CHAPTER I

PRELIMINARY

The Algebra of Quaternions

Let 1.i.j and k denote the elements of the standard basis for $\mathbb{R}^4$. The quaternion product on $\mathbb{R}^4$ is then the $\mathbb{R}$-bilinear product with 1 as its multiplicative identity by the formulae $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $jk = i = -kj$ and $ki = j = -ik$. In this thesis we shall denote the $\mathbb{R}$-algebra of all quaternions by " $\mathbb{H}$ ". See [3].

Each quaternions $q = a_0.1 + a_1.i + a_2.j + a_3.k$ ($a_n \varepsilon \mathbb{R}$ for all n) is uniquely expressible in the form Re(q) + Pu(q), where Re(q) $= a_0.1 \varepsilon \mathbb{R}$ and $Pu(q) = a_1.i + a_2.j + a_3.k \varepsilon \mathbb{R}^3$, Re(q) being called the real quaternion part of q and Pu(q) the pure quaternion part of q.

The conjugate $\bar{q}$ of a quaternion q is defined to be the quaternion Re(q) - pu(q). Hence $\overline{a+b} = \bar{a}+\bar{b}$, $\overline{\lambda a} = \lambda\bar{a}$, $\bar{\bar{a}} = a$ and $\overline{ab} = \bar{b}\bar{a}$ for all a, b ε $\mathbb{H}$ and $\lambda \varepsilon \mathbb{R}$. Moreover, a ϵ $\mathbb{R}$ if and only if $\bar{a} = a$, while $Re(a) = \frac{1}{2}(a+\bar{a})$ and $Pu(a) = \frac{1}{2}(a-\bar{a})$. See [3].

Let $a = a_0 + a_1.i + a_2.j + a_3.k$ the non-negative number $|a| = (a\ddot{a})^{\frac{1}{2}} = (\sum_{n=0}^{3} a_n^2)^{\frac{1}{2}}$ is called the absolute value of the quaternion a. If $a \neq 0$, then $|a| \neq 0$ and $\frac{a\bar{a}}{|a|^2} = \frac{\bar{a}a}{|a|^2} = 1$. So we have :

Proposition 1.1 $\mathbb{H}$ is a division ring.

Proof. See [3]. #

Proposition 1.2 $|a.b| = |a||b|$ for all a, b ϵ $\mathbb{H}$.

Proof. See [3]. #

Proposition 1.3 $\mathbb{H}$ is complete with respect to this absolute value.

Proof. Standard. #

Linear algebra over $\mathbb{H}$.

Definition 1.4 A left vector space X over $\mathbb{H}$ is a set of elements in which the operations of addition and scalar multiplication on the left are defined such that 1) X is an abelian group under addition and if $x, y \in X$ and $\alpha, \beta \in \mathbb{H}$ then

2) $\alpha(x + y) = \alpha x + \alpha y$,

3) $\alpha(\beta x) = (\alpha\beta)x$,

4) $1.x = x$,

5) $(\alpha + \beta)x = \alpha x + \beta x$.

From now on unless otherwise specified a vector space over $\mathbb{H}$ means a left vector space over $\mathbb{H}$. Examples of vector spaces over $\mathbb{H}$

(i) $\mathbb{H}^n$ (ii) $S = \{ (z_n)_{n \in \mathbb{N}}$ in $\mathbb{H} \}$

(iii) $C = \{(z_n)_{n \in \mathbb{N}} \mid z_n \in \mathbb{H}$ for all $n \in \mathbb{N}$ and $(z_n)_{n \in \mathbb{N}}$ converges $\}$

Definition 1.5 Let X be a vector space over $\mathbb{H}$ and $\emptyset \neq A \subseteq X$. Then A is said to be a vector subspace or subspace of X if and only if $\alpha x + \beta y \in A$ for all $x, y \in A$ and for all $\alpha, \beta \in \mathbb{H}$.

Definition 1.6 Let X be a vector space over $\mathbb{H}$ and $A \subseteq X$. The span of A, written by $\langle A \rangle$, is the set of all (finite) linear combinations of A.

Definition 1.7 Let X be a vector space over $\mathbb{H}$ and $A \subseteq X$. Then A is called convex if $sA + tA \subseteq A$ for $o \leq s, t \leq 1$, $s + t = 1$: balanced if $ta \in A$ for $|t| \leq 1$; and absorbing if for every $x \in X$ there exists an

$\varepsilon > 0$ such that $tx \in A$ for $|t| < \varepsilon$. For a balanced convex and absorbing set A, define $\|x\| = \inf \{ t > 0 \mid x \in tA \}$. $\|.\|$ is called the gauge of A.

Definition 1.8 Let X be a vector space over $\mathbb{H}$. A vector subspace S of X is called maximal if and only if $S \neq X$ and $X = S + \langle x \rangle$ for some $x \in X$.

Definition 1.9 Let X be a vector space over $\mathbb{H}$. A subset $(V_\alpha)_{\alpha \in I}$ of X is said to be linearly independent if and only if for any finite $V_{\alpha_1}, V_{\alpha_2}, \ldots, V_{\alpha_n}$, $\sum_{m=1}^{n} \beta_m V_{\alpha_m} = 0$ implies that $\beta_m = 0$ for all m.

$(V_\alpha)_{\alpha \in I}$ is linearly dependent if and only if it is not independent.

Definition 1.10 A linear independent set spanning a vector space X is called a basis or base of X.

Definition 1.11 Let X,Y be vector spaces over $\mathbb{H}$ and $f : X \to Y$ a map. Then f is said to be linear map if and only if $f(ax + by) = af(x) + bf(y)$ for all $x, y \in X$ and for all $a, b \in \mathbb{H}$.

Topological Prerequisites

Definition 1.12 A pseudometric on a set X is a real valued function d on the set $X \times X$ such that

(1) $d(x, y) = d(y, x) \geq 0$ for all $x, y \in X$,

(2) $d(x, x) = 0$ for all $x \in X$,

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

If also, (4) $d(x, y) = 0$ implies that $x = y$, d is called a metric.

Defintion 1.13   Let X be a topological space. $S \subseteq X$ is said to be of the first category in X if it is the union of a sequence of closed sets each of which has empty interior. Tf S is not the first cateqory in X it is said to be of the second category in X.

Theorem 1.14   (The Baire category theorem) A complete pseudometric space X is of the second category in itself.

Proof : See [1]. #

Definition 1.15   Let $(x_\delta)_{\delta \in D}$ be a net in a topological space X, $a \in X$. We say $x_\delta \to a$ if and only if for each neighberhood U of a there exists a $\delta \in D$ such that $\delta' \geq \delta$ implies that $x_{\delta'} \in U$.

Theorm 1.16   Let X, Y be topological spaces and $f : X \to Y$. Then f is continuous at $a \in X$ if and only if $x_\delta \to a$ implies that $f(x_\delta) \to f(a)$ for each net $(x_\delta)_{\delta \in D}$ in X.

Proof : See [1]. #

Corollary 1.17 Let T, T′ be topologies on a set X such that for any net $(x_\delta)_{\delta \in D}$ in X, $x_\delta \to a$ in (X, T) implies that $x_\delta \to a$ in (X, T′). then $T \supseteq T'$.

Proof : By Theorem 1.16, the identity map $i : (X, T) \to (X, T)$ is continuous. #

Theorem 1.18   Let   be a collection of topologies for a set X. Then there exists a unique topology, denoted by $v\,\Phi$ (= the set of all unions of finite intersections of members in Φ), such that for any net $(x_\delta)_{\delta \in D}$ in X, $x_\delta \to a$ in $(X, v\Phi)$ if and only if $x_\delta \to a$ in (X, T) for each $T \in \Phi$. For any topological space Z, a function $f : Z \to (X, v\Phi)$ is continuous if and only if $f : Z \to (X, T)$ is continuous for each $T \in \Phi$.

<u>Proof</u> : See [1]. #

<u>Theorem 1.19</u> Let X be a set and $F = \{ f_\alpha : X \to Y_{f_\alpha} \mid \alpha \in I\}$ where for each $\alpha \in I$, Y_{f_α} is a topological space. Then there exists a unique topology on X, denoted by wF, such that for any net $(x_\delta)_{\delta \in D}$ in X, $x_\delta \to a$ in (X, wF) if and only if $f_\alpha(x_\delta) \to f_\alpha(a)$ in Y_{f_α} for each $\alpha \in I$. For any topological space Z, a function $g : Z \to (X, wF)$ is continuous if and only if $f_\alpha \circ g$ is continuous for each $\alpha \in I$.

<u>Proof</u> : Suppose $F = \{f\}$ where $f : X \to Y$. Let $wf = \{ f^{-1}(G) \mid G$ is an open set in $Y\}$. wf is a topology for X and $f : (X, wf) \to Y$ is continuous, so for any net $(x_\delta)_{\delta \in D}$ in X such that $x_\delta \to a$ in (X, wf) we get that $f(x_\delta) \to f(a)$ in Y. Conversely, suppose that $f(x_\delta) \to f(a)$ in Y where $(x_\delta)_{\delta \in D}$ is a net in (X, wf) and $a \in X$. We must show that $x_\delta \to a$ in (X, wf). Let U be an open neighberhood of a in (X, wf). Then $U = f^{-1}(V)$ for some neighberhood V of f(a) in Y. Since $f(x_\delta) \to f(a)$ and $V \ni f(a)$, there exists a $\delta' \in D$ such that $\delta \geq \delta'$ implies that $f(x_\delta) \in V$. Hence $x_\delta \in f^{-1}(f(x_\delta)) \subseteq f^{-1}(V) = U$. Since $U \ni a$ was arbitrary, $x_\delta \to a$ in (X, wf). Let Z be any topological space. Suppose that $g : Z \to (X, wf)$ is continuous. $f \circ g$ is the composition of continuous maps. Then $f \circ g$ is continuous. Conversely, suppose $f \circ g$ is continuous. We must show that $g : Z \to (X, wf)$ is continuous. Let $a \in Z$ be arbitrary. Let $(x_\delta)_{\delta \in D}$ be a net converging to a. Then $f(g(x_\delta)) \to f(g(a))$ therefore,

$g(x_\delta) \to g(a)$ in (X, wf). Hence g is continuous.

In the qeneral case, let $wF = v\{ wf_\alpha \mid f_\alpha \in F, \alpha \in I\}$ = the set of all unions of finite intersections of members of $\bigcup_{\alpha \in I} \{ wf_\alpha\}$. By Theorem 1.18, wF is a topology on X. By Theorem 1.18,

$x_\delta \to a$ in (X,wF) if and only if $x_\delta \to a$ in (X, wf_α) for all $\alpha \in I$. By the one function case, $x_\delta \to a$ in (X, wf_α) for all $\alpha \in I$ if and only if $f(x_\alpha) \to f(a)$ in Y_{f_α} for all $\alpha \in I$. The uniqueness of wF come from Corollary 1.17. We shall now prove the rest of the theorem. Let Z be any topological space. Suppose $g : Z \to (X, wF)$ is continuous. Let $\alpha \in I$. Since f_α is continuous in (X, wf_α), $f_\alpha o g : Z \to Y_{f_\alpha}$ is continuous. Conversely, suppose $f_\alpha o g$ is continuous for each $\alpha \in I$. We must show that $g : Z \to (X, wF)$ is continuous. Let $a \in Z$. Let $(x_\delta)_{\delta \in D}$ be a net in Z such that $x_\delta \to a$. Since $f_\alpha o g$ is continuous for each $\alpha \in I$, $f_\alpha(g(x_\delta)) \to f_\alpha(g(a))$ for each $\alpha \in I$. By the one function case, $g(x_\delta) \to g(a)$ in (X, wf_α) for each $\alpha \in I$. By Theorem 1.18, $g(x_\delta) \to g(a)$ in (X, wF). Hence $g : Z \to (X, wF)$ is continuous. #

<u>Definition 1.20</u> Let $\{X_\alpha\}_{\alpha \in I}$ be a family of topological spaces. <u>The product ΠX_α</u> is the set of all functions $x : I \to \bigcup_{\alpha \in I} X_\alpha$ such that $x_\alpha \in X_\alpha$ for each $\alpha \in I$. [we write x_α for $x(\alpha)$]. For two spaces X, Y we write the product as X x Y. By X^I we mean $\Pi\{X_\alpha \mid \alpha \in I\}$ with $X_\alpha = X$ for each $\alpha \in I$. For each $\alpha \in I$, define $P_\alpha : \Pi X_\beta \to X_\alpha$ by $P_\alpha(x) = x_\alpha$. Given $\alpha \in I$, P_α is called the projection on the α th factor.

<u>Theorem 1.21</u> Let $\{X_\alpha \mid \alpha \in I\}$ be a family of topological spaces. There exists a unique topology on ΠX_α (called the product topology) such that for any net $(x_\delta)_{\delta \in D}$ in ΠX_α, $x^\delta \to a$ if and only if $x^\delta_\alpha \to a_\alpha$ for

each $\alpha \in I$. For any topological space Z and function $g : Z \rightarrow \Pi X_{\alpha}$, g is continuous if and only if $P_{\alpha} o g$ is continuous for each $\alpha \in I$.

Proof : See [1]. #