CHAPTER II

FEYNMAN PATH INTEGRALS

It is known that the operator formalism in Schrödinger and Heisenberg representations of quantum mechanics and quantum statistics may not always yield the most transparent understanding of quantum phenomena. There exists another equivalent formalism in which operators are avoided and replaced by the use of infinite products of integrals call *Feynman path integrals* [Feynman and Hibbs 1965, Schulman 1981, Kleinert 1990]. In contrast to the Schrödinger equation, which is a differential equation specifying the properties of a state at a time given its knowledge at an infinitesimal time earlier, the path integral constitutes an all-time global approach to the calculation of quantum mechanical amplitudes.

Path Integral Representation

The path integral approach to quantum mechanics was developed by Feynman [Feynman 1948]. In its original form it was valid only for a point particle moving in a cartesian coordinate system and served to calculate the transition amplitudes of the time displacement operator between localized states of the particle,

$$(x_b t_b \,|\, x_a t_a) \;=\; \langle x_b \,|\, \hat{U}(t_b, t_a) \,|\, x_a \rangle\,, \qquad t_b > t_a\,. \tag{2.1}$$

For simplicity, we shall consider at first only a point particle in one cartesian dimension and shall be interested only in causal or retarded time displacement amplitudes as described in Chapter I.

Feynman realized that due to the fundamental composition law of the time displacement operator , the amplitude (2.1) could be sliced into a large number, say $N+1$, of time displacement operators, each acting across an infinitesimal time slice of width $\varepsilon \equiv t_n - t_{n-1} = (t_b - t_a)/(N+1) > 0$,

$$(x_b t_b | x_a t_a) = \langle x_b | \widehat{U}(t_b, t_N)\widehat{U}(t_N, t_{N-1}) \ldots$$
$$\ldots \widehat{U}(t_n, t_{n-1}) \ldots \widehat{U}(t_2, t_1)\widehat{U}(t_1, t_a)| x_a \rangle . \tag{2.2}$$

When inserting a complete set of states between each pair of $\widehat{U}$'s ,

$$\int_{-\infty}^{\infty} dx_n | x_n \rangle \langle x_n | = 1, \quad n = 1, \ldots , N , \tag{2.3}$$

the amplitude becomes a product of N integrals

$$(x_b t_b | x_a t_a) = \prod_{n=1}^{N} \left[\int_{-\infty}^{\infty} dx_n \right] \prod_{n=1}^{N+1} (x_n t_n | x_{n-1} t_{n-1}) , \tag{2.4}$$

where we have identified $x_b \equiv x_{N+1}, x_a \equiv x_0, t_b \equiv t_{N+1}, t_a \equiv t_0$. The amplitudes for the infinitesimal time intervals are given by

$$(x_n t_n | x_{n-1} t_{n-1}) = \left\langle x_n \left| e^{-i\varepsilon \widehat{H}(t_n)/\hbar} \right| x_{n-1} \right\rangle , \tag{2.5}$$

with the abbreviation for the Hamiltonian operator

$$\widehat{H}(t) \equiv H(\widehat{p}, \widehat{x}, t) . \tag{2.6}$$

The further development becomes simplest under the assumption that the Hamiltonian $H(\widehat{p}, \widehat{x}, t)$ be of the standard form, consisting of a sum of a kinetic and a potential energy

$$H(p, x, t) = T(p, t) + V(x, t). \tag{2.7}$$

For sufficiently small ε, the time displacement operator

$$e^{-i\varepsilon\hat{H}/\hbar} = e^{-i\varepsilon(\hat{T}+\hat{V})/\hbar} \tag{2.8}$$

is factorizable according to the *Baker-Hausdorff formula* as follows

$$e^{-i\varepsilon(\hat{T}+\hat{V})/\hbar} = e^{-i\varepsilon\hat{V}/\hbar}\, e^{-i\varepsilon\hat{T}/\hbar}\, e^{-\varepsilon^2\hat{X}/\hbar^2}, \tag{2.9}$$

where the operator $\hat{X}$ has the expansion

$$\hat{X} \equiv \frac{i}{2}\left[\hat{V},\hat{T}\right] - \frac{\varepsilon}{\hbar}\left(\frac{1}{6}\left[\hat{V},\left[\hat{V},\hat{T}\right]\right] - \frac{1}{3}\left[\left[\hat{V},\hat{T}\right],\hat{T}\right]\right) + \ldots . \tag{2.10}$$

The omitted terms of order ε^4, ε^5, ... contain higher and higher commutators of $\hat{V}$ and $\hat{T}$. If we neglect, for a moment, also the $\hat{X}$ term of order ε^2 we calculate for the local matrix elements of $e^{-i\varepsilon\hat{H}/\hbar}$ the following simple expression

$$\left\langle x_n\right| e^{-i\varepsilon\hat{H}(t_n)/\hbar}\left| x_{n-1}\right\rangle \approx \int_{-\infty}^{\infty} dx \left\langle x_n\right| e^{-i\varepsilon V(\hat{x},t_n)/\hbar}\left| x\right\rangle \left\langle x\right| e^{-i\varepsilon T(\hat{p},t_n)/\hbar}\left| x_{n-1}\right\rangle$$

$$= \int_{-\infty}^{\infty} dx \left\langle x_n\right| e^{-i\varepsilon V(\hat{x},t_n)/\hbar}\left| x\right\rangle \int_{-\infty}^{\infty} \frac{dp_n}{2\pi\hbar}\, e^{\,ip_n(x-x_{n-1})/\hbar}\, e^{-i\varepsilon T(p_n,t_n)/\hbar} . \tag{2.11}$$

With the local matrix elements

$$\left\langle x_n\right| e^{-i\varepsilon V(\hat{x},t_n)/\hbar}\left| x\right\rangle = \delta(x_n - x)\, e^{-i\varepsilon V(x_n,t_n)/\hbar} , \tag{2.12}$$

this becomes

$$\left\langle x_n\right| e^{-i\varepsilon V(\hat{x},t_n)/\hbar}\left| x_{n-1}\right\rangle \approx$$

$$\int_{-\infty}^{\infty} \frac{dp_n}{2\pi\hbar}\, exp\left\{ip_n(x_n - x_{n-1})/\hbar - i\varepsilon\left[\, T(p_n,t_n) + V(x_n,t_n)\right]/\hbar\right\} . \tag{2.13}$$

Inserting this back into (2.4) we obtain the multiple integral

$$(x_b t_b | x_a t_a) \approx \prod_{n=1}^{N} \left[\int_{-\infty}^{\infty} dx_n \right] \prod_{n=1}^{N+1} \left[\int_{-\infty}^{\infty} \frac{dp_n}{2\pi\hbar} \right] exp \left\{ \frac{i}{\hbar} S^N \right\}, \tag{2.14}$$

where S^N is the sum

$$S^N = \sum_{n=1}^{N+1} \left[p_n (x_n - x_{n-1}) - \varepsilon H (p_n, x_n, t_n) \right]. \tag{2.15}$$

First we observe that in the *continuum limit* $N \to \infty$, $\varepsilon \to 0$, the sum S^N in (2.15) tends towards the integral

$$S[p, x] = \int_{t_a}^{t_b} dt \left[p(t) \dot{x}(t) - H(p(t), x(t), t) \right]. \tag{2.16}$$

This is recognized as the *classical canonical action* for the path $x(t), p(t)$ in phase space. Since the position variables x_{N+1} and x_0 are fixed at the initial and final values x_b and x_a, the paths satisfy the boundary condition $x(t_b) = x_b, x(t_a) = x_a$. In the same limit, the product of infinitely many integrals in (2.14) will be called a path integral and written as follows

$$\lim_{N \to \infty} \prod_{n=1}^{N} \left[\int_{-\infty}^{\infty} dx_n \right] \prod_{n=1}^{N+1} \left[\int_{-\infty}^{\infty} \frac{dp_n}{2\pi\hbar} \right] \equiv \int \mathcal{D}'x \int \frac{\mathcal{D}p}{2\pi\hbar}. \tag{2.17}$$

By definition, there is always one more p_n than x_n integral in this product since there is one p_n for every pair of x_n's, while the two x_n's at the end are held fixed. This fact is recorded by the prime on the functional integral $\mathcal{D}'x$. With this definition, the amplitude can be written in the short form

$$(x_b t_b \,|\, x_a t_a) \;=\; \int \mathcal{D}'x \int \frac{\mathcal{D}p}{2\pi\hbar}\, e^{\,iS[p,\,x]/\hbar}\,. \tag{2.18}$$

The path integral representation of the quantum mechanical amplitude has a simple intuitive interpretation: The path integration corresponds to a sum over all path histories along which a general physical system can possibly evolve. The exponential $e^{\,iS[p,\,x]/\hbar}$ is the quantum analog of the Boltzmann factor e^{-E/k_BT} in statistical mechanics. Instead of an exponential probability, however, it assigns a pure phase to each possible history. The total amplitude for going from x_a, t_a to x_b, t_b is obtained by adding up the phases for all these histories,

$$(x_b t_b \,|\, x_a t_a) \;=\; \sum_{\text{all histories}} e^{\,iS[p,\,x]/\hbar}\,, \tag{2.19}$$

where the sum comprises all paths in phase space with fixed end points x_b, x_a in x-space.

Of course, the above observed asymmetry in the functional integrals over x and p is a result of keeping the end points fixed in position space. It is possible to proceed alternatively and keep the initial and final momenta p_b and p_a fixed. The time displacement amplitude can be derived by going through the same steps as before but working in the momentum space representation of the Hilbert space.

Actually, in his original paper Feynman did not give the path integral formula in the above general phase space formulation. Since the kinetic energy in (2.7) has usually the form $T(p,t) \;=\; p^2/2M$, he considered right-away the Hamiltonian

$$H \;=\; \frac{p^2}{2M} + V(x,t)\,. \tag{2.20}$$

Then, in the above phase space formulation, the action in the time-sliced form (2.15) becomes

$$S^N = \sum_{n=1}^{N+1} \left\{ p_n (x_n - x_{n-1}) - \varepsilon \frac{p_n^2}{2M} - \varepsilon V(x_n, t_n) \right\}. \tag{2.21}$$

This can be quadratically completed to

$$S^N = \sum_{n=1}^{N+1} \left\{ -\frac{\varepsilon}{2M} \left(p_n - \frac{x_n - x_{n-1}}{\varepsilon} M \right)^2 + \frac{M}{2} \varepsilon \left(\frac{x_n - x_{n-1}}{\varepsilon} \right)^2 - \varepsilon V(x_n, t_n) \right\}. \tag{2.22}$$

The momentum integrals in (2.14) can then be performed using the *Fresnel integral formula* (by $\sqrt{i}$ we shall always mean $\equiv e^{i\pi/4}$)

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi i}} \exp\left\{ i \frac{a}{2} x^2 \right\} = \frac{1}{\sqrt{|a|}} \begin{cases} \sqrt{i} & a > 0, \\ 1/\sqrt{i} & a < 0. \end{cases} \tag{2.23}$$

The phases follow from an appropriate analytic continuation of the *Gauss formula*

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \exp\left\{ -\frac{\alpha}{2} x^2 \right\} = \frac{1}{\sqrt{\alpha}} \qquad \text{Re}\, \alpha > 0 \tag{2.24}$$

to imaginary $\alpha = \pm i|a|$, to be approached inside the range of validity of the formula, which is from the right of the imaginary α-axis. To characterize this more clearly one sets $\alpha = \pm i|a| + \eta$ with an infinitesimal positive quantity η. Since the Fresnel formula is special analytically continued case of the Gauss formula, we shall in the sequel always speak of Gauss formulas and use Fresnel's name only if the imaginary nature of the quadratic exponent is to be emphasized.

With the Fresnel formula, the momentum integral in (2.14) gives.

$$\int_{-\infty}^{\infty} \frac{dp_n}{2\pi\hbar} \exp\left\{ -\frac{i}{\hbar} \frac{\varepsilon}{2M} \left(p_n - \frac{x_n - x_{n-1}}{\varepsilon} \right)^2 \right\} = \frac{1}{\sqrt{2\pi\hbar i\varepsilon / M}}, \tag{2.25}$$

and we remain with the alternative representation

$$(x_b t_b | x_a t_a) \approx \frac{1}{\sqrt{2\pi\hbar i\varepsilon / M}} \prod_{n=1}^{N} \left[\int_{-\infty}^{\infty} \frac{dx_n}{\sqrt{2\pi\hbar i\varepsilon / M}} \right] exp \left\{ \frac{i}{\hbar} S^N \right\}, \qquad (2.26)$$

where S^N is now the sum

$$S^N = \varepsilon \sum_{n=1}^{N+1} \left[\frac{M}{2} \left(\frac{x_n - x_{n-1}}{\varepsilon} \right)^2 - V(x_n, t_n) \right], \qquad (2.27)$$

with with $x_{N+1} = x_b, x_0 = x_a$. Here the integrals cover all paths in *configuration space* rather than phase space.

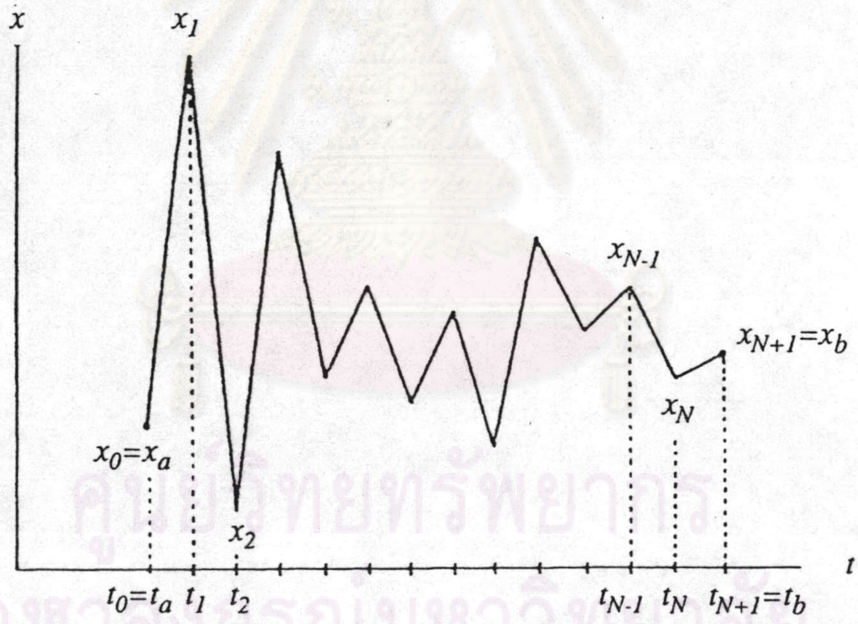

Figure 2.1 The zigzag paths along which a point particle propagates all possible ways of reaching the point x_b at a time t_b starting from x_a at a time t_a

The particle explores all possible ways of reaching a given final point x_b starting from a given initial point x_a, the amplitude of each path being $exp\{i S^N / \hbar\}$. See Fig. 2.1

for a geometric illustration of the path integration. In the continuum limit, the sum (2.27) becomes the action in the Lagrangian form, expressed as a functional of x, $\dot{x}$.

$$S[x, \dot{x}] = \int_{t_a}^{t_b} dt\, L(x, \dot{x}) = \int_{t_a}^{t_b} dt \left[\frac{M}{2}\dot{x}^2 - V(x, t)\right]. \tag{2.28}$$

In the the continuum limit, we shall write the amplitude (2.26) as a path integral

$$(x_b t_b \,|\, x_a t_a) \equiv \int \mathcal{D}x\, e^{i S[x, \dot{x}]/\hbar}. \tag{2.29}$$

This is Feynman's original formula for the quantum mechanical amplitude (2.1), summing paths in configuration space and using the x, $\dot{x}$ form of the action.

We have used the same integration symbol $\mathcal{D}x$ for the pure configuration space path measure as in the different phase space path measures since the there is no danger of confusion. Notice that the extra dp_n integration in the phase space formula (2.14) results now in one extra $1/\sqrt{2\pi\hbar i\varepsilon / M}$ factor in (2.26) which is not accompanied by a dx_n integration.

The Feynman amplitude can be used to calculate the quantum mechanical partition function as a configuration space path integral

$$Z_{QM} = \int_{-\infty}^{\infty} dx\, (x\, t_b \,|\, x t_a) \approx \int_{x(t_b) = x(t_a)} \mathcal{D}x\, e^{iS[x, \dot{x}]/\hbar}. \tag{2.30}$$

Here the integration symbol $\int \mathcal{D}x$ has yet a different meaning; it stands for

$$\int \mathcal{D}x = \prod_{n=1}^{N+1} \int_{-\infty}^{\infty} dx_n \,/\, \sqrt{2\pi i \hbar\varepsilon / M}. \tag{2.31}$$

It contains no extra $1/\sqrt{2\pi i\hbar\varepsilon / M}$ factor, as in (2.26), (2.29), due to the integration over the initial (=final) position $x_b = x_a$ which represents the quantum mechanical trace.

The use of the same symbol $\int \mathcal{D}x$ cannot lead to a confusion since its present meaning is easily recognized by the absence of x_b, x_a coordinates on the left-hand side of an equation.

Exact Solution for the Free Particle

In order to develop some experience with Feynman's path integral formula let us consider in detail the simplet case of a free particle, which in the canonical form reads [Feynman and Hibbs 1965, Schulman 1981, Kleinert 1990]

$$(x_b\, t_b | x_a\, t_a) = \int \mathcal{D}'x \int \frac{\mathcal{D}p}{2\pi\hbar} exp \left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} dt \left(p\dot{x} - \frac{p^2}{2M} \right) \right\}, \quad (2.32)$$

and in the pure configuration form,

$$(x_b\, t_b | x_a\, t_a) = \int \mathcal{D}x\ exp \left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} dt\, \frac{M}{2} \dot{x}^2 \right\}. \quad (2.33)$$

The problem can be solved most easily in the configuration form. The time-sliced expression to be integrated is given by Eqs. (2.26), (2.27) with vanishing potential $V(x)$. This is a product of Gauss integrals which can easily be done successively using formula (2.23) with the result

$$(x_b\, t_b | x_a\, t_a) = \frac{1}{\sqrt{2\pi i \hbar (N+1)\varepsilon / M}} exp \left\{ \frac{i}{\hbar} \frac{M}{2} \frac{(x_b - x_a)^2}{(N+1)\varepsilon} \right\}$$

$$= \frac{1}{\sqrt{2\pi i \hbar (t_b - t_a) / M}} exp \left\{ \frac{i}{\hbar} \frac{M}{2} \frac{(x_b - x_a)^2}{t_b - t_a} \right\}. \quad (2.34)$$

Notice that the result happens to be independent of the number of time slice.

There exists another method of calculating the free-particle amplitude which is somewhat more involved but which will turn out to be useful in a generalized form when trying to treat nontrivial path integrals. In this method we count all paths with respect to the classical path, i.e., we split all paths into the *classical path*

$$x_{cl}(t) = x_a + \frac{x_b - x_a}{t_b - t_a}(t - t_a), \tag{2.35}$$

along which the free particle would run following the equation of motion

$$\ddot{x}_{cl}(t) = 0, \tag{2.36}$$

plus deviations $\delta x(t)$ which vanish at the ends:

$$\delta x(t_a) = \delta x(t_b) = 0. \tag{2.37}$$

The deviations $\delta x(t)$ will be called *quantum fluctuations.* Inserting the decomposition

$$x(t) = x_{cl}(t) + \delta x(t) \tag{2.38}$$

into the action we observe, that due to the equation of motion (2.36) for the classical path, the action separates into the sum of a classical and a purely quadratic fluctuation term

$$\frac{M}{2}\int_{t_a}^{t_b} dt\ \left\{\dot{x}_{cl}^2(t) + 2\dot{x}_{cl}(t)\,\delta\dot{x}(t) + \left[\delta\dot{x}(t)\right]^2\right\}$$

$$= \frac{M}{2}\int_{t_a}^{t_b} dt\ \dot{x}_{cl}^2 + M\,\dot{x}\delta x\Big|_{t_a}^{t_b} - M\int_{t_a}^{t_b} dt\ \ddot{x}_{cl}\ \delta\dot{x} + \frac{M}{2}\int_{t_a}^{t_b} dt\ \left(\delta\dot{x}\right)^2$$

$$= \frac{M}{2}\left[\int_{t_a}^{t_b} dt\ \dot{x}_{cl}^2 + \int_{t_a}^{t_b} dt\ \left(\delta\dot{x}\right)^2\right] \tag{2.39}$$

In fact, this is a general consequence of the extremality property of the classical path,

$$\delta S\Big|_{x(t)=x_{cl}(t)} = 0, \tag{2.40}$$

which implies that a quadratic *fluctuation expansion* around it can have no linear term in $\delta x(t)$, i.e., it must start as follows,

$$S = S_{cl} + \frac{1}{2}\int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt' \frac{\delta^2 S}{\delta x(t)\,\delta x(t')}\,\delta x(t)\,\delta x(t')\Big|_{x(t)=x_{cl}(t)} + \ldots, \tag{2.41}$$

where S_{cl} denotes the action of the classical path

$$S_{cl} \equiv S[x_{cl}, \dot{x}_{cl}]. \tag{2.42}$$

As a consequence, the amplitude factorizes into the product of a classical amplitude $e^{iS_{cl}/\hbar}$ and a fluctuation factor $F_0(t_b - t_a)$,

$$(x_b\, t_b | x_a\, t_a) = \int \mathcal{D}x e^{iS[x,\dot{x}]/\hbar} = e^{iS_{cl}/\hbar} F_0(t_b, t_a). \tag{2.43}$$

where S_{cl} is now the classical action of the free particle

$$S_{cl} = \int_{t_a}^{t_b} dt\, \frac{M}{2}\dot{x}_{cl}^2. \tag{2.44}$$

and $F_0(t_b - t_a)$ is given by the path integral

$$F_0(t_b - t_a) = \int \mathcal{D}\delta x(t)\, exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} dt\, \frac{M}{2}(\delta\dot{x})^2\right\}. \tag{2.45}$$

Due to the vanishing of $\delta x(t)$ at the end points this does not depend on x_b, x_a but only on the instal and final times t_b, t_a. The time translational invariance reduces this dependence further to the time difference $t_b - t_a$. The subscript 0 of $F_0(t_b - t_a)$ indicates the free-particle nature of the fluctuation factor. Using (2.35) we find immediately the classical action

$$S_{cl} \quad = \quad \frac{M}{2}\frac{(x_b, x_a)^2}{t_b - t_a}. \tag{2.46}$$

The fluctuation factor, on the other hand, requires in the time-sliced definition of the path integral the evaluation of the multiple integral

$$F_0^N(t_b - t_a) \quad = \quad \frac{1}{\sqrt{2\pi\hbar i\varepsilon / M}} \prod_{n=1}^{N} \left[\int_{-\infty}^{\infty} \frac{d\delta x_n}{\sqrt{2\pi\hbar i\varepsilon / M}} \right] exp\left\{ \frac{i}{\hbar} S_{fl}^N \right\} \tag{2.47}$$

where S_{fl}^N is the time-sliced action of configurational fluctuations

$$S_{fl}^N \quad = \quad \frac{M}{2}\varepsilon \sum_{n=1}^{N+1} \left(\frac{\delta x_n - \delta x_{n-1}}{\varepsilon} \right)^2. \tag{2.48}$$

At the end we have to take the continuum limit

$$N \to \infty, \quad \varepsilon \quad = \quad (t_b - t_a)/(N+1) \to 0. \tag{2.49}$$

The remainder of this section will be devoted to calculating the fluctuation factor (2.47). For this it is useful to develop first a few general techniques for dealing with such multiple integrals. Because of the frequent appearance of the fluctuating δx we shall drop all δ 's, for brevity.

A useful device for manipulating expressions on a sliced time axis such as (2.48) is the difference operator ∇ and its conjugate $\overline{\nabla}$. They are defined as follows,

$$\nabla x(t) \quad \equiv \quad \frac{1}{\varepsilon}[x(t+\varepsilon) - x(t)],$$

$$\overline{\nabla} x(t) \quad \equiv \quad \frac{1}{\varepsilon}[x(t) - x(t+\varepsilon)]. \tag{2.50}$$

They are two different discrete versions of the time derivative ∂_t, to which they both reduce in the continuum limit $\varepsilon \to 0$

$$\nabla \xrightarrow{\varepsilon \to 0} \partial_t. \tag{2.51}$$

For the coordinates $x_n = x(t_n)$ at the discrete times t_n we shall therefore write

$$\nabla x_n = \frac{1}{\varepsilon}(x_{n+1} - x_n), \qquad N \geq n \geq 0,$$

$$\overline{\nabla} x_n = \frac{1}{\varepsilon}(x_n - x_{n-1}), \qquad N+1 \geq n \geq 1. \tag{2.52}$$

The time-sliced action (2.60) can then be expressed in term of $\overline{\nabla}$ as follows (writing as announced x_n instead of δx_n)

$$S_{fl}^{N} = \frac{M}{2}\varepsilon \sum_{n=1}^{N+1} \left(\overline{\nabla} x_n\right)^2. \tag{2.53}$$

In this notation, the limit $\varepsilon \to 0$ is most obvious: The sum $\varepsilon \Sigma_n$ goes into the time integral , and $\left(\nabla_t x\right)^2$ tends to $\dot{x}^{\,2}$, so that

$$S_{fl}^{N} = \varepsilon \sum_n \frac{M}{2}\left(\overline{\nabla} x_n\right)^2 \to \int_{t_a}^{t_b} dt \, \frac{M}{2} \dot{x}^{\,2}. \tag{2.54}$$

The time-sliced action becomes the Lagrangian action. Since the discretized time axis with $N+1$ steps constitutes a one-dimensional lattice, the derivatives ∇, $\overline{\nabla}$ are called *lattice derivatives.*

In the problem at hand, the quantum fluctuations $x_n \left(= \delta x_n\right)$ vanish at the ends so that we can write

$$\sum_{n=1}^{N+1} \left(\overline{\nabla} x_n\right)^2 = \sum_{n=0}^{N} \left(\nabla x_n\right)^2 = -\sum_{n=0}^{N} x_n \, \nabla\overline{\nabla} x_n. \tag{2.55}$$

The right-hand side is a short notation for the matrix expression

$$-\sum_{n=0}^{N} x_n \, \nabla\overline{\nabla} \, x_n = -\sum_{n,n'=0}^{N} x_n \left(\nabla\overline{\nabla}\right)_{nn'} x_{n'}, \tag{2.56}$$

with the $(N+1)\times(N+1)$ matrix,

$$\nabla\bar{\nabla} \equiv \bar{\nabla}\nabla \equiv \frac{1}{\varepsilon^2}\begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}. \tag{2.57}$$

This is obviously the lattice version of the double time derivative ∂_t^2 to which it reduces in the continuum limit $\varepsilon \rightarrow 0$. It will therefore be called *lattice Laplacian.*

A further common property of lattice and ordinary derivatives is that they can both be diagonalized by going to Fourier components. If we expand

$$x(t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} x(\omega) \tag{2.58}$$

and apply the lattice derivative ∇ we find

$$\begin{aligned} \nabla x(t_n) &= \int_{-\infty}^{\infty} d\omega\, \frac{1}{\varepsilon}\left(e^{-i\omega(t_n+\varepsilon)} - e^{-i\omega t_n}\right) x(\omega) \\ &= \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t_n} \frac{1}{\varepsilon}\left(e^{-i\omega\varepsilon} - 1\right) x(\omega). \end{aligned} \tag{2.59}$$

Hence, on the Fourier components, ∇ has the eigenvalues

$$\frac{1}{\varepsilon}\left(e^{-i\omega\varepsilon} - 1\right). \tag{2.60}$$

In the continuum limit $\varepsilon \rightarrow 0$ this becomes, of course, the eigenvalue of the ordinary time derivative ∂_t, i.e., $-i$ times the frequency of the Fourier component ω. As a reminder of this we shall denote the eigenvalue of $i\nabla$ by Ω and have

$$i\nabla x(\omega) = \Omega x(\omega) \equiv \frac{i}{\varepsilon}\left(e^{-i\omega\varepsilon} - 1\right) x(\omega). \tag{2.61}$$

For the conjugate lattice derivative we find similarly

$$i\,\overline{\nabla}x(\omega) \;=\; \overline{\Omega}\,x(\omega) \;\equiv\; -\frac{i}{\varepsilon}\left(e^{\,i\omega\varepsilon}-1\right)x(\omega)\,. \tag{2.62}$$

where $\overline{\Omega}$ is the complex conjugate number of Ω, i.e., $\overline{\Omega} \equiv \Omega^{*}$, which has the same continuum limit ω. As a consequence, the eigenvalues of the negative lattice Laplacian $-\nabla\overline{\nabla} \equiv \overline{\nabla}\nabla$ are real and nonnegative,

$$-\nabla\overline{\nabla}\,x(\omega) \;=\; \frac{i}{\varepsilon}\left(e^{\,-i\omega\varepsilon}-1\right)\frac{i}{\varepsilon}\left(1-e^{\,i\omega\varepsilon}\right) \;=\; \frac{1}{\varepsilon^{2}}\left[2-2\cos(\omega\varepsilon)\right] \;\geq\; 0\,. \tag{2.63}$$

When decomposing the quantum fluctuations $x(t)\;[=\delta x(t)]$ into its Fourier components, not all eigenfunctions occur. Since $x(t)$ vanishes at the initial time $t = t_a$, we can restrict the decomposition to the sine functions

$$x(t) \;=\; \int_{-\infty}^{\infty} d\omega\,\sin\omega(t-t_a)\,x(\omega)\,. \tag{2.64}$$

The vanishing at the final time $t = t_b$ is enforced by a restriction of the frequencies ω to the discrete values

$$\upsilon_m \;=\; \frac{\pi m}{t_b - t_a} \;=\; \frac{\pi m}{(N+1)\,\varepsilon}\,. \tag{2.65}$$

Thus we are dealing with the Fourier series

$$x(t) \;=\; \sum_{m=1}^{\infty}\sqrt{\frac{2}{N+1}}\,\sin\upsilon_m(t-t_a)\,x(\upsilon_m)\,. \tag{2.66}$$

With $x(t)$ also the Fourier components $x(\omega)$ are real. A further restriction comes from the fact that for finite ε, the series has to represent $x(t)$ only at the discrete points $x(t_n)$, $n = 0, \ldots, N+1$. It is therefore sufficient to carry the sum only up to $m = N$ and expand $x(t_n)$ as follows

$$x(t_n) \;=\; \sum_{m=1}^{N}\sqrt{\frac{2}{N+1}}\,\sin\upsilon_m(t_n-t_a)\,x(\upsilon_m)\,. \tag{2.67}$$

The set of the expansion functions is orthogonal and complete in the sense

$$\frac{2}{N+1}\sum_{n=1}^{N} \sin \upsilon_m (t_n - t_a) \sin \upsilon_{m'} (t_n - t_a) = \delta_{mm'}, \tag{2.68}$$

$$\frac{2}{N+1}\sum_{m=1}^{N} \sin \upsilon_m (t_n - t_a) \sin \upsilon_m (t_{n'} - t_a) = \delta_{nn'}, \tag{2.69}$$

respectively (where $0 < m, m' < N + 1$). The orthogonality relation follows directly from rewriting the left-hand side of (2.68) as

$$\frac{2}{N+1}\frac{1}{2} Re \sum_{n=0}^{N+1} \left\{ exp\left[\frac{i\pi(m - m')}{N+1} n\right] - exp\left[\frac{i\pi(m + m')}{N+1} n\right]\right\}, \tag{2.70}$$

where we have extended the sum by one trivial term at each end without harm. Being of the geometric type this can be calculated right-away. For $m = m'$ it obviously adds up to 1 while for $m \neq m'$ it becomes

$$\frac{2}{N+1}\frac{1}{2} Re \left\{ \frac{1 - e^{\,i\pi(m-m')} e^{\,i\pi(m-m')/(N+1)}}{1 - e^{\,i\pi(m-m')/(N+1)}} - (m' \rightarrow - \mathrm{m}') \right\}. \tag{2.71}$$

The first expression in curly brackets is equal to 1 for even $m - m' \neq 0$ and imaginary for odd $m - m'$. For the second term the same thing is true for even and odd $m + m' \neq 0$, respectively. Since $m - m'$ and $m + m'$ are either both even or both odd, the right-hand side of of (2.68) vanishes for $m \neq m'$ [remember that both m and m' are $\varepsilon[0, N + 1]$ in the expansion (2.67) and thus in (2.68)]. The proof of the completeness relation (2.69) is completely analogous.

Inserting now the expansion (2.67) into the time-sliced fluctuation action (2.48) and using the orthogonality relation (2.68) we can write

$$S_{fl}^{N} = \frac{M}{2}\sum_{n=0}^{N} (\overline{\nabla} x_n)^2 = \frac{M}{2}\sum_{m=1}^{N+1} x(\upsilon_m)\, \Omega_m \overline{\Omega}_m x(\upsilon_m). \tag{2.72}$$

Thus the action decomposes into a sum of independent quadratic terms, with the discrete set of eigenvalues

$$\Omega_m \overline{\Omega}_m = \frac{1}{\varepsilon^2}[2 - 2\cos(\upsilon_m \varepsilon)] = \frac{1}{\varepsilon^2}\left[2 - 2\cos\left(\frac{\pi m}{N+1}\right)\right]. \qquad (2.73)$$

With the action being quadratic, the fluctuation factor (2.47) becomes a product of Gauss integrals

$$F_0^N(t_b - t_a) = \frac{1}{\sqrt{2\pi\hbar i\varepsilon / M}} \prod_{n=1}^{N} \left[\int_{-\infty}^{\infty} \frac{dx_n}{\sqrt{2\pi\hbar i\varepsilon / M}}\right]$$

$$\times \prod_{m=1}^{N} exp\left\{\frac{i}{\hbar}\frac{M}{2}\Omega_m \overline{\Omega}_m [x(\upsilon_m)]^2\right\}. \qquad (2.74)$$

To perform these we must transform the measure of integration from the local x_n to the Fourier components $x(\upsilon_m)$. Due to the orthogonality relation (2.68), the transformation has a unit determinant. Hence we can write directly

$$\prod_{n=1}^{N} dx_n = \prod_{m=1}^{N} dx(\upsilon_m). \qquad (2.75)$$

After this, we use the Fresnel formula (2.83) and see that (2.74) becomes

$$F_0^N(t_b - t_a) = \frac{1}{\sqrt{2\pi\hbar i\varepsilon / M}} \prod_{m=1}^{N} \frac{1}{\sqrt{\varepsilon^2 \Omega_m \overline{\Omega}_m}}. \qquad (2.76)$$

To calculate the product we use the formula [Gradsteyn and Ryznik 1980]

$$\prod_{m=1}^{N} \left(1 + x^2 - 2x\cos\frac{m\pi}{N+1}\right) = \frac{x^{2(N+1)} - 1}{x^2 - 1}. \qquad (2.77)$$

Taking the limit $x \rightarrow 1$ gives

$$\prod_{m=1}^{N} \varepsilon^{2}\Omega_m \bar{\Omega}_m = \prod_{m=1}^{N} 2\left(1 - \cos\frac{m\pi}{N+1}\right) = N + 1 . \tag{2.78}$$

Hence we obtain for the time-sliced fluctuation factor of a free particle

$$F_0^N(t_b - t_a) = \frac{1}{\sqrt{2\pi\hbar i(N+1)\varepsilon/M}} . \tag{2.79}$$

Expressing this in terms of $t_b - t_a$ gives

$$F_0(t_b - t_a) = \frac{1}{\sqrt{2\pi\hbar i(t_b - t_a)/M}} . \tag{2.80}$$

We have dropped the superscript N of $F_0^N(t_b - t_a)$ since the result is independent of the number of time slices, as as before the amplitude (2.34) Note that the dimension of the fluctuation factor is 1/length. In fact, we might introduce a length scale

$$l(t_b - t_a) \equiv \sqrt{2\pi\hbar(t_b - t_a)/M} \tag{2.81}$$

and write

$$F_0(t_b - t_a) = \frac{1}{\sqrt{i}\, l(t_b - t_a)} . \tag{2.82}$$

Together with (2.46), the full time displacement amplitude of a free particle (2.43) is therefore given again by

$$(x_b t_b | x_a t_a) = \frac{1}{\sqrt{2\pi i\hbar(t_b - t_a)/M}} \exp\left\{\frac{i}{\hbar}\frac{M}{2}\frac{(x_b - x_a)^2}{t_b - t_a}\right\} . \tag{2.83}$$

Exact Solution for the Harmonic Oscillator

A further problem that can be solved along similar lines is the time displacement amplitude of the linear oscillator,

$$(x_b t_b | x_a t_a) = \int \mathcal{D}'x \int \frac{\mathcal{D}p}{2\pi\hbar} exp\left\{\frac{i}{\hbar} S[p, x]\right\}$$

$$= \int \mathcal{D}x \, exp\left\{\frac{i}{\hbar} S[x, \dot{x}]\right\}, \quad (2.84)$$

with the canonical action

$$S[p, x] = \int_{t_a}^{t_b} dt \left(p\dot{x} - \frac{1}{2M} p^2 - \frac{M\omega^2}{2} x^2\right), \quad (2.85)$$

and the Lagrangian one

$$S[x, \dot{x}] = \int_{t_a}^{t_b} \frac{M}{2} \left(\dot{x}^2 - \omega^2 x^2\right). \quad (2.86)$$

As before, we shall proceed with the latter action, with the time-sliced form

$$S^N = \varepsilon \frac{M}{2} \sum_{n=1}^{N+1} \left[\left(\overline{\nabla} x_n\right)^2 - \omega^2 x_n^2\right]. \quad (2.87)$$

The path integral is again a product of Gauss integrals which can be done successively. In contrast to the free-particle case, however, this direct method is now more complicated than the fluctuation expansion in which the paths are split into classical path $x_{cl}(t)$ plus fluctuations $\delta x(t)$. The fluctuation expansion makes use of the fact that the action it is quadratic in $x = x_{cl} + \delta x$, and decomposes into the sum of a classical part

$$S_{cl} = \int_{t_a}^{t_b} dt \, \frac{M}{2} \left(\dot{x}_{cl}^2 - \omega^2 x_{cl}^2\right), \quad (2.88)$$

and a fluctuation part

$$S_{fl} = \int_{t_a}^{t_b} dt \frac{M}{2} \left[(\delta \dot{x})^2 - \omega^2 (\delta x)^2\right], \tag{2.89}$$

with the boundary condition

$$\delta x(t_a) = \delta x(t_b) = 0. \tag{2.90}$$

There is no mixed term, due to the extremality of the classical action. The equation of motion is

$$\ddot{x}_{cl} = -\omega^2 x_{cl}. \tag{2.91}$$

Thus, as in the free-particle case, the total time displacement amplitude splits into a classical and a fluctuation factor

$$(x_b t_b | x_a t_a) = \int \mathcal{D}x\, e^{iS[x,\dot{x}]/\hbar} = e^{iS_{cl}/\hbar} F_\omega(t_b - t_a). \tag{2.92}$$

The subscript ω of F_ω records the frequency of the oscillator.

The classical orbit connecting initial and final points is obviously

$$x_{cl}(t) = \frac{x_b \sin \omega(t - t_a) + x_a \sin \omega(t_b - t)}{\sin \omega(t_b - t_a)}. \tag{2.93}$$

Notice that this equation makes sense only if $t_b - t_a$ is not equal to integer multiples of π/θ which we shall always assume from now on.

By a partial integration we can rewrite the classical action S_{cl} as

$$S_{cl} = \int_{t_a}^{t_b} dt \frac{M}{2} \left[x_{cl}\left(-\ddot{x}_{cl} - \omega^2 x_{cl}\right)\right] + \frac{M}{2} x_{cl} \dot{x}_{cl}\Big|_{t_a}^{t_b}. \tag{2.94}$$

The first piece vanishes due to the equation of motion (2.91), so that we obtain the simple expression

$$S_{cl} = \frac{M}{2}[x_b \dot{x}_{cl}(t_b) - x_a \dot{x}_{cl}(t_a)]. \tag{2.95}$$

Since

$$\dot{x}_{cl}(t_a) = \frac{\omega}{\sin\omega(t_b - t_a)}[x_b - x_a \cos\omega(t_b - t_a)],$$

$$\dot{x}_{cl}(t_b) = \frac{\omega}{\sin\omega(t_b - t_a)}[x_b \cos\omega(t_b - t_a) - x_a], \tag{2.96}$$

we can rewrite the classical action as

$$S_{cl} = \frac{M\omega}{2\sin\omega(t_b - t_a)}[(x_b^2 + x_a^2)\cos\omega(t_b - t_a) - 2x_b x_a]. \tag{2.97}$$

We now turn to the fluctuation factor. After time slicing we use the matrix notation for the operator $-\nabla\overline{\nabla} - \omega^2$ and have to evaluate the multiple integral

$$F_\omega^N(t_b, t_a) = \frac{1}{\sqrt{2\pi\hbar i\varepsilon/M}} \prod_{n=1}^{N} \left[\int_{-\infty}^{\infty} \frac{d\delta x_n}{\sqrt{2\pi\hbar i\varepsilon/M}}\right]$$

$$\times \exp\left\{\frac{i}{\hbar}\frac{M}{2}\varepsilon\sum_n d\delta x_n\left[-\nabla\overline{\nabla} - \omega^2\right]_{nn'}\delta x_{n'}\right\}. \tag{2.98}$$

By going again to the Fourier components, the integrations factorize in the same way as for the free-particle expression (2.74). the only difference lies in the eigenvalues of the fluctuation operator which are now

$$\Omega_m\overline{\Omega}_m - \omega^2 = \frac{1}{\varepsilon^2}[2 - 2\cos(\upsilon_m\varepsilon)] - \omega^2 \tag{2.99}$$

stead of $\Omega_m\overline{\Omega}_m$. Hence we obtain directly

$$F_\omega^N(t_b, t_a) = \frac{1}{\sqrt{2\pi\hbar i\varepsilon/M}} \prod_{n=1}^{N} \frac{1}{\sqrt{e^2\Omega_m\overline{\Omega}_m - e^2\omega^2}}. \tag{2.100}$$

The product of these eigenvalues is found by introducing an auxiliary frequency $\tilde{\omega}$ as follows

$$\sin\frac{\tilde{\omega}\varepsilon}{2} \equiv \frac{\omega\varepsilon}{2}. \tag{2.101}$$

Then we can decompose the product as

$$\begin{aligned}\prod_{m=1}^{N}\left[\varepsilon^2\Omega_m\bar{\Omega}_m - \varepsilon^2\omega^2\right] &= \prod_{m=1}^{N}\left[\varepsilon^2\Omega_m\bar{\Omega}_m\right]\prod_{m=1}^{N}\left[\frac{\varepsilon^2\Omega_m\bar{\Omega}_m - \varepsilon^2\omega^2}{\varepsilon^2\Omega_m\bar{\Omega}_m}\right] \\ &= \prod_{m=1}^{N}\left[\varepsilon^2\Omega_m\bar{\Omega}_m\right]\left[\prod_{m=1}^{N}\left(1-\frac{\sin^2\frac{\varepsilon\tilde{\omega}}{2}}{\sin^2\frac{m\pi}{2(N+1)}}\right)\right].\end{aligned} \tag{2.102}$$

The first factor is equal to $(N+1)$ from (2.78). The second factor, the product of ratios of eigenvalues, is found from the standard formula [Gradsteyn and Ryznik 1980]

$$\prod_{m=1}^{N}\left(1-\frac{\sin^2 x}{\sin^2\frac{m\pi}{2(N+1)}}\right) = \frac{1}{\sin 2x}\frac{\sin[2(N+1)x]}{(N+1)}. \tag{2.103}$$

Hence we arrive at the fluctuation determinant

$$\det{}_N\left(-\varepsilon^2\nabla\bar{\nabla} - \varepsilon^2\omega^2\right) = \prod_{m=1}^{N}\left[\varepsilon^2\Omega_m\bar{\Omega}_m - \varepsilon^2\omega^2\right] = \frac{\sin\tilde{\omega}(t_b - t_a)}{\sin\tilde{\omega}\varepsilon}, \tag{2.104}$$

and the fluctuation factor is given by

$$F_{\omega}^{N}(t_b, t_a) = \frac{1}{\sqrt{2\pi i\hbar/M}}\sqrt{\frac{\sin\tilde{\omega}\varepsilon}{\varepsilon\sin\tilde{\omega}(t_b - t_a)}}, \quad t_b - t_a < \pi/\tilde{\omega}. \tag{2.105}$$

We have agreed earlier [see Eq. (2.35)] that $\sqrt{i}$ should always mean $e^{i\pi/4}$. This is why result (2.124) is valid only for

$$t_b - t_a < \pi / \widetilde{\omega} . \tag{2.106}$$

Let us now take the continuum limit, $\varepsilon \to 0, N \to \infty$. Then the auxiliary frequency $\widetilde{\omega}$ tends to ω and the fluctuation determinant becomes

$$det_N \left(- \varepsilon^2 \nabla \overline{\nabla} - \varepsilon^2 \omega^2 \right) \xrightarrow{\varepsilon \to 0} \frac{\sin \omega (t_b - t_a)}{\omega \varepsilon} . \tag{2.107}$$

The fluctuation factor $F_{\omega}^{N}(t_b - t_a)$ goes over into

$$F_{\omega}(t_b - t_a) = \frac{1}{\sqrt{2\pi i \hbar / M}} \sqrt{\frac{\omega}{\sin \omega (t_b - t_a)}} \tag{2.108}$$

with the phase for $t_b - t_a > \pi / \omega$ determined from the $i\eta$ -prescription.

In the limit $\omega \to 0$, both fluctuation factors agree, of course, with the free-particle result (2.80).

In the continuum limit, the ratios of eigenvalues in (2.102) can also be calculated in the following simpler way. We perform the limit $\varepsilon \to 0$ directly in each factor. This gives

$$\frac{\varepsilon^2 \Omega_m \overline{\Omega}_m - \varepsilon^2 \omega^2}{\varepsilon^2 \Omega_m \overline{\Omega}_m} = 1 - \frac{\varepsilon^2 \omega^2}{2 - 2\cos(\omega_m \varepsilon)}$$

$$\xrightarrow{\varepsilon \to 0} 1 - \frac{\omega^2 (t_b - t_a)^2}{\pi^2 m^2} . \tag{2.109}$$

In the limit $\varepsilon \to 0$, the number N goes to infinity so that we wind up with an infinite product of these factors. We can then use the well-known infinite-product formula for the sine function.

$$\sin x = x \prod_{m=1}^{\infty} \left(1 - \frac{x^2}{m^2 \pi^2} \right), \tag{2.110}$$

and find directly

$$\prod_m \frac{\Omega_m\bar{\Omega}_m}{\bar{\Omega}_m\Omega_m - \omega^2} \xrightarrow{\varepsilon \to 0} \prod_{m=1}^{\infty} \frac{\upsilon_m^2}{\upsilon_m^2 - \omega^2} = \to \frac{\omega(t_b - t_a)}{\sin \omega(t_b - t_a)}, \tag{2.111}$$

so that the fluctuation factor in the continuum becomes again (2.108).

Combining the fluctuation factor with the classical amplitude, the total amplitude of the linear oscillator in the continuum reads.

$$\begin{aligned}(x_b t_b | x_a t_a) &= \int \mathcal{D}x(t) \exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} dt \frac{M}{2}\left(\dot{x}^2 - \omega^2 x^2\right)\right\} \\ &= \frac{1}{\sqrt{2\pi i \hbar / M}} \sqrt{\frac{\omega}{\sin \omega(t_b - t_a)}} \\ &\quad \times \exp\left\{\frac{i}{2\hbar}\frac{M\omega}{\sin \omega(t_b - t_a)}\left[\left(x_b^2 + x_a^2\right)\cos \omega(t_b - t_a) - 2x_b x_a\right]\right\}.\end{aligned} \tag{2.112}$$

Path Integrals with the External Source

Important information on a quantum mechanical system is carried by the correlation functions of the path $x(t)$, which are defines as the functional averages of a product of the path positions at various times, $x(t_1) \ldots x(t_n)$. Objects of this type are often observable in simple scattering experiments. The most efficient way of extracting them from a theory proceeds by extending the action by a external source term and studying the response of the system to a disturbance. In this chapter we shall do this with the harmonic action treated in the last chapter, with a source term linear in the path $x(t)$ coupled to a so-called *current* or *external force* $j(t)$. Such a term does not destroy the solvability of the path integral. The resulting amplitude is a simple functional of $j(t)$ which serves to calculate the temporal correlation functions of the system. It is the celebrated generating functional of the theory. Now we consider a harmonic oscillator with an action[Feynman and Hibbs 1965, Schulman 1981, Kleinert 1990]

$$S_0 = \int_{t_a}^{t_b} dt \frac{M}{2} \left(\dot{x}^2 - \omega^2 x^2\right), \tag{2.113}$$

and suppose that it is driven by an external source current or an external force $j(t)$ coupled linearly to the particle coordinate $x(t)$, with the action

$$S_{source} = \int_{t_a}^{t_b} dt\, x(t) j(t) \tag{2.114}$$

Since the total action

$$S = S_0 + S_{source} . \tag{2.115}$$

is at most quadratic in $x, \dot{x}$, it is easy to solve the path integral also in the presence of this source term. In particular, the source term does not destroy the factorization property (2.92) of the time displacement amplitude into a classical amplitude $e^{iS_{cl,j}/\hbar}$ and a fluctuation factor $F_{\omega,j}(t_b, t_a)$,

$$(x_b t_b | x_a t_a) = e^{(i/\hbar) S_{cl,j}} F_{\omega,j}(t_b, t_a) \tag{2.116}$$

(see Eq. (2.92)). Here $S_{cl,j}$ is the action for the classical orbit $x_{cl,j}(t)$ which minimizes the total action S in the presence of the source term, with the equation of motion

$$\ddot{x}_{cl,j}(t) + \omega^2 x_{cl,j}(t) = j(t) . \tag{2.117}$$

In what follows we shall first work with the classical orbit $x_{cl}(t)$ which extremizes the action without the source term,

$$x_{cl}(t) = \frac{x_b \sin \omega(t - t_a) + x_a \sin \omega(t_b - t)}{\sin \omega(t_b - t_a)}. \qquad (2.118)$$

We separate all paths into a sum of this classical orbit, $x_{cl}(t)$, plus a fluctuating part $\delta x(t)$,

$$x(t) = x_{cl}(t) + \delta x(t). \qquad (2.119)$$

The action separates into

$$S = S_0 + S_{source} = (S_{0,cl} + S_{source,cl}) + (S_{0,fl} + S_{source,fl})$$
$$\equiv S_{cl} + S_{fl}. \qquad (2.120)$$

The time displacement amplitude can then be expressed as follows

$$(x_b t_b | x_a t_a) = e^{(i/\hbar)S_{cl}} \int \mathcal{D}x \exp\left\{\frac{i}{\hbar} S_{fl}\right\}$$
$$= e^{(i/\hbar)(S_{0,cl} + S_{source,cl})} \int \mathcal{D}x \exp\left\{\frac{i}{\hbar}(S_{0,fl} + S_{source,fl})\right\}. \qquad (2.121)$$

The classical action $S_{0,cl}$ is known from Section 2.3, Eq. (2.97),

$$S_{0,cl} = \frac{M\omega}{2 \sin\omega(t_b - t_a)}\left[(x_b^2 + x_a^2)\cos \omega(t_b - t_a) - 2x_b x_a\right], \qquad (2.123)$$

and the classical part of the source term is from (2.118),

$$S_{source,cl} = \int_{t_a}^{t_b} dt\, x_{cl}(t) j(t)$$
$$= \frac{1}{\sin \omega(t_b - t_a)} \int_{t_a}^{t_b} dt[x_a \sin \omega(t_b - t) + x_b \sin \omega(t - t_a)] j(t). \qquad (2.124)$$

consider now the fluctuation part of the action, $S_{fl} = S_{0fl} + S_{source,fl}$. Since $x_{cl}(t)$ extremizes only the action S_0 without the source, S_{fl} will contain a term linear in $\delta x(t)$. After a partial integration [which respects the vanishing of $\delta x(t_a)$ at the ends], it can be written as

$$S_{fl} = \frac{M}{2}\int_{t_a}^{t_b} dt\, dt'\, \delta x(t)\, \mathcal{D}(t,t')\, \delta x(t') + \int_{t_a}^{t_b} dt\, \delta x(t) j(t), \qquad (2.125)$$

where $\mathcal{D}(t,t')$ is the differential operator

$$\mathcal{D}(t,t') = \left(-\partial_t^2 - \omega^2\right)\delta(t-t'), \quad t,t' \in (t_a, t_b). \qquad (2.126)$$

It is a functional matrix in the space of the t-dependent functions vanishing at the ends t_a, t_b. The functional inverse of this matrix, $\mathcal{D}^{-1}(t,t')$, is formally defined by the relation

$$\int_{t_a}^{t_b} dt'\, \mathcal{D}(t'',t')\, \mathcal{D}^{-1}(t',t) = \delta(t''-t), \quad t'',t \in (t_a, t_b). \qquad (2.127)$$

It is the standard classical Green function of the harmonic oscillator

$$G(t,t') \equiv \mathcal{D}^{-1}(t,t') = \left(-\partial_t^2 - \omega^2\right)^{-1}\delta(t,t'), \quad t,t' \in (t_a, t_b). \qquad (2.128)$$

This definition is not unique since it leaves room for the addition to $\mathcal{D}^{-1}(t',t)$ of an arbitrary solution $H(t,t')$ of the homogeneous equation $\int_{t_a}^{t_b} dt'\, \mathcal{D}(t'',t') H(t',t) = 0$. Appropriate boundary conditions, however, will remove this freedom. In the fluctuation action (2.125), we now perform a quadratic completion via a shift in $\delta x(t)$ to

$$\delta\tilde{x}(t) \equiv \delta x(t) + \frac{1}{2m}\int_{t_a}^{t_b} dt'\, G(t,t')j(t'). \tag{2.129}$$

Then the action becomes diagonal in $\delta\tilde{x}$ and j,

$$S_{fl} = \int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt \left[\frac{M}{2}\delta\tilde{x}(t)\mathcal{D}(t,t')\,\delta\tilde{x}(t') - \frac{1}{2M} j(t)\,G(t,t')j(t')\right]. \tag{2.130}$$

The Green function obeys the same boundary condition as the fluctuations $\delta x(t)$,

$$G(t,t') = 0 \quad for \quad \begin{cases} t = t_b, & t' \; arbitrary, \\ t \; arbitrary, & t' = t_a. \end{cases} \tag{2.131}$$

Therefore, the shifted fluctuations $\delta\tilde{x}(t)$ of (2.129) vanish at the ends and run through the same functional space as the original $\delta x(t)$. The measure of path integration $\mathcal{D}\delta x(t)$ is obviously unchanged by the simple shift (2.129). Hence the path integral $\mathcal{D}\delta\tilde{x}$ over $e^{iS_{fl}/\hbar}$ with the action (2.130) gives, by the first term in S_{fl}, the usual fluctuation factor $F_\omega(t_b - t_a)$ calculated in (2.108)

$$F_\omega(t_b - t_a) = \frac{1}{\sqrt{2\pi i\hbar/M}}\sqrt{\frac{\omega}{\sin\omega(t_b - t_a)}}. \tag{2.132}$$

The only effect of the source term is a multiplicative factor coming from the second term in (2.130) quadratic in the source j, plus the classical contribution (2.124). The total time displacement amplitude in the presence of a source term can therefore be written as the product

$$(x_b t_b | x_a t_a)_j = (x_b t_b | x_a t_a)_0 F_{source,cl}[j]\, F^{qu}_{source,fl}[j], \tag{2.133}$$

where $(x_b t_b | x_a t_a)_0$ is the source-free time displacement amplitude

$$(x_b t_b | x_a t_a)_0 = e^{(i/\hbar)S_{0,cl}} F_\omega(t_b - t_a) = \frac{1}{\sqrt{2\pi i\hbar/M}}\sqrt{\frac{\omega}{\sin\omega(t_b - t_a)}}$$

$$\times \exp\left\{\frac{i}{2\hbar}\frac{M\omega}{\sin\omega(t_b - t_a)}\left[\left(x_b^2 + x_a^2\right)\cos\omega(t_b - t_a) - 2x_b x_a\right]\right\}, \quad (2.134)$$

and $F_{source,cl}[j]$ the amplitude involving the classical source action

$$F_{source,cl}[j] = e^{(i/\hbar)S_{source,cl}} =$$

$$\exp\left\{\frac{i}{\hbar}\frac{1}{\sin\omega(t_b - t_a)}\int_{t_a}^{t_b} dt\left[x_a \sin\omega(t_b - t) + x_b \sin\omega(t - t_a)\right]j(t)\right\}, \quad (2.135)$$

while $F^{qu}_{source,fl}[j]$ is the exponential of the quadratic source terms caused by the fluctuations,

$$F_{source,fl}[j] = \exp\left\{\frac{i}{\hbar}S^{qu}_{source,fl}[j]\right\}, \quad (2.136)$$

with the action being the second term in (4.17) :

$$S^{qu}_{source,fl} = -\frac{1}{2M}\int_{t_a}^{t_b} dt\int_{t_a}^{t_b} dt' j(t)\,G(t,t')\,j(t'). \quad (2.137)$$

To complete the result it remains to calculate the Green function $G(t, t')$.

We now calculate the Green function of the differential operator $-\partial_t^2 - \omega^2$ appearing in the amplitude (2.127). By Eq. (2.128) it is formally obtained by inverting the second-order differential operator $-\partial_t^2 - \omega^2$,

$$G(t,t') = \left(-\partial_t^2 - \omega^2\right)^{-1}\delta(t - t'), \quad t, t' \in (t_a, t_b), \quad (2.138)$$

except for the ambiguity by solutions of the homogeneous equation. The boundary conditions for $G(t, t')$ are those of the fluctuations $\delta x(t)$, i.e. $G(t, t')$ has to vanish if either t or t' or both hit and end point t_a or t_b (Dirichlet boundary condition). The Green function is symmetric in t and t'. There are several ways of finding such a $G(t, t')$.

The simplest is by the following direct construction the result of which will be referred to as the *Wronskian construction* . For different time arguments, $t > t'$ or $t < t'$, $G(t, t')$ has to solve the homogeneous differential equations

$$\left(-\partial_t^2 - \omega^2\right)G(t, t') = 0, \quad \left(-\partial_{t'}^2 - \omega^2\right)G(t, t') = 0. \tag{2.139}$$

The Green function $G(t, t')$ must therefore be a linear combination of two independent solutions of the homogeneous differential equation in t as well as t', with the boundary conditions to vanish as the end points. For $t > t'$ we see that $G(t, t')$ must be proportional to $\sin \omega(t_b - t)$ at well as to $\sin \omega(t' - t_a)$ which vanish at the upper and at lower end, respectively. This leaves only the product solution

$$G(t, t') = C \sin \omega(t_b - t) \sin \omega(t' - t_a), \quad t > t'. \tag{2.140}$$

For $t < t'$ we obtain similarly

$$G(t, t') = C \sin \omega(t_b - t') \sin \omega(t - t_a), \quad t < t'. \tag{2.141}$$

The two cases can be written concisely as

$$G(t, t') = C \sin \omega(t_b - t_>) \sin \omega(t_< - t_a), \tag{2.142}$$

where the symbols $t_{>,<}$ denote the larger and smaller of the two times t, t', respectively. The unknown constant C is fixed by considering the limit of coincident times, $t = t'$. There the time derivative of $G(t, t')$ has a discontinuity. For $t > t'$ it is given by

$$\partial_t G(t, t') = -C\omega \cos\omega(t_b - t) \sin \omega(t' - t_a). \tag{2.143}$$

and for $t < t'$ by

$$\partial_t G(t, t') = C\omega \sin\omega(t_b - t') \cos\omega(t - t_a), \tag{2.144}$$

At $t = t'$ we find the discontinuity

$$\partial_t G(t,t')\big|_{t=t'+\varepsilon} - \partial_t G(t,t')\big|_{t=t'-\varepsilon} = -C\omega\ \sin\omega(t_b - t_a)\,. \qquad (2.145)$$

This implies that $-\partial_t^2 G(t,t')$ is proportional to a δ-function,

$$-\partial_t^2 G(t,t') = C\omega\ \sin\omega(t_b - t_a)\,\delta(t-t'). \qquad (2.146)$$

Hence the desired Green function is determined to be

$$G(t,t') = \frac{1}{\omega\ \sin\omega(t_b - t_a)} \sin\omega(t_> - t_b)\sin\omega(t_< - t_b). \qquad (2.147)$$

This exists only if $t_b - t_a$ is not equal to an integer multiple of π/ω-, just as the amplitude without external sources. This completes the calculation of the time displacement amplitude in the presence of an external source, (2.133).

Time Displacement Amplitude with a Source Term

Give the Green function $G(t,t')$, the quadratic source contribution to the fluctuation factor (2.136) is given explicitly by

$$S^{qu}_{source,fl} = -\frac{1}{2M}\int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt' G(t,t') j(t) j(t') =$$

$$-\frac{1}{M}\frac{1}{\omega \sin\omega(t_b - t_a)}\int_{t_a}^{t_b} dt \int_{t_a}^{t} dt' \sin\omega(t_b - t)\sin\omega(t' - t_a) j(t) j(t'). \qquad (2.148)$$

Altogether, the path integral in the presence of an external source $j(t)$ reads

$$(x_b t_b | x_a t_a)_j = \int \mathcal{D}x(t) exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b} dt \left[\frac{M}{2}\left(\dot{x}^2 - \omega^2 x^2\right) + jx\right]\right\} =$$

$$\frac{\sqrt{M\omega}}{\sqrt{2\pi i\hbar \sin\omega(t_b - t_a)}} exp\left\{\frac{i}{2\hbar}\frac{M\omega}{\sin\omega(t_b - t_a)}\left[\left(x_b^2 + x_a^2\right)\cos\omega(t_b - t_a) - 2x_b x_a\right]\right\}$$

$$\times exp \left\{\frac{i}{\hbar}\frac{1}{sin\ \omega(t_b - t_a)}\int_{t_a}^{t_b} dt[x_a\ sin\ \omega(t_b - t) + x_b\ sin\ \omega(t - t_a)]j(t)\right.$$

$$\left.-\frac{i}{\hbar}\frac{1}{M\omega sin\ \omega(t_b - t_a)}\int_{t_a}^{t_b} dt \int_{t_a}^{t} dt' sin\ \omega(t_b - t)\ sin\ \omega(t' - t_a)j(t)j(t')\right\}. \tag{2.149}$$

If the source does not depend on time, the last two exponents in (2.149) can be integrated in time and become

$$\frac{i}{\hbar}\left\{\frac{1}{\omega sin\ \omega(t_b - t_a)}[1 - cos\omega(t_b - t_a)](x_b + x_a)j\right.$$

$$\left.+\frac{1}{2M\omega^4}\left(\omega(t_b - t_a) + 2\frac{cos\omega(t_b - t_a) - 1}{sin\omega(t_b - t_a)}j\right)^2\right\}. \tag{2.150}$$

Actually, this $j =$ const result could have been obtained simpler by taking the potential plus source term in the action,

$$-\int_{t_a}^{t_b} dt\left(\frac{M}{2}\omega^2 x^2 - xj\right), \tag{2.151}$$

and quadratically completing it to the form

$$-\int_{t_a}^{t_b} dt\ \frac{M}{2}\omega^2\left(x - \frac{j}{M\omega^2}\right)^2 + \frac{t_b - t_a}{2M\omega^2}j^2, \tag{2.152}$$

displaying a harmonic potential shifted in x by $-j/M\omega^2$. The time displacement amplitude can plus immediately be written down as follows

$$(x_b t_b | x_a t_a)_{j = const} = \frac{\sqrt{M\omega}}{\sqrt{2\pi i\ \hbar sin\ \omega(t_b - t_a)}} exp\left\{\frac{i}{2\hbar}\frac{M\omega}{sin\ \omega(t_b - t_a)}\right.$$

$$\times\left[\left(\left(x_b - \frac{j}{M\omega^2}\right)^2 + \left(x_a - \frac{j}{M\omega^2}\right)^2\right)cos\omega(t_b - t_a)\right.$$

$$\left.\left.-2\left(x_b - \frac{j}{M\omega^2}\right)\left(x_a - \frac{j}{M\omega^2}\right)\right] + \frac{i}{\hbar}\frac{t_b - t_a}{2M\omega^2}j^2\right\}. \tag{2.153}$$

From this we read off the total source action

$$S^{eff}_{source} \equiv S_{source,\,cl} + S^{qu}_{source,\,fl} =$$

$$\frac{1-\cos\omega(t_b-t_a)}{\omega\sin\omega(t_b-t_a)}(x_b+x_a)_j + \frac{1}{2M\omega^2}\left(t_b-t_a+2\frac{\cos\omega(t_b-t_a)-1}{\omega\sin\omega(t_b-t_a)}\right)j^2. \quad (2.154)$$

In the limit of a free particle, $\omega \rightarrow 0$, the j = const result becomes particularly simple

$$(x_b t_b | x_a t_a)_{j=const} = \frac{1}{\sqrt{2\pi i\hbar(t_b-t_a)/M}} \exp\left\{\frac{i}{\hbar}\frac{M}{2}\frac{(x_b-x_a)^2}{t_b-t_a}\right\}$$
$$\times \exp\left\{\frac{i}{\hbar}\left[\frac{1}{2}(x_b+x_a)(t_b-t_a)_j - \frac{1}{12}(t_b-t_a)^3\frac{j^2}{M}\right]\right\}. \quad (2.155)$$