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SOME TYPES OF CONVERGENCE OF SEQUENCES OF FUNCTIONS

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Csaszar และ Laczkovich ได้นิยามและศึกษาการลู่เข้าของลำดับของฟังก์ชันของจำนวนจริงแบบเท่ากันและแบบวิยุต ต่อมา Papanastassiou ได้นิยามและศึกษาการลู่เข้าของลำดับ แบบเท่ากันอย่างเอกรูปและแบบวิยุตอย่างเอกรูป ยิ่งไปกว่านั้นเราได้ศึกษาถึงความสัมพันธ์และคุณสมบัติของการลู่เข้าของลำดับของฟังก์ชันทั้ง 4 รูปแบบนี้ พร้อมทั้งคุณสมบัติของคลาสของฟังก์ชัน ซึ่งเป็นลิมิตเท่ากันอย่างเอกรูปและลิมิตวิยุตอย่างเอกรูป

ต่อมาได้นิยามและศึกษาการลู่เข้าของลำดับ แบบแอลฟา(การลู่เข้าแบบต่อเนื่อง) บนปริภูมิอิงระยะทาง แล้วทำให้ได้ผลสรุปที่ทราบกันดีว่า ถ้า X เป็นปริภูมิอิงระยะทางกระชับแล้ว $f_n \xrightarrow{\alpha} f$ จะได้ว่า $f_n \xrightarrow{u} f$ ซึ่ง u แทนการลู่เข้าอย่างเอกรูป ในบทกลับ Hola และ Salat ได้พิสูจน์โดยใช้คุณสมบัติของปริภูมิอิงระยะทางกระชับ จึงทำให้ได้ว่า X เป็นปริภูมิอิงระยะทางกระชับก็ต่อเมื่อ $f_n \xrightarrow{\alpha} f$ จะได้ว่า $f_n \xrightarrow{u} f$ เป็นทฤษฎีบทที่เป็นจริง จากนั้นได้นิยามการลู่เข้าของลำดับแบบแอลฟาเท่ากันอย่างเอกรูป และได้ศึกษาถึงความสัมพันธ์และคุณสมบัติของการลู่เข้าของลำดับของฟังก์ชันทั้ง 6 รูปแบบนี้ พร้อมทั้งศึกษาว่าการลู่เข้าแบบใดที่สามารถนำไปแทนใน u แล้วยังคงทำให้ทฤษฎีบทที่กล่าวไปนั้นเป็นจริง

ภาควิชา............คณิตศาสตร์.......... ลายมือชื่อนิสิต..

สาขาวิชา..........คณิตศาสตร์.......... ลายมือชื่อที่ปรึกษาวิทยานิพนธ์หลัก..

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Csaszar and Laczkovich defined and studied discrete convergence as well as the equal convergence for sequences of real-valued functions. Then definitions of uniformly equal convergence and uniformly discrete convergence were proposed by Papanastassiou as well. Nonetheless, the properties and relations of these convergent sequences of functions are not studied yet. In this project we will study the relations and the properties of the convergent sequences of functions. In addition, the properties of classes of functions which are uniformly equal limits and uniformly discrete limits will be then studied. The notion of α - convergence for sequences of real-valued functions on a metric space will be introduced. The result is that if X is a compact metric space then $f_n \xrightarrow{\alpha} f$ implies $f_n \xrightarrow{u} f$ where u denotes uniform convergence. Conversely, Hola and Salat showed that is true only if X is a compact metric space, thus characterizing a compactness in terms of these convergences. Subsequently, we define new types of convergence called α - uniform equal convergence. The relations of these convergences are then studied. The properties of these convergences are also investigated in order that we could obtain sufficient conditions for a metric space to be compact in term of it.

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Principal Advisor's Signature : ....

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CONTENTS

CHAPTER I
VARIOUS TYPES OF CONVERGENCE

Let X be a non-empty set. By a function on X, we mean a real-valued function on X. Let Φ be an arbitrary class of functions defined on X. Then we have the following definitions.

Definition 1.1. *A sequence of functions (f_n) in Φ is said to* ***converge uniformly*** *to a function f in Φ (written as $f_n \xrightarrow{u.} f$) if for every $\varepsilon > 0$, there exists a natural number n_0 such that $n \geq n_0$ implies $|f_n(x) - f(x)| < \varepsilon$, for all $x \in X$. That is*

$$f_n \xrightarrow{u.} f \iff \forall \varepsilon > 0 \, \exists n_0 \in \mathbb{N} \, \forall n \geq n_0 \, \forall x \in X, |f_n(x) - f(x)| < \varepsilon.$$

Definition 1.2. *A sequence of functions (f_n) in Φ is said to* ***converge equally*** *to a function f in Φ (written as $f_n \xrightarrow{e.} f$) if there exists a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ of positive reals converging to zero such that, for each $x \in X$, there exists a natural number $n(x)$ satisfying $|f_n(x) - f(x)| < \varepsilon_n$, for each $n \geq n(x)$. That is*

$$f_n \xrightarrow{e.} f \iff \exists (\varepsilon_n) \to 0 \, \forall x \in X \, \exists n(x) \in \mathbb{N} \, \forall n \geq n(x), |f_n(x) - f(x)| < \varepsilon_n.$$

Also, (f_n) in Φ is said to ***converge discretely*** *to a function f in Φ (written as $f_n \xrightarrow{d.} f$) if, for every $x \in X$, there exists $n(x) \in \mathbb{N}$ such that $f_n(x) = f(x)$ for all $n \geq n(x)$. That is*

$$f_n \xrightarrow{d.} f \iff \forall x \in X \, \exists n(x) \in \mathbb{N} \, \forall n \geq n(x), f_n(x) = f(x).$$

Example 1.3. Let $f_n(x) = x^n$ for $x \in [0, 1)$. Then the sequence (f_n) converges equally to the zero function on $[0, 1)$.

Proof. Let $x \in [0, 1)$ and $(\varepsilon_n) = (\frac{1}{n})$. Note that if $\alpha > 1$, then there exists $n_0 \in \mathbb{N}$ such that $\alpha^n > n$ for all $n \geq n_0$. Since $\frac{1}{x} > 1$, there exists $n(x) \in \mathbb{N}$ such that $(\frac{1}{x})^n > n$, for each $n \geq n(x)$. Thus $x^n < \frac{1}{n}$, for each $n \geq n(x)$. Hence $|f_n(x)| < \varepsilon_n$, for each $n \geq n(x)$. Therefore $f_n \xrightarrow{e.} 0$ □

Example 1.4. Let

$$f_n(x) = \begin{cases} 0 & \text{for } x \in (-\infty, n-1]; \\ x-n+1 & \text{for } x \in [n-1, n]; \\ 1 & \text{for } x \in [n-1, \infty). \end{cases}$$

Then the sequence (f_n) converges discretely to the zero function on $[0, 1)$.

Proof. Let $x \in \mathbb{R}$. Then there exists $n(x) \in \mathbb{N}$ such that $n(x) - 2 \leq x \leq n(x) - 1$. Since $x \in (-\infty, n(x) - 1)$, we have $f_n(x) = 0$, for all $n \geq n(x)$. Hence $f_n(x) = 0$, for all $n \geq n(x)$. Therefore $f_n \xrightarrow{d.} 0$ □

Definition 1.5. *A sequence of functions (f_n) in Φ is said to* ***converge uniformly equally*** *to a function f in Φ (written as $f_n \xrightarrow{u.e.} f$) if there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals converging to zero and a natural number n_0 such that the cardinality of the set $\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\}$ is at most n_0, for each $x \in X$. That is*

$$f_n \xrightarrow{u.e.} f \iff \exists(\varepsilon_n) \to 0 \, \exists n_0 \in \mathbb{N} \, \forall x \in X, |\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\}| \leq n_0.$$

Example 1.6. Let $0 < \delta < 1$ and $f_n(x) = x^n$ for $x \in [0, \delta]$. Then the sequence (f_n) converges uniformly equally to the zero function on $[0, \delta]$.

Proof. For $(\varepsilon_n) = (\delta^n + \frac{1}{n})$, we have

$$|\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\}| \leq n_0 = 1, \text{ for each } x \in [0, \delta]$$

(Since $f_n(x) = x^n \leq \delta^n < \delta^n + \frac{1}{n}$, for each $n \in \mathbb{N}$ and $x \in [0, \delta]$.) Hence $f_n \xrightarrow{u.e.} 0$ with witnessing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$. □

Example 1.7. Let $f_n(x)$ be the characteristic function of the interval $[n-1, n]$, $n \in \mathbb{N}$ given by

$$f_n(x) = \begin{cases} 1 & \text{for } x \in [n-1, n]; \\ 0 & \text{for } x \notin [n-1, n]. \end{cases}$$

Then the sequence (f_n) converges uniformly equally to the zero function on $\mathbb{R}$.

Proof. For $(\varepsilon_n) = (\frac{1}{n})$, we have $\left|\{n \in \mathbb{N} : |f_n(x)| \geq \frac{1}{n}\}\right| \leq n_0 = 1$, for each $x \in \mathbb{R}$.
Case : 1. Let $x \in [n-1, n]$. Then $f_n(x) = 1$. Thus

$$\left|\left\{n \in \mathbb{N} : |f_n(x)| \geq \frac{1}{n}\right\}\right| \leq 1$$

Case : 2. Let $x \notin [n-1, n]$. Then $f_n(x) = 0$. Hence $|f_n(x) - f(x)| = 0$. Thus $\left|\{n \in \mathbb{N} : |f_n(x)| \geq \frac{1}{n}\}\right| = 0$. From cases 1, 2, we have

$$|\{n \in \mathbb{N} : |f_n(x)| \geq \varepsilon_n\}| \leq n_0 = 1 \qquad \text{for each } x \in \mathbb{R}.$$

Hence $f_n \xrightarrow{u.e.} 0$ with witnessing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$. □

Example 1.8. Let $f_n : [0,1) \to \mathbb{R}$ be defined by $f_n(x) = x^n, n \in \mathbb{N}$ and $f \equiv 0$. Then the sequence (f_n) does not converge uniformly equally to the zero function on $[0,1)$.

Proof. Suppose on the contrary that $f_n \xrightarrow{u.e.} 0$. Then there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals converging to zero and a natural number n_0 such that the cardinality of the set $\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\}$ is at most n_0. Since $(\varepsilon_n) \to 0$, there exists $n_1 \in \mathbb{N}$ such that $\epsilon_n < \frac{1}{2}$ for all $n \geq n_1$. Then $(\frac{1}{2})^{\frac{1}{n_1+2n_0}} \in [0,1)$ and $(\frac{1}{2})^{\frac{m}{n_1+2n_0}} > \frac{1}{2} > \varepsilon_m$ for $n_1 \leq m \leq n_1 + 2n_0 - 1$. Hence $\left|f_m((\frac{1}{2})^{\frac{1}{n_1+2n_0}})\right| > \varepsilon_m$ for $n_1 \leq m \leq n_1 + 2n_0 - 1$. Then

$$\left|\left\{n \in \mathbb{N} : \left|f_n((\frac{1}{2})^{\frac{1}{n_1+2n_0}})\right| > \varepsilon_n\right\}\right| \geq n_1 + 2n_0 - 1 - n_1 + 1 = 2n_0 > n_0,$$

which is a contradiction. □

Definition 1.9. *A sequence of functions (f_n) in Φ is said to* ***converge uniformly discretely*** *to a function f in Φ (written as $f_n \xrightarrow{u.d.} f$) if there exists a natural number n_0 such that the cardinality of the set $\{n \in \mathbb{N} : |f_n(x) - f(x)| > 0\}$ is at most n_0, for each $x \in X$. That is*

$$f_n \xrightarrow{u.d.} f \iff \exists n_0 \in \mathbb{N}\, \forall x \in X, |\{n \in \mathbb{N} : |f_n(x) - f(x)| > 0\}| \leq n_0.$$

Example 1.10. Let f_n be the characteristic function of the interval $[n-1, n]$, $n \in \mathbb{N}$ given by

$$f_n(x) = \begin{cases} 1 & \text{for } x \in [n-1, n]; \\ 0 & \text{for } x \notin [n-1, n]. \end{cases}$$

Then the sequence (f_n) converges uniformly discretely to the zero function on $\mathbb{R}$.

Proof. It is obvious that there exists n_0 such that

$$|\{n \in \mathbb{N} : |f_n(x)| > 0\}| \leq n_0 = 1 \qquad \text{for each } x \in \mathbb{R}.$$

Hence the sequence (f_n) converges uniformly discretely to the zero function on $\mathbb{R}$. □

Example 1.11. Let $0 < \delta < 1$ and $f_n(x) = x^n$ for $x \in [0, \delta]$. Then the sequence (f_n) does not converge uniformly discretely to the zero function on $[0, \delta]$.

Proof. It is obvious that the set $\{n \in \mathbb{N} : \delta^n > 0\}$ is unbounded. Hence the sequence (f_n) does not converge uniformly discretely to the zero function on $[0,\delta]$. □

Example 1.12. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{1}{n}$ for $n \in \mathbb{N}$. Then the sequence (f_n) does not converge uniformly discretely to the zero function on $\mathbb{R}$.

Proof. It is obvious. □

Example 1.13. Let

$$f_n(x) = \begin{cases} 0 & \text{for } x \in (-\infty, n-1]; \\ x - n + 1 & \text{for } x \in [n-1, n]; \\ 1 & \text{for } x \in [n, \infty). \end{cases}$$

Then the sequence (f_n) does not converge uniformly discretely to the zero function on $\mathbb{R}$.

Proof. Suppose on the contrary that $f_n \xrightarrow{u.d.} f$. Then there exists a natural number n_0 such that the cardinality of the set $\{n \in \mathbb{N} : |f_n(x) - f(x)| > 0\}$ is at most n_0, for each $x \in X$. Since $f_1(n_0 + 1) = 1, f_2(n_0 + 1) = 1, \ldots, f_{n_0}(n_0 + 1) = 1, f_{n_0+1}(n_0 + 1) = 1$, we obtain $|\{n \in \mathbb{N} : |f_n(n_0 + 1)| > 0\}| = n_0 + 1$, which is a contradiction. □

For a sequence of functions in Φ, we will show implications any among various kinds of convergence.

Theorem 1.14. *Uniform convergence implies uniform equal convergence.*

Proof. Let $f_n \xrightarrow{u} f$, then $\|f_n - f\|_\infty \to 0$. For $(\varepsilon_n) = (\|f_n - f\|_\infty + \frac{1}{n}) \to 0$, we have

$$|\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \|f_n - f\|_\infty \geq \varepsilon_n\}| \leq n_0 = 1, \text{for each } x \in X.$$

Hence $f_n \xrightarrow{u.e.} f$ with witnessing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$. □

Example 1.15. The converse of the above implication fails. Let f_n be the characteristic function of the interval $[n-1, n], n \in \mathbb{N}$ and given by

$$f_n(x) = \begin{cases} 1 & \text{for } x \in [n-1, n]; \\ 0 & \text{for } x \notin [n-1, n]. \end{cases}$$

Then the sequence (f_n) converges uniformly equally to the zero function on $\mathbb{R}$ but not uniformly.

Proof. From Example 1.7, we have $f_n \xrightarrow{u.e.} f$. Since $\|f_n - f\|_\infty = 1$, we have that $\|f_n - f\|_\infty$ does not converge to the zero function on $\mathbb{R}$. Hence $\|f_n - f\|_\infty$ does not converge uniformly to the zero function in $\mathbb{R}$. □

Theorem 1.16. *Uniform equal convergence implies equal convergence.*

Proof. Let $f_n \xrightarrow{u.e.} f$. Then there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals converging to zero and $n_0 \in \mathbb{N}$ such that

$$|\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\}| \leq n_0 \quad \text{for each } x \in X.$$

Hence for each $x \in X$, we have

$$|f_n(x) - f(x)| < \varepsilon_n \quad \text{for each } n \geq \max\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\}.$$

Therefore $f_n \xrightarrow{e} f$ with witnessing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$. □

Example 1.17. The converse of the above implication fails. Let $f_n : [0,1) \to \mathbb{R}$ be defined by $f_n(x) = x^n, n \in \mathbb{N}$. Then the sequence (f_n) converges equally to the zero function on $[0,1)$ but not uniformly equally.

Theorem 1.18. *Uniform discrete convergence implies both discrete and uniform equal convergence.*

Proof. Suppose that $f_n \xrightarrow{u.d.} f$. Then there exists $n_0 \in \mathbb{N}$ such that

$$|\{n \in \mathbb{N} : |f_n(x) - f(x)| > 0\}| \leq n_0 \qquad \text{for each } x \in X.$$

For $(\varepsilon_n) = (\frac{1}{n}) \to 0$, we have that $|\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\}| \leq n_0$, for each $x \in X$. Hence $f_n \xrightarrow{u.e.} f$ with witnessing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$. Moreover, for each $x \in X$, we have $|f_n(x) - f(x)| = 0$, for all $n \geq \max\{n \in \mathbb{N} : |f_n(x) - f(x)| > 0\}$. Therefore $f_n \xrightarrow{d.} f$. □

Example 1.19. The converse of the above implication fails.

(1). Let

$$f_n(x) = \begin{cases} 0 & \text{for } x \in (-\infty, n-1]; \\ x - n + 1 & \text{for } x \in [n-1, n]; \\ 1 & \text{for } x \in [n, \infty). \end{cases}$$

Then the sequence (f_n) converges discretely to the zero function on $\mathbb{R}$ but not uniformly discretely.

(2) Let $0 < \delta < 1$ and $f_n(x) = x^n$ for $x \in [0, \delta]$. Then the sequence (f_n) converges uniformly equally to the zero function on $[0, \delta]$, but not uniformly discretely.

We give an alternative definition of uniformly equal convergence in the following theorem:

Theorem 1.20. *Let $f_n, f : X \to \mathbb{R}$, and $n \in \mathbb{N}$. Then $f_n \xrightarrow{u.e.} f$ if and only if there exists an unbounded sequence $(\rho_n)_{n\in\mathbb{N}}$ of positive integers such that*

$$\rho_n |f_n - f| \xrightarrow{u.e.} 0.$$

Proof. Suppose that $f_n \xrightarrow{u.e.} f$. Then there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals converging to zero and $n_0 \in \mathbb{N}$ such that

$$|\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\}| \leq n_0 \qquad \text{for each } x \in X.$$

Note that $\left|\left\{n \in \mathbb{N} : \rho_n \left|f_n(x) - f(x)\right| \geq \sqrt{\varepsilon_n}\right\}\right| \leq n_0$ for each $x \in X$, where $(\rho_n) = \left(\left[\frac{1}{\sqrt{\varepsilon_n}}\right]\right)$ is an unbounded sequence of positive integers and hence $\rho_n |f_n - f| \xrightarrow{u.e.} 0$.

Conversely, if $\rho_n |f_n - f| \xrightarrow{u.e.} 0$, where (ρ_n) is an unbounded sequence of positive integers, then there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of positive integers converging to zero and $n_1 \in \mathbb{N}$ such that

$$|\{n \in \mathbb{N} : \rho_n |f_n(x) - f(x)| \geq \lambda_n\}| \leq n_1 \qquad \text{for each } x \in X.$$

Since $\rho_n \geq 1$, for all $n \in \mathbb{N}$, we have that $0 < \frac{1}{\rho_n} \leq 1$, for each $n \in \mathbb{N}$. Then $0 < {}^{\lambda_n}/_{\rho_n} \leq \lambda_n$, for each $n \in \mathbb{N}$. Thus $\left({}^{\lambda_n}/_{\rho_n}\right) \to 0$. For $(\theta_n) = \left({}^{\lambda_n}/_{\rho_n}\right)$, we have

$$|\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \theta_n\}| \leq n_1 \qquad \text{for each } x \in X.$$

Hence $f_n \xrightarrow{u.e.} f$ with witnessing sequence (θ_n). □

We first observe the following Lemmas.

Lemma 1.21. *Let $f_n : X \to \mathbb{R}$ for each $n \in \mathbb{N}$. If $f_n \xrightarrow{u.e.} 0$, then $f_n^2 \xrightarrow{u.e.} 0$.*

Proof. If $(\lambda_n)_{n\in\mathbb{N}}$ is a witnessing sequence for uniform equal convergence of (f_n) to zero, then $(\lambda_n^2)_{n\in\mathbb{N}}$ is a witnessing sequence for uniform equal convergence of (f_n^2) to zero. □

Lemma 1.22. *Let $f_n : X \to \mathbb{R}$ for each $n \in \mathbb{N}$. If f is bounded and $f_n \xrightarrow{u.e.} f$, then $f_n \cdot f \xrightarrow{u.e.} f^2$.*

Proof. Let $M > 0$ be such that $|f(x)| < M$ for each $x \in X$. Since $f_n \xrightarrow{u.e.} f$, there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals converging to zero and $n_0 \in \mathbb{N}$ such that

$$|\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\}| \leq n_0 \qquad \text{for each } x \in X.$$

Hence

$$\left|\left\{n \in \mathbb{N} : \left|(f_n \cdot f)(x) - f^2(x)\right| \geq \varepsilon_n \cdot M\right\}\right| \leq n_0 \qquad \text{for each } x \in X.$$

Thus $f_n \cdot f \xrightarrow{u.e.} f^2$.

Claim Let

$$A = \{n \in \mathbb{N} : \left|(f_n \cdot f)(x) - f^2(x)\right| \geq \varepsilon_n \cdot M\},$$
$$B = \{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\}.$$

Then $A \subseteq B$, for each $x \in X$. Let $x \in X$ and $m \in B^c$. Then $m \in \mathbb{N}$ and $|f_m(x) - f(x)| < \varepsilon_m$. Since $|f(x)| < M$, then $|f(x)|\,|f_m(x) - f(x)| < \varepsilon_m \cdot M$. Hence $|(f_m \cdot f)(x) - f^2(x)| < \varepsilon_m \cdot M$. Thus $m \in A^c$. Hence $A \subseteq B$. □

Theorem 1.23. *Let $f, g : X \to \mathbb{R}$ be bounded functions and $f_n, g_n : X \to \mathbb{R}$ for each $n \in \mathbb{N}$ be such that $f_n \xrightarrow{u.e.} f$ and $g_n \xrightarrow{u.e.} g$. Then $f_n \cdot g_n \xrightarrow{u.e.} f \cdot g$.*

Proof. Since $f_n \xrightarrow{u.e.} f$ and $g_n \xrightarrow{u.e.} g$, we have $f_n + g_n \xrightarrow{u.e.} f + g$ and $f_n - g_n \xrightarrow{u.e.} f - g$. Now using Lemmas 1.21 and 1.22, we get

$$f_n \cdot g_n = \frac{(f_n + g_n)^2 - (f_n - g_n)^2}{4} \xrightarrow{u.e.} \frac{(f + g)^2 - (f - g)^2}{4} = f \cdot g.$$

Note : Since $(f_n + g_n) \xrightarrow{u.e.} (f + g)$, $(f_n + g_n) - (f + g) \xrightarrow{u.e.} 0$. By Lemma 1.21, we have

$$[(f_n + g_n) - (f + g)]^2 \xrightarrow{u.e.} 0. \tag{1.24}$$

Since $(f_n + g_n) \xrightarrow{u.e.} (f + g)$, $f + g$ bounded and Lemma 1.22, we have

$$(f_n + g_n)(f + g) \xrightarrow{u.e.} (f + g)^2. \tag{1.25}$$

Consider $(f_n + g_n)^2 - (f + g)^2$

$$= [(f_n + g_n) - (f + g)]^2 + 2(f_n + g_n)(f + g) - 2(f + g)^2.$$

From (1.24), (1.25) we have $(f_n + g_n)^2 - (f + g)^2 \xrightarrow{u.e.} 0$. Hence $(f_n + g_n)^2 \xrightarrow{u.e.} (f + g)^2$. Similarly $(f_n - g_n)^2 \xrightarrow{u.e.} (f - g)^2$. □

Now, we study the properties of uniformly discrete convergence in the following result follows from definition:

(i) If $f_n \xrightarrow{u.d.} f$, then $|f_n| \xrightarrow{u.d.} |f|$.

(ii) If $f_n \xrightarrow{u.d.} f$, $g_n \xrightarrow{u.d.} g$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f_n + \beta g_n \xrightarrow{u.d.} \alpha f + \beta g$.

(iii) Let $f_n : X \to \mathbb{R}, n \in \mathbb{N}$. If $f_n \xrightarrow{u.d.} 0$, then $f_n^2 \xrightarrow{u.d.} 0$.

(iv) Let $f_n : X \to \mathbb{R}, n \in \mathbb{N}$. If f is bounded and $f_n \xrightarrow{u.d.} f$, then $f_n \cdot f \xrightarrow{u.d.} f^2$.

(v) Let $f, g : X \to \mathbb{R}$ be bounded functions and $f_n, g_n : X \to \mathbb{R}, n \in \mathbb{N}$ be such that $f_n \xrightarrow{u.d.} f$ and $g_n \xrightarrow{u.d.} g$, then $f_n \cdot g_n \xrightarrow{u.d.} f \cdot g$.

CHAPTER II
LATTICES OF UNIFORM EQUAL AND UNIFORM DISCRETE LIMITS

Definition 2.1. *Let Φ be a class of functions on X.*

(a) *Φ is called a **lattice** if Φ contains all constants and $f, g \in \Phi$ implies $\max\{f, g\} \in \Phi$ and $\min\{f, g\} \in \Phi$.*

(b) *Φ is called a **translation lattice** if it is a lattice and $f \in \Phi$, $c \in \mathbb{R}$ implies $f + c \in \Phi$.*

(c) *Φ is called a **congruence lattice** if it is a translation lattice and $f \in \Phi$ implies $-f \in \Phi$.*

(d) *Φ is called a **weakly affine lattice** if it is a congruence lattice and there is a set $C \subseteq (0, \infty)$ such that C is not bounded and $f \in \Phi$, $c \in C$ implies $cf \in \Phi$.*

(e) *Φ is called an **affine lattice** if it is a congruence lattice and $f \in \Phi$, $c \in \mathbb{R}$ implies $cf \in \Phi$.*

(f) *Φ is called a **subtractive lattice** if it is a lattice and $f, g \in \Phi$ implies $f - g \in \Phi$.*

(g) *Φ is called an **ordinary class** if it is a subtractive lattice, $f, g \in \Phi$ implies $f \cdot g \in \Phi$ and $f \in \Phi$, $f(x) \neq 0$, for all $x \in X$ implies $1/f \in \Phi$.*

We denote by $\Phi^{u.e.}$, the set of all functions on X which are **uniformly equally** limits of sequences of functions in Φ. Similarly $\Phi^{u.d.}$ denotes the set of all functions on X which are **uniformly discretely** limits of sequences of functions in Φ.

Note 2.2. One can observe that if $f \in \Phi^{u.e.}$, then for any sequence $(\lambda_n)_{n\in\mathbb{N}}$ of positive reals converging to zero, there exists a sequence of functions in Φ which converges uniformly equally to f with witnessing sequence $(\lambda_n)_{n\in\mathbb{N}}$.

Theorem 2.3. *Let Φ be a class of functions on X. If Φ is a lattice, a translation lattice, a congruence lattice, a weakly affine lattice, an affine lattice or a subtractive lattice, then so is $\Phi^{u.e.}$.*

Proof. We will show that if Φ is a lattice, then so is $\Phi^{u.e.}$. Suppose that Φ is a lattice.

<u>Claim</u>: If Φ contains constant functions, then $\Phi^{u.e.}$ contains constant functions. Let f be a constant function in Φ. Let $f_n(x) = f(x), x \in X$ and $n \in \mathbb{N}$. Choose $(\varepsilon_n) = \left({}^1\!/_n\right) \to 0$, we have $|\{n \in \mathbb{N} : |f_n(x) - f(x)| = 0 \geq \varepsilon_n\}| = |\varnothing| = 0 \leq 1$, for all $x \in X$. Hence $f \in \Phi^{u.e.}$ by definition. It follows that if $f_n \xrightarrow{u.e.} f$, then $|f_n| \xrightarrow{u.e.} |f|$ and if $f_n \xrightarrow{u.e.} f, g_n \xrightarrow{u.e.} g$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f_n + \beta g_n \xrightarrow{u.e.} \alpha f + \beta g$. Let $f, g \in \Phi^{u.e.}$, then $f_n \xrightarrow{u.e.} f, g_n \xrightarrow{u.e.} g$. Hence

$$\left(\frac{f_n + g_n}{2}\right) + \frac{|f_n - g_n|}{2} \xrightarrow{u.e.} \frac{f+g}{2} + \frac{|f-g|}{2}.$$

Which implies that $\max\{f, g\} \in \Phi^{u.e.}$. Similarly $\min\{f, g\} \in \Phi^{u.e.}$. Thus $\Phi^{u.e.}$ is a lattice. □

It is easy to observe that if Φ is a translation lattice, a congruence lattice, a weakly affine lattice, an affine lattice or a subtractive lattice, then so is $\Phi^{u.e.}$.

Theorem 2.4. *Let Φ be an ordinary class of functions on X. Let $f \in \Phi^{u.e.}$ be bounded and such that $f(x) \neq 0$ for each $x \in X$. If $1/f$ is bounded on X, then $1/f \in \Phi^{u.e.}$.*

Proof. Let λ be such that $f^2(x) > \lambda > 0$ for each $x \in X$. Since $f \in \Phi^{u.e.}$ and f is bounded, by Theorem 1.23, $f^2 \in \Phi^{u.e.}$ and hence by Note 2.2, there exists $f_n \in \Phi, n \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that

$$\left|\left\{n \in \mathbb{N} : \left|f_n(x) - f^2(x)\right| \geq \frac{1}{n^3}\right\}\right| \leq n_0 \qquad \text{for each } x \in X.$$

Let $g_n(x) = \max\{f_n(x), \frac{1}{n}\}$, $x \in X$ and $n \in \mathbb{N}$. Then $g_n \in \Phi$ for each $n \in \mathbb{N}$. Note that

$$\left|\left\{n \in \mathbb{N} : g_n(x) = f_n(x) \text{ and } |g_n(x) - f^2(x)| \geq \frac{1}{n^3}\right\}\right| \leq n_0$$

and

$$\left|\left\{n \in \mathbb{N} : g_n(x) = \frac{1}{n} \text{ and } |g_n(x) - f^2(x)| \geq \frac{1}{n^3}\right\}\right| \leq n^* + n_0,$$

where $n^* = \left[\frac{1}{\lambda}\right] + 1$.

Using the fact that

$$\left\{n \in \mathbb{N} : |g_n(x) - f^2(x)| \geq \frac{1}{n^3}\right\}$$

$$= \left\{n \in \mathbb{N} : g_n(x) = f_n(x) \text{ and } |g_n(x) - f^2(x)| \geq \frac{1}{n^3}\right\}$$

$$\cup \left\{n \in \mathbb{N} : g_n(x) = \frac{1}{n} \text{ and } |g_n(x) - f^2(x)| \geq \frac{1}{n^3}\right\}$$

we have

$$\left|\left\{n \in \mathbb{N} : |g_n(x) - f^2(x)| \geq \frac{1}{n^3}\right\}\right| \leq n_0 + (n_0 + n^*) \equiv n_1 \qquad \text{for each } x \in X.$$

Therefore

$$\left|\{n \in \mathbb{N} : |\frac{1}{g_n(x)} - \frac{1}{f^2(x)}| \geq \frac{1}{n^3} \cdot n \cdot \frac{1}{\lambda}\}\right| = \left|\{n \in \mathbb{N} : \frac{|g_n(x) - f^2(x)|}{|g_n(x)|\,|f^2(x)|} \geq \frac{1}{n^3} \cdot n \cdot \frac{1}{\lambda}\}\right|$$

$$\leq \left|\{n \in \mathbb{N} : |g_n(x) - f^2(x)| \geq \frac{1}{n^3}\}\right| \leq n_1$$

for each $x \in X$. Thus $f^{-2} \in \Phi^{u.e.}$ and so $f \cdot f^{-2} = 1/f \in \Phi^{u.e.}$. □

Theorem 2.5. *If Φ is a lattice, a translation lattice, a congruence lattice, a weakly affine, an affine lattice or a subtractive lattice, then so is $\Phi^{u.d.}$.*

We have the following result for a function class Φ which is an ordinary class.

Theorem 2.6. *Let Φ be an ordinary class of functions on X. Then $f, g \in \Phi^{u.d.}$ implies $f \cdot g \in \Phi^{u.d.}$. Also, if $f \in \Phi^{u.d.}$ is such that $f(x) \neq 0$ for each $x \in X$ and $1/f$ is bounded on X, then $1/f \in \Phi^{u.d.}$.*

Proof. Let $f, g \in \Phi^{u.d.}$. Then there exist sequences (f_n) and (g_n) in Φ such that $f_n \overset{u.d.}{\to} f$, $g_n \overset{u.d.}{\to} g$. It follows from the definition that $f_n \cdot g_n \overset{u.d.}{\to} f \cdot g$. Let f satisfy the assumptions. Choose λ such that $f^2(x) > \lambda > 0$ for each $x \in X$. We first show that $f^{-2} \in \Phi^{u.d.}$. Since $f \in \Phi^{u.d.}$, then there exist sequences (f_n) in Φ such that $f_n \overset{u.d.}{\to} f$. Since Φ is an ordinary class, $f_n^2 \in \Phi, n \in \mathbb{N}$. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive reals converging to zero and $g_n = \max\{f_n^2, \varepsilon_n\}$. Then $g_n \in \Phi$. Since $f_n \overset{u.d.}{\to} f$, there exists $n_0 \in \mathbb{N}$ satisfying $|\{n \in \mathbb{N} : f_n(x) \neq f(x)\}| \leq n_0$ for each $x \in X$. Hence $|\{n \in \mathbb{N} : g_n(x) \neq \max\{f^2(x), \varepsilon_n\}\}| \leq n_0$ for each $x \in X$, which implies

$$\left|\left\{n \in \mathbb{N} : \frac{1}{g_n(x)} \neq \frac{1}{\max\{f^2(x), \varepsilon_n\}}\right\}\right| \leq n_0 \qquad \text{for each } x \in X. \tag{2.7}$$

Since $(\varepsilon_n)_{n\in\mathbb{N}}$ converges to zero, there exists $n^* \in \mathbb{N}$ satisfying $\varepsilon_n < \lambda$ for all $n \geq n^*$. Hence

$$\left|\left\{n \in \mathbb{N} : \frac{1}{\max\{f^2(x), \varepsilon_n\}} \neq \frac{1}{f^2(x)}\right\}\right| < n^* \qquad \text{for each } x \in X. \tag{2.8}$$

Now using equations (2.7) and (2.8)

$$\left|\left\{n \in \mathbb{N} : \frac{1}{g_n(x)} \neq \frac{1}{f^2(x)}\right\}\right| < n_0 + n^* \qquad \text{for each } x \in X.$$

Hence $f^{-2} \in \Phi^{u.d.}$, consequently $f \cdot f^{-2} = f^{-1} \in \Phi^{u.d.}$. □

CHAPTER III
α-UNIFORM EQUAL CONVERGENCE

Definition 3.1. *Let X be a metric space and f_n, f, $n \in \mathbb{N}$ be real valued functions defined on X. Then (f_n) is α -* ***convergent*** *to f (written as $f_n \xrightarrow{\alpha} f$) if for any $x \in X$ and for any sequence (x_n) of points of X converging to x, $(f_n(x_n))$ converges to $f(x)$.*

The notion of α - convergence (known as continuous convergence) for sequences of real valued functions on a metric space turned out to be useful for characterizing compactness in metric spaces. It is known that if X is a metric space and f, f_n : $X \to \mathbb{R}$ and $n \in \mathbb{N}$ are such that $f_n \xrightarrow{\alpha} f$ (i.e. (f_n) is α - convergent to f), then f is continuous. Also, if X is a compact metric space, then $f_n \xrightarrow{\alpha} f$ implies $f_n \xrightarrow{u} f$, where u denotes uniform convergence.

In [2] Hola and Salat have obtained the following characterization of compact metric spaces.

Theorem 3.2. *A metric space (X, d) is compact if and only if for f_n, $f : X \to \mathbb{R}$ $f_n \xrightarrow{\alpha} f$ implies $f_n \xrightarrow{u} f$.*

Definition 3.3. *Let (X, d) be a metric space and $f_n, f : X \to \mathbb{R}$ for any $n \in \mathbb{N}$. Then (f_n) converges α -* ***uniformly equally*** *to f (written as $f_n \xrightarrow{\alpha-u.e.} f$) if there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals converging to zero and an $n_0 \in \mathbb{N}$ such that*

$$|\{n \in \mathbb{N} : |f_n(x_n) - f(x)| \geq \varepsilon_n\}| \leq n_0 \qquad \text{for each } x \in X \text{ and } x_n \to x.$$

$$f_n \xrightarrow{\alpha-u.e.} f \iff \exists(\varepsilon_n) \to 0\, \exists n_0 \in \mathbb{N}\, \forall x \in X\, \forall(x_n),$$

$$x_n \to x \Rightarrow |\{n \in \mathbb{N} : |f_n(x_n) - f(x)| \geq \varepsilon_n\}| \leq n_0$$

Remark 3.4. It is clear from this definition that α - $u.e.$ convergence implies both α - convergence and $u.e.$ convergence. However, the following examples show that the converse of each of the above implications fails.

Example 3.5. Let f_n be the characteristic function of the interval $\left[n, n+\frac{1}{n}\right]$ for any $n \in \mathbb{N}$. Then $f_n \xrightarrow{u.e.} 0$. For if $(\varepsilon_n)_{n\in\mathbb{N}}$ is a sequence of positive reals converging to zero then $|\{n \in \mathbb{N} : f_n(x) \geq \varepsilon_n\}| \leq 1$, for each $x \in \mathbb{R}$. Also $f_n \xrightarrow{\alpha} f$. For if $x_0 \in \mathbb{R}$ then there exists $n^* \in \mathbb{N}$ such that for all $n > n^*$ we have $x_0 < n$ and this implies $f_n(x_0) = 0$, for all $n \geq \max\{n^*, n_0(\varepsilon)\}$. Therefore $f_n(x_n) \to f(x_0) = 0$. Hence $f_n \xrightarrow{\alpha} f$.

Now we can observe that if

$$x_n = \begin{cases} n + \frac{1}{2n} & \text{if } n \leq m; \\ x_0 - \frac{1}{n} & \text{if } n > m. \end{cases}$$

where $m \in \mathbb{N}$ is fixed and $\varepsilon_n < 1$ for all $n \in \mathbb{N}$, then

$$|\{n \in \mathbb{N} : |f_n(x_n) - f(x_0)| \geq \varepsilon_n\}| = m.$$

Hence (f_n) does not converge α - uniformly equally to the zero function.

Example 3.6. Let (f_n) be the piecewise linear function supported on $\left[n-1, n+1+\frac{1}{n}\right]$ and give by

$$f_n(x) = \begin{cases} x + 1 - n & \text{for } x \in [n-1, n]; \\ 1 & \text{for } x \in [n, n+\frac{1}{n}]; \\ n + 1 + \frac{1}{n} - x & \text{for } x \in [n+\frac{1}{n}, n+1+\frac{1}{n}]. \end{cases}$$

Then we see that (f_n) α-converges to the zero function but not α-uniformly equally.

In the above examples, in fact $f_n \xrightarrow{u.d.} 0$. Therefore $u.d.$ - convergence need not imply α - $u.e.$ - convergence. Moreover, the following example shows that α - $u.e.$ - convergence also need not imply $u.d.$ - convergence.

Example 3.7. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{1}{n}$ for any $n \in \mathbb{N}$ on $\mathbb{R}$. Then note that $f_n \xrightarrow{\alpha - u.e.} f$ but $f_n \overset{u.d.}{\not\to} f$.

Remark 3.8. (*i*) Let (X, d) be a metric space and $f_n, f : X \to \mathbb{R}$ for any $n \in \mathbb{N}$ such that $f_n \xrightarrow{\alpha-u.e.} f$. Then $f_n \xrightarrow{\alpha} f$ and hence f is continuous even if f_n are not. Thus α - *u.e.* convergence implies that the limit function is continuous.
(*ii*) In general, uniform convergence need not imply α - uniform equal convergence. For example if f is a discontinuous from X to $\mathbb{R}$ and $f_n = f$, for all $n \in \mathbb{N}$, then $f_n \xrightarrow{u} f$ but since f is discontinuous, f_n does not converge α - uniformly equally to f. However, we have the following result.

Theorem 3.9. *Let X be a metric space and $f_n : X \to \mathbb{R}$ for any $n \in \mathbb{N}$. If the sequence (f_n) converges uniformly to the zero function, then the sequence (f_n) converges α - uniformly equally to the zero function.*

Proof. Since $f_n \xrightarrow{u} 0$, there exists a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ of positive reals converging to zero and $n_0 \in \mathbb{N}$ such that $|f_n(x)| < \varepsilon_n$, for all $n > n_0$ and for each $x \in X$. This gives $|\{n \in \mathbb{N} : |f_n(x_n)| \geq \varepsilon_n\}| \leq n_0$ for every converging sequence (x_n) in X. Hence $f_n \xrightarrow{\alpha-u.e.} 0$. □

In the opposite direction, we have the following result.

Theorem 3.10. *Let (X, d) be a compact metric space and f_n, $f : X \to \mathbb{R}$ for any $n \in \mathbb{N}$. Then $f_n \xrightarrow{\alpha-u.e.} f$ implies $f_n \xrightarrow{u} f$.*

Proof. It follows from the fact that $f_n \xrightarrow{\alpha-u.e.} f$ implies $f_n \xrightarrow{\alpha} f$ which again implies $f_n \xrightarrow{u} f$, as X is a compact metric space. □

The following example shows that α - convergence need not imply uniform equal convergence.

Example 3.11. Let $f_n : (0, 1) \to \mathbb{R}$ be defined by $f_n(x) = x^n$ for each $n \in \mathbb{N}$. Then $f_n \xrightarrow{\alpha} f$. Let $0 < \delta < 1$ and $x_n \in (0, 1)$ be such that $x_n \to \delta$. If $\delta < \vartheta < 1$, then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $x_n < \vartheta$. But then $f_n(x_n) = x_n^n < \vartheta^n$. So $f_n(x_n) \to 0 = f(\delta)$, since $\vartheta^n \to 0$. Hence $f_n \xrightarrow{\alpha} 0$. However $f_n \overset{u.e.}{\not\to} f$. For if $(\varepsilon_n)_{n \in \mathbb{N}}$ is a sequence of positive reals converging to zero and $0 < \varepsilon < 1$, then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0, \varepsilon_n < \varepsilon$. Consequently we have

$$\{n \in \mathbb{N} : x^n \geq \varepsilon\} \cap [n_0, +\infty) \subset \{n \in \mathbb{N} : x^n \geq \varepsilon_n\} \cap [n_0, +\infty).$$

But the function $n_\varepsilon(x) = |\{n \in \mathbb{N} : x^n \geq \varepsilon\}|, x \in (0,1)$ is unbounded and therefore the function $n(x) = |\{n \in \mathbb{N} : x^n \geq \varepsilon_n\}|, x \in (0,1)$ is unbounded. Hence $f_n \overset{u.e.}{\not\to} 0$.

Note 3.12. (i) The previous example also shows that α-convergence need not imply α-uniform equal convergence. In addition we have that the sequence (f_n) converges equally to 0 on $(0,1)$, since $(0,1) = \cup_{k=2}^{\infty} \left[\frac{1}{k}, 1 - \frac{1}{k}\right]$ and (f_n) converges uniformly to 0 on $\left[\frac{1}{k}, 1 - \frac{1}{k}\right]$ for every $k \geq 2$. So we have here a simple example which distinguishes α-convergence from α-uniformly equally convergence, and at the same time the equal convergence from uniformly equal convergence.
(ii) Examples in Remark 3.8 (ii) show that uniform equal convergence need not imply α-convergence (since f being discontinuous, $f_n \overset{\alpha}{\not\to} f$). Also the following example shows the same.

Example 3.13. Let

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{for } x \in \left[0, \frac{1}{n}\right]; \\ 0 & \text{for } x \in \left(\frac{1}{n}, 1\right); \\ 1 & \text{for } x = 1. \end{cases}$$

for any $n \in \mathbb{N}$ and $f : [0,1] \to [0,1]$ be defined by

$$f(x) = \begin{cases} 1 & \text{for } x = 1; \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that $f_n \xrightarrow{u} f$ and hence $f_n \xrightarrow{u.e.} f$ but, f being discontinuous, $f_n \overset{\alpha}{\not\to} f$.

We now obtain a characterization of compact metric spaces using α-uniform equal convergence.

Theorem 3.14. *A metric space (X,d) is compact if and only if the α - convergence of a sequence (f_n) of real valued functions defined on X to the zero function implies the α - uniform equal convergence of the sequence (f_n) to the zero function.*

Proof. If X is compact metric space, then by Theorem 3.2, $f_n \xrightarrow{\alpha} 0$ implies $f_n \xrightarrow{u} 0$ and hence by Theorem 3.9 $f_n \xrightarrow{\alpha-u.e.} 0$. Conversely, suppose (X,d) is not compact metric space. We first recall the construction of maps f_p^* in Theorem 3.1 in [2]. Since X is not compact, there exists a sequence (x_k) of distinct points of X such that there exists no convergent subsequence of (x_k). Since every point of the set $\{x_1, x_2, ..., x_m, ...\}$ is an isolated point of the set $\{x_1, x_2, ..., x_m, ...\}$, there exist $\delta_k > 0, k = 1, 2, ...$ such that $\delta_k \to 0$ as $k \to \infty$ and the closed balls $B[x_k, \delta_k] = \{x \in X \mid d(x, x_k) \le \delta_k\}, k = 1, 2, ...$ are pairwise disjoint. Then $H = \cup_{k=1}^{\infty} B[x_k, \delta_k]$ is a closed set. Define a sequence f_p of real valued functions on the set $\{x_1, x_2, ..., x_m, ...\}$ by $f_1(x_n) = 0$ for any $n \in \mathbb{N}$ and for $p > 1, f_p(x_m) = (1-1/m)^{p-1}$ if $1 \le m \le p$ and $f_1(x_{p+j}) = f_p(x_p)$ for $j = 1, 2, ...$ Define for $p \in \mathbb{N}, f_p^*(x) = 0$ if $x \notin H$ and $f_p^*(x) = f_p(x_j) \cdot (\delta_j - d(x, x_j))/\delta_j$, if $x \in B[x_j, \delta_j](j = 1, 2, ..)$. Then as proved in [2], $f_p^* \xrightarrow{\alpha} 0$. However, the fact that $f_p^*(x_p) = (1 - 1/p)^{p-1}$ is decreasing and converges to e^{-1}, where e is Euler number, implies that (f_p^*) does not converge uniformly equally to the zero function. For if $(\varepsilon_n)_{n \in \mathbb{N}}$ is a null sequence of positive reals, then there exists $n_0 \in \mathbb{N}$ such that, for all $n \ge n_0$ we have $\varepsilon_n < e^{-1}$. Let $\varepsilon \in (\varepsilon_{n_0}, e^{-1})$. Then for all $p \in \mathbb{N}$,

$$|\{n \in \mathbb{N} : f_n^*(x_p) \ge \varepsilon\}| \ge p - 2.$$

Also we have

$$\{n \in \mathbb{N}, n \ge n_0 : f_n^*(x_p) \ge \varepsilon\} \subset \{n \in \mathbb{N}, n \ge n_0 : f_n^*(x_p) \ge \varepsilon_n\}.$$

Therefore for all $p \in \mathbb{N}$

$$|\{n \in \mathbb{N}, n \ge n_0 : f_n^*(x_p) \ge \varepsilon_n\}| \ge p - 2 - n_0.$$

Hence $f_p^* \overset{u.e.}{\not\to} 0$. Now, since α - $u.e.$ convergence implies $u.e.$ convergence, we have that the sequence (f_p^*) does not converges α - uniformly equally to the zero function.

□

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