CHAPTER I

PRELIMINARIES

A semigroup is a pair $(S,\cdot)$ consisting of a nonempty set S and a binary operation $\cdot$ on S such that $a\cdot(b\cdot c) = (a\cdot b)\cdot c$ for all $a,b,c \in S$. A nonempty subset T of a semigroup S is called a subsemigroup of S if T is closed under the same operation of S.

A semigroup S is said to be commutative if $xy = yx$ for all $x,y \in S$. A maximal commutative subsemigroup of a semigroup S is a commutative subsemigroup of S which is not contained properly in any commutative subsemigroup of S.

An element e of a semigroup S is called an idempotent of S if $e^2 = e$. A semilattice is a commutative semigroup in which all elements are idempotents.

An element a of a semigroup S is said to be regular if $a = axa$ for some $x \in S$. A semigroup S is said to be regular if every element of S is regular. Then every idempotent of a semigroup S is a regular element of S. It is known that a semigroup S is a group if and only if S is a regular semigroup containing exactly one idempotent.

A triple $(S,+,\cdot)$ is called a semiring if

(i) $(S,+)$ is a semigroup,

(ii) $(S,\cdot)$ is a semigroup,

(iii) $x\cdot(y+z) = x\cdot y + x\cdot z$ and $(y+z)\cdot x = y\cdot x + z\cdot x$ for all $x,y,z \in S$.

If (S,+,•) is a semiring, the operations + and • are usually called the addition and the multiplication of the semiring (S,+,•), respectively. A semiring (S,+,•) is said to be additively [multiplicatively] commutative if (S,+) [(S,•)] is a commutative semigroup, and it is said to be commutative if it is both additively commutative and multiplicatively commutative.

Let S = (S,+,•) be a semiring.

An element 0 of S is called a zero of the semiring S if $x+0 = 0+x = x$ and $x\cdot 0 = 0\cdot x = 0$ for every $x \in S$. An element 1 of S is called an identity of the semiring S if $x\cdot 1 = 1\cdot x = x$ for every $x \in S$. An element a of the semiring S is called an additive [multiplicative] idempotent of the semiring S if a is an idempotent of the semigroup (S,+) [(S,•)]. If the semiring S has a zero 0 [an identity 1] and $x,y \in S$ are such that $x+y = y+x = 0$ [$x\cdot y = y\cdot x = 1$], then y is called an additive [multiplicative] inverse of x. If the semiring S has identity 1, an element x of S ia said to be (multiplicatively) invertible if x has a multiplicative inverse in S. If the semiring S has a zero 0, then a nonzero element $x \in S$ is called a zero divisor of the semiring S if there exists a nonzero element $y \in S$ such that $xy = yx = 0$. Then a semiring S with zero has no zero divisors if and only if for $x,y \in S$, $xy = 0$ implies $x = 0$ or $y = 0$.

If we say that S is a semiring with 0 [with 1], we mean S is a a semiring having 0 [1] as its zero [identity]. If a semiring S has a zero 0 and an identity 1, then $0 = 1$ if and only if $|S| = 1$ where $|S|$ denotes the cardinality of S. If we say that S is a semiring with 0,1 we always mean S is a semiring having 0 and 1 as a zero and an identity, respectively, and $0 \neq 1$.

A semiring $(S,+,\cdot)$ is called a <u>regular semiring</u> if $(S,+)$ and $(S,\cdot)$ are regular semigroups. Also, a semiring $(S,+,\cdot)$ is called a <u>semilattice semiring</u> if $(S,+)$ and $(S,\cdot)$ are semilattices.

A <u>Boolean algebra</u> is a triple $(B,+,\cdot)$ such that

(i) $(B,+)$ and $(B,\cdot)$ are commutative semigroups,

(ii) $a\cdot(b+c) = a\cdot b + a\cdot c$ and $a+b\cdot c = (a+b)\cdot(a+c)$ for all $a,b,c \in B$,

(iii) there exists 2 elements 0 and 1 of B such that $0 \neq 1$, $0+a = a$ and $1\cdot a = a$ for every $a \in B$,

(iv) for every $a \in B$, there exists an element $a' \in B$ such that $a+a' = 1$ and $a\cdot a' = 0$.

Every Boolean algebra is a semilattice semiring with 0,1, A semilattice semiring $(S,+,\cdot)$ with 0,1 is a Boolean algebra if $a+b\cdot c = (a+b)\cdot(a+c)$ for all $a,b,c \in S$ and for every element $a \in S$, there exists an element $a' \in S$ such that $a+a' = 1$ and $a\cdot a' = 0$.

If S is an additively commutative semiring and n is a positive integer, let $M_n(S)$ be the set of all $n\times n$ matrices over S, so $M_n(S)$ is a semigroup under matrix multiplication.

Let S be an additively commutative semiring and n a positive integer. If $A \in M_n(S)$, then for $i,j \in \{1,2,...,n\}$ the notation A_{ij} denotes the element of the matrix A in the <u>ith</u> row and <u>jth</u> column, and $A = (a_{ij})$ denotes

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

For $A \in M_n(S)$, A^T denotes the transpose of A. Then, $(A^T)^T = A$ for every $A \in M_n(S)$. And, if S is a commutative semiring, then for $A,B \in M_n(S)$, $(AB)^T = B^TA^T$.

It is known that if R is a ring and n is a positive integer, then the matrix semigroup $M_n(R)$ is regular if and only if R is a regular ring [3, Theorem 24 of Part II.] In particular, any matrix semigroup $M_n(F)$ with F a field and n a positive integer is always regular.

If n is a positive integer such that $n \geqslant 2$, let $\mathcal{S}_n$ denote the permutation group (the symmetric group) of degree n, let $\mathcal{A}_n$ denote the alternating group of degree n (that is, $\mathcal{A}_n = \{\sigma \in \mathcal{S}_n \mid \sigma$ is an even permutation$\}$) and let $\mathcal{B}_n = \mathcal{S}_n \setminus \mathcal{A}_n$ (that is, $\mathcal{B}_n = \{\sigma \in \mathcal{S}_n \mid \sigma$ is an odd permutation$\}$).

Let S be a commutative semiring and n a positive integer such that $n \geqslant 2$. For $A \in M_n(S)$, the <u>positive determinant</u> of A, $\det^+A$, and the <u>negative determinant</u> of A, $\det^-A$, are defined by

$$\det^+A = \sum_{\sigma\in\mathcal{A}_n} A_{1\sigma(1)}A_{2\sigma(2)}\cdots A_{n\sigma(n)}, \det^-A = \sum_{\sigma\in\mathcal{B}_n} A_{1\sigma(1)}A_{2\sigma(2)}\cdots A_{n\sigma(n)}.$$

Then $\det^+A = \det^+(A^T)$ and $\det^-A = \det^-(A^T)$ for every $A \in M_n(S)$. For convenience, if S is a commutative semiring with 0 and A is a 1×1 matrix over S, let $\det^+A$ be the element of A and $\det^-A = 0$. Therefore, If R is a commutative ring and A is a square matrix over R, then the determinant of A, det A, is $\det^+A - \det^-A$.

Let S be an additively commutative semiring with 0,1 and n a positive integer. Let I_n denote the $n\times n$ identity matrix over S. Then I_n is the identity of the matrix semigroup $M_n(S)$. For $A \in M_n(S)$, A is called a <u>permutation</u> <u>matrix</u> over S if every element of A is either 0 or 1 and every row and every column of A has exactly one element which is 1. For $A \in M_n(S)$, A is called an <u>invertible</u> <u>matrix</u> over S if $AB = BA = I_n$ for some $B \in M_n(S)$.

We know from the theory of matrices over a field that a square matrix A over a field is invertible if and only if $\det A \neq 0$. The theory of matrices over a ring gives a generalization of this result that a square matrix over a commutative ring with identity is invertible if and only if detA is an invertible element of the ring R [2; Theorem 4 of Chapter 5].

Reutenauer and Straubing proved in [8; Lemma 1] that if S is a commutative semiring with 0,1, n is a positive integer and $n \geqslant 2$, then for any $A,B \in M_n(S)$, there exists an element $r \in S$ such that

$$\det{}^+(AB) = (\det{}^+A)(\det{}^+B)+(\det{}^-A)(\det{}^-B)+r,$$

$$\det{}^-(AB) = (\det{}^+A)(\det{}^-B)+(\det{}^-A)(\det{}^+B)+r,$$

and then they proved an another important theorem concerning invertible matrices which states that if S is a commutative semiring with 0,1 and n is a positive integer, then for $A,B \in M_n(S)$, $AB = I_n$ implies $BA = I_n$ [8; Theorem].

In this thesis, we let $\mathbb{N}$, $\mathbb{Q}^+$ and $\mathbb{R}^+$ denote the set of all natural numbers (the set of all positive integers), the set of all positive rational numbers and the set of all positive real numbers, respectively.