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ENERGY OF UNITARY CAYLEY GRAPHS AND GCD-GRAPHS

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งานวิจัยนี้มีแนวความคิดมาจากอีลิค ซึ่งหาพลังงานของกราฟเคย์เลย์ยูนิแทรี โดยเราศึกษาค่าลักษณะเฉพาะของกราฟเคย์เลย์ยูนิแทรีบนริงจำกัดสลับที่และกราฟตัวหารร่วมมากบางกราฟตลอดจนคำนวณพลังงานของกราฟเหล่านั้น ยิ่งกว่านั้นเราประยุกต์ผลที่ได้มาคำนวณพลังงานของส่วนเติมเต็มของกราฟเคย์เลย์ยูนิแทรีและพลังงานของกราฟเคย์เลย์ยูนิแทรีที่กำกัดบนส่วนตกค้างกำลังสอง

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This work is based on ideas of Ilić on the energy of unitary Cayley graph. We study the eigenvalues of the unitary Cayley graph of a finite commutative ring and some gcd-graphs and compute their energy. Moreover, we apply these results to obtain the energy of the complement of unitary Cayley graphs and of the restricted unitary Cayley graphs on quadratic residues.

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CONTENTS

CHAPTER I

UNITARY CAYLEY GRAPHS AND THEIR ENERGY

1.1 Unitary Cayley Graphs

The study of algebraic properties of graphs has become an exciting research topic in the last twenty years, leading to many fascinating results and questions. There are many articles on assigning a graph to a ring such as [1], [2] and [20].

Let R be a finite commutative ring with unity $1 \neq 0$. Its unit group of all invertible elements is denoted by $R^\times$. The *unitary Cayley graph of* R, $G_R = \mathrm{Cay}(R, R^\times)$, is the Cayley graph whose vertex set is R and edge set is $\{\{a, b\} : a, b \in R \text{ and } a-b \in R^\times\}$. For some other recent papers on unitary Cayley graphs, we refer the reader to [14], [19], [20] and [21].

For two graphs G and H, their *tensor product* $G \otimes H$ is the graph with vertex-set $V(G)\times V(H)$, where $((u, v), (u', v')) \in E(G\otimes H)$ if and only if $((u, u'), (v, v')) \in E(G) \times E(H)$. Recall that a *local ring* is a commutative ring which has a unique maximal ideal, and a finite commutative ring is a product of finite local rings (Theorem 8.7 of [3]). Furthermore, if R is a local ring with a unique maximal ideal M, then $R^\times = R \smallsetminus M$.

Example 1.1.1. (i) It is easy to see that every field is a local ring with maximal ideal $\{0\}$.

(ii) The ring of integers modulo p^s, $\mathbb{Z}_{p^s} = \mathbb{Z}/p^s\mathbb{Z}$, where p is a prime number and $s \geq 1$, is a local ring with maximal ideal $p\mathbb{Z}/p^s\mathbb{Z}$.

We have the following results.

Proposition 1.1.2. *[2] Let R be a finite commutative ring.*

(i) G_R is a regular graph of degree $|R^\times|$.

(ii) If $R \cong R_1 \times \cdots \times R_s$ is a product of local rings, then $G_R = \bigotimes_{i=1}^{s} G_{R_i}$.

(iii) If R is a local ring with maximal ideal M, then G_R is a complete multipartite graph whose partite sets are the cosets of M.

The *complement of a graph* G, denoted by $\bar{G}$, is the graph with the same vertex set as G such that two vertices of $\bar{G}$ are adjacent if and only if they are not adjacent in G.

Let G be a graph. The *eigenvalues* [*resp. eigenvectors*] *of* G are defined to be the eigenvalues [resp. eigenvectors] of its adjacency matrix $A(G)$. The set of all eigenvalues of G is called the *spectrum of* G. The eigenvalues of G and its complement $\bar{G}$ are studied in the next proposition.

Proposition 1.1.3. *[10, 24] If a graph G with n vertices is k-regular, then G and $\bar{G}$ have the same eigenvectors. The eigenvalue associated with n-vector $\vec{1}_n$, whose entry are all 1, is k for G and $n-k-1$ for $\bar{G}$. If $x \neq \vec{1}$ is an eigenvector of G for eigenvalue λ of G, then its associated eigenvalue in $\bar{G}$ is $-1-\lambda$.*

Akhtar et al. [2] studied and obtained all eigenvalues of the unitary Cayley graph G_R. We now present these eigenvalues with multiplicities. As is standard, if $\lambda_1, \dots, \lambda_k$ are eigenvalues of a graph G of respective multiplicities $m_1, \dots, m_k$, we use the notation $\operatorname{Spec} G = \begin{pmatrix} \lambda_1 & \dots & \lambda_k \\ m_1 & \dots & m_k \end{pmatrix}$ to describe the spectrum of G.

Proposition 1.1.4. *Let R be a finite local ring with maximal ideal M of size m. Then*

$$\operatorname{Spec} G_R = \begin{pmatrix} |R^\times| & -m & 0 \\ 1 & \frac{|R|}{m} - 1 & \frac{|R|}{m}(m-1) \end{pmatrix} = \begin{pmatrix} |R^\times| & -m & 0 \\ 1 & \frac{|R^\times|}{m} & \frac{|R|}{m}(m-1) \end{pmatrix}.$$

In particular, if F is the field with q elements, then

$$\operatorname{Spec} G_F = \begin{pmatrix} q-1 & -1 \\ 1 & q-1 \end{pmatrix} = \begin{pmatrix} |F^\times| & -1 \\ 1 & |F^\times| \end{pmatrix}.$$

Proof. Since R is a local ring with maximal ideal M, by Proposition 1.1.2 (iii) G_R is a complete multipartite graph with $|R|/m$ partite sets, each of size $m = |M|$. In view of the regularity of G_R, by Proposition 1.1.3, if $\lambda_1, \dots, \lambda_n$ are eigenvalues for $A(G_R)$, that is not associated with $\vec{1}$, then $-1-\lambda_1, \dots, -1-\lambda_n$ are eigenvalues for $A(\bar{G}_R)$. However, $\bar{G}_R$ is a disjoint union of $|R|/m$ cliques, each of size m. For the eigenvector $\vec{1}$, its eigenvalue for $\bar{G}_R$ is $|R| - |R^\times| - 1 = m - 1$, so its eigenvalue for G_R is $|R^\times|$. Therefore, $\operatorname{Spec} G_R = \begin{pmatrix} |R^\times| & -m & 0 \\ 1 & \frac{|R|}{m} - 1 & \frac{|R|}{m}(m-1) \end{pmatrix}$. □

1.2 Energy of Unitary Cayley Graphs

We first recall another fact.

Proposition 1.2.1. *Let G and H be graphs. Suppose that $\lambda_1, \dots, \lambda_n$ are the eigenvalues of G and $\mu_1, \dots, \mu_m$ are the eigenvalues of H (repetition is possible). Then the eigenvalues of $G \otimes H$ are $\lambda_i \mu_j$, where $1 \le i \le n$ and $1 \le j \le m$.*

Proof. The result follows immediately from the well known fact that $A(G \otimes H)$ is the tensor product of the matrices $A(G)$ and $A(H)$, and that the eigenvalues of a tensor product of matrices may be found by taking products of the eigenvalues of the factors. $\square$

Applying Propositions 1.1.2 and 1.2.1, we obtain the following lemma.

Lemma 1.2.2. *Let R be a finite commutative ring, where $R = R_1 \times R_2 \times \cdots \times R_s$ and R_i is a local ring with maximal ideal M_i of size m_i for all $i \in \{1, 2, \ldots, s\}$. Then the eigenvalues of G_R are*

(i) $(-1)^{|C|} \dfrac{|R^\times|}{\prod_{j \in C} |R_j^\times|/m_j}$ *with multiplicity* $\prod_{j \in C} |R_j^\times|/m_j$ *for all subsets C of $\{1, 2, \ldots, s\}$, and*

(ii) 0 *with multiplicity* $|R| - \prod_{i=1}^{s} \left(1 + \dfrac{|R_i^\times|}{m_i}\right)$.

The sum of absolute values of all eigenvalues of a graph G is called the *energy of G* and denoted by Engy G. The energy is a graph parameter stemming from the Hückel molecular orbital approximation for the total π-electron energy (for survey on molecular graph energy see e.g., [6] and [12]). This concept was introduced by Gutman [11]. Later, the energy of graph was studied intensively in many literatures (see e.g., [12], [13], [17] and [18]). Note that it follows directly from Proposition 1.2.1 that:

Proposition 1.2.3. *Let G and H be graphs. Then*

$$\operatorname{Engy} G \otimes H = \operatorname{Engy} G \ \operatorname{Engy} H.$$

We next proceed to compute the energy of the unitary Cayley graph of a finite commutative ring R.

Theorem 1.2.4. *Let R be a finite commutative ring, where $R = R_1 \times R_2 \times \cdots \times R_s$ and R_i is a local ring with maximal ideal M_i of size m_i for all $i \in \{1, 2, \ldots, s\}$. Then*

$$\operatorname{Engy} G_R = 2^s |R^\times|.$$

Proof. Recall from Proposition 1.1.2 (ii) that $G_R = \bigotimes_{i=1}^s G_{R_i}$. In addition, $\operatorname{Engy} G_{R_i} = 2|R_i^\times|$ for all $i \in \{1, 2, \ldots, s\}$ by Proposition 1.1.4. Thus, Proposition 1.2.3 implies

$$\operatorname{Engy} G_R = \prod_{i=1}^s \operatorname{Engy} G_{R_i} = 2^s \prod_{i=1}^s |R_i^\times| = 2^s |R^\times|$$

as desired. □

Remark. The above result generalizes Theorem 2.3 of Ilić [14] on the unitary Cayley graph $\operatorname{Cay}(\mathbb{Z}_n, \mathbb{Z}_n^\times)$. His proof used some results on eigenvalues from [19] and the fact that this graph is circulant and applied the Gauss sum for computing its energy.

A graph G with n vertices is said to be *hyperenergetic* if its energy exceeds the energy of the complete graph K_n, or equivalently if $\operatorname{Engy} G > 2n - 2$. Hyperenergetic graphs are important because molecular graphs with maximum energy pertain to maximality stable π-electron systems. It has been proved in [6] that for every $n \geq 8$, there always exists a hyperenergetic graph of order n. Moreover, Ilić [14] characterized all hyperenergetic unitary Cayley graphs when $R = \mathbb{Z}_n$.

Let R be a finite commutative ring, where $R = R_1 \times R_2 \times \cdots \times R_s$ and R_i is a local ring with maximal ideal M_i of size m_i for all $i \in \{1, 2, \ldots, s\}$. Then $R^\times = R_1^\times \times R_2^\times \times \cdots \times R_s^\times$. Since each R_i is a local ring, $R_i^\times = R_i \smallsetminus M_i$ for all i. Thus, we have

$$|R^\times| = \prod_{i=1}^{s}(|R_i| - m_i) = |R| \prod_{i=1}^{s} \left(1 - \frac{1}{|R_i|/m_i}\right).$$

Recall that $|R_i|/m_i \geq 2$ for all $i \in \{1, 2, \ldots, s\}$. It follows that G_R is hyperenergetic if and only if $2^{s-1}|R^\times| \geq |R|$, which is equivalent to have the inequality

$$2^{s-1} \geq \frac{|R|}{|R^\times|} = \frac{|R|}{|R| \prod_{i=1}^{s} \frac{|R_i|/m_i - 1}{|R_i|/m_i}} = \frac{\prod_{i=1}^{s} |R_i|/m_i}{\prod_{i=1}^{s} (|R_i|/m_i - 1)}. \tag{1.2.1}$$

We conclude criteria to determine if G_R is hyperenergetic as follows.

Theorem 1.2.5. *Let R be a finite commutative ring, where $R = R_1 \times R_2 \times \cdots \times R_s$ and R_i is a local ring with maximal ideal M_i of size m_i for all $i \in \{1, 2, \ldots, s\}$. Assume that*

$$|R_1|/m_1 \leq |R_2|/m_2 \leq \cdots \leq |R_s|/m_s.$$

(i) For $s = 1$, G_R is not hyperenergetic.

(ii) For $s = 2$, G_R is hyperenergetic if and only if $|R_1|/m_1 \geq 3$ and $|R_2|/m_2 \geq 4$.

(iii) For $s \geq 3$, G_R is hyperenergetic if and only if $(|R_{s-2}|/m_{s-2} \geq 3)$ or $(|R_{s-1}|/m_{s-1} \geq 3$ and $|R_s|/m_s \geq 4)$.

Proof. Suppose that G_R is hyperenergetic. It follows from inequality (1.2.1) that $s \geq 2$. If $s = 2$, we have

$$2 \geq \frac{|R_1|/m_1}{(|R_1|/m_1 - 1)} \frac{|R_2|/m_2}{(|R_2|/m_2 - 1)},$$

and so $|R_1|/m_1 \geq 3$ and $|R_2|/m_2 \geq 4$.

Next, we assume that $s \geq 3$ and $|R_{s-2}|/m_{s-2} < 3$. Then $|R_i|/m_i = 2$ for all $i \in \{1, 2, \ldots, s-2\}$. By (1.2.1), we get

$$2 \geq \frac{|R_{s-1}|/R_{s-1}}{(|R_{s-1}|/m_{s-1} - 1)} \frac{|R_s|/m_s}{(|R_s|/m_s - 1)}.$$

Hence, we obtain the same conclusion $|R_{s-1}|/m_{s-1} \geq 3$ and $|R_s|/m_s \geq 4$ as before. Another direction easily follows from substitutions and computations using inequality (1.2.1). □

Example 1.2.6.

1. Let $R = \mathbb{Z}[i]/(2+i)^3$. We know that $|R| = N(2+i)^3 = 125$, $R^\times \cong \mathbb{Z}_{5^3-5^2}$ and $|R^\times| = 100$. Then $\operatorname{Engy} G_R = 2(100) = 200 \leq 248 = 2(125) - 2 = 2|R| - 2$ which shows that G_R is not hyperenergetic.

2. Let $R = \mathbb{Z}[i]/(5)^2 \cong \mathbb{Z}[i]/(2+i)^2 \times \mathbb{Z}[i]/(2-i)^2 \cong R_1 \times R_2$. Then $|R| = N(5)^2 = 625$, $R^\times \cong \mathbb{Z}_{5^2-5} \times \mathbb{Z}_{5^2-5}$ which make $|R^\times| = 20 \times 20 = 400$ and $m_1 = m_2 = \frac{|\mathbb{Z}[i]/(2+i)^2|}{|\mathbb{Z}[i]/(2+i)|} = 5$. Hence, $\frac{|R_1|}{m_1} = \frac{|R_2|}{m_2} = 5$. By Theorem 1.2.4 we have $\operatorname{Engy} G_R = 2^2(400) = 1,600 > 1,248 = 2(625) - 2 = 2|R| - 2$. Therefore, G_R is hyperenergetic.

3. Let $R = \mathbb{Z}[i]/(1+i)^3(2+i)^2 \cong \mathbb{Z}[i]/(1+i)^3 \times \mathbb{Z}[i]/(2+i)^2 \cong R_1 \times R_2$. Then $|R| = N(1+i)^3 \times N(2+i)^2 = 8 \times 25 = 200$, $R^\times \cong \mathbb{Z}_4 \times \mathbb{Z}_{5^2-5}$ which make $|R^\times| = 4 \times 20 = 80$, $m_1 = \frac{|\mathbb{Z}[i]/(1+i)^3|}{|\mathbb{Z}[i]/(1+i)|} = 4$ and $m_2 = \frac{|\mathbb{Z}[i]/(2+i)^2|}{|\mathbb{Z}[i]/(2+i)|} = 5$. Hence, $\frac{|R_1|}{m_1} = \frac{8}{4} = 2$ and $\frac{|R_2|}{m_2} = 5$. By Theorem 1.2.4 we have $\operatorname{Engy} G_R = 2^2(80) = 320 < 398 = 2(200) - 2 = 2|R| - 2$. Hence, G_R is not hyperenergetic.

4. Let $R = \mathbb{Z}[i]/(2+3i)(5) \cong \mathbb{Z}[i]/(2+3i) \times \mathbb{Z}[i]/(2+i) \times \mathbb{Z}[i]/(2-i) \cong R_1 \times R_2 \times R_3$. Then $|R| = N(2+3i) \times N(2+i) \times N(2-i) = 13 \times 5 \times 5 = 325$,

$R^{\times} \cong \mathbb{Z}_{12} \times \mathbb{Z}_4 \times \mathbb{Z}_4$ which make $|R^{\times}| = 12 \times 4 \times 4 = 192$ and $m_1 = m_2 = m_3 = 1$. Hence, $\frac{|R_1|}{m_1} = 13$ and $\frac{|R_2|}{m_2} = \frac{|R_3|}{m_3} = 5$. By Theorem 1.2.4 we have $\operatorname{Engy} G_R = 2^3(192) = 1,536 > 648 = 2(325) - 2 = 2|R| - 2$. Thus G_R is hyperenergetic.

5. Let $R = \mathbb{Z}[i]/(1+i)(5) \cong \mathbb{Z}[i]/(1+i) \times \mathbb{Z}[i]/(2+i) \times \mathbb{Z}[i]/(2-i) \cong R_1 \times R_2 \times R_3$. Then $|R| = N(1+i) \times N(2+i) \times N(2-i) = 2 \times 5 \times 5 = 50$, $R^{\times} \cong \mathbb{Z}_1 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ which make $|R^{\times}| = 4 \times 4 = 16$ and $m_1 = m_2 = m_3 = 1$. Hence, $\frac{|R_1|}{m_1} = 2$ and $\frac{|R_2|}{m_2} = \frac{|R_3|}{m_3} = 5$. By Theorem 1.2.4 we have $\operatorname{Engy} G_R = 2^3(16) = 128 > 98 = 2(50) - 2 = 2|R| - 2$ which shows that G_R is not hyperenergetic.

Remark. We can use Theorem 1.2.5 to determine the above example directly.

CHAPTER II

GCD-GRAPHS AND COMPLEMENT OF UNITARY CAYLEY GRAPHS

2.1 GCD-Graphs

Throughout this section, we consider a unique factorization domain D. Let $c \in D$ be a nonzero nonunit element. We have the quotient ring $D/(c) = \{x + (c) : x \in D\}$ is a commutative ring. Assume that this ring is finite. Let $\mathcal{C}$ be a set of proper divisors of c. Define the *gcd-graph*, $D_c(\mathcal{C})$, to be a graph whose vertex set is $D/(c)$ and edge set is

$$\{\{x + (c), y + (c)\} : x, y \in D \text{ and } \gcd(x - y, c) \in \mathcal{C}\}.$$

The gcd considered here is unique up to associate. We refer the reader to basic abstract algebra textbooks such as [9] for more details on quotient rings and the gcd of elements in a unique factorization domain. It is easy to see that $D_c(\{1\}) = G_{D/(c)} = \text{Cay}(D/(c), D/(c)^\times)$ previously studied in the first chapter.

The definition above generalizes gcd-graphs or integral circulant graphs (i.e., its adjacency matrix is circulant and all eigenvalues are integers) defined over $\mathbb{Z}$ (see [19] and [23]). For further development on integral circulant graphs, see [5], [15], [16] and [4]. Note that the gcd-graphs are circulant if and only if $D/(c)$ is cyclic under addition. This is the case for $D = \mathbb{Z}$ and we can apply the Gauss sum to compute the energy [23]. However, $D/(c)$ may not be cyclic in general. Fortunately, Theorem 1.2.4 can be used to determine the energy of our gcd-graphs.

Theorem 2.1.1. *Let $c = p_1^{a_1} \dots p_n^{a_n}$ be factored as a product of irreducible elements and assume that $D/(c)$ is finite. For $1 \le i \le n$, if $a_i = 1$, then we have*

$$\text{Engy}\, D_c(\{1, p_i\}) = 2^{n-1}|D/(p_i)||D/(c/p_i)^\times|.$$

Proof. Let $1 \le i \le n$ and assume that $a_i = 1$. We first observe that the edge set

$$\begin{aligned} E(D_c(1,p_i)) &= \{\{x+(c), y+(c)\} : x, y \in D \text{ and } \gcd(x-y, c) = 1 \text{ or } p_i\} \\ &= \{\{x+(c), y+(c)\} : x, y \in D \text{ and } \gcd(x-y, c/p_i) = 1\} \\ &\cong \{\{(x+(p_i), x+(c/p_i)), (y+(p_i), y+(c/p_i))\} : x, y \in D \text{ and} \\ &\qquad\qquad \gcd(x-y, c/p_i) = 1\}. \end{aligned}$$

Thus, the graph $D_c(\{1, p_i\})$ is isomorphic to the graph $\mathring{K}_{|D/(p_i)|} \otimes G_{D/(c/p_i)}$, where $\mathring{K}_{|D/(p_i)|}$ is the $|D/(p_i)|$-complete graph with a loop on each vertex and $G_{D/(c/p_i)}$ denotes the unitary Cayley graph of the ring $D/(c/p_i)$. Since $A(\mathring{K}_{|D/(p_i)|})$ is the $|D/(p_i)| \times |D/(p_i)|$ matrix whose entry are all 1, we have

$$\text{Spec}\, \mathring{K}_{|D/(p_i)|} = \begin{pmatrix} |D/(p_i)| & 0 \\ 1 & |D/(p_i)| - 1 \end{pmatrix}.$$

Hence,

$$\text{Engy}\, D_c(\{1, p_i\}) = |D/(p_i)|\, \text{Engy}\, G_{D/(c/p_i)} = 2^{n-1}|D/(p_i)||D/(c/p_i)^\times|$$

by Theorem 1.2.4. $\square$

The Cartesian product of two graphs G and H is the graph $G \square H$ such that $V(G \square H) = V(G) \times V(H)$ and any two vertices (u, u') and (v, v') are adjacent in

$G\square H$ if and only if either $u = v$ and u' is adjacent with v' in H, or $u' = v'$ and u is adjacent with v in G. Next, we recall that $A(G\square H) = A(G) \otimes I + I \otimes A(H)$ which implies our next proposition.

Proposition 2.1.2. *Let G and H be two graphs. Suppose that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of G and $\mu_1, \ldots, \mu_m$ are the eigenvalues of H (repetition is possible). Then the eigenvalues of the graph $G\square H$ are $\lambda_i + \mu_j$, where $1 \leq i \leq n$ and $1 \leq j \leq m$.*

This proposition results in the computation of energy for another gcd-graph.

Lemma 2.1.3. *Let D be a UFD. If p_1 and p_2 are non-associate primes in D such that $D/(p_1)$ and $D/(p_2)$ are finite, then*

$$\operatorname{Engy}(G_{D/(p_1)}\square G_{D/(p_2)}) = 2^2|D/(p_1)^\times||D/(p_2)^\times|.$$

Proof. Recall that $D/(p_1)$ and $D/(p_2)$ are finite fields. Then by Proposition 1.1.4, we have $\operatorname{Spec} G_{D/(p_1)} = \begin{pmatrix} |D/(p_1)^\times| & -1 \\ 1 & |D/(p_1)^\times| \end{pmatrix}$ and

$\operatorname{Spec} G_{D/(p_2)} = \begin{pmatrix} |D/(p_2)^\times| & -1 \\ 1 & |D/(p_2)^\times| \end{pmatrix}$.

Thus, we obtain from Proposition 2.1.2 that $\operatorname{Spec}(G_{D/(p_1)}\square G_{D/(p_2)})$ is given by

$$\begin{pmatrix} |D/(p_1)^\times| + |D/(p_2)^\times| & |D/(p_1)^\times| - 1 & |D/(p_2)^\times| - 1 & -2 \\ 1 & |D/(p_2)^\times| & |D/(p_1)^\times| & |D/(p_1)^\times||D/(p_2)^\times| \end{pmatrix}.$$

Consequently,

$$\begin{aligned}\operatorname{Engy}(G_{D/(p_1)}\Box G_{D/(p_2)}) &= (|D/(p_1)^{\times}| + |D/(p_2)^{\times}|) + |D/(p_2)^{\times}|(|D/(p_1)^{\times}| - 1)\\ &\quad + |D/(p_1)^{\times}|(|D/(p_2)^{\times}| - 1) + 2|D/(p_1)^{\times}||D/(p_2)^{\times}|\\ &= 2^2|D/(p_1)^{\times}||D/(p_2)^{\times}|\end{aligned}$$

as desired. □

Theorem 2.1.4. *Let $c = p_1 \dots p_k p_{k+1}^{a_{k+1}} \dots p_n^{a_n}$ be factored as a product of irreducible elements, where $a_l > 1$ for all $l \in \{k+1, \dots, n\}$. Assume that $D/(c)$ is finite. For $1 \le i < j \le k$, we have*

$$\operatorname{Engy} D_c(\{p_i, p_j\}) = 2^n|D/(c)^{\times}|.$$

Proof. Let $1 \le i < j \le k$. Note that

$$\begin{aligned}E(D_c(\{p_i, p_j\})) &= \{\{x + (c), y + (c)\} : x, y \in D \text{ and } \gcd(x - y, c) = p_i \text{ or } p_j\}\\ &= \{\{x + (c), y + (c)\} : x, y \in D \text{ and } \gcd(x - y, c/p_ip_j) = 1 \text{ and }\\ &\qquad \gcd(x - y, c) = p_i \text{ or } p_j\}\\ &\cong \{\{(x + (c/p_ip_j), x + (p_ip_j)), (y + (c/p_ip_j), y + (p_ip_j))\} : x, y \in D \text{ and }\\ &\qquad \gcd(x - y, c/p_ip_j) = 1 \text{ and } \gcd(x - y, c) = p_i \text{ or } p_j\}.\end{aligned}$$

Then $D_c(\{p_i, p_j\})$ is isomorphic to $G_{D/(c/p_ip_j)} \otimes G$, where G is the graph whose vertex set $V(G) = D/(p_ip_j) \cong D/(p_1) \times D/(p_2)$ by the Chinese remainder theorem, and edge set

$$\begin{aligned} E(G) &= \{\{x+(p_ip_j), y+(p_ip_j)\} : x, y \in D \text{ and } \gcd(x-y, c) = p_i \text{ or } p_j\} \\ &= \{\{(x+(p_i), x+(p_j)), (y+(p_i), y+(p_j))\} : x, y \in D \text{ and} \\ &\qquad\qquad x - y \in (p_i, p_j) - (p_ip_j)\} \\ &\cong \{\{(x+(p_i), x+(p_j)), (y+(p_i), y+(p_j))\} : x, y \in D \text{ and} \\ &\qquad [(x-y \in (p_i) \text{ and } x-y \notin (p_j)) \text{ or } (x-y \in (p_j) \text{ and } x-y \notin (p_i))]\}. \end{aligned}$$

This implies that the graph G is isomorphic to the product $G_{D/(p_i)} \square G_{D/(p_j)}$. Hence,

$$\begin{aligned} \operatorname{Engy} D_c(\{p_i, p_j\}) &= \operatorname{Engy} G_{D/(c/p_ip_j)} \operatorname{Engy} G \\ &= \operatorname{Engy} G_{D/(c/p_ip_j)} \operatorname{Engy}(G_{D/(p_i)} \square G_{D/(p_j)}) \\ &= (2^{n-2}|D/(c/p_ip_j)^\times|)(2^2|D/(p_i)^\times||D/(p_j)^\times|) \\ &= 2^n|D/(c)^\times| \end{aligned}$$

by Theorem 1.2.4 and Proposition 2.1.2. □

Remark. Theorems 2.1.1 and 2.1.4 extend the work in Section 4 of [14]. Again, our computational approach is different and straightforward.

2.2 Complement of Unitary Cayley Graphs

This final section covers the energy of the complement of unitary Cayley graphs. Recall from Proposition 1.1.3 that the spectrum of $\bar{G}_R$ consists of eigenvalues $|R|-|R^\times|-1, -1-\lambda_2, \ldots, -1-\lambda_{|R|}$, where λ_i is an eigenvalue of G_R not associated to $\vec{1}$ for all $i \in \{2, 3, \ldots, |R|\}$.

Theorem 2.2.1. *Let R be a finite ring, where $R = R_1 \times R_2 \times \cdots \times R_s$, and R_i is a local ring with maximal ideal M_i of size m_i for all $i \in \{1, 2, \dots, s\}$. Then*

$$\operatorname{Engy} \bar{G}_R = 2|R| - 2 + (2^s - 2)|R^\times| - \prod_{i=1}^{s} |R_i|/m_i + \prod_{i=1}^{s} (2 - |R_i|/m_i).$$

Proof. Let $\lambda_1 = |R^\times|, \lambda_2, \dots, \lambda_{|R|}$ be the eigenvalues of G_R and $N = \{1, 2, \dots s\}$. By Lemma 1.2.2 (i), we first verify the sum

$$\begin{aligned}
\sum_{\substack{\lambda_i \neq 0 \\ i \neq 1}} |\lambda_i + 1| &= \sum_{\substack{C \subseteq N \\ C \neq \varnothing}} \prod_{j \in C} \frac{|R_j^\times|}{m_j} \left| (-1)^{|C|} \frac{|R^\times|}{\prod_{j \in C} |R_j^\times|/m_j} + 1 \right| \\
&= \sum_{\substack{C \subseteq N \\ C \neq \varnothing}} \left| (-1)^{|C|} |R^\times| + \prod_{j \in C} |R_j^\times|/m_j \right| \\
&= \sum_{\substack{C \subseteq N \\ C \neq \varnothing}} |R^\times| + \sum_{\substack{C \subseteq N \\ C \neq \varnothing}} (-1)^{|C|} \prod_{j \in C} |R_j^\times|/m_j \\
&= (2^s - 1)|R^\times| + (-1 + \prod_{i=1}^{s} (1 - |R_i^\times|/m_i)) \\
&= (2^s - 1)|R^\times| - 1 + \prod_{i=1}^{s} (2 - |R_i|/m_i) \qquad (2.2.1)
\end{aligned}$$

because $|R_i^\times| = |R_i \setminus M_i| = |R_i| - m_i$ for all $i \in \{1, 2, \dots, s\}$. Hence,

$$\begin{aligned}
\operatorname{Engy} \bar{G}_R &= (|R| - |R^\times| - 1) + \sum_{i \neq 1} |-1 - \lambda_i| \\
&= (|R| - |R^\times| - 1) + \sum_{i \neq 1} |\lambda_i + 1| \\
&= (|R| - |R^\times| - 1) + \sum_{\substack{i \neq 1 \\ \lambda_i \neq 0}} |\lambda_i + 1| + \operatorname{nullity} G_R,
\end{aligned}$$

where $\operatorname{nullity} G_R$ is the multiplicity of zero as the eigenvalue. Thus, Lemma 1.2.2 (ii) implies that

$$\operatorname{nullity} G_R = |R| - \prod_{i=1}^{s} \left(1 + \frac{|R_i^\times|}{m_i}\right) = |R| - \prod_{i=1}^{s} |R_i|/m_i.$$

Together with Eq. (2.2.1), we finally reach

$$\begin{aligned}\operatorname{Engy} \bar{G}_R &= (|R| - |R^\times| - 1) + \Big((2^s - 1)|R^\times| - 1 + \prod_{i=1}^{s}(2 - |R_i|/m_i)\Big) \\ &\quad + \Big(|R| - \prod_{i=1}^{s}|R_i|/m_i\Big) \\ &= 2|R| - 2 + (2^s - 2)|R^\times| - \prod_{i=1}^{s}|R_i|/m_i + \prod_{i=1}^{s}(2 - |R_i|/m_i).\end{aligned}$$

This completes the proof. □

Corollary 2.2.2. *Let D be a UFD and $c \in D$. Assume that $c = p_1^{a_1} p_2^{a_2} \dots p_s^{a_s}$ is factored as a product of irreducible elements and $D/(c)$ is finite. Then*

$$\operatorname{Engy}(\bar{G}_{D/(c)}) = 2|D/(c)| - 2 + (2^s - 2)|D/(c)^\times| - \prod_{i=1}^{s} D/(p_i) + \prod_{i=1}^{s}(2 - |D/(p_i)|).$$

Proof. The Chinese remainder theorem implies that

$$D/(c) \cong D/(p_1^{a_1}) \times D/(p_2^{a_2}) \times \cdots \times D/(p_s^{a_s}).$$

Moreover, we have the isomorphism

$$D/(p_l^{a_l})/(p_l)/(p_l^{a_l}) \cong D/(p_l)$$

for all $l \in \{1, 2, \dots, s\}$. Hence, Theorem 2.2.1 directly gives the desired result. □

Remark. The above corollary generalizes Theorem 3.1 of [14].

CHAPTER III

ENERGY OF THE RESTRICTED UNITARY CAYLEY GRAPHS ON QUADRATIC RESIDUES

This final chapter consists of two sections. They present results on the energy of the restricted unitary Cayley graphs on quadratic residues of a positive integer $n > 1$ and of a non-constant polynomial f over finite fields. The computations make use of the energy of the unitary Cayley graphs discovered in Section 1.2.

3.1 Quadratic Residues of n

Let $n > 1$ be a positive integer. The unitary Cayley graph of $\mathbb{Z}_n$, $G_n := G_{\mathbb{Z}_n} = \mathrm{Cay}(\mathbb{Z}_n, \mathbb{Z}_n^\times)$, is the Cayley graph whose vertex set is $\mathbb{Z}_n$ and edge set is $\{\{a, b\} : a, b \in \mathbb{Z}_n \text{ and } a - b \in \mathbb{Z}_n^\times\}$. Here, $\mathbb{Z}_n^\times$ denotes the unit group of $\mathbb{Z}_n$.

Consider the exact sequence of groups

$$1 \longrightarrow K_n \longrightarrow \mathbb{Z}_n^\times \xrightarrow{\theta} (\mathbb{Z}_n^\times)^2 \longrightarrow 1, \tag{3.1.1}$$

where $\theta : a \mapsto a^2$ is the square mapping on $\mathbb{Z}_n^\times$ with kernel $K_n = \{a \in \mathbb{Z}_n^\times : a^2 = 1\}$ and $(\mathbb{Z}_n^\times)^2 = \{a^2 : a \in \mathbb{Z}_n^\times\}$ is the set of quadratic residues of n. Let $T_n = K_n(\mathbb{Z}_n^\times)^2$. Define the subgraph H_n of the unitary Cayley graphs by $H_n = \mathrm{Cay}(\mathbb{Z}_n, T_n)$, in which two vertices are adjacent if and only if their difference is in T_n. Observe that H_n is undirected. The quadratic unitary Cayley graph

$\mathrm{Cay}(\mathbb{Z}_n, (\mathbb{Z}_n^\times)^2)$ was introduced by Beaudrap [7]. He bounded the diameter of such graphs and characterized the conditions on n for $\mathrm{Cay}(\mathbb{Z}_n, (\mathbb{Z}_n^\times)^2)$ to be perfect. However, sometimes his graphs are directed.

In what follows, we study the structure of the graph H_n and obtain its eigenvalues. In addition, we compute the energy of H_n in our final theorem.

Let p be an odd prime and $s \geq 1$. We recall that $\mathbb{Z}_{p^s}^\times$ is cyclic, so it has a unique element of order two, namely -1. Then $K_{p^s} = \{a \in \mathbb{Z}_{p^s} : a^2 = 1\} = \{1, -1\}$. Thus, $T_{p^s} = \pm(\mathbb{Z}_{p^s}^\times)^2$, and hence Lemma 2 of [7] gives the next lemma.

Lemma 3.1.1. *For $s \geq 1$ and an odd prime p, we have*

$$H_{p^s} \cong H_p \otimes \overset{\circ}{K}_{p^{s-1}},$$

where $\overset{\circ}{K}_{p^{s-1}}$ is the p^{s-1}-complete graph with a loop on each vertex.

Let $G = (V, E)$ be a regular graph with v vertices and degree k. G is said to be *strongly regular* if there are also integers λ and μ such that:

(i) every two adjacent vertices have λ common neighbours, and

(ii) every two non-adjacent vertices have μ common neighbours.

A graph of this kind is sometimes said to be a strongly regular graph with parameters (v, k, λ, μ). We can explicitly determine the eigenvalues of a strongly regular graph as follows:

Lemma 3.1.2. *[10] A strongly regular graph with parameters (v, k, λ, μ) has exactly three eigenvalues:*

(i) k *whose multiplicity is* 1*,*

(ii) $\frac{1}{2}[(\lambda-\mu)+\sqrt{(\lambda-\mu)^2+4(k-\mu)}]$ *whose multiplicity is* $\frac{1}{2}[(v-1)-\frac{2k+(v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}}]$*, and*

(iii) $\frac{1}{2}[(\lambda-\mu)-\sqrt{(\lambda-\mu)^2+4(k-\mu)}]$ *whose multiplicity is* $\frac{1}{2}[(v-1)+\frac{2k+(v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}}]$*.*

Let r be a prime power such that $r \equiv 1 \mod 4$. Note that this implies that the unique finite field of order r, $\mathbb{F}_r$, contains a square root of -1. The *Paley graph* is the graph whose vertex set is $\mathbb{F}_r$ and edge set is $\{\{a,b\} : a,b \in \mathbb{F}_r \text{ and } a-b \in (\mathbb{F}_r^\times)^2\}$.

Lemma 3.1.3. *The Paley graph over the finite field* $\mathbb{F}_r$ *is strongly regular with parameters* $(r, \frac{r-1}{2}, \frac{r-5}{4}, \frac{r-1}{4})$*.*

Proof. Define the map $\chi : \mathbb{F}_r \to \{-1,0,1\}$ by

$$\chi(a) = \begin{cases} 0, & \text{if } a = 0; \\ 1, & \text{if } a \in (\mathbb{F}_r^\times)^2; \\ -1, & \text{otherwise.} \end{cases}$$

Clearly, χ is a homomorphism from $\mathbb{F}_r^\times$ onto $\{-1,1\}$. Note that $\chi(a-b)=1$ if and only if a is adjacent to b. Let $a,b \in \mathbb{F}_r$. To count the number of x in $\mathbb{F}_r$ such that $\chi(a-x)=\chi(b-x)$, we first consider

$$\sum_{x\neq a,b} \chi[(a-x)(b-x)] = \sum_{\substack{x\neq a,b \\ \chi(a-x)=\chi(b-x)}} 1 - \sum_{\substack{x\neq a,b \\ \chi(a-x)\neq\chi(b-x)}} 1.$$

For $x \neq b$, $\chi(b-x) = \chi(b-x)^{-1}$, so the sum on the left can be written as

$$\sum_{x \neq a,b} \chi\left(\frac{a-x}{b-x}\right) = \sum_{x \neq a,b} \chi\left(1 + \frac{a-b}{b-x}\right) = \sum_{x \neq 0,1} \chi(x) = -1,$$

since exactly half of the non-zero elements of $\mathbb{F}_r$ are quadratic residues. This same reason also gives us that $k = \frac{r-1}{2}$. Now suppose that a adjacent to b. Then $\sum_{x \neq a,b} \chi(a-x) = \sum_{x \neq a,b} \chi(b-x) = -1$. We have four equations in four unknowns: define α to be the number of times that $\chi(x-a) = 1$ and $\chi(x-b) = 1$, β to be the number of times that $\chi(x-a) = 1$ and $\chi(x-b) = -1$ and γ and δ similarly in case $\chi(x-a) = -1$. Thus, $\alpha + \beta$ is just the total number of times $\chi(x-a) = 1$, which is $\frac{r-3}{2}$, and $\beta + \gamma$ is the number of times $\chi(x-a)$ and $\chi(x-b)$ have different signs, which is $\frac{r-1}{2}$. Solving these and the other two equations give $\lambda = \frac{1}{4}(r-5)$. On the other hand, if a is not adjacent to b, then we can solve again to get $\mu = \frac{1}{4}(r-1)$. $\square$

We know that the adjacency matrix of the p^{s-1}-complete graph with a loop on each vertex, $\overset{\circ}{K}_{p^{s-1}}$, is the $p^{s-1} \times p^{s-1}$ matrix of all 1s, and hence

$$\operatorname{Spec} \overset{\circ}{K}_{p^{s-1}} = \begin{pmatrix} p^{s-1} & 0 \\ 1 & p^{s-1}-1 \end{pmatrix}.$$

Moreover, if $p \equiv 1 \mod 4$, then -1 is a quadratic residue of p, so $T_p = (\mathbb{Z}_p^\times)^2$. Thus, H_p is the Paley graph which is strongly regular with parameters $(p, (p-1)/2, (p-5)/4, (p-1)/4)$ by Lemma 3.1.3. Hence, from Lemma 3.1.2,

$$\operatorname{Spec} H_p = \begin{pmatrix} \frac{p-1}{2} & \frac{-1+\sqrt{p}}{2} & \frac{-1-\sqrt{p}}{2} \\ 1 & \frac{p-1}{2} & \frac{p-1}{2} \end{pmatrix}.$$

By Proposition 1.2.1, this leads to our first theorem.

Theorem 3.1.4. *Let p be a prime. If $p \equiv 1 \mod 4$, then*

$$\operatorname{Spec} H_{p^s} = \begin{pmatrix} \frac{p^{s-1}(p-1)}{2} & \frac{p^{s-1}(-1+\sqrt{p})}{2} & \frac{p^{s-1}(-1-\sqrt{p})}{2} & 0 \\ 1 & \frac{p-1}{2} & \frac{p-1}{2} & p^s - p \end{pmatrix}$$

for all $s \geq 1$.

Next, we assume that q is a prime and $q \equiv 3 \mod 4$. Then -1 is a quadratic non-residue of q, so of q^s. Thus, $(-1)(\mathbb{Z}_{q^s}^{\times})^2 \cap (\mathbb{Z}_{q^s}^{\times})^2 = \varnothing$. This implies

$$\begin{aligned} |T_{q^s}| &= |(\mathbb{Z}_{q^s}^{\times})^2 \cup (-1)(\mathbb{Z}_{q^s}^{\times})^2| \\ &= |(\mathbb{Z}_{q^s}^{\times})^2| + |(-1)(\mathbb{Z}_{q^s}^{\times})^2| \\ &= 2|(\mathbb{Z}_{q^s}^{\times})^2| \\ &= \frac{2|\mathbb{Z}_{q^s}^{\times}|}{|K_{q^s}|} \\ &= |\mathbb{Z}_{q^s}^{\times}| \end{aligned}$$

from the exactness of (3.1.1). Since $T_{q^s} \subseteq \mathbb{Z}_{q^s}^{\times}$, $T_{q^s} = \mathbb{Z}_{q^s}^{\times}$. Hence, H_{q^s} is the unitary Cayley graph G_{q^s} and we may obtain its eigenvalues from Proposition 1.1.4.

Theorem 3.1.5. *Let q be a prime. If $q \equiv 3 \mod 4$, then H_{q^s} is the unitary Cayley graph G_{q^s} and*

$$\operatorname{Spec} H_{q^s} = \begin{pmatrix} (q-1)q^{s-1} & -q^{s-1} & 0 \\ 1 & q-1 & q^s - q \end{pmatrix}$$

for all $s \geq 1$.

Theorem 3.1.6. *Assume that $p_1, \dots, p_s$ are primes congruent to 1 modulo 4 and $q_1, \dots, q_t$ are primes congruent to 3 modulo 4. Then the following statements hold.*

(i) If $n = p_1^{a_1} \dots p_s^{a_s} q_1^{b_1} \dots q_t^{b_t}$ *for all* $a_i \geq 1$ *and* $b_j \geq 1$*, then*

$$H_n \cong H_{p_1^{a_1} \dots p_s^{a_s}} \otimes H_{q_1^{b_1} \dots q_t^{b_t}}.$$

(ii) $H_{p_1^{a_1} \dots p_s^{a_s}} \cong H_{p_1^{a_1}} \otimes \cdots \otimes H_{p_s^{a_s}}$ *for all* $a_i \geq 1$.

(iii) $H_{q_1^{b_1} \dots q_t^{b_t}} \cong G_{q_1^{b_1}} \otimes \cdots \otimes G_{q_t^{b_t}} \cong G_{q_1^{b_1} \dots q_t^{b_t}}$ *for all* $b_j \geq 1$.

Proof. Note that $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{a_1} \dots p_s^{a_s}} \times \mathbb{Z}_{q_1^{b_1} \dots q_t^{b_t}}$ induces the isomorphisms $\mathbb{Z}_n^\times \cong \mathbb{Z}_{p_1^{a_1} \dots p_s^{a_s}}^\times \times \mathbb{Z}_{q_1^{b_1} \dots q_t^{b_b}}^\times$ and $(\mathbb{Z}_n^\times)^2 \cong (\mathbb{Z}_{p_1^{a_1} \dots p_s^{a_s}}^\times)^2 \times (\mathbb{Z}_{q_1^{b_1} \dots q_t^{b_t}}^\times)^2$. In addition, $K_n \cong K_{p_1^{a_1} \dots p_s^{a_s}} \times K_{q_1^{b_1} \dots q_t^{b_t}}$. Thus, $H_n \cong H_{p_1^{a_1} \dots p_s^{a_s}} \times H_{q_1^{b_1} \dots q_t^{b_t}}$. Since $(\mathbb{Z}_{p_1^{a_1} \dots p_s^{a_s}}^\times)^2 \cong (\mathbb{Z}_{p_1^{a_1}}^\times)^2 \times \cdots \times (\mathbb{Z}_{p_s^{a_s}}^\times)^2$ and $K_{p_1^{a_1} \dots p_s^{a_s}} \cong K_{p_1^{a_1}} \times \cdots \times K_{p_s^{a_s}}$, we have

$$\begin{aligned} T_{p_1^{a_1} \dots p_s^{a_s}} &= K_{p_1^{a_1} \dots p_s^{a_s}} (\mathbb{Z}_{p_1^{a_1} \dots p_s^{a_s}}^\times)^2 \\ &\cong K_{p_1^{a_1}} (\mathbb{Z}_{p_1^{a_1}}^\times)^2 \times \cdots \times K_{p_s^{a_s}} (\mathbb{Z}_{p_s^{a_s}}^\times)^2 \\ &= T_{p_1^{a_1}} \times \cdots \times T_{p_s^{a_s}}, \end{aligned}$$

Similarly, $T_{q_1^{b_1} \dots q_t^{b_t}} \cong T_{q_1^{b_1}} \times \cdots \times T_{q_t^{b_t}}$ which equals $\mathbb{Z}_{q_1^{b_1}}^\times \times \cdots \times \mathbb{Z}_{q_t^{b_t}}^\times$ because $q_j \equiv 3$ mod 4. Hence, $T_{q_1^{b_1} \dots q_t^{b_t}} = \mathbb{Z}_{q_1^{b_1} \dots q_t^{b_t}}^\times$, and so $H_{q_1^{b_1} \dots q_t^{b_t}} \cong G_{q_1^{b_1}} \otimes \cdots \otimes G_{q_t^{b_t}} \cong G_{q_1^{b_1} \dots q_t^{b_t}}$ as desired. □

Moreover, it follows from Proposition 1.2.1 that Engy $G \otimes H =$ Engy G Engy H. A direct computation from Theorems 3.1.4, 3.1.5 and 3.1.6 gives a formula for the energy of the graph H_n, where n is odd.

Theorem 3.1.7. *Assume that* $p_1, \dots, p_s$ *are primes congruent to* 1 *modulo* 4 *and* $q_1, \dots, q_t$ *are primes congruent to* 3 *modulo* 4*. Then the following statements hold.*

(i) If $n = p_1^{a_1} \dots p_s^{a_s} q_1^{b_1} \dots q_t^{b_t}$ *for all* $a_i \geq 1$ *and* $b_j \geq 1$*, then*

$$\text{Engy}\, H_n = (\text{Engy}\, H_{p_1^{a_1} \dots p_s^{a_s}})(\text{Engy}\, H_{q_1^{b_1} \dots q_t^{b_t}}).$$

(ii) $\text{Engy}\, H_{p_1^{a_1} \dots p_s^{a_s}} = \prod_{i=1}^{s} \text{Engy}\, H_{p_i^{a_i}} = 2^{-s} \prod_{i=1}^{s} (p_i^{a_i} - p_i^{a_i - 1})(1 + \sqrt{p_i}).$

(iii) $\text{Engy}\, H_{q_1^{b_1} \dots q_t^{b_t}} = \text{Engy}\, G_{q_1^{b_1} \dots q_t^{b_t}} = 2^t \prod_{j=1}^{t} (q_j^{b_j} - q_j^{b_j - 1}).$

3.2 Quadratic Residues of f

Let $\mathbb{F}_q$ be the finite field with $q = p^s$ elements of characteristic odd prime p. Let $A = \mathbb{F}_q[T]$, and let $f \in A$ be a non-constant polynomial. Consider the exact sequence of groups

$$1 \longrightarrow K_f \longrightarrow (A/fA)^\times \xrightarrow{\theta} ((A/fA)^\times)^2 \longrightarrow 1, \tag{3.2.1}$$

where $\theta : a \mapsto a^2$ is the square mapping on $(A/fA)^\times$ with kernel $K = \{a \in (A/fA)^\times : a^2 = 1\}$ and $((A/fA)^\times)^2 = \{a^2 : a \in (A/fA)^\times\}$.

Let $T_f = K_f((A/fA)^\times)^2$. Define the graph $H_f = \text{Cay}(A/fA, T_f)$, in which two vertices are adjacent if and only if their difference is in T_f. Observe that H_f is undirected, so its adjacency matrix is symmetric. In this section, we study the structure of the graph H_f and obtain its eigenvalues. Furthermore, we compute the energy of H_f.

Let $P \in A$ be an irreducible polynomial and $e \geq 1$. Write $|P|$ for $q^{\deg P}$. We recall that the group $(A/P^eA)^\times$ is an abelian group of order $(|P| - 1)|P|^{e-1}$. It follows from the theory of finite abelian groups that as a group $(A/P^eA)^\times$ is a product of cyclic group of order $|P| - 1$ (isomorphic to $(A/PA)^\times$) and a p-group

$\mathcal{P}$. Hence, $(A/P^eA)^\times$ has a unique element of order two, namely -1 which is $(-1, 1)$ in $(A/PA)^\times \times \mathcal{P}$. Then $K_{P^e} = \{a \in (A/P^eA)^\times : a^2 = 1\} = \{1, -1\}$. Thus, $T_{P^e} = \pm((A/P^eA)^\times)^2$. Next, we proceed by recalling Theorem 1.10 of [22] that:

Theorem 3.2.1. *[22] Let d be a positive integer such that $d \mid (|P| - 1)$. Then $x^d \equiv a \mod P^e$ has a solution if and only if $a^{\frac{|P|-1}{d}} \equiv 1 \mod P$ in A.*

Therefore, to determine the case when -1 is a quadratic residue of P^e, we consider when $(-1)^{\frac{|P|-1}{2}} \equiv 1 \mod P$ in A, so

$$1 \equiv (-1)^{\frac{|P|-1}{2}} \equiv (-1)^{\frac{q^{\deg P}-1}{2}} \equiv (-1)^{\frac{p^{s(\deg P)}-1}{2}} \mod P,$$

which makes $-1 \in ((A/P^eA)^\times)^2$ whenever ($p \equiv 1 \mod 4$) or ($p \equiv 3 \mod 4$ and $s(\deg P)$) is even.

Lemma 3.2.2. *For $e \geq 1$ and an irreducible polynomial P in A, we have*

$$H_{P^e} \cong H_P \otimes \overset{\circ}{K}_{|P|^{e-1}},$$

where $\overset{\circ}{K}_{|P|^{e-1}}$ is the $|P|^{e-1}$-complete graph with a loop on each vertex.

Proof. Since $(A/P^eA)^\times \cong (A/PA)^\times \times \mathcal{P}$ for some p-group $\mathcal{P}$ of order $|P|^{e-1}$, we can write each element $a \in (A/P^eA)^\times$ as $(a_1, a_2) \in (A/PA)^\times \times \mathcal{P}$. Then the adjacency condition becomes $a - b \in ((A/P^eA)^\times)^2$ if and only if $a_1 - b_1 \in ((A/PA)^\times)^2$. Thus, we have $H_{P^e} \cong H_P \otimes \overset{\circ}{K}_{|P|^{e-1}}$ as desired. □

Since the adjacency matrix of $\overset{\circ}{K}_{|P|^{e-1}}$ is the $|P|^{e-1} \times |P|^{e-1}$ matrix of all 1s, we get

$$\text{Spec } \overset{\circ}{K}_{|P|^{e-1}} = \begin{pmatrix} |P|^{e-1} & 0 \\ 1 & |P|^{e-1} - 1 \end{pmatrix}.$$

Moreover, if -1 is a quadratic residue of P, then $T_P = ((A/PA)^\times)^2$. Thus, H_P is the Paley graph which is strongly regular with parameters $(|P|, (|P|-1)/2, (|P|-5)/4, (|P|-1)/4)$ by Lemma 3.1.3. Hence, from Lemma 3.1.2

$$\operatorname{Spec} H_P = \begin{pmatrix} \frac{|P|-1}{2} & \frac{-1+\sqrt{|P|}}{2} & \frac{-1-\sqrt{|P|}}{2} \\ 1 & \frac{|P|-1}{2} & \frac{|P|-1}{2} \end{pmatrix}.$$

By Proposition 1.2.1, this brings us to the following theorem.

Theorem 3.2.3. *Let $P \in A$ be irreducible. Assume that ($p \equiv 1 \mod 4$) or ($p \equiv 3 \mod 4$ and $s(\deg P)$ is even). Then*

$$\operatorname{Spec} H_{P^e} = \begin{pmatrix} \frac{|P|^{e-1}(|P|-1)}{2} & \frac{|P|^{e-1}(-1+\sqrt{|P|})}{2} & \frac{|P|^{e-1}(-1-\sqrt{|P|})}{2} & 0 \\ 1 & \frac{|P|-1}{2} & \frac{|P|-1}{2} & |P|^e - |P| \end{pmatrix}$$

for all $e \geq 1$.

Next, for the finite field $\mathbb{F}_q$ with $q = p^s$ elements of characteristic p, we assume that $p \equiv 3 \mod 4$ and $s(\deg P)$ is odd. Then -1 is a quadratic non-residue modulo P^e. Thus, $(-1)((A/P^eA)^\times)^2 \cap ((A/P^eA)^\times)^2 = \varnothing$. This implies

$$\begin{aligned} |T_{P^e}| &= |((A/P^eA)^\times)^2 \cup (-1)((A/P^eA)^\times)^2| \\ &= |((A/P^eA)^\times)^2| + |((A/P^eA)^\times)^2| \\ &= 2|((A/P^eA)^\times)^2| \\ &= \frac{2|(A/P^eA)^\times|}{|K_{P^e}|} \\ &= |(A/P^eA)^\times| \end{aligned}$$

from the exactness of (3.2.1). Since $T_{P^e} \subseteq (A/P^eA)^\times$, we have $T_{P^e} = (A/P^eA)^\times$. Hence, H_{P^e} is the unitary Cayley graph $G_{P^e} := \operatorname{Cay}(A/P^eA, (A/P^eA)^\times)$ over the finite ring A/P^eA and we can obtain its eigenvalues from Proposition 1.1.4.

Theorem 3.2.4. *Let $P \in A$ be irreducible. Assume that $p \equiv 3 \mod 4$, $s(\deg P)$ is odd. Then H_{P^e} is the unitary Cayley graph G_{P^e} and*

$$\operatorname{Spec} H_{P^e} = \begin{pmatrix} (|P|-1)|P|^{e-1} & -|P|^{e-1} & 0 \\ 1 & |P|-1 & |P|^e - |P| \end{pmatrix}.$$

Theorem 3.2.5. *Let $P_1, \dots, P_{r+t} \in A$ be irreducible. Assume that $p \equiv 3 \mod 4$, s is odd, $\deg P_1, \dots, \deg P_r$ are even and $\deg P_{r+1}, \dots, \deg P_{r+t}$ are odd. Then the following statements hold.*

(i) If $f = P_1^{e_1} \dots P_r^{e_r} P_{r+1}^{l_1} \dots P_{r+t}^{l_t}$, then

$$H_f \cong H_{P_1^{e_1} \dots P_r^{e_r}} \otimes H_{P_{r+1}^{l_1} \dots P_{r+t}^{l_t}}.$$

(ii) $H_{P_1^{e_1} \dots P_r^{e_r}} \cong H_{P_1^{e_1}} \otimes \dots \otimes H_{P_r^{e_r}}$.

(iii) $H_{P_{r+1}^{l_1} \dots P_{r+t}^{l_t}} \cong G_{P_{r+1}^{l_1}} \otimes \dots \otimes G_{P_{r+t}^{l_t}} \cong G_{P_{r+1}^{l_1} \dots P_{r+t}^{l_t}}$.

Proof. Note that $A/fA \cong A/(P_1^{e_1} \dots P_r^{e_r})A \times A/(P_{r+1}^{l_1} \dots P_{r+t}^{l_t})A$ induces the isomorphisms

$$(A/fA)^\times \cong (A/(P_1^{e_1} \dots P_r^{e_r})A)^\times \times (A/(P_{r+1}^{l_1} \dots P_{r+t}^{l_t})A)^\times$$

and

$$((A/fA)^\times)^2 \cong ((A/(P_1^{e_1} \dots P_r^{e_r})A)^\times)^2 \times ((A/(P_{r+1}^{l_1} \dots P_{r+t}^{l_t})A)^\times)^2.$$

In addition, $K_f \cong K_{P_1^{e_1} \dots P_r^{e_r}} \times K_{P_{r+1}^{l_1} \dots P_{r+t}^{l_t}}$. Thus, $H_f \cong H_{P_1^{e_1} \dots P_r^{e_r}} \times H_{P_{r+1}^{l_1} \dots P_{r+t}^{l_t}}$. Since

$$((A/(P_1^{e_1} \dots P_r^{e_r})A)^\times)^2 \cong ((A/(P_1^{e_1})A)^\times)^2 \times \dots \times ((A/(P_r^{e_r})A)^\times)^2$$

and

$$K_{P_1^{e_1}\dots P_r^{e_r}} \cong K_{P_1^{e_1}} \times \cdots \times K_{P_r^{e_r}},$$

we have

$$\begin{aligned} T_{P_1^{e_1}\dots P_r^{e_r}} &= K_{P_1^{e_1}\dots P_r^{e_r}}((A/(P_1^{e_1}\dots P_r^{e_r})A)^{\times})^2 \\ &\cong K_{P_1^{e_1}}((A/(P_1^{e_1})A)^{\times})^2 \times \cdots \times K_{P_r^{e_r}}((A/(P_r^{e_r})A)^{\times})^2 \\ &= T_{P_1^{e_1}} \times \cdots \times T_{P_r^{e_r}}, \end{aligned}$$

Similarly, $T_{P_{r+1}^{l_1}\dots P_{r+t}^{l_t}} \cong T_{P_{r+1}^{l_1}} \times \cdots \times T_{P_{r+t}^{l_t}}$ which equals $(A/(P_1^{l_1})A)^{\times} \times \cdots \times (A/(P_t^{l_t})A)^{\times}$ because $\deg P_j, j \geq r+1$ is odd. Hence,

$$T_{P_{r+1}^{l_1}\dots P_{r+t}^{l_t}} = (A/(P_{r+1}^{l_1}\dots P_{r+t}^{l_t})A)^{\times},$$

and so $H_{P_{r+1}^{l_1}\dots P_{r+t}^{l_t}} \cong G_{P_{r+1}^{l_1}} \otimes \cdots \otimes G_{P_{r+t}^{l_t}} \cong G_{P_{r+1}^{l_1}\dots P_{r+t}^{l_t}}$ as desired. □

Finally, a direct computation from Theorems 3.2.4 and 3.2.5 gives a formula for the energy of the graph H_f.

Theorem 3.2.6. *Let $P_1, \dots, P_{r+t} \in A$ be irreducible. Assume that $p \equiv 3 \mod 4$, s is odd,* $\deg P_1, \dots, \deg P_r$ *are even and* $\deg P_{r+1}, \dots, \deg P_{r+t}$ *are odd. Then the following statements hold.*

(i) If $f = P_1^{e_1}\dots P_r^{e_r}P_{r+1}^{l_1}\dots P_{r+t}^{l_t}$, then

$$\text{Engy}\, H_f = (\text{Engy}\, H_{P_1^{e_1}\dots P_r^{e_r}})(\text{Engy}\, H_{P_{r+1}^{l_1}\dots P_{r+t}^{l_t}}).$$

(ii) $\text{Engy}\, H_{P_1^{e_1}\dots P_r^{e_r}} = \prod_{i=1}^{r} \text{Engy}\, H_{P_i^{e_i}} = 2^{-r}\prod_{i=1}^{r}(|P_i|^{e_i} - |P_i|^{e_i-1})(1+\sqrt{|P_i|}).$

(iii) $\text{Engy}\, H_{P_{r+1}^{l_1}\dots P_{r+t}^{l_t}} = \text{Engy}\, G_{P_{r+1}^{l_1}\dots P_{r+t}^{l_t}} = 2^t\prod_{j=1}^{t}(|P_{r+j}|^{l_j} - |P_{r+j}|^{l_j-1}).$

Corollary 3.2.7. *Let* $P_i \in A$ *be irreducible and* $e_i \geq 1$ *for all* i*. Assume that* $(p \equiv 1 \mod 4)$ *or* $(p \equiv 3 \mod 4$ *and* s *is even*$)$*. Then*

$$\operatorname{Engy} H_{P_1^{e_1} \dots P_r^{e_r}} = 2^{-r} \prod_{i=1}^{r} (|P_i|^{e_i} - |P_i|^{e_i - 1})(1 + \sqrt{|P_i|}).$$

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