CHAPTER IV

THE OPEN MAPPING AND CLOSED GRAPH THEOREMS

<u>Definition 4.1</u> X is called a <u>Frechet space</u> over $\mathbb{H}$ (abbreviated by FS ($\mathbb{H}$) if and only if X is a separated complete paranormed space over $\mathbb{H}$.

<u>Example 4.2</u> (1) Every Banach space over $\mathbb{H}$ is a FS($\mathbb{H}$).

(2) Let $w = \{ z = (z_n)_{n \in \mathbb{N}} | z_n \in \mathbb{H}$ for each $n \in \mathbb{N}\}$ with $\|z\| = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|z_n|}{1+|z_n|}$ where $z = (z_n)_{n \in \mathbb{N}} \in w$. Then $(w, \|\cdot\|)$ is a FS($\mathbb{H}$) and $(w, \|\cdot\|)$ is not a Banach space over $\mathbb{H}$.

<u>Proof</u> : For each $n \in \mathbb{N}$, let $P_n(z) = |z_n|$, $z = (z_n)_{n \in \mathbb{N}} \in w$. It is clear that P_n is a paranorm on w for each $n \in \mathbb{N}$. By Theorem 2.5, we have shown that $\|\cdot\|$ is a paranorm on w. It is clear that $\|\cdot\|$ is not a norm ; hence $(w, \|\cdot\|)$ is not a Banach space over $\mathbb{H}$. By Theorem 2.5, we have the property that for any net z in w, $z \to 0$ if and only if $P_n(z) \to 0$ for each $n \in \mathbb{N}$. Let $\mathbb{H}^{\mathbb{N}}$ denote the space of infinite tuples of quaternions, let $T_{\|\cdot\|}$ be the paranorm topology on w and let T be the product topology for $\mathbb{H}^{\mathbb{N}}$. We must show that $T = T_{\|\cdot\|}$. Let $z = (z^{\delta})_{\delta \in D}$ be a net in w such that $z^{\delta} \to 0$ in $(w, T_{\|\cdot\|})$. Then $z^{\delta} \to 0$ in $\mathbb{H}$ so $P_n(z^{\delta}) \to 0$ in $\mathbb{H}$ for each $n \in \mathbb{N}$. Hence $|z_n^{\delta}| \to 0$ for each $n \in \mathbb{N}$ so $|z_n^{\delta}| \to 0$ in $\mathbb{H}$ for each $n \in \mathbb{N}$ therefore $z = (z^{\delta})_{\delta \in D} \to 0$ in (w, T). By Corollary 1.17, $T_{\|\cdot\|} \supseteq T$. We can prove similarly that $T \supseteq T_{\|\cdot\|}$

so $T = T_{\|\cdot\|}$ and hence the paranorm $\|\cdot\|$ gives the product topology i.e. $(w, T_{\|\cdot\|}) = (\mathbb{H}^{\mathbb{N}}, T)$. Since $\mathbb{H}$ is complete, by Theorem 3.55, $\mathbb{H}^{\mathbb{N}}$ is complete so $(w, \|\cdot\|)$ is complete. Let $z_0 \in w \setminus \{0\}$. Then $\|z_0\| > 0$. Choose $U = \{ z \in w \mid \|z\| < \|z_0\|/_2 \}$. Then U is a neighborhood of 0 in w and $z \in U$ so w is separated. Hence w is a separated paranorm space over $\mathbb{H}$ so $(w, \|\cdot\|)$ is a $FS(\mathbb{H})$ which is not a Banach space over $\mathbb{H}$. #

<u>Definition 4.3</u> Let $(X, \|\cdot\|)$ a be paranormed space over $\mathbb{H}$. A series $\sum_{n=1}^{\infty} x_n$ in X is said to be absolutely convergent if and only if $\sum_{n=1}^{\infty} \|x_n\| < \infty$.

<u>Theorem 4.4</u> A paranormed space over $\mathbb{H}$ is complete if and only if every absolutely convergent series is convergent.

<u>Proof</u> : Let $(X, \|\cdot\|)$ be a paranormed space over $\mathbb{H}$.

($\Longrightarrow$) Assume that X is complete. Let $\sum_{n=1}^{\infty} x_n$ be an absolutely convergent series. We must show that $\sum_{n=1}^{\infty} x_n$ converges. Claim that $\{ \sum_{i=1}^{n} x_i \}_{n \in \mathbb{N}}$ is cauchy. Let $\varepsilon > 0$ be given. We must show that there exists an $N \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$, $m, n \geq N$ implies that $\| \sum_{i=1}^{m} x_i - \sum_{i=1}^{n} x_i \| < \varepsilon$. Since $\sum_{n=1}^{\infty} \|x_n\|$ converges, there exists an $L \in \mathbb{R}$ and an $N \in \mathbb{N}$ such that $n \geq N$ implies that $| \sum_{i=1}^{n} \|x_i\| - L | < \varepsilon/_2$. Let $m, n \in \mathbb{N}$ be such that $m, n \geq N$. Suppose that $m \geq n$. Then $\| \sum_{i=1}^{m} x_i - \sum_{i=1}^{n} x_i \| = \| \sum_{i=n+1}^{m} x_i \| \leq \sum_{i=n+1}^{m} \|x_i\|$

$= \sum_{i=1}^{m}\|x_i\| - \sum_{i=1}^{n}\|x_i\| \leq \left|\sum_{i=1}^{m}\|x_i\| - \sum_{i=1}^{n}\|x_i\|\right| = \left|\left(\sum_{i=1}^{m}\|x_i\| - L\right) + \left(L - \sum_{i=1}^{n}\|x_i\|\right)\right|$

$\leq \left|\sum_{i=1}^{m}\|x_i\| - L\right| + \left|L - \sum_{i=1}^{n}\|x_i\|\right| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ so we have the claim. Since X is complete, $\{\sum_{i=1}^{n} x_i\}_{n\in\mathbb{N}}$ converges ; hence $\sum_{i=1}^{\infty} x_i$ converges.

($\Leftarrow$) Suppose that X is not complete. Let $(y_n)_{n\in\mathbb{N}}$ be a nonconvergent Cauchy sequence. Let $k \in \mathbb{N}$. Then there exists $N_k \in \mathbb{N}$ such that $m, n \geq N_k$ implies that $\|y_m - y_n\| < 2^{-k}$ and $N_{k+1} > N_k$. Let $x_k = y_{N_{k+1}} - y_{N_k}$. Then $\sum_{k=1}^{\infty}\|x_k\| = \sum_{k=1}^{\infty}\|y_{N_{k+1}} - y_{N_k}\| \leq \sum_{k=1}^{\infty} 2^{-k} = 1$. By the comparision test, $\sum_{k=1}^{\infty}\|x_k\|$ converges. Since $(y_n)_{n\in\mathbb{N}}$ has no convergent subsequence, $(x_k)_{k\in\mathbb{N}}$ does not converge so $\sum_{k=1}^{\infty} x_k$ does not converge. #

Definition 4.5 Let X, Y be TVS($\mathbb{H}$)'s. A linear map $f : X \to Y$ is called almost open if and only if for each $U \in N(X)$, $\overline{f(U)} \in N(Y)$.

Lemma 4.6 Let X, Y be TVS($\mathbb{H}$)'s. Let $f : X \to Y$ be linear and suppose that for each $U \in N(X)$, $\overline{f(U)}$ has nonempty interior. Then f is almost open.

Proof : Let $U \in N(X)$. Then there exists a $V \in N(X)$ such that $V - V \subseteq U$. Let $y \in \mathrm{Int}(\overline{f(V)})$. Since f is linear, $f(V) - f(V) = f(V-V) \subseteq f(U)$. By Theorem 3.24, $\overline{f(V)} - \overline{f(V)} \subseteq \overline{f(V) - f(V)} \subseteq \overline{f(U)}$ so $\overline{f(V)} - y = \{x - y \mid x \in \overline{f(V)}\} \subseteq \{x - z \mid x, z \in \overline{f(V)}\} = \overline{f(V)} - \overline{f(V)} \subseteq \overline{f(U)}$. Since $0 \in \mathrm{Int}(\overline{f(V)} - y \in N(X)$, $\overline{f(U)} \in N(Y)$ therefore f is almost open. #

Lemma 4.7 Let X, Y be TVS($\mathbb{H}$)'s and let $f : X \to Y$ be linear. Suppose that $f(X)$ is of second category in Y. Then f is almost open.

Proof : Let $U \in N(X)$. We must show that $f(U) \in N(Y)$. Let $V = f(U)$. Claim that $X = \bigcup_{n \in \mathbb{N}} \{nU\}$. Let $x \in X$. Since U is absorbing, there exists an $\varepsilon > 0$ such that $tx \in U$ for $|t| < \varepsilon$. Choose $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \varepsilon$. Since $|\frac{1}{n_0}| < \varepsilon$, $\frac{1}{n_0} x \in U$ so $x \in n_0 U$ therefore $x \in \bigcup_{n \in \mathbb{N}} \{nU\}$ so we have the claim. Hence $f(X) = f(\bigcup_{n \in \mathbb{N}} \{nU\}) = \bigcup_{n \in \mathbb{N}} \{nf(U)\} = \bigcup_{n \in \mathbb{N}} \{nV\}$. Since $f(X)$ is of second category, there exists an $m \in \mathbb{N}$ such that $\mathrm{Int}(\overline{mV}) \neq \emptyset$. We must show that $\mathrm{Int}(\bar{V}) \neq \emptyset$. Let $y \in \mathrm{Int}(\overline{mV})$. Then there exists an open set $W \ni y$ such that $W \subseteq \overline{mV} = \overline{mf(U)}$. Claim that $W \subseteq m\overline{f(U)}$. Suppose not. Then there exists a $z \in W$ such that $z \notin m\overline{f(U)}$ so $\frac{z}{m} \notin \overline{f(U)}$. Hence there exists an open set $M \ni \frac{z}{m}$ such that $M \cap f(U) = \emptyset$ therefore $mM \cap mf(U) = \emptyset$. Now $mM \ni z$ and since M is open, mM is open therefore $z \in mM$ and $mM \cap mf(U) = \emptyset$. Hence $z \notin \overline{mf(U)}$, a contradiction. So we have the claim. Since $W \ni y$ and $W \subseteq \overline{mf(U)}$, $y/_m \in W/_m \subseteq \overline{f(U)}$ so $\mathrm{Int}(\bar{V}) = \mathrm{Int}(\overline{f(U)}) \neq \emptyset$. By Lemma 4.6, we can conclude that f is almost open. #

Lemma 4.8 Let $(X, \|.\|)$ be a FS($\dot{\mathbb{H}}$) and (Y, p) is a separated paranormed space over $\mathbb{H}$. Let $f : X \to Y$ be linear, continuous and almost open. Then f is open .

Proof : Let U be an open set in $N(X)$. We must show that $f(U) \in N(Y)$. Since $U \in N(X)$, $U \supseteq M_\varepsilon = \{x \mid \|x\| < \varepsilon\}$ for some $\varepsilon > 0$.

Let $V = M_{\varepsilon/2}$. We must show that $\overline{f(V)} \subseteq f(U)$. Let $z \in \overline{f(V)}$. Since f is almost open, for all $n \in \mathbb{N}$, $\overline{f(M_{\varepsilon/2^n})} \in N(Y)$; hence there exists a $\delta_n' > 0$ such that $\overline{f(M_{\varepsilon/2^n})} \supseteq \{y \in Y \mid p(y) < \delta_n'\}$. Let $\delta_n = \min\{\delta_1', \delta_2', \ldots, \delta_n'\}$

Then $\{\delta_n\}_{n \in \mathbb{N}}$ is a decreasing sequence of positive real numbers such that $\overline{f(M_{\varepsilon/2^n})} \supseteq \{y \in Y \mid p(y) < \delta_n\}$ for $n = 1,2,\ldots$ Let $B = \{y \in Y \mid p(z-y) < \delta_2\}$. Since $z \in B$, $B \neq \emptyset$ so B is a nonempty open set. Since $z \in \overline{f(M_{\varepsilon/2})}$ and $B \ni z$, $B \cap f(M_{\varepsilon/2}) \neq \emptyset$. Choose $x_1 \in M_{\varepsilon/2}$ such that $p(z - f(x_1)) < \delta_2$. Then $z - f(x_1) \in \overline{f(M_{\varepsilon/4})}$. Choose $x_2 \in M_{\varepsilon/4}$ such that $p(z - f(x_1) - f(x_2)) < \delta_3$. Repeating this argument, we get that for each $n \in N$, there exists an $x_n \in M_{\varepsilon/2^n}$ such that $p(z - \sum_{i=1}^{n} f(x_i)) < \delta_{n+1}$. Since $\|x_n\| < \varepsilon/2^n$, by Theorem 4.4, $x = \sum_{n=1}^{\infty} x_n$ converges . Since $\|.\|$ is continuous, $\|x\| \leq \sum_{n=1}^{\infty} \|x_n\| < \varepsilon$; so $x \in U$.

Since $p(z - f(\sum_{i=1}^{n} x_i)) = p(z - \sum_{i=1}^{n} f(x_i)) < \delta_{n+1}$ for all $n = 1,2,\ldots$ and $(\delta_n)_{n \in \mathbb{N}}$ is decreasing, $z = f(\sum_{i=1}^{\infty} x_i) = f(x) \in f(U)$ so $\overline{f(V)} \subseteq f(U)$.

Since $\overline{f(V)} \in N(Y)$, $f(U) \in N(Y)$ therefore f is open. #

<u>Theorem 4.9</u> (Open mapping theorem)

Let X, Y be FS($\mathbb{H}$)'s. Let $f : X \to Y$ be linear, continuous and onto. Then f is open.

<u>Proof</u> : Claim that Y is of second category. Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of closed sets in Y with empty interior.

Given $n \in \mathbb{N}$, let $G_n = Y \setminus A_n$. Since $\operatorname{Int}(Y \setminus G_n) = \operatorname{Int}(A_n) = \emptyset$, G_n is a dense open set for all $n \in \mathbb{N}$. We must show that $\bigcap_{n \in \mathbb{N}} G_n = \emptyset$. Let d be the metric of Y. Since G_1 is an open nonempty subset of Y there exists a $y_1 \in G_1$ and an $\gamma_1 > 0$ such that $D(y_1, \gamma_1) = \{y \mid d(y, y_1) \leq \gamma_1\} \subseteq G_1$. Since G_2 is dense in Y, $G_2 \cap D(y_1, \gamma_1) \neq \emptyset$. Choose $y_2 \in G_2 \cap D(y_1, \gamma_1)$. Then there exists an $\gamma_2' > 0$ such that $D(y_1, \gamma_2') = \{ y \mid d(y, y_2) < \gamma_2' \} = G_2 \cap D(y_1, \gamma_1)$. Let $\gamma_2 = \min\{ \gamma_2', \frac{\gamma_1}{2^2} \}$ Then $D(y_2, \gamma_2) \subseteq G_2 \cap D(y_1, \gamma_1)$ and $0 < \gamma_2 < \frac{\gamma_1}{2}$. Continuing this mehtod, we obtain a sequence $(\gamma_n)_{n \in \mathbb{N}}$ with $0 < \gamma_{n+1} < \gamma_n/2$ and $D(y_{n+1}, \gamma_{n+1}) \subseteq D(y_n, \gamma_n) \cap G_n$. Let $m, n \in \mathbb{N}$ be such that $m > n$. Then $y_m \in D(y_m, \gamma_m) \subseteq D(y_n, \gamma_n)$; so $d(y_m, y_n) \leq r_n$ therefore $(y_n)_{n \in \mathbb{N}}$ is a Cauchy sequence, Since Y is complete, $(y_n)_{n \in \mathbb{N}}$ converges to a point in Y, say y. Since $y_m \in D(y_n, \gamma_n)$ for all $m > n$ and $D(y_n, \gamma_n)$ is closed in Y for each n, $y \in D(y_n, \gamma_n)$ for each n. Hence $y \in G_n$ for all n so $\bigcap_{n \in \mathbb{N}} G_n \neq \emptyset$. Hence $\bigcup_{n \in \mathbb{N}} A_n \neq Y$ so Y is of the second category. By Lemma 4.7, f is almost open. By Lemma 4.8, f is open. #

<u>Theorem 4.10</u> Let X be a FS($\mathbb{H}$), Y a separated paranormed space over $\mathbb{H}$, and $f : X \to Y$ linear and continuous. If f(X) is of second category in Y then f is an open map from X onto Y.

<u>Proof</u> : Since f is linear and f(X) is of second category in Y, by Lemma 4.7, f is almost open. By Lemma 4.8, since f is linear continuous and almost open, f is open. We must show that f is onto.

Let $y \in Y$. Since $f(X) \in N(Y)$ and Y is a TVS(IH), $f(X)$ is absorbing so there exists an $\varepsilon > 0$ such that $ty \in f(X)$ for $|t| < \varepsilon$. Thus $(\varepsilon/2)y \in f(X)$; hence $y \in \frac{2}{\varepsilon} f(X) \subseteq f(X)$ so f is onto. #

Corollary 4.11 Let X, Y be Frechet spaces over IH and $f : X \to Y$ a linear, continuous, one-to-one and onto map. Then f is a homeomorphism.

Proof : By Theorem 4.10, f is open. Then f^{-1} is continuous ; so f is homeomorphism. #

Corollary 4.12 Two comparable Frechet topologies for a vector space over IH must be equal.

Proof : Let X be a vector space over IH with two comparable Frechet topologies, say T_1 and T_2. Suppose $T_1 \supseteq T_2$. Let $i : (X, T_1) \to (X, T_2)$ be the identity map. Then i is linear, continuous, 1 - 1 and onto. By Corollary 4.11, i is a homeomorphism. Let $Q \in T_1$ be open. Then $i(Q) = Q$ is open in T_2. Hence $Q \in T_2$. Then $T_1 \subseteq T_2$. Hence $T_1 = T_2$. #

Theorem 4.13 (Closed graph Theorem).
Let X, Y be FS(IH)'s. Let $f : X \to Y$ be a linear map with a closed graph G. Then f is continuous.

Proof : Claim 1 $X \times Y$ is a FS(IH)

Define $\|\cdot\| : X \times Y \to$ IH by $\|(x, y)\| = p_1(x) + p_2(y)$ where p_1, p_2 are the paranorms on X and Y respectively. We must show that $\|\cdot\|$ is a paranorm on $X \times Y$. Let $x \in X$ and $y \in Y$. Then $\|-(x, y)\| = \|(-x, -y)\|$ $= p_1(-x) + p_2(-y) = p_1(x) + p_2(y) = \|(x, y)\|$. Let $(x, y), (u, v) \in X \times Y$.

Then $\|(x, y) + (u, v)\| = \|(x+u, y+v)\| = p_1(x+u) + p_2(y+v) \leq p_1(x) + p_1(u) + p_2(y) + p_2(v) = (p_1(x) + p_2(y)) + (p_1(u) + p_2(v)) = \|(x, y)\| + \|(u, v)\|$. Let (t_n) be a sequence of elements in $\mathbb{H}$ such that $t_n \to t$ for some $t \in \mathbb{H}$ and $((x_n, y_n))_{n \in \mathbb{N}} \subseteq X \times Y$ with $\|(x_n, y_n) - (x, y)\| \to 0$ for some $(x, y) \in X \times Y$. We must show that $\|t_n(x_n, y_n) - t(x, y)\| \to 0$. Since $p_1(t_n x_n - tx) \to 0$, $p_2(t_n y_n - ty) \to 0$ and + is continuous,

$\|t_n(x_n, y_n) - t(x, y)\| = \|(t_n x_n - tx, t_n y_n - ty)\| = p_1(t_n x_n - tx) + p_2(t_n y_n - ty) \to 0 + 0 = 0$ as $n \to \infty$. Hence $\|\cdot\|$ is a paranorm on $X \times Y$. Let $(x, y) \in X \times Y$ be such that $\|(x, y)\| = 0$. Then $p_1(x) + p_2(y) = 0$; so $p_1(x) = 0$ and $p_2(y) = 0$. Since p_1 and p_2 are total, $x = y = 0$ so $\|\cdot\|$ is total. Let $(x, y) \in X \times Y \setminus \{(0, 0)\}$. Then $\|(x, y)\| > 0$. Let $0 < r < \|(x, y)\|$. Then $(x, y) \notin B((0, 0), r)$ so $X \times Y$ is separated. We must show that $X \times Y$ is complete. Let $(x_\delta, y_\delta)_{\delta \in D}$ be a Cauchy net in $X \times Y$. Claim that $(x_\delta)_{\delta \in D}$ is a Cauchy net in X and $(y_\delta)_{\delta \in D}$ is a Cauchy net in Y. Let $U \in N(X)$ and $V \in N(Y)$. Since $(x_\delta, y_\delta)_{\delta \in D}$ is Cauchy net in $(X \times Y)$, there exists a $\delta' \in D$ such that $\alpha \geq \delta'$, $\beta \geq \delta'$ implies that $(x_\alpha, y_\alpha) - (x_\beta, y_\beta) \in U \times V$; that is, $(x_\alpha - x_\beta, y_\alpha - y_\beta) \in U \times V$; hence $x_\alpha - x_\beta \in U$ and $y_\alpha - y_\beta \in V$ for all $\alpha \geq \delta'$, $\beta \geq \delta'$ so we have the claim. Since X and Y are complete, $x_\delta \to x'$, $y_\delta \to y'$ for some $x' \in X$, $y' \in Y$. Then $(x_\delta, y_\delta) \to (x', y') \in X \times Y$ so $X \times Y$ is complete. Hence $(X \times Y, \|\cdot\|)$ ia a separated paranormed space so $(X \times Y, \|\cdot\|)$ is a $FS(\mathbb{H})$ and the topology induced by $\|\cdot\|$ is the product topology of $X \times Y$.

Claim 2 G is a $FS(\mathbb{H})$ with respect to the relative topology Let $\|\cdot\|$ be paranorm on $X \times Y$. Then $(G, \|\cdot\|)$ is a paranormed space over $\mathbb{H}$ with respect to the relative topology. Since $\{(0, 0)\}$ is closed in $X \times Y$ and G is closed in $X \times Y$, $\{(0, 0)\} = \{(0, 0)\} \cap G$. Thus $(G, \|.\|)$ is a separated paranormed space over $\mathbb{H}$. We must show that $(G, \|.\|)$ is complete. Let $(x_\delta, f(x_\delta))_{\delta \in D}$ be a Cauchy net in G. Then $(x_\delta)_{\delta \in D}$ is a Cauchy net in X and $(f(x_\delta))_{\delta \in D}$ is a Cauchy net net in $f(X)$. Since X is complete, $x_\delta \to x_0$ for some $x_0 \in X$. Since $(f(x_\delta))_{\delta \in D}$ is Cauchy in $f(X)$, it is Cauchy in Y. Since Y is complete, $f(x_\delta) \to y_0$ for some $y_0 \in Y$. Since G is closed, $y_0 \in f(X)$ so G is complete and thus G is a $FS(\mathbb{H})$ with respect to the relative topology

Let $T_1 : G \to X$ be defined by $T_1(x, y) = x$. Clearly, T_1 is linear, continuous, one-to-one and onto. Since G is a Frechet space T_1 is a homeomorphism. Also the map $T_2 : G \to Y$ given by $T_2(x, y) = y$ is continuous. Since $f = T_2 \circ T_1^{-1}$ and T_2, T_1 are continuous, f is continuous. #

<u>Theorem 4.14</u> Let X, Y be $TVS(\mathbb{H})$'s and let $f : X \to Y$ be a linear map. Then f has a closed graph if and only if for each net $(x_\delta)_{\delta \in D}$ in X, if $x_\delta \to 0$ and $f(x_\delta) \to y$ then $y = 0$.

<u>Proof</u> : ($\Longrightarrow$) Suppose that f has a closed graph, say G. Let $(x_\delta)_{\delta \in D}$ be a net in X such that $x_\delta \to 0$ and $f(x_\delta) \to y$ for some $y \in Y$. We must show that $y = 0$. Since $(x_\delta, f(x_\delta)) \to (0, y)$, $(0, y) \in \bar{G} = G$ so $y = f(0) = 0$.

($\Longleftarrow$) Let G be the graph of f. Let $(a, b) \in \bar{G}$. Then there exists a net $(g_\delta)_{\delta \in D}$ in G such that $g_\delta \to (a, b)$ in $X \times Y$, say $g_\delta = (x_\delta, f(x_\delta))$ where $(x_\delta)_{\delta \in D}$ is a net in X. By assumption, $x_\delta \to a$ and $f(x_\delta) \to b$. Then $x_\delta - a \to 0$ and $f(x_\delta - a) = f(x_\delta) - f(a) \to b - f(a)$. Hence $b - f(a) = 0$, so $f(a) = b$. Hence $(a, b) \in G$ so G is closed. #

<u>Corollary 4.15</u> Let X, Y be TVS($\mathbb{H}$)'s with Y separated. Then a continuous linear map $f : X \to Y$ must have closed graph.

<u>Proof</u> : Let $(x_\delta)_{\delta \in D}$ be a net in X such that $x_\delta \to 0$ and $f(x_\delta) \to y$ for some $y \in Y$. Since f is continuous and $x_\delta \to 0$, $f(x_\delta) \to f(0) = 0$; hence $y = 0$. By Theorem 4.14, f has closed graph. #

<u>Theorem 4.16</u> Let X, Y be FS($\mathbb{H}$)'s. Let $f : X \to Y$ be linear and onto with closed graph. Then f is continuous and open.

<u>Proof</u> : Since f is linear and has a closed graph, by Theorem 4.13, f is continuous. Since f is onto, by Theorem 4.9, f is open. #

<u>Definition 4.17</u> Let X be a TVS($\mathbb{H}$) and let A, B be vector subspaces of X. A and B are <u>algebraically complementary</u> if and only if $A + B = X$ and $A \cap B = \{0\}$. Define $P : X \to A$ by $P(x) = a$ where $x = a + b$, $a \in A$ $b \in B$. P is called the <u>projection on A</u>.

<u>Theorem 4.18</u> Let X be a TVS($\mathbb{H}$) and A, B be vector subspaces of X. which are algebraically complementary. Then A, B are closed if and only if the projection on A has closed graph.

<u>Proof</u> : ($\Longrightarrow$) Suppose A and B are closed. Let P be the projection on A. We must show that P has a closed graph. Let $(x_\delta)_{\delta \in D}$ be a net in X such that $x_\delta \to 0$ and $P(x_\delta) \to y$ for some $y \in A$.

Since $A + B = X$, $x_\delta - P(x_\delta) \in B$ and $x_\delta - P(x_\delta) \to 0 - y = -y$. Since B is a closed vector subspace of X and $-y \in B$, $y \in B$. Since $P(x_\delta) \in A$, $P(x_\delta) \to y$ and since A is closed in X, $y \in A$. Hence $y \in A \cap B = \{0\}$; so $y = 0$. By Theorem 4.14, P has closed graph.

($\Leftarrow$) Let $(a_\delta)_{\delta \in D}$ be a net in A such that $a_\delta \to a$ for some $a \in X$. We must show that $a \in A$. Since $a_\delta \to a$, $a_\delta - a \to 0$; hence $P(a_\delta - a) = a_\delta - P(a) \to a - P(a)$. Since P has closed graph, by Theorem 4.14, $a - P(a) = 0$ so $a = P(a) \in A$ therefore A is closed. Similarly, we can show that B is closed in X. #

<u>Definition 4.19</u> Let X be a TVS($\mathbb{H}$) and let A, B be algebraically complementary of X. Then A, B are called <u>topologically complemantary</u> if and only if f the projection on A is continuous. A is said to be <u>complemented in X</u> if and only if A is one of a pair of topologically complementary subspaces A, B in which case we write $X = A + B$.

<u>Theorem 4.20</u> Let X be a Frechet space over $\mathbb{H}$ and let A, B be algebraically complementary closed subspaces of X. Then A and B are topologically complementary.

<u>Proof</u> : Let P be the projection on A. We must show that P is continuous. Since A and B are closed, by Theorem 4.18, P has a closed graph. Since P is linear and has a closed graph, by the closed graph theorem, P is continuous. #

<u>Definition 4.21</u> Let X be a TVS($\mathbb{H}$). A <u>basis for X</u> is a sequence $(b^n)_{n \in \mathbb{N}}$ such that every $x \in X$ has a unique representation $x = \sum_{n=1}^{\infty} t_n b^n$.

For example, let c_0 = the set of all sequences converging to 0 with $\|x\|_\infty = \sup\{\,|x_n\| \; n \in \mathbb{N}\,\}$. Then $(\delta^n)_{n \in \mathbb{N}}$ is a basis of c_0 where δ^n is the sequence with $x_n = 1$ and $x_k = 0$ if $k \neq n$.

<u>Definition 4.22</u> Let X be a TVS($\mathbb{H}$). Let $(b^n)_{n \in \mathbb{N}}$ be a basis of X. Define $\ell_n : X \to \mathbb{H}$ by $\ell_n(x) = t_n$ where $x = \sum_{n=1}^{\infty} t_n b^n$. For all $n \in \mathbb{N}$, ℓ_n is a linear functional on X and ℓ_n has the property of forming with $(b^n)_{n \in \mathbb{N}}$ a biorthogonal system, that is, $\ell_n(b^k) = 0$ if $n \neq k$ and $\ell_n(b^n) = 1$. For each $n \in \mathbb{N}$, ℓ_n is called a <u>coordinate functional</u> and $(b^n)_{n \in \mathbb{N}}$ is called a <u>Schauder basis</u> if and only if each $\ell_n \in X'$.

<u>Definition 4.23</u> A <u>K - space over $\mathbb{H}$</u> is a vector space over $\mathbb{H}$ of sequences which has a topology such that each P_n is continuous, where $P_n(x) = x_n$, $x = (x_n)_{n \in \mathbb{N}}$.

<u>Remark 4.24</u> Let X be a sequence space over $\mathbb{H}$ with basis $B = (\delta^n)_{n \in \mathbb{N}}$. Then X is a K-space over $\mathbb{H}$ if and only if B is a Schauder basis.

<u>Proof</u> : ($\Rightarrow$) Suppose that X is a K-space over $\mathbb{H}$. Then X has a topology such that P_n is continuous for each $n \in \mathbb{N}$. Since P_n is linear and continuous and $P_n(\delta^k) = 0$ if $k \neq n$ and $P_n(\delta^n) = 1$, B is a Schauder basis.

($\Leftarrow$) Since B is a Schauder basis for each $n \in \mathbb{N}$ there exists an $\ell_n \in X'$ such that $\ell_n(b^k) = 0$ if $k \neq n$ and $\ell_n(b^n) = 1$. Then $\ell_n(x) = \ell_n(\sum_{n=1}^{\infty} x_n \delta^n) = x_n$. Since $\ell_n \in X'$, there exists a topology in X which makes ℓ_n continuous for all $n \in \mathbb{N}$. Thus X is K-space over $\mathbb{H}$. #

<u>Remark 4.25</u> Let X be a TVS($\mathbb{H}$) with basis $(b^n)_{n\in\mathbb{N}}$. Then X is linearly homeomorphic to a sequence space Y with basis $(\delta^n)_{n\in\mathbb{N}}$. Furthermore, Y is a K-space over $\mathbb{H}$ if and only if $(b^n)_{n\in\mathbb{N}}$ is a Schauder basis of X.

<u>Proof</u> : Let X be a TVS($\mathbb{H}$) with basis $(b^n)_{n\in\mathbb{N}}$. We must show that there exists a sequence space Y with basis $(\delta^n)_{n\in\mathbb{N}}$ and there exists a $u : X \to Y$ such that u is a linear homeomorphism . Let $Y = \{t = (t_n)_{n\in\mathbb{N}} \mid \sum_{n=1}^{\infty} t_n b^n$ converges in X $\}$. Since $Y \ni 0$, $Y \neq \emptyset$.

Then $Y = \{t = (t_n)_{n\in\mathbb{N}} \mid t_n = \ell_n(x)$ for some $x \in X\}$. Let $u : X \to Y$ be defined as follows : Let $x \in X$. Then $x = \sum_{n=1}^{\infty} t_n b^n$, define $u(x) = (t_n)_{n\in\mathbb{N}}$. Clearly, u is linear and bijective .

Let $T = \{ M \subseteq Y \mid$ there exists an open set G in X such that $M = u(G) \}$. Clearly T is a topology on Y. Also u is continuous with respect to T so u is a linear homeomorphism of X onto Y. Also, $u(b^n) = \delta^n$ and $\ell_n = P_n \circ u$ where $P_n(t) = P_n((t_n)_{n\in\mathbb{N}}) = t_n$, $n \in \mathbb{N}$. Thus $P_n \in Y'$ if and only if $\ell_n \in X'$, By Remark 4.24, we can conclude that Y is a K-space over $\mathbb{H}$ if and only if $(b^n)_{n\in\mathbb{N}}$ is a Schauder basis for X. #

<u>Lemma 4.26</u> Let (X, p) be a paranormed space over $\mathbb{H}$ with basis $(\delta^n)_{n\in\mathbb{N}}$. Let $q(x) = \sup\{ p(\sum_{i=1}^{n} x_i \delta^i) \mid n \in \mathbb{N}$ and $x = \sum_{i=1}^{\infty} x_i \delta^i \}$. Then q is a paranorm for X and $q \geq p$.

<u>Proof</u> : For $x \in X$, let $u^n = u^n(x) = \sum_{i=1}^{n} x_i \delta^i = (x_1, x_2, \ldots, x_n, 0, 0, \ldots, 0)$. Then $q(x) = \sup \{p(u^n) \mid n \in \mathbb{N}\}$. We must show that q is a paranorm on X. It is clear that $q(x) \geq 0$, $q(x) = q(-x)$ for all $x \in X$, $q(0) = 0$ and $q(x+y) \leq q(x) + q(y)$ for all x, $y \in X$. We must show that scalar multiplication is continuous.

Claim 1 Let $(r_k)_{k \in \mathbb{N}}$ be a sequence of elements in $\mathbb{H}$ such that $r_k \to 0$. Then $q(r_k x) \to 0$ for each $x \in X$. ________ (1)

Let $(r_k)_{k \in \mathbb{N}}$ be a sequence of elements in $\mathbb{H}$ such that $r_k \to 0$. Let $x \in X$. We must show that $q(r_k x) \to 0$. Let $\varepsilon > 0$ be given. Since $(u^n)_{n \in \mathbb{N}} = (u^n(x))_{n \in \mathbb{N}}$ converges to x in (X, p), $(u^n)_{n \in \mathbb{N}}$ is bounded ; so there exists a $\delta > 0$ such that $|t| < \delta$ implies that $p(tu^n) < \varepsilon/_2$ for all $n \in \mathbb{N}$. Since $r_k \to 0$, there exists an $N \in \mathbb{N}$ such that $k > N$ implies that $|r_k| < \delta$. Let $k \in \mathbb{N}$ be such that $k > N$. Then $p(r_k u^n) < \varepsilon/_2$ for all $n \in \mathbb{N}$; so $q(r_k x) = \sup \{p(r_k u^n) \mid n \in \mathbb{N}\} \leq \varepsilon/_2 < \varepsilon$ so we have claim 1.

Claim 2. Let $k \in \mathbb{N}$. Let $(x_n^k)_{n \in \mathbb{N}}$ be a sequence in X. Then $q(x^k) \to 0$ implies that $q(tx^k) \to 0$ for all $t \in \mathbb{H}$. (2)

Let $t \in \mathbb{H}$, let $x^k = (x_n^k)_{n \in \mathbb{N}}$ and let $u^{k,n} = (x_1^k, x_2^k, \ldots, x_n^k, 0, 0, \ldots)$. Let $\varepsilon > 0$ be given. Let $U \in N(X, p)$ be such that $p(tu) < \varepsilon/_2$ for all $u \in U$. Since $q(x^k) \to 0$ and $p(u^{k,n}) = p((x_1^k, x_2^k, \ldots, x_n^k, 0, 0, \ldots)) \leq \sup \{ p(u^{k,n}) \} \leq q(x^k)$, there exists an $N \in \mathbb{N}$ such that $k > N$ implies that $u^{k,n} = (x_1^k, x_2^k, \ldots, x_n^k, 0, 0, \ldots) \in U$ for all $n \in \mathbb{N}$. Let $k \in \mathbb{N}$ be such that $k > N$. Since $u^{k,n} \in U$ and $p(tu) < \varepsilon/_2$ for all $u \in U$, $p(tu^{k,n}) < \varepsilon/_2$

for all $n \in \mathbb{N}$. Then $q(tx^k) = \sup\{p(tu^{k,n}) \mid n \in \mathbb{N}\} \leq \varepsilon_{/2} < \varepsilon$ so we have claim 2.

Claim 3. Let $(r_k)_{k \in \mathbb{N}}$ be a sequence of elements in $\mathbb{H}$ and $(x^k)_{k \in \mathbb{N}} \subseteq X$. If $r_k \to 0$ and $q(x^k) \to 0$ then $q(r_k x^k) \to 0$. ——(3)

Let $\varepsilon > 0$ be given. Let U be a balanced neighberhood of 0 in (X, p) with $p(u) < \varepsilon_{/2}$ for all $u \in U$. Then there exists an $N \in \mathbb{N}$ such that $k > N$ implies that $\|r_k\| < 1$ and $u^{k,n} \in U$ for all $n \in \mathbb{N}$. Let $k \in \mathbb{N}$ be such that $k > N$. Since $\|r_k\| < 1$ and U is balanced, $r_k u^{k,n} \in r_k U \subseteq U$ for all $n \in \mathbb{N}$ and so $p(r_k u^{k,n}) < \varepsilon_{/2}$ for all $n \in \mathbb{N}$. Thus $q(r_k x^k)$ $= \sup\{p(r_k u^{k,n}) \mid n \in \mathbb{N}\} \leq \varepsilon_{/2}$ so we have claim 3. Let $(t_k)_{k \in \mathbb{N}}$ be a sequence of elements in $\mathbb{H}$ such that $t_k \to t$ for some $t \in \mathbb{H}$ and let $(x^k)_{k \in \mathbb{N}}$ be a sequence of elements in X such that $q(x^k - x) \to 0$ for some $x \in X$. We must show that $q(t_k x^k - tx) \to 0$. Let $r_k = t_k - t$ for all $k \in \mathbb{N}$ and $y^k = x^k - x$. Then $q(r_k x^k - tx) = q(r_k(x^k - x) + r_k x + t(x^k - x) - tx^k) \leq q(r_k y^k) + q(r_k x) + q(ty^k) + q(tx_k)$. By claim 3, $q(r_k y^k) \to 0$ and $q(tx^k) \to 0$. By claim 1, $q(r_k x) \to 0$. By claim 2, $q(ty^k) \to 0$ hence $q(r_k x^k - tx) \to 0$ so scalar multiplication is continuous therefore q is a paranorm on X. Let $x \in X$. Then $x = \sum_{i=1}^{\infty} x_i \delta^i$ so $p(x) = p(\sum_{i=1}^{\infty} x_i \delta^i)$ $\leq \sup\{p(\sum_{i=1}^{n} x_i \delta^i) \mid n \in \mathbb{N}\} = q(x)$ therefore $p \leq q$. #

<u>Theorem 4.27</u> Every basis of a FS($\mathbb{H}$) is a Schauder basis.

<u>Proof</u> : Let (X, p) be a FS($\mathbb{H}$). By Remark 4.25, we can take (X, p) to be a sequence space with basis $(\delta^n)_{n\in\mathbb{N}}$. For each $x \in X$, $x = \sum_{i=1}^{\infty} x_i\delta^i$, let $q(x) = \sup\{p(\sum_{i=1}^{n} x_i\delta^i) \mid n = 1,2,3,\ldots\}$. By Lemma 4.26, q is a paranorm for X and $q \geq p$. For each $n \in \mathbb{N}$ and for all $x \in X$, let $u^n = u^n(x) = \sum_{i=1}^{n} x_i\delta^i = (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$. Claim that u^n is continuous on (X, q) for each $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ and $a \in X$. Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon$. Let $x \in X$ be such that $q(x-a) < \delta$. Then $q(u^n(x) - u^n(a)) = q(u^n(x-a)) = q(\sum_{i=1}^{n}(x_i - a_i)\delta^i)$

$= \sup\{p(\sum_{i=1}^{k}(x_i - a_i)\delta^i) \mid k = 1,2,\ldots,n\} \leq \sup\{p(\sum_{i=1}^{k}(x_i - a_i)\delta^i) \mid k = 1,2,3,\ldots\} = q(x-a) < \delta = \varepsilon$. So u^n is continuous at $x = a$. Since a was arbitrary, u^n is continuous on (X, q) for each n. Since $u^n(x) - u^{n-1}(x) = x_n\delta^n = P_n(x)\delta^n$, by Corollary 3.48, P_n is continuous on (X, q) for each n. Claim that (X, q) is complete. Let $(x^n)_{n\in\mathbb{N}}$ be a q - Cauchy sequence. Since P_k is linear and continuous on (X, q) for each k, P_k is uniformly continuous on (X, q) for each k ; hence $(x_k^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{H}$ for each k. Let $x_k = \lim_{n\to\infty} x_k^n$. We shall show that $\sum_{k=1}^{\infty} x_k\delta^k$ converges in (X, p). Since p is continuous on (X, p), $p(\sum_{k=u}^{v} x_k\delta^k) = \lim_{n\to\infty} p(\sum_{k=u}^{v} x_k^n\delta^k)$ for all $u, v \in \mathbb{N}$.

Let $\varepsilon > 0$ be given. Since $(x^n)_{n\in\mathbb{N}}$ is q - Cauchy, there exists an $N \in \mathbb{N}$ such that $n > m \geq N$ implies that $q(x^m - x^n) < \varepsilon/3$.

Let $n \in \mathbb{N}$ be such that $n > N$. Then $p(\sum_{k=u}^{v} x_k^n \delta^k)$

$$= p(\sum_{k=u}^{v} (x_k^n - x_k^N)\delta^k + \sum_{k=u}^{v} x_k^N \delta^k)$$

$$\leq p(\sum_{k=u}^{v} (x_k^n - x_k^N)\delta^k) + p(\sum_{k=u}^{v} x_k^N \delta^k)$$

$$= p(\sum_{k=1}^{v} (x_k^n - x_k^n)\delta^k - \sum_{k=1}^{u-1} (x_k^n - x_k^N)\delta^k) + p(\sum_{k=u}^{v} x_k^N \delta^k)$$

$$\leq p(\sum_{k=1}^{v} (x_k^n - x_k^N)\delta^k) + p(\sum_{k=1}^{u-1} (x_k^n - x_k^N)\delta^k) + p(\sum_{k=1}^{v} x_k^N \delta^k)$$

$$\leq q(\sum_{k=1}^{v} (x_k^n - x_k^N)\delta^k) + q(\sum_{k=1}^{u-1} (x_k^n - x_k^N)\delta^k) + q(\sum_{k=u}^{v} x_k^N \delta^k)$$

$$< \varepsilon/_3 + \varepsilon/_3 + \varepsilon/_3 = \varepsilon$$

Hence $(\sum_{k=1}^{n} x_k \delta^k)_{n \in \mathbb{N}}$ is p - Cauchy. Since (X, p) is complete, $\sum_{k=1}^{\infty} x_k \delta^k$ converges in (X, p). Let $a = \sum_{k=1}^{\infty} x_k \delta^k \in X$ taken with the p metric. We shall show that $q(x^n - a) \to 0$. Let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $n > m \geq N$ implies that $q(x^m - x^n) < \varepsilon$. Let $n, m \in \mathbb{N}$ be such that $n > m \geq N$. Then $p(\sum_{k=1}^{r} (x_k^n - x_k^m)\delta^k) \leq q(x^n - x^m) < \varepsilon$ for any r. Thus for any such r, $p(\sum_{k=1}^{r} (x_k - x_k^m)\delta^k) \leq \varepsilon$ whenever $m \geq N$. Thus $q(a - x^m) \leq \varepsilon$ for $m \geq N$. Hence $q(x^n - a) \to 0$. Therefore, $(x^n)_{n \in \mathbb{N}}$ converges in (X, q). So (X, q) is complete. Hence (X, q) is a FS($\mathbb{H}$). Since $p \leq q$, by Corollory 4.12, the topolgy induced by p equals the topology induced by q. Hence P_n is continuous in (X, p) for each n so $(\delta^n)_{n \in \mathbb{N}}$ is a Schauder basis for X. #