-1}$, and $1 = \det I_n = (\det D)(\det(B^{-1}A)) = \det D$ which implies that $\det D^{-1} = 1$. #

To prove the first two theorems, we need the following lemma.

Lemma 4.2. Let $(F,+,\cdot)$ be a field and S a matrix semigroup over F satisfying the following properties :

(i) $\{\det A \mid A \in S\} = F$ or $F \smallsetminus \{0\}$.

(ii) For $A \in S$, if A is nonsingular, then $A^{-1} \in S$.

Assume that the semigroup $(F,\cdot)$ has a proper dense subsemigroup. Then S has a proper dense subsemigroup.

Proof : Let n be a positive integer such that $S \subseteq M_n(F)$.

Let U be a proper dense subsemigroup of the semigroup $(F,\cdot)$. Since $F \smallsetminus \{0\}$ is a subgroup of $(F,\cdot)$, by Theorem 1.2, $\mathrm{Dom}(F \smallsetminus \{0\},F) = F \smallsetminus \{0\}$ (under the multiplication of F). If $0 \notin U$, then U is a subsemigroup of the group $(F \smallsetminus \{0\},\cdot)$, so $F = \mathrm{Dom}(U,F) \subseteq \mathrm{Dom}(F \smallsetminus \{0\},F) = F \smallsetminus \{0\}$, a contradiction. Hence $0 \in U$. Under the multiplication of F, we have by Theorem 1.2 that $\mathrm{Dom}(\{0\},F) = \{0\} \neq F$. This implies that $U \cap (F \smallsetminus \{0\}) \neq \emptyset$. Hence $\{A \in S \mid \det A \in U\} \neq \emptyset$ since $F \smallsetminus \{0\} \subseteq \{\det A \mid A \in S\}$ by assumption. Let

$$\overline{U} = \{A \in S \mid \det A \in U\}.$$

If A and $B \in S$ are such that $\det A, \det B \in U$, then $AB \in S$ and $\det(AB) = (\det A)(\det B) \in U$, so $AB \in \overline{U}$. Since $U \subsetneq F$, there is an

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element $a \in F \setminus U$. Then $a \in F \setminus \{0\} \subseteq \{\det A \mid A \in S\}$. Hence there is a matrix $C \in S$ such that $\det C = a \notin U$. Thus $C \in S \setminus \overline{U}$. Hence $\overline{U}$ is a proper subsemigroup of S.

We claim that $\overline{U}$ is dense in S. To prove the claim, let $A \in S$. If $\det A \in U$, then $A \in \overline{U} \subseteq \mathrm{Dom}(\overline{U},S)$. Assume that $\det A \notin U$. Since U is dense in $(F,\cdot)$, there are $u_0,u_1,\ldots,u_{2m} \in U$, $x_1,x_2,\ldots,x_m$, $y_1,y_2,\ldots,y_m \in F$ such that

$$\begin{aligned}
\det A &= u_0y_1, \\
&= x_1u_1y_1, \quad u_0 = x_1u_1, \\
&= x_1u_2y_2, \quad u_1y_1 = u_2y_2, \\
&= x_2u_3y_2, \quad x_1u_2 = x_2u_3, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&= x_mu_{2m-1}y_m, \quad x_{m-1}u_{2m-2} = x_mu_{2m-1}, \\
&= x_mu_{2m}, \quad u_{2m-1}y_m = u_{2m}.
\end{aligned}$$

Since $\det A \notin U$, $\det A \neq 0$ which implies that all of $u_0,u_1,\ldots,u_{2m}$, $x_1,x_2,\ldots, x_m$, $y_1,y_2,\ldots, y_m$ are elements of $F \setminus \{0\}$. But $F \setminus \{0\} \subseteq \{\det A \mid A \in S\}$, so there are matrices $B_0,B_1,\ldots,B_{2m}$, $C_1,C_2,\ldots, C_m$, $D_1,D_2,\ldots, D_m$ in S such that

$$\begin{aligned}
\det B_i &= u_i \quad \text{for } i = 0,1,\ldots,2m, \\
\det C_i &= x_i \quad \text{for } i = 1,2,\ldots,m \text{ and} \\
\det D_i &= y_i \quad \text{for } i = 1,2,\ldots,m .
\end{aligned}$$

Then all of $B_0,B_1,\ldots,B_{2m}$, $C_1,C_2,\ldots,C_m$, $D_1,D_2,\ldots,D_m$ are nonsingular, $B_0,B_1,\ldots,B_{2m} \in \overline{U}$ and

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$$\begin{aligned}
\det A &= \det(B_0D_1), \\
&= \det(C_1B_1D_1), \quad \det B_0 = \det(C_1B_1), \\
&= \det(C_1B_2D_2), \quad \det(B_1D_1) = \det(B_2D_2), \\
&= \det(C_2B_3D_2), \quad \det(C_1B_2) = \det(C_2B_3), \\
&\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&= \det(C_mB_{2m-1}D_m), \quad \det(C_{m-1}B_{2m-2}) = \det(C_mB_{2m-1}), \\
&= \det(C_mB_{2m}), \quad \det(B_{2m-1}D_m) = \det(B_{2m}).
\end{aligned}$$

Since $\det A = \det(B_0D_1)$ and $\det B_0 = \det(C_1B_1)$, by Lemma 4.1, there are $X_1, Y_1 \in M_n(F)$ such that $A = B_0D_1Y_1$, $B_0 = C_1B_1X_1$ and $\det X_1 = \det Y_1 = 1$. But for each $i \in \{1,2,\ldots,m-1\}$, $\det(C_iB_{2i}) = \det(C_{i+1}B_{2i+1})$, so for each $i \in \{1,2,\ldots,m-1\}$, there is X_{i+1} in $M_n(F)$ such that $\det X_{i+1} = 1$ and $C_iB_{2i} = C_{i+1}B_{2i+1}X_{i+1}$. By the assumption (ii), we see that $X_i, Y_1 \in S$ for all $i \in \{1,2,\ldots,m\}$. Since for each $i \in \{0,1,\ldots,m-1\}$, $B_{2i+1}X_{i+1} \in S$ and $\det(B_{2i+1}X_{i+1}) = \det B_{2i+1} = u_{2i+1} \in U$, we have that $B_{2i+1}X_{i+1} \in \overline{U}$ for all $i \in \{0,1,\ldots,m-1\}$. Now, we have that

$$A = B_0D_1Y_1 \quad \text{and} \quad D_1Y_1 \in S \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and

$$\left.\begin{aligned}
B_0 &= C_1B_1X_1, \\
C_iB_{2i} &= C_{i+1}B_{2i+1}X_{i+1}, \ i = 1,2,\ldots,m-1 \text{ and} \\
B_{2i+1}X_{i+1} &\in \overline{U}, \ i = 0,1,\ldots,m-1.
\end{aligned}\right\} \ldots.(2)$$

Since $\det(B_1X_1D_1Y_1) = \det(B_1D_1) = \det(B_2D_2)$, there exists $Y_2 \in M_n(F)$ such that $\det Y_2 = 1$ and $B_1X_1D_1Y_1 = B_2D_2Y_2$. It follows from the assumption (ii) that $Y_2 \in S$. Because $\det(B_3X_2D_2Y_2) = \det(B_3D_2) = = \det(B_4D_3)$, there exists $Y_3 \in M_n(F)$ such that $\det Y_3 = 1$ and

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$B_3X_2D_2Y_2 = B_4D_3Y_3$. Then $Y_3 \varepsilon S$ by the assumption (ii). By this process, we have that there are $Y_1, Y_2, \ldots, Y_m$ in S such that

$$\left.\begin{array}{l} B_{2i-1}X_iD_iY_i = B_{2i}D_{i+1}Y_{i+1} \ , \ i = 1,2,\ldots,m-1 \\ \text{and thus} \\ D_iY_i \ \varepsilon \ S \ , \ i = 1,2,\ldots,m. \end{array}\right\} \ldots\ldots\ldots(3)$$

Since $\det(B_{2m-1}X_mD_mY_m) = \det(B_{2m-1}D_m) = \det B_{2m}$, there exists $Y \varepsilon M_n(F)$ such that $\det Y = 1$ and $B_{2m-1}X_mD_mY_m = B_{2m}Y$. By assumption (ii), $Y \varepsilon S$. Since $\det(B_{2m}Y) = \det B_{2m} = u_{2m} \varepsilon U$, we have that $B_{2m}Y \varepsilon \overline{U}$. Hence we have $Y \varepsilon S$ and

$$B_{2m-1}X_mD_mY_m = B_{2m}Y \varepsilon \overline{U} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Thus, we have from (1)-(4) that

$$\begin{aligned}
A &= B_0(D_1Y_1),\ B_0 \varepsilon \overline{U},\ D_1Y_1 \varepsilon S \quad \text{(from (1))}, \\
&= C_1(B_1X_1)(D_1Y_1),\ B_1X_1 \varepsilon \overline{U},\ C_1 \varepsilon S,\ B_0 = C_1(B_1X_1) \text{ (from (2))}, \\
&= C_1B_2(D_2Y_2),\ B_2 \varepsilon \overline{U},\ D_2Y_2 \varepsilon S,\ B_1X_1D_1Y_1 = B_2D_2Y_2 \text{ (from (3))}, \\
&= C_2(B_3X_2)(D_2Y_2),\ B_3X_2 \varepsilon \overline{U},\ C_2 \varepsilon S,\ C_1B_2 = C_2(B_3X_2) \text{ (from (2))}, \\
&= C_2B_4(D_3Y_3),\ B_4 \varepsilon \overline{U},\ D_3Y_3 \varepsilon S,\ B_3X_2D_2Y_2 = B_4D_3Y_3 \text{ (from (3))}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&= C_{m-1}B_{2m-2}(D_mY_m),\ B_{2m-2} \varepsilon \overline{U},\ D_mY_m \varepsilon S,\ B_{2m-3}X_{m-1}D_{m-1}Y_{m-1} \\
&\qquad = B_{2m-2}D_mY_m \text{ (from (3))}, \\
&= C_m(B_{2m-1}X_m)(D_mY_m),\ B_{2m-1}X_m \varepsilon \overline{U},\ C_{m-1}B_{2m-2} = C_m(B_{2m-1}X_m) \\
&\qquad \text{(from (2))}, \\
&= C_m(B_{2m}Y),\ B_{2m}Y \varepsilon \overline{U},\ (B_{2m-1}X_m)(D_mY_m) = B_{2m}Y \quad \text{(from (4))}.
\end{aligned}$$

Thus by Theorem 1.1, $A \varepsilon \text{Dom}(\overline{U},S)$.

This proves that $\overline{U}$ is a proper dense subsemigroup of S, as required. #

Let F be a field and n a positive integer. It is clearly seen that the matrix semigroups $M_n(F)$ and $G_n(F)$ satisfy the properties (i) and (ii) of Lemma 4.2. If for $x \in F \smallsetminus \{0\}$, the $n \times n$ matrix

$$\begin{bmatrix} x & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

is an element of $D_n(F)$ and also of $U_n(F)$ $[L_n(F)]$ whose determinant is x. Hence the matrix semigroups $D_n(F)$ and $U_n(F)$ $[L_n(F)]$ satisfy the property (i) of Lemma 4.2. If A is an $n \times n$ diagonal matrix which is nonsingular, then A^{-1} is clearly a diagonal matrix. It was proved in [5], page 410 that if A is an $n \times n$ upper triangular matrix which is nonsingular, then A^{-1} is upper triangular. Hence $D_n(F)$ and $U_n(F)$ $[L_n(F)]$ satisfy the property (ii) of Lemma 4.2.

Therefore, by Lemma 4.2, we obtain the following theorem.

Theorem 4.3. Let $F = (F,+,\cdot)$ be a field and n a positive integer. If $(F,\cdot)$ has a proper dense subsemigroup, then each of the matrix semigroups $M_n(F)$, $G_n(F)$, $D_n(F)$, $U_n(F)$, $L_n(F)$ has a proper dense subsemigroup.

Theorem 4.3 and the lemma given below show that all matrix semigroups mentioned above over the field of real numbers or the field of complex numbers always have proper dense subsemigroups.

Lemma 4.4. The multiplicative semigroup of real numbers, $(\mathbb{R},\cdot)$ and the multiplicative semigroup of complex numbers, $(\mathbb{C},\cdot)$ have proper dense subsemigroups.

Proof : Let S be any one of $(\mathbb{R},\cdot)$ or $(\mathbb{C},\cdot)$, and let

$$U = \{x \varepsilon S \mid |x| \leqslant 1\}.$$

Then U is a proper subsemigroup of S and $1 \varepsilon U$. To show that U is dense in S, let $x \varepsilon S$. If $x \varepsilon U$, then $x \varepsilon \mathrm{Dom}(U,S)$. Assume that $x \notin U$. Then $|x| > 1$, so $\frac{1}{x} \varepsilon S$ and $|\frac{1}{x}| < 1$. Thus $\frac{1}{x} \varepsilon U$ and

$$\begin{aligned} x &= 1\cdot x, \quad 1 \varepsilon U, \quad x \varepsilon S, \\ &= x\cdot\frac{1}{x}\cdot x, \quad \frac{1}{x} \varepsilon U, \ x \varepsilon S, \ 1 = x\cdot\frac{1}{x}, \\ &= x\cdot 1, \quad 1 \varepsilon U, \quad \frac{1}{x}\cdot x = 1 \end{aligned}$$

which implies by Theorem 1.1 that $x \varepsilon \mathrm{Dom}(U,S)$. #

Theorem 4.5. For any positive integer n, each of the matrix semigroups $M_n(\mathbb{R})$, $M_n(\mathbb{C})$, $G_n(\mathbb{R})$, $G_n(\mathbb{C})$, $U_n(\mathbb{R})$, $U_n(\mathbb{C})$, $L_n(\mathbb{R})$, $L_n(\mathbb{C})$, $D_n(\mathbb{R})$, $D_n(\mathbb{C})$ has a proper dense subsemigroup.

In the last part of this chapter, we shall show that over any finite field F, each of the matrix semigroups $M_n(F)$, $G_n(F)$, $D_n(F)$, $U_n(F)$, $L_n(F)$ has no proper dense subsemigroup for every positive integer n.

If F is a finite field and n is a positive integer, then $G_n(F)$ is a finite group, so by Theorem 2.5, it has no proper dense subsemigroup.

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<u>Theorem 4.6</u>. For any positive integer n, if F is a finite field, then the matrix group $G_n(F)$ has no proper dense subsemigroup.

To prove the theorem for the case of $M_n(F)$, we need the two following lemmas. The first one is also used to prove the theorems for the cases of $D_n(F)$, $U_n(F)$, $L_n(F)$.

<u>Lemma 4.7</u>. Let n be a positive integer, F a field and $S = M_n(F)$, $D_n(F)$, $U_n(F)$ or $L_n(F)$. Let U be a dense subsemigroup of S. Then $U \cap G_n(F) \neq \emptyset$, and if F is finite, then $G_n(F) \cap S \subseteq U$.

<u>Proof</u> : It is clearly seen that the set $\{A \in S \mid A \text{ is singular}\}$ is an ideal of S, so it is not dense in S (see Chapter I, page 9). Then, if U does not contain any nonsingular matrices, then $U \subseteq \{A \in S \mid A \text{ is singular}\}$, so U is not dense in S. Thus $U \cap G_n(F) \neq \emptyset$. Then $U \cap G_n(F)$ is subsemigroup of the matrix group $G_n(F)$.

Claim that $\mathrm{Dom}(U \cap G_n(F), G_n(F) \cap S) = G_n(F) \cap S$. Let $A \in G_n(F) \cap S$. If $A \in U$, then $A \in U \cap G_n(F) \subseteq \mathrm{Dom}(U \cap G_n(F), G_n(F) \cap S)$. Assume that $A \notin U$. Since U is dense in S, there are $B_0, B_1, \ldots, B_{2m} \in U$, $C_1, C_2, \ldots, C_m$, $D_1, D_2, \ldots, D_m \in S$ such that

$$\begin{aligned} A &= B_0D_1, \\ &= C_1B_1D_1, \quad B_0 = C_1B_1, \\ &= C_1B_2D_2, \quad B_1D_1 = B_2D_2, \\ &= C_2B_3D_2, \quad C_1B_2 = C_2B_3, \\ &\quad \cdots\cdots\cdots\cdots \\ &= C_mB_{2m-1}D_m, \quad C_{m-1}B_{2m-2} = C_mB_{2m-1}, \\ &= C_mB_{2m}, \quad B_{2m-1}D_m = B_{2m} . \end{aligned}$$

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The nonsingularity of A implies that each of $B_0, B_1, \ldots, B_{2m}$, $C_1, C_2, \ldots, C_m$, $D_1, D_2, \ldots, D_m$ is nonsingular. Then by Theorem 1.1, we have that $A \in \mathrm{Dom}(U \cap G_n(F), G_n(F) \cap S)$. Hence we have the claim.

It is clear that $I_n \in S$. It is known that for a nonsingular matrix $A \in M_n(F)$, if A is diagonal, then so is A^{-1}, and if A is upper [lower] triangular, then so is A^{-1}. These imply that $G_n(F) \cap S$ is a subgroup of $G_n(F)$.

Assume that F is a finite field. Then $G_n(F) \cap S$ is a finite group which implies that $U \cap G_n(F)$ is a subgroup of the group $G_n(F) \cap S$. Therefore by Theorem 1.2, $\mathrm{Dom}(U \cap G_n(F), G_n(F) \cap S) = U \cap G_n(F)$. But it was proved that $\mathrm{Dom}(U \cap G_n(F), G_n(F) \cap S) = G_n(F) \cap S$, so we have $U \cap G_n(F) = G_n(F) \cap S$, and hence $G_n(F) \cap S \subseteq U$, as required. #

For an $n \times n$ matrix A over a field and for $i, j \in \{1,2,\ldots,n\}$, the notation A_{ij} is used to denote the element (entry) of A in i^{th} row and j^{th} column.

Lemma 4.8. Let F be a field and n a positive integer. Let S be a subsemigroup of the matrix semigroup $M_n(F)$ such that $G_n(F) \subseteq S$. If S contains a matrix in $M_n(F)$ of rank k with $k < n$, then S contains all matrices in $M_n(F)$ of rank $\leqslant k$. In particular, if S contains a matrix in $M_n(F)$ of rank n-1, then $S = M_n(F)$.

Proof : Let $A \in S$ be such that rank $(A) = k$, $k \leqslant n-1$. If $B \in M_n(F)$ is such that rank $(B) = k$, then $B = PAQ$ for some $P, Q \in G_n(F)$, so we have that $B \in S$ since $G_n(F) \subseteq S$ and $A \in S$. Thus S contains all matrices in $M_n(F)$ of rank k. If $k = 0$, we have nothing to do. Assume $k \geqslant 1$. Let C and D be matrices in $M_n(F)$ defined by

$$C_{ij} = \begin{cases} 1 & \text{if } i = j \leqslant k , \\ 0 & \text{otherwise} \end{cases}$$

and

$$D_{ij} = \begin{cases} 1 & \text{if } 2 \leqslant i = j \leqslant k+1 \\ 0 & \text{otherwise .} \end{cases}$$

Then rank (C) = rank $(D) = k$, so $C, D \in S$. Let $E = CD$. Then $E \in S$ and

$$E_{ij} = \sum_{\ell=1}^{n} C_{i\ell} D_{\ell j} = \begin{cases} 1 & \text{if } 2 \leqslant i = j \leqslant k, \\ 0 & \text{otherwise.} \end{cases}$$

Thus rank $(E) = k-1$. If $B \in M_n(F)$ is of rank $k-1$, then $B = PEQ$ for some $P, Q \in G_n(F)$ which implies that $B \in S$. Thus S contains all matrices in $M_n(F)$ of rank $k-1$. By induction process, we have that S contains all matrices in $M_n(F)$ of rank $\leqslant k$. #

<u>Theorem 4.9</u>. For any positive integer n, if F is a finite field, then the matrix semigroup $M_n(F)$ has no proper dense subsemigroup.

<u>Proof</u> : Let U be a dense subsemigroup of $M_n(F)$. By Lemma 4.7, $G_n(F) \subseteq U$. We shall prove that U contains a matrix in $M_n(F)$ of rank $n-1$. Suppose that U does not contain any matrices in $M_n(F)$ of rank $n-1$. Let $A \in M_n(F)$ of rank $n-1$. Then $A \notin U$. Since U is dense in $M_n(F)$, then there are $B_0, B_1, \ldots, B_{2m} \in U$, $C_1, C_2, \ldots, C_m$, $D_1, D_2, \ldots$ $\ldots, D_m \in M_n(F)$ such that

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$$\begin{aligned} A &= B_0D_1, \\ &= C_1B_1D_1, \quad B_0 = C_1B_1, \\ &= C_1B_2D_2, \quad B_1D_1 = B_2D_2, \\ &= C_2B_3D_2, \quad C_1B_2 = C_2B_3, \\ &\quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \\ &= C_mB_{2m-1}D_m, \quad C_{m-1}B_{2m-2} = C_mB_{2m-1}, \\ &= C_mB_{2m}, \qquad B_{2m-1}D_{2m} = B_{2m} . \end{aligned}$$

Then rank (A) $\leqslant$ rank (B_i) for i = 0,1,...,2m. But rank (A) = n-1 and U does not contain any matrices in $M_n(F)$ of rank n-1, it follows that rank (B_i) = n for i = 0,1,...,2m. Thus all $B_0, B_1, \ldots, B_{2m}$ are nonsingular. Since $B_0 = C_1B_1$, C_1 is nonsingular. From $C_iB_{2i} = C_{i+1}B_{2i+1}$ for i = 1,2,...,m-1, we have that $C_{i+1} = C_iB_{2i}B_{2i+1}^{-1}$ for i = 1,2,...,m-1. The nonsingularity of $B_1, B_2, \ldots, B_{2m}$, C_1 implies that C_i is nonsingular for all i = 1,2,...,m. In particular, C_m is nonsingular. Hence C_mB_{2m} = A is nonsingular. It is a contradiction since rank (A) = n-1.

Thus, U contains a matrix in $M_n(F)$ of rank n-1. It follows from Lemma 4.8 that U = $M_n(F)$. This proves that $M_n(F)$ has no proper dense subsemigroup. #

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Lemma 4.7 and the following lemma are used to prove that for every positive integer n, $D_n(F)$ has no proper dense subsemigroup if F is a finite field.

Lemma 4.10. Let F be a field and n a positive integer. Let S be a subsemigroup of the matrix semigroup $D_n(F)$. For a positive integer k with k $\leqslant$ n-1, if S contains all matrices in $D_n(F)$ of rank k, then

S contains all matrices in $D_n(F)$ of rank $\leqslant k$.

Proof : Assume that k is a nonnegative integer, $k \leqslant n-1$ and S contains all matrices in $D_n(F)$ of rank k. If $k = 0$, then we have nothing to do. Assume that $k \geqslant 1$. Let $A \in D_n(F)$ be of rank k-1. Then $A_{ij} = 0$ if $i \neq j$ and the number of all $i \in \{1,2,\ldots,n\}$ such that $A_{ii} \neq 0$ is k-1. Since $k-1 \leqslant n-2$, there exist $\ell, m \in \{1,2,\ldots,n\}$ such that $\ell \neq m$, $A_{\ell\ell} = A_{mm} = 0$. Define the $n \times n$ matrices B and C over F by

$$B_{ij} = \begin{cases} A_{ii} & \text{if } i = j \neq \ell, \\ 1 & \text{if } i = j = \ell, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$C_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } A_{ii} \neq 0, \\ 1 & \text{if } i = j = m, \\ 0 & \text{otherwise}. \end{cases}$$

Then $B, C \in D_n(F)$ and rank (B) = rank (C) = k. It is clearly seen that $A = BC$. Since $B, C \in S$, $A \in S$. Hence S contains all matrices in $D_n(F)$ of rank k-1. By induction process, we have that S contains all matrices in $D_n(F)$ of rank $\leqslant k$. #

Theorem 4.11. For any positive integer n, if F is a finite field, then the matrix semigroup $D_n(F)$ has no proper dense subsemigroup.

Proof : Let U be a dense subsemigroup of $D_n(F)$. By Lemma 4.7, $\dot{G}_n(F) \cap D_n(F) \subseteq U$.

Let $\ell \in \{1,2,\dots,n\}$. Suppose that U does not contain any matrices $A \in D_n(F)$ with rank $(A) = n-1$ and $A_{\ell\ell} = 0$. Let $E \in D_n(F)$ be such that rank $(E) = n-1$ and $E_{\ell\ell} = 0$. Then $E_{ij} = 0$ if $i \neq j$ and $E_{ii} \neq 0$ for every $i \neq \ell$, and $E \notin U$. Since U is dense in $D_n(F)$, there are $B^{(0)}, B^{(1)}, \dots, B^{(2m)} \in U$, $C^{(1)}, C^{(2)}, \dots, C^{(m)}$, $D^{(1)}, D^{(2)}, \dots, D^{(m)} \in D_n(F)$ such that

$$\begin{aligned}
E &= B^{(0)}D^{(1)}, \\
&= C^{(1)}B^{(1)}D^{(1)}, \quad B^{(0)} = C^{(1)}B^{(1)}, \\
&= C^{(1)}B^{(2)}D^{(2)}, \quad B^{(1)}D^{(1)} = B^{(2)}D^{(2)}, \\
&= C^{(2)}B^{(3)}D^{(2)}, \quad C^{(1)}B^{(2)} = C^{(2)}B^{(3)}, \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&= C^{(m)}B^{(2m-1)}D^{(m)}, \quad C^{(m-1)}B^{(2m-2)} = C^{(m)}B^{(2m-1)}, \\
&= C^{(m)}B^{(2m)}, \quad B^{(2m-1)}D^{(m)} = B^{(2m)}.
\end{aligned}$$

Then we have

$$\begin{aligned}
E_{\ell\ell} &= B^{(0)}_{\ell\ell}D^{(1)}_{\ell\ell}, \\
&= C^{(1)}_{\ell\ell}B^{(1)}_{\ell\ell}D^{(1)}_{\ell\ell}, \quad B^{(0)}_{\ell\ell} = C^{(1)}_{\ell\ell}B^{(1)}_{\ell\ell}, \\
&= C^{(1)}_{\ell\ell}B^{(2)}_{\ell\ell}D^{(2)}_{\ell\ell}, \quad B^{(1)}_{\ell\ell}D^{(1)}_{\ell\ell} = B^{(2)}_{\ell\ell}D^{(2)}_{\ell\ell}, \\
&= C^{(2)}_{\ell\ell}B^{(3)}_{\ell\ell}D^{(2)}_{\ell\ell}, \quad C^{(1)}_{\ell\ell}B^{(2)}_{\ell\ell} = C^{(2)}_{\ell\ell}B^{(3)}_{\ell\ell}, \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&= C^{(m)}_{\ell\ell}B^{(2m-1)}_{\ell\ell}D^{(m)}_{\ell\ell}, \quad C^{(m-1)}_{\ell\ell}B^{(2m-2)}_{\ell\ell} = C^{(m)}_{\ell\ell}B^{(2m-1)}_{\ell\ell}, \\
&= C^{(m)}_{\ell\ell}B^{(2m)}_{\ell\ell}, \quad B^{(2m-1)}_{\ell\ell}D^{(m)}_{\ell\ell} = B^{(2m)}_{\ell\ell}.
\end{aligned}$$

But $E_{\ell\ell} = 0$ and U does not contain any matrices $A \in D_n(F)$ with rank $n-1$ and $A_{\ell\ell} = 0$, it follows that $B^{(i)}_{\ell\ell} \neq 0$ for $i = 0,1,\dots,2m$. Since

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$B^{(0)}_{\ell\ell} = C^{(1)}_{\ell\ell}B^{(1)}_{\ell\ell}$, $C^{(1)}_{\ell\ell} \neq 0$. From $C^{(i)}_{\ell\ell}B^{(2i)}_{\ell\ell} = C^{(i+1)}_{\ell\ell}B^{(2i+1)}_{\ell\ell}$ for $i = 1,2,\ldots,m-1$ and $B^{(1)}_{\ell\ell}, B^{(2)}_{\ell\ell},\ldots, B^{(2m)}_{\ell\ell}, C^{(1)}_{\ell\ell} \neq 0$, it follows that $C^{(i)}_{\ell\ell} \neq 0$ for $i = 1,2,\ldots,m$. In particular, $C^{(m)}_{\ell\ell} \neq 0$. Hence $E_{\ell\ell} = C^{(m)}_{\ell\ell}B^{(2m)}_{\ell\ell} \neq 0$. It is a contradiction since $E_{\ell\ell} = 0$. This proves that U contains a matrix $A \varepsilon D_n(F)$ such that rank $(A) = n-1$ and $A_{\ell\ell} = 0$. Claim that U contains all matrices B in $D_n(F)$ of rank $n-1$ and $B_{\ell\ell} = 0$. Let $B \varepsilon D_n(F)$ be such that rank $(B) = n-1$ and $B_{\ell\ell} = 0$. Let $C, D \varepsilon D_n(F)$ be defined by

$$C_{ij} = \begin{cases} A^{-1}_{ij} & \text{if } i = j \neq \ell , \\ 1 & \text{if } i = j = \ell , \\ 0 & \text{otherwise ,} \end{cases}$$

and

$$D_{ij} = \begin{cases} B_{ij} & \text{if } i = j \neq \ell , \\ 1 & \text{if } i = j = \ell , \\ 0 & \text{otherwise .} \end{cases}$$

Then $C, D \varepsilon G_n(F) \cap D_n(F)$, so $C, D \varepsilon U$. Therefore $ACD \varepsilon U$. It is easy to see that $B = ACD$, so $B \varepsilon U$. Hence we have the claim. But since ℓ is arbitrary in $\{1,2,\ldots,n\}$, it follows that U contains all matrices in $D_n(F)$ of rank $n-1$. From $G_n(F) \cap D_n(F) \subseteq U$ and Lemma 4.10, we have that $U = D_n(F)$.

Therefore, $D_n(F)$ has no proper dense subsemigroup. #

The last theorem of this chapter shows that for any positive integer n and for any finite field F, $U_n(F)$ has no proper dense

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subsemigroup. To prove this theorem, Lemma 4.7 and the following two lemmas are required.

Lemma 4.12. Let F be a field, n a positive integer and S a subsemigroup of the matrix semigroup $U_n(F)$. For a positive integer k with $k \leqslant n-1$, if S contains every matrix in $U_n(F)$ having the number of zero entries on its main diagonal $\leqslant k$, then S contains every matrix in $U_n(F)$ having exactly $k+1$ zero entries on its main diagonal.

Proof : Assume that k is a positive integer and S contains every matrix in $U_n(F)$ which has the number of zero entries on its main diagonal $\leqslant k$. Let $A \in U_n(F)$ be such that A has exactly $k+1$ zero entries on its main diagonal. Let $i_1, i_2, \ldots, i_{k+1} \in \{1,2,\ldots,n\}$ be such that $i_1 < i_2 < \ldots < i_{k+1}$ and $A_{i_1 i_1} = A_{i_2 i_2} = \ldots = A_{i_{k+1} i_{k+1}} = 0$. Then $A_{ii} \neq 0$ for all $i \in \{1,2,\ldots,n\} \smallsetminus \{i_1, i_2, \ldots, i_{k+1}\}$.

Case 1 : $i_1 = 1$. We define the $n \times n$ matrices B and C over F by

$$B_{ij} = \begin{cases} 1 & \text{if } i = j = 1 , \\ 0 & \text{if } i = 1 \text{ and } j > 1 , \\ A_{ij} & \text{otherwise} \end{cases}$$

and

$$C_{ij} = \begin{cases} A_{ij} & \text{if } i = 1 , \\ 1 & \text{if } i = j,\ i \neq 1 , \\ 0 & \text{otherwise} , \end{cases}$$

so $B, C \in U_n(F)$. By defining B and C, we see that for $i \in \{1,2,\ldots,n\}$, $B_{ii} = 0$ if and only if $i \in \{i_2, i_3, \ldots, i_{k+1}\}$, and $C_{ii} = 0$ if and only

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if i = 1. Then B has exactly k zero entries on its main diagonal and C has exactly one zero entry on its main diagonal. Hence by assumption B, C ε S. Since B and C are upper triangular, it follows that $(BC)_{ij} = 0 = A_{ij}$ for all i, j ε {1,2,...,n}, i > j and $(BC)_{ii} = B_{ii}C_{ii}$ for all i ε {1,2,...,n}. It is clearly seen that $B_{ii}C_{ii} = A_{ii}$ for all i ε {1,2,...,n}. Then $(BC)_{ii} = A_{ii}$ for all i ε {1,2,...,n}. For i, j ε {1,2,...,n}, i < j, we have that

$$(BC)_{ij} = B_{ii}C_{ij} + B_{i,i+1}C_{i+1,j} + \dots + B_{ij}C_{jj}$$

since B and C are upper triangular. Thus, for j = 2,3,...,n,

$$\begin{aligned}(BC)_{1j} &= B_{11}C_{1j} + B_{12}C_{2j} + \dots + B_{1j}C_{jj}\\ &= A_{1j} + 0 + \dots + 0 = A_{1j}\ .\end{aligned}$$

If i, j ε {1,2,...,n}, i < j, i $\neq$ 1, then j $\neq$ 1, so we have that

$$\begin{aligned}(BC)_{ij} &= B_{ii}C_{ij} + B_{i,i+1}C_{i+1,j} + \dots + B_{ij}C_{jj}\\ &= 0 + 0 + \dots + 0 + A_{ij} = A_{ij}\ .\end{aligned}$$

Hence A = BC ε S.

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<u>Case 2</u> : $i_1 > 1$. Define the n × n matrices B and C over F by

$$B_{ij} = \begin{cases} 1 & \text{if } i = j \leqslant i_1\ , \\ 0 & \text{if } i \leqslant i_1 - 1 \text{ and } i \neq j\ , \\ A_{ij} & \text{otherwise} \end{cases}$$

and

and

$$C_{ij} = \begin{cases} A_{ij} & \text{if } i \leqslant i_1 - 1 \text{ ,} \\ 1 & \text{if } i = j \text{ and } i > i_1 \text{ ,} \\ 0 & \text{otherwise .} \end{cases}$$

Then B, C ε $U_n(F)$. By defining of B and C, we have that for $i \ \varepsilon \ \{1,2,\ldots,n\}$, $B_{ii} = 0$ if and only if $i \ \varepsilon \ \{i_2,i_3,\ldots,i_{k+1}\}$, and $C_{ii} = 0$ if and only if $i = i_1$. Then B has exactly k zero entries on its main diagonal and C has exactly one zero entry on its main diagonal. Hence by assumption, B, C ε S. Claim that BC = A. Because B and C are upper triangular, $(BC)_{ij} = 0 = A_{ij}$ for all $i, j \ \varepsilon \ \{1,2,\ldots,n\}$, $i > j$. Since $B_{ii} = 1$ and $C_{ii} = A_{ii}$ if $i \leqslant i_1$ and $B_{ii} = A_{ii}$ and $C_{ii} = 1$ if $i > i_1$, we have that $B_{ii}C_{ii} = A_{ii}$ for all $i \ \varepsilon \ \{1,2,\ldots,n\}$. Then $(BC)_{ii} = A_{ii}$ for all $i \ \varepsilon \ \{1,2,\ldots,n\}$.

For $i, j \ \varepsilon \ \{1,2,\ldots,n\}$, if $i < j$ and $i \leqslant i_1 - 1$, then

$$(BC)_{ij} = B_{ii}C_{ij} + B_{i,i+1}C_{i+1,j} + \ldots + B_{ij}C_{jj}$$

$$= A_{ij} + 0 + \ldots + 0 = A_{ij} \text{ .}$$

If $j = i_1 + 1, i_1 + 2, \ldots, n$,

$$(BC)_{i_1j} = B_{i_1i_1}C_{i_1j} + B_{i_1,i_1+1}C_{i_1+1,j} + \ldots + B_{i_1j}C_{jj}$$

$$= 0 + 0 + \ldots + A_{i_1j} = A_{i_1j} \text{ .}$$

If $i, j \ \varepsilon \ \{1,2,\ldots,n\}$, $i_1 < i < j$, then

$$(BC)_{ij} = B_{ii}C_{ij} + B_{i,i+1}C_{i+1,j} + \ldots + B_{ij}C_{jj}$$

$$= 0 + 0 + \ldots + A_{ij} = A_{ij} \text{ .}$$

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Thus $A = BC \varepsilon S$.

Hence S contains every matrix in $U_n(F)$ having exactly $k+1$ zero entries on its main diagonal. #

<u>Lemma 4.13.</u> Let F be a field, n a positive integer and $A \varepsilon U_n(F)$. If $\ell \varepsilon \{1,2,\ldots,n\}$ is such that $A_{\ell\ell} \neq 0$, then there exists $P \varepsilon G_n(F) \cap U_n(F)$ such that $(PA)_{\ell\ell} = 1$, $(PA)_{i\ell} = 0$ for all $i < \ell$ and $(PA)_{ij} = A_{ij}$ if $i \leqslant j < \ell$.

<u>Proof</u> : Let $\ell \varepsilon \{1,2,\ldots,n\}$ and $A_{\ell\ell} \neq 0$. Define the $n \times n$ matrix P over F by

$$P_{ij} = \begin{cases} 1 & \text{if } i = j \neq \ell, \\ A_{\ell\ell}^{-1} & \text{if } i = j = \ell, \\ -A_{i\ell}A_{\ell\ell}^{-1} & \text{if } i < \ell,\ j = \ell, \\ 0 & \text{otherwise.} \end{cases}$$

Then $P \varepsilon G_n(F) \cap U_n(F)$ and $(PA)_{\ell\ell} = P_{\ell\ell}A_{\ell\ell} = A_{\ell\ell}^{-1}A_{\ell\ell} = 1$.

For $i < \ell$, we have

$$\begin{aligned}(PA)_{i\ell} &= P_{ii}A_{i\ell} + P_{i,i+1}A_{i+1,\ell} + \ldots + P_{i\ell}A_{\ell\ell} \\ &= A_{i\ell} + 0 + \ldots + 0 + (-A_{i\ell}A_{\ell\ell}^{-1})A_{\ell\ell} \\ &= 0.\end{aligned}$$

If $i \leqslant j < \ell$, then

$$\begin{aligned}(PA)_{ij} &= P_{ii}A_{ij} + P_{i,i+1}A_{i+1,j} + \ldots + P_{ij}A_{jj} \\ &= A_{ij} + 0 + \ldots + 0 \\ &= A_{ij}.\end{aligned}$$ #

<u>Theorem 4.14.</u> For any positive integer n, if F is a finite field, then the matrix semigroup $U_n(F)$ has no proper dense subsemigroup.

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Proof : Let U be a dense subsemigroup of $U_n(F)$. By Lemma 4.7, $G_n(F) \cap U_n(F) \subseteq U$.

Let $\ell \varepsilon \{1,2,\ldots,n\}$. Suppose that U does not contain any matrix A in $U_n(F)$ with $A_{\ell\ell} = 0$ and $A_{ii} \neq 0$ for all $i \neq \ell$. Let $E \varepsilon U_n(F)$ be such that $E_{\ell\ell} = 0$ and $E_{ii} \neq 0$ for all $i \neq \ell$. Then $E \not\in U$. Since U is dense in $U_n(F)$, there are $B^{(0)}, B^{(1)}, \ldots, B^{(2m)} \varepsilon U$, $C^{(1)}, C^{(2)}, \ldots, C^{(m)}, D^{(1)}, D^{(2)}, \ldots, D^{(m)} \varepsilon U_n(F)$ such that

$$\begin{aligned}
E &= B^{(0)}D^{(1)},\\
&= C^{(1)}B^{(1)}D^{(1)}, \quad B^{(0)} = C^{(1)}B^{(1)},\\
&= C^{(1)}B^{(2)}D^{(2)}, \quad B^{(1)}D^{(1)} = B^{(2)}D^{(2)},\\
&= C^{(2)}B^{(3)}D^{(2)}, \quad C^{(1)}B^{(2)} = C^{(2)}B^{(3)},\\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&= C^{(m)}B^{(2m-1)}D^{(m)}, \quad C^{(m-1)}B^{(2m-2)} = C^{(m)}B^{(2m-1)},\\
&= C^{(m)}B^{(2m)}, \quad B^{(2m-1)}D^{(m)} = B^{(2m)}.
\end{aligned}$$

Then we have

$$\begin{aligned}
E_{\ell\ell} &= B^{(0)}_{\ell\ell}D^{(1)}_{\ell\ell},\\
&= C^{(1)}_{\ell\ell}B^{(1)}_{\ell\ell}D^{(1)}_{\ell\ell}, \quad B^{(0)}_{\ell\ell} = C^{(1)}_{\ell\ell}B^{(1)}_{\ell\ell},\\
&= C^{(1)}_{\ell\ell}B^{(2)}_{\ell\ell}D^{(2)}_{\ell\ell}, \quad B^{(1)}_{\ell\ell}D^{(1)}_{\ell\ell} = B^{(2)}_{\ell\ell}D^{(2)}_{\ell\ell},\\
&= C^{(2)}_{\ell\ell}B^{(3)}_{\ell\ell}D^{(2)}_{\ell\ell}, \quad C^{(1)}_{\ell\ell}B^{(2)}_{\ell\ell} = C^{(2)}_{\ell\ell}B^{(3)}_{\ell\ell},\\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&= C^{(m)}_{\ell\ell}B^{(2m-1)}_{\ell\ell}D^{(m)}_{\ell\ell}, \quad C^{(m-1)}_{\ell\ell}B^{(2m-2)}_{\ell\ell} = C^{(m)}_{\ell\ell}B^{(2m-1)}_{\ell\ell},\\
&= C^{(m)}_{\ell\ell}B^{(2m)}_{\ell\ell}, \quad B^{(2m-1)}_{\ell\ell}D^{(m)}_{\ell\ell} = B^{(2m)}_{\ell\ell}.
\end{aligned}$$

Since U does not contain any matrices A in $U_n(F)$ with $A_{\ell\ell} = 0$ and

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$A_{ii} \neq 0$ for all $i \neq \ell$, it follows that $B^{(i)}_{\ell\ell} \neq 0$ for $i = 0,1,...,2m$. Since $B^{(0)}_{\ell\ell} = C^{(1)}_{\ell\ell}B^{(1)}_{\ell\ell}$, $C^{(1)}_{\ell\ell} \neq 0$. From the fact that $C^{(1)}_{\ell\ell} \neq 0$, $B^{(i)}_{\ell\ell} \neq 0$ for all $i \in \{1,2,...,2m\}$ and $C^{(i)}_{\ell\ell}B^{(2i)}_{\ell\ell} = C^{(i+1)}_{\ell\ell}B^{(2i+1)}_{\ell\ell}$ for all $i \in \{1,2,...,m-1\}$, we have that $C^{(i)}_{\ell\ell} \neq 0$ for all $i \in \{1,2,...,m\}$. In particular, $C^{(m)}_{\ell\ell} \neq 0$. Hence $C^{(m)}_{\ell\ell}B^{(2m)}_{\ell\ell} = E_{\ell\ell} \neq 0$. It is a contradiction since $E_{\ell\ell} = 0$. Then there exists $B \in U$ be such that $B_{\ell\ell} = 0$ and $B_{ii} \neq 0$ for all $i \neq \ell$. Note that if $P \in G_n(F) \cap U_n(F)$, then $P_{ii} \neq 0$ for all $i \in \{1,2,...,n\}$ which implies that $(PB)_{ii} \neq 0$ for $i \neq \ell$. Hence it follows from Lemma 4.13 that there exist $P^{(1)}, P^{(2)},..., P^{(n-1)} \in G_n(F) \cap U_n(F)$ such that

$$(P^{(n-1)} \cdots P^{(2)}P^{(1)}B)_{ij} = \begin{cases} 1 & \text{if } i = j \neq \ell, \\ 0 & \text{if } i < j \neq \ell. \end{cases}$$

Let $C = P^{(n-1)} \cdots P^{(2)}P^{(1)}B$. Since $B \in U$ and $G_n(F) \cap U_n(F) \subseteq U$, we have $C \in U$ and

$$C_{ij} = \begin{cases} 1 & \text{if } i = j \neq \ell, \\ 0 & \text{if } i = j = \ell, \\ 0 & \text{if } i < j \neq \ell. \end{cases}$$

It is clearly seen that

$$(CC)_{ij} = \begin{cases} 1 & \text{if } i = j \neq \ell, \\ 0 & \text{if } i = j = \ell, \\ 0 & \text{if } i < j \neq \ell. \end{cases}$$

Define the $n \times n$ matrix Q over F by

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$$Q_{ij} = \begin{cases} -1 & \text{if } i = j = \ell, \\ C_{ij} & \text{otherwise.} \end{cases}$$

Then $Q \varepsilon G_n(F) \cap U_n(F)$, so $Q \varepsilon U$. Therefore $CQ \varepsilon U$ and

$$(CQ)_{ij} = (CC)_{ij} \quad \text{for all } i,\ j \varepsilon \{1,2,\ldots,n\},\ j \neq \ell.$$

And if $i < \ell$, then

$$\begin{aligned} (CQ)_{i\ell} &= C_{ii}Q_{i\ell} + C_{i,i+1}Q_{i+1,\ell} + \ldots + C_{i\ell}Q_{\ell\ell} \\ &= 1C_{i\ell} + 0 + \ldots + 0 + C_{i\ell}(-1) = 0. \end{aligned}$$

Hence

$$(CQ)_{ij} = \begin{cases} 1 & \text{if } i = j \neq \ell, \\ 0 & \text{otherwise.} \end{cases}$$

Let $D = CQ$. Then $D \varepsilon U$ and

$$D_{ij} = \begin{cases} 1 & \text{if } i = j \neq \ell, \\ 0 & \text{otherwise.} \end{cases}$$

Next, we claim that for any matrix $A \varepsilon U_n(F)$, if $A_{\ell\ell} = 0$ and $A_{ii} \neq 0$ for all $i \neq \ell$, then $A \varepsilon U$. To prove this, let $A \varepsilon U_n(F)$ be such that $A_{\ell\ell} = 0$ and $A_{ii} \neq 0$ for all $i \neq \ell$. By the previous proof, there exist $R, S \varepsilon G_n(F) \cap U_n(F)$ such that $RAS = D$. Since $D \varepsilon U$ and $G_n(F) \cap U_n(F) \subseteq U$, it follows that $A = R^{-1}DS^{-1} \varepsilon U$.

This proves that U contains every matrix $A \varepsilon U_n(F)$ having $A_{\ell\ell} = 0$ and $A_{ii} \neq 0$ for all $i \neq \ell$. But since ℓ is arbitrary in $\{1,2,\ldots,n\}$, it follows that U contains every matrix in $U_n(F)$ having

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exactly one zero on its main diagonal. By Lemma 4.12, U contains every matrix in $U_n(F)$ having some zeros on its main diagonal. But we have that $G_n(F) \cap U_n(F) \subseteq U$, this implies that U contains every matrix in $U_n(F)$ having nonzero entries on its main diagonal. Hence $U = U_n(F)$. #

The following theorem is a consequence of Theorem 4.6, 4.9, 4.11 and 4.14.

Theorem 4.15. Let F be a field, n a positive integer and $S = G_n(F)$, $M_n(F)$, $D_n(F)$, $U_n(F)$ or $L_n(F)$. If S has a proper dense subsemigroup, then F is infinite.

It is natural to ask whether the converses of Theorem 4.3 and Theorem 4.15 are true. They cannot be answered in this research yet. We leave them as conjectures as follows : Let $F = (F,+,\cdot)$ be a field, n a positive integer and $S = M_n(F)$, $G_n(F)$, $D_n(F)$, $U_n(F)$ or $L_n(F)$.

Conjecture 1. If S has a proper dense subsemigroup, then $(F,\cdot)$ has a proper dense subsemigroup.

Conjecture 2. If F is an infinite field, then S has a proper dense subsemigroup.

Observe that the " if " parts and " only if " parts of Conjecture 1 and Conjecture 2 are true if F is the field of real numbers or the field of complex numbers.

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