ทฤษฎีบทสมสัณฐานสำหรับแวเรียนต์ของกึ่งกรุปบางชนิด

นางสาวเรืองลักษณ์ จงโชตินนท์

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2554

ISOMORPHISM THEOREMS FOR VARIANTS OF SOME SEMIGROUPS

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A Dissertation Submitted in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy Program in Mathematics
Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Academic Year 2011

Thesis Title	ISOMORPHISM THEOREMS FOR VARIANTS OF SOME SEMIGROUPS
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Field of Study	Mathematics
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Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Doctoral Degree

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เรืองลักษณ์ จงโชตินนท์ : ทฤษฎีบทสมสัณฐานสำหรับแวเรียนต์ของกึ่งกรุปบางชนิด. (ISOMORPHISM THEOREMS FOR VARIANTS OF SOME SEMI-GROUPS) อ.ที่ปรึกษาวิทยานิพนธ์หลัก : ผศ. ดร. สุรีย์พร ชาวแพรกน้อย, อ. ที่ปรึกษาวิทยานิพนธ์ร่วม : ศ. ดร. ยุพาภรณ์ เข็มประสิทธิ์, 45 หน้า.

*แวเรียนต์*ของกึ่งกรุป S *โดย* a ใน S คือกึ่งกรุป $(S,\circ)$ โดยที่ $x\circ y = xay$ สำหรับทุก x, y ใน S และเราแทน $(S,\circ)$ ด้วย (S,a) ในการวิจัยนี้เราแสดงว่าสำหรับ $a,b\in\mathbb{Z}$, $((\mathbb{Z}_n,\cdot),\overline{a})\cong((\mathbb{Z}_n,\cdot),\overline{b})$ ก็ต่อเมื่อ $(a,n)=(b,n)$ ให้ $L_F(V)$ เป็นกึ่งกรุปภายใต้การประกอบของการแปลงเชิงเส้นทั้งหมดจากปริภูมิเวกเตอร์ V บนฟีลด์ F ไปยังตัวเอง เราแสดงว่าถ้า V มีมิติจำกัดและ F เป็นฟีลด์จำกัด แล้วสำหรับ $\theta_1,\theta_2\in L_F(V)$, $(L_F(V),\theta_1)\cong(L_F(V),\theta_2)$ ก็ต่อเมื่อ $\operatorname{rank}\theta_1=\operatorname{rank}\theta_2$ เราได้ผลที่ตามมาอันหนึ่งว่า สำหรับ $P_1,P_2\in M_n(F)$, $((M_n(F),\cdot),P_1)\cong((M_n(F),\cdot),P_2)$ ก็ต่อเมื่อ $\operatorname{rank}P_1=\operatorname{rank}P_2$ โดยที่ $M_n(F)$ คือเซตของเมทริกซ์ $n\times n$ บน F ทั้งหมด ยิ่งไปกว่านั้นเราศึกษาทฤษฎีบทสมสัณฐานสำหรับแวเรียนต์ของกึ่งกรุปต่อไปนี้ด้วย กึ่งกรุปของจำนวนเต็มภายใต้การคูณและการบวก กึ่งกรุปของการแปลงบนเซตบางชนิด และกึ่งกรุปของการแปลงเชิงเส้นอื่น ๆ บางชนิด

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต ....

ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก ....

สาขาวิชา คณิตศาสตร์ ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์ร่วม ....

ปีการศึกษา 2554

##5173849023 : MAJOR MATHEMATICS
KEYWORDS : ISOMORPHISM THEOREM / VARIANT OF A SEMIGROUP / THE MULTIPLICATIVE SEMIGROUP $\mathbb{Z}_n$ / SEMIGROUP OF TRANSFORMATIONS OF A SET / SEMIGROUP OF LINEAR TRANSFORMATIONS

RUANGLAK JONGCHOTINON : ISOMORPHISM THEOREMS FOR VARIANTS OF SOME SEMIGROUPS. ADVISOR : ASST. PROF. SUREEPORN CHAOPRAKNOI, Ph.D. CO-ADVISOR : PROF. YUPAPORN KEMPRASIT, Ph.D., 45 pp.

A *variant* of a semigroup S *induced by* $a \in S$ is the semigroup $(S, \circ)$ where $x \circ y = xay$ for all $x, y \in S$ and $(S, \circ)$ is denoted by (S, a). In this research, it is shown that for $a, b \in \mathbb{Z}$, $((\mathbb{Z}_n, \cdot), \bar{a}) \cong ((\mathbb{Z}_n, \cdot), \bar{b})$ if and only if $(a, n) = (b, n)$. Let $L_F(V)$ be the semigroup under composition of all linear transformations from a vector space V over a field F into itself. We show that if V is finite-dimensional and F is a finite field, then for $\theta_1, \theta_2 \in L_F(V), (L_F(V), \theta_1) \cong (L_F(V), \theta_2)$ if and only if rank $\theta_1 =$ rank θ_2. We have a consequence that for $P_1, P_2 \in M_n(F), ((M_n(F), \cdot), P_1) \cong ((M_n(F), \cdot), P_2)$ if and only if rank $P_1 =$ rank P_2 where $M_n(F)$ is the set of all $n \times n$ matrices over F. In addition, isomorphism theorems for the variants of the following semigroups are studied: multiplicative and additive semigroups of integers, some semigroups of transformations of sets and some other semigroups of linear transformations.

Department : Mathematics and Computer Science .... Student's Signature ....

Field of Study : Mathematics .... Advisor's Signature ....

Academic Year : 2011 .... Co-advisor's Signature ....

ACKNOWLEDGEMENTS

I am greatly indebted to Assistant Professor Dr. Sureeporn Chaopraknoi, my thesis advisor and Professor Dr. Yupaporn Kemprasit, my thesis co-advisor, for their valuable suggestions, helpfulness and encouragement throughout the preparation of my dissertation. I am very thankful to my thesis committee: Associate Professor Dr. Amorn Wasanawichit, Assistant Professor Dr. Sajee Pianskool, Dr. Samruam Baupradist and Assistant Professor Dr. Knograt Savettaseranee. Moreover, I would like to thank all the lecturers for their previous valuable lectures during my study.

I acknowledge to the Development and Promotion of Science and Technology Talents Project (DPST) for a long term financial support until I got a Ph.D. in mathematics at Chulalongkorn University.

Finally, my sincere appreciation goes to my beloved father and mother for their kind encouragement throughout my study.

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CHAPTER I

INTRODUCTION

If S is a semigroup and $a \in S$, the semigroup $(S, \circ)$ defined by $x \circ y = xay$ for all $x, y \in S$ is called the *variant* of S *induced by* a and it is denoted by (S, a). Variants of abstract semigroups were first studied by Hickey [2] in 1983. In fact, variants of concrete semigroups of relations were earlier considered by Magill [9] in 1967. Hickey [1, 2, 3, 4, 5] introduced various results relating to variants of semigroups. Khan and Lawson [8] determined an element a in a regular semigroup and an inverse semigroup such that (S, a) is a regular semigroup.

Isomorphism theorems are considered important in every algebraic structure. It is interesting to know when two variants of a certain semigroup are isomorphic. It is clear that if S is a semigroup with identity and a is a unit of S, then $(S, a) \cong S$ through the mapping $x \mapsto ax$. In particular, any variant of a group G is isomorphic to G.

For a nonempty set X, let $T(X), P(X)$ and $I(X)$ denote the full transformation semigroup, the partial transformation semigroup and the symmetric inverse semigroup on X, respectively. Notice that $T(X)$ and $I(X)$ are subsemigroups of $P(X)$. If X is a finite set containing n elements, let T_n, P_n and I_n stand for $T(X), P(X)$ and $I(X)$, respectively. For $\theta \in P_n$ and $k \in \{1, \ldots, n\}$, let

$$t_k = |\{y \in \operatorname{ran}\theta \mid |y\theta^{-1}| = k\}|.$$

The n-tuple $(t_1, t_2, \ldots, t_n)$ is called the *type of* θ. In 2003-2004, Tsyaputa [12, 13] provided the remarkable results on the variants of I_n, T_n and P_n as follows: for $\theta_1, \theta_2 \in I_n, (I_n, \theta_1) \cong (I_n, \theta_2)$ if and only if $|\operatorname{ran}\theta_1| = |\operatorname{ran}\theta_2|$; for $\theta_1, \theta_2 \in T_n, (T_n, \theta_1) \cong (T_n, \theta_2)$ if and only if θ_1 and θ_2 have the same type and this is also true for the variants of P_n.

The purpose of this research is to give necessary and/or sufficient conditions for two variants of the semigroups of our interest to be isomorphic.

This research is organized as follows:

Chapter II contains basic definitions, notations and quoted results which are needed for this research.

Chapter III deals with some multiplicative and additive semigroups of integers. We give necessary and sufficient conditions for two variants of the following semigroups to be isomorphic:

$$(\mathbb{N},\cdot),\ (\mathbb{N}_k,+)\ \text{ and }\ (k\mathbb{N},+)$$

where $\mathbb{N}$ is the set of all natural numbers (positive integers) and $\mathbb{N}_k = \{k, k+1, k+2, \ldots\}$. Note that $\mathbb{N} = \mathbb{N}_1 = 1\mathbb{N}$. It is shown that $((\mathbb{N},\cdot),a) \cong ((\mathbb{N},\cdot),b)$ if and only if either $a = b = 1$ or $a = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$ and $b = q_1^{r_1}q_2^{r_2}\cdots q_k^{r_k}$ for some distinct primes $p_1, p_2, \ldots, p_k$ and some distinct primes $q_1, q_2, \ldots, q_k$ and some $r_1, r_2, \ldots, r_k \in \mathbb{N}$. We show that $((\mathbb{N}_k,+),a) \cong ((\mathbb{N}_k,+),b)$ if and only if $a = b$. This is also true for the variants of $(k\mathbb{N},+)$. In addition, necessary conditions for being isomorphic of two variants of $(\mathbb{Z},\cdot)$ are provided where $\mathbb{Z}$ is the set of all integers.

It is shown in Chapter IV that $((\mathbb{Z}_n,\cdot),\bar{a}) \cong ((\mathbb{Z}_n,\cdot),\bar{b})$ if and only if $(a,n) = (b,n)$. Dirichlet's theorem for primes in arithmetic progression in number theory is useful to prove this fact. We also show that if $((k\mathbb{Z}_n,\cdot),k\bar{a}) \cong ((k\mathbb{Z}_n,\cdot),k\bar{b})$, then $(k^3a,n) = (k^3b,n)$.

In Chapter V, the following semigroups of transformations of X are considered where X is a nonempty set which need not be finite:

$$I(X),\ M(X),\ E(X)\ \text{ and }\ T(X)$$

where

$$M(X) = \{\alpha \in T(X) \mid \alpha \text{ is 1-1}\},$$

$$E(X) = \{\alpha \in T(X) \mid \alpha \text{ is onto}\}.$$

We consider when two variants of $I(X)$ are isomorphic. Using the technique of the proof given in [12] and the generalized continuum hypothesis, we obtain necessary conditions as given in [12] as follows: for $\theta_1, \theta_2 \in I(X)$, if $(I(X),\theta_1) \cong$

$(I(X),\theta_2)$, then $|\operatorname{ran}\theta_1| = |\operatorname{ran}\theta_2|$. Moreover, we give an example to show that the converse is not true in general. However, we also give sufficient conditions for two variants of $I(X)$ are isomorphic as follows: for $\theta_1,\theta_2 \in I(X)$, if $|\operatorname{ran}\theta_1| = |\operatorname{ran}\theta_2|$, $|X \smallsetminus \operatorname{ran}\theta_1| = |X \smallsetminus \operatorname{ran}\theta_2|$ and $|X \smallsetminus \operatorname{dom}\theta_1| = |X \smallsetminus \operatorname{dom}\theta_2|$, then $(I(X),\theta_1) \cong (I(X),\theta_2)$. Sufficient conditions for any two variants of the $M(X), E(X)$ and $T(X)$ to be isomorphic are provided. The following results are shown for an infinite set X. If $\theta_1,\theta_2 \in M(X)$ and $|X \smallsetminus \operatorname{ran}\theta_1| = |X \smallsetminus \operatorname{ran}\theta_2|$, then $(M(X),\theta_1) \cong (M(X),\theta_2)$. If $\theta_1,\theta_2 \in E(X)$ and the partition of X induced by θ_1 and the partition of X induced by θ_2 are equivalent, then $(E(X),\theta_1) \cong (E(X),\theta_2)$. For $\theta_1,\theta_2 \in T(X)$, $(T(X),\theta_1) \cong (T(X),\theta_2)$ if both the above sufficient conditions are satisfied. Note that the *partition* of X *induced by* $\theta \in T(X)$ is $\{x\theta^{-1} \mid x \in \operatorname{ran}\theta\}$ and the partition of X induced by θ_1 and the partition of X induced by θ_2 are said to be *equivalent* if there exists a bijection $\varphi : \operatorname{ran}\theta_2 \to \operatorname{ran}\theta_1$ such that $|x\theta_2^{-1}| = |(x\varphi)\theta_1^{-1}|$ for all $x \in \operatorname{ran}\theta_2$.

In the last chapter, the semigroup $L_F(V)$ under composition of all linear transformations from a vector space V over a field F into itself is considered. Tsyaputa's works mentioned above motivate us to consider variants of $L_F(V)$ where V is finite-dimensional and F is finite. The following result is obtained. If V is finite-dimensional and F is a finite field, then for $\theta_1,\theta_2 \in L_F(V), (L_F(V),\theta_1) \cong (L_F(V),\theta_2)$ if and only if rank $\theta_1 =$ rank θ_2. We obtain the following result as a consequence. For a finite field F, a positive integer n and $P_1, P_2 \in M_n(F)$, $((M_n(F),\cdot),P_1) \cong ((M_n(F),\cdot),P_2)$ if and only if rank $P_1 =$ rank P_2 where $M_n(F)$ denotes the set of all $n \times n$ matrices over F. From a lemma of the proof of the main result of this chapter, we obtain sufficient conditions for two variants of $L_F(V)$ to be isomorphic as follows: for $\theta_1,\theta_2 \in L_F(V)$, if rank $\theta_1 =$ rank θ_2, $\dim_F \ker\theta_1 = \dim_F \ker\theta_2$ and $\dim_F(V/\operatorname{ran}\theta_1) = \dim_F(V/\operatorname{ran}\theta_2)$, then $(L_F(V),\theta_1) \cong (L_F(V),\theta_2)$. In particular, if V is a finite-dimensional vector space over F and rank $\theta_1 =$ rank θ_2, then $(L_F(V),\theta_1) \cong (L_F(V),\theta_2)$. We obtain as a consequence of this fact that if $P_1, P_2 \in M_n(F)$ are such that rank $P_1 =$ rank P_2, then $((M_n(F),\cdot),P_1) \cong ((M_n(F),\cdot),P_2)$. In addition, we give sufficient conditions for two variants of the following subsemigroups of $L_F(V)$ to be isomorphic:

$$\begin{aligned} M_F(V) &= \{\alpha \in L_F(V) \mid \alpha \text{ is 1-1}\} \\ &\quad (= \{\alpha \in L_F(V) \mid \ker\alpha = \{0\}\}), \\ E_F(V) &= \{\alpha \in L_F(V) \mid \alpha \text{ is onto}\} \\ &\quad (= \{\alpha \in L_F(V) \mid \operatorname{ran}\alpha = V\}). \end{aligned}$$

For an infinite-dimensional vector space V, we obtain the following results: if $\theta_1, \theta_2 \in M_F(V)$ are such that $\dim_F(V/\operatorname{ran}\theta_1) = \dim_F(V/\operatorname{ran}\theta_2)$, then $(M_F(V), \theta_1) \cong (M_F(V), \theta_2)$; if $\theta_1, \theta_2 \in E_F(V)$ are such that $\dim_F \ker\theta_1 = \dim_F \ker\theta_2$, then $(E_F(V), \theta_1) \cong (E_F(V), \theta_2)$.

CHAPTER II

PRELIMINARIES

The cardinality of a set X is denoted by $|X|$. The value of a mapping α at x in the domain of α shall be written as $x\alpha$. The notation $\dot{\cup}$ stands for a disjoint union. The identity mapping on a set A is denoted by 1_A.

Denote by $\mathbb{N}$ and $\mathbb{Z}$ the set of all natural numbers (positive integers) and the set of all integers, respectively. For $a, b \in \mathbb{Z}$ and $a \neq 0$, $a \,|\, b$ means that b is divisible by a. In this research, we use the generalized continuum hypothesis on cardinal numbers. It follows that if a and b are cardinal numbers such that $2^a = 2^b$, then $a = b$ ([11], p. 142).

In a semigroup S, we can adjoin an extra element 0 and define $0x = x0 = 0$ for all $x \in S$. Then $S \cup \{0\}$ becomes a semigroup with zero 0. For a semigroup S, we let

$$S^0 = \begin{cases} S \cup \{0\} & \text{if } |S| = 1 \text{ or } S \text{ has no zero,} \\ S & \text{otherwise.} \end{cases}$$

A semigroup S is called a *left* [*right*] *zero semigroup* if every element of S is a left [right] zero, i.e., $xy = x$ [$xy = y$] for all $x, y \in S$. A semigroup S with zero 0 is called a *zero semigroup* if $xy = 0$ for all $x, y \in S$.

A *Kronecker semigroup* is a semigroup S with zero 0 such that for all $x, y \in S$,

$$xy = \begin{cases} x & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

If S is a semigroup with identity 1 and $a \in S$, then a is called a *unit* of S if $ab = ba = 1$ for some $b \in S$. We can see that the element b is unique and it is denoted by a^{-1}. Note that the set of all units of S forms a subgroup of S, which is the greatest subgroup of S containing 1 and it is called the *group of units* of S.

An element a of a semigroup S is called an *idempotent* if $a^2 = a$. The identity of a group G is exactly one idempotent of G. We denote the set of all idempotents of a semigroup S by $\mathrm{E}(S)$.

An element a of a semigroup S is called *regular* if $a = axa$ for some $x \in S$ and S is called a *regular semigroup* if every element of S is regular. A semigroup S is called an *inverse semigroup* if for every $x \in S$, there is the unique x^{-1} in S such that $x = xx^{-1}x$ and $x^{-1} = x^{-1}xx^{-1}$.

If S is a semigroup and $a \in S$, then the semigroup $(S, \circ)$ defined by $x \circ y = xay$ for all $x, y \in S$ is called the *variant* of S *induced by* a and it is denoted by (S, a). It is clear that if S has a zero 0, then 0 is the zero of the variant (S, a) of S.

For semigroups S and S', $S \cong S'$ means that S is isomorphic to S', i.e., there exists a bijection $\varphi : S \to S'$ such that $(xy)\varphi = (x\varphi)(y\varphi)$ for all $x, y \in S$. Notice that we also have $S' \cong S$ through φ^{-1}. Therefore we have that for a, b in a semigroup S, $(S, a) \cong (S, b)$ if and only if there is a bijection $\varphi : S \to S$ such that $(xay)\varphi = (x\varphi)b(y\varphi)$ for all $x, y \in S$. In addition, for semigroups S, S' and S'', if $S \cong S'$ and $S' \cong S''$ through φ and φ', respectively, then $S \cong S''$ through $\varphi\varphi'$. The notation $S \ncong S'$ means that S and S' are not isomorphic.

The following facts relate to being isomorphic of variants which will be used later.

Proposition 2.1. *Let S be a semigroup with identity and a, b units of S. Then $(S, a) \cong (S, b)$ through the mapping $x \mapsto axb^{-1}$. In particular, $(S, a) \cong S$ through the mapping $x \mapsto ax$ and $S \cong (S, a)$ through the mapping $x \mapsto xa^{-1}$. Hence for any group G, $(G, a) \cong G$ for all $a \in G$.*

Proof. Define $\varphi : S \to S$ by $x\varphi = axb^{-1}$ for all $x \in S$. Since a and b are units, φ is clearly 1-1. If $x \in S$, then $(a^{-1}xb)\varphi = a(a^{-1}xb)b^{-1} = x$. If $x, y \in S$, then

$$(xay)\varphi = a(xay)b^{-1} = (axb^{-1})b(ayb^{-1}) = (x\varphi)b(y\varphi).$$

Hence φ is an isomorphism from (S, a) onto (S, b), as desired. $\square$

Proposition 2.2. *Let S be a semigroup with identity and $a, b \in S$. If there are units u, v in S such that $uav = b$, then $(S, a) \cong (S, b)$.*

Proof. Define $\varphi : S \to S$ by $x\varphi = v^{-1}xu^{-1}$ for all $x \in S$. Since u and v are units, φ is clearly 1-1. If $x \in S$, then $(vxu)\varphi = v^{-1}(vxu)u^{-1} = x$. If $x, y \in S$, then

$$\begin{aligned}(xay)\varphi &= v^{-1}(xay)u^{-1}\\ &= (v^{-1}xu^{-1})uav(v^{-1}yu^{-1})\\ &= (v^{-1}xu^{-1})b(v^{-1}yu^{-1})\\ &= (x\varphi)b(y\varphi).\end{aligned}$$

Thus φ is an isomorphism from (S, a) onto (S, b). □

Theorem 2.3 ([2])**.** *If S is a semigroup and $a \in S$ such that (S, a) has an identity, then*

(i) S has an identity,

(ii) a is a unit and

(iii) $(S, a) \cong S$.

Hence for a semigroup S with identity and $a \in S, (S, a) \cong S$ if and only if a is a unit.

Note that Hickey [2] proved Theorem 2.3 by using a fact of Green's relations on semigroups.

For a nonempty set X, let $T(X), P(X)$ and $I(X)$ denote the full transformation semigroup, the partial transformation semigroup and the symmetric inverse semigroup (the 1-1 partial transformation semigroup) on X, respectively. Note that $T(X)$ and $I(X)$ are subsemigroups of $P(X)$. The domain and the range (image) of α in $P(X)$ are denoted by $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$, respectively. We have that for $\alpha, \beta \in P(X)$,

$$\begin{aligned}\operatorname{dom}(\alpha\beta) &= (\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\alpha^{-1} \subseteq \operatorname{dom}\alpha,\\ \operatorname{ran}(\alpha\beta) &= (\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\beta \subseteq \operatorname{ran}\beta \quad \text{and}\end{aligned}$$

$$\text{for } x \in X, x \in \operatorname{dom} \alpha\beta \Leftrightarrow x \in \operatorname{dom} \alpha \text{ and } x\alpha \in \operatorname{dom} \beta.$$

It is well-known that $P(X)$ and $T(X)$ are regular semigroups and $I(X)$ is an inverse semigroup ([6], p. 4). We see that 1_X (the identity mapping on X) is the identity of $P(X), T(X)$ and $I(X)$ and the empty transformation 0 is the zero of $P(X)$ and $I(X)$. For $\alpha \in P(X), \alpha$ is an idempotent of $P(X)$, i.e., $\alpha^2 = \alpha$, if and only if $\operatorname{ran} \alpha \subseteq \operatorname{dom} \alpha$ and $x\alpha = x$ for all $x \in \operatorname{ran} \alpha$. It follows that for $\alpha \in I(X), \alpha$ is an idempotent of $I(X)$ if and only if α is the identity mapping on $\operatorname{dom} \alpha$, i.e., $\alpha = 1_{\operatorname{dom} \alpha}$.

If X is finite and $|X| = n$, let T_n, P_n and I_n stand for $T(X), P(X)$ and $I(X)$, respectively. For $\theta \in P_n$ and $k \in \{1, 2, \ldots, n\}$, let

$$t_k = |\{y \in \operatorname{ran} \theta \mid |y\theta^{-1}| = k\}|,$$

i.e.,

$$t_k = |\{y \in \operatorname{ran} \theta \mid |\{x \in \operatorname{dom} \theta \mid x\theta = y\}| = k\}|.$$

The n-tuple $(t_1, t_2, \ldots, t_n)$ is called the *type of* θ. The following remarkable isomorphism theorems for the variants of I_n, T_n and P_n were given by Tsyaputa [12, 13].

Theorem 2.4 ([12])**.** *For $\theta_1, \theta_2 \in I_n, (I_n, \theta_1) \cong (I_n, \theta_2)$ if and only if $|\operatorname{ran} \theta_1| = |\operatorname{ran} \theta_2|$.*

Theorem 2.5 ([12])**.** *For $\theta_1, \theta_2 \in T_n, (T_n, \theta_1) \cong (T_n, \theta_2)$ if and only if θ_1 and θ_2 have the same type.*

Theorem 2.6 ([13])**.** *For $\theta_1, \theta_2 \in P_n, (P_n, \theta_1) \cong (P_n, \theta_2)$ if and only if θ_1 and θ_2 have the same type.*

For convenience, we may write $\alpha \in P(X)$, by using a bracket notation. For example,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ stands for the mapping } \alpha \text{ with } \operatorname{dom} \alpha = \{a, b\}, \operatorname{ran} \alpha = \{c, d\},$$

$a\alpha = c$ and $b\alpha = d$,

$\begin{pmatrix} A & x \\ a & x' \end{pmatrix}_{x \in X \smallsetminus A}$ stands for the mapping β with $\operatorname{dom} \beta = X$,

$$\operatorname{ran} \beta = \{a\} \cup \{x' \mid x \in X \smallsetminus A\} \text{ and } x\beta = \begin{cases} a & \text{if } x \in A, \\ x' & \text{if } x \in X \smallsetminus A. \end{cases}$$

By the above notations, a mapping α can be written as $\alpha = \begin{pmatrix} x\alpha^{-1} \\ x \end{pmatrix}_{x \in \operatorname{ran} \alpha}$.

We shall give some examples of being isomorphic for the variants of the following semigroups: left zero semigroups, right zero semigroups, zero semigroups and Kronecker semigroups.

Example 2.7. (1) If S is a left zero semigroup, $a \in S$ and φ is a bijection on S, then for all $x, y \in S$, $(xay)\varphi = x\varphi = (x\varphi)(y\varphi)$. Hence φ is an isomorphism from (S, a) onto S. This shows that $(S, a) \cong S$ for all $a \in S$. It follows that for all $a, b \in S$, $(S, a) \cong (S, b)$.

(2) It can be shown dually to (1) that for a right zero semigroup S, $(S, a) \cong S$ for all $a \in S$. Consequently, $(S, a) \cong (S, b)$ for all $a, b \in S$.

(3) If S is a zero semigroup with zero 0, $a \in S$ and φ is a bijection on S such that $0\varphi = 0$, then for all $x, y \in S, (xay)\varphi = 0\varphi = 0 = (x\varphi)(y\varphi)$. Therefore φ is an isomorphism from (S, a) onto S. This shows that $(S, a) \cong S$ for all $a \in S$ and hence $(S, a) \cong (S, b)$ for all $a, b \in S$.

(4) Let S be a Kronecker semigroup with zero 0, i.e.,

$$xy = \begin{cases} x & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

Assume that $|S| > 1$. Since $(S, 0)$ is a zero semigroup, it follows that $(S, 0) \ncong S$. Claim that for $a \in S \smallsetminus \{0\}$, $(S, a) \cong S$ if and only if $|S| = 2$. Let $\varphi : (S, a) \to S$ be an isomorphism and assume that $|S| > 2$. Let $b \in S \smallsetminus \{0, a\}$. Then $0\varphi = (bab)\varphi = (b\varphi)(b\varphi) = b\varphi$, so $b = 0$, a contradiction. This shows that if $(S, a) \cong S$, then $|S| = 2$. If $|S| = 2$, then it is clearly seen that the identity mapping on S is an isomorphism from (S, a) onto S.

Next, assume that $|S| > 2$. Claim that for all $a, b \in S \smallsetminus \{0\}$, $(S, a) \cong (S, b)$. Let $a, b \in S \smallsetminus \{0\}$ and define $\varphi : S \to S$ by

$$\varphi = \begin{pmatrix} a & b & x \\ b & a & x \end{pmatrix}_{x \in S \smallsetminus \{a,b\}}.$$

Then φ is a bijection on S and for $x, y \in S$,

$$\begin{aligned} (xay)\varphi &= \begin{cases} a\varphi & \text{if} \quad x = a = y, \\ 0 & \text{otherwise}, \end{cases} \\ &= \begin{cases} b & \text{if} \quad x = a = y, \\ 0 & \text{otherwise}, \end{cases} \\ &= \begin{cases} b & \text{if} \quad x\varphi = b = y\varphi, \\ 0 & \text{otherwise}, \end{cases} \\ &= (x\varphi)b(y\varphi). \end{aligned}$$

Hence φ is an isomorphism from (S, a) onto (S, b), as desired.

For $k \in \mathbb{N}$, let

$$\mathbb{N}_k = \{k, k+1, k+2, \ldots\} = \{k + l \mid l \in \mathbb{N} \cup \{0\}\}.$$

Then $(\mathbb{N}_k, +)$ and $(k\mathbb{N}, +)$ are ideals of $(\mathbb{N}, +)$. Note that $(k\mathbb{N}, +)$ is the infinite cyclic semigroup generated by k. Recall that all of the infinite cyclic semigroups are isomorphic.

For $a, b \in \mathbb{Z}$, not both 0, let (a, b) denote the g.c.d. of a and b in $\mathbb{Z}$. For $n \in \mathbb{N}$, let $\mathbb{Z}_n$ be the set of integers modulo n. For $x \in \mathbb{Z}$, let $\overline{x}$ be the congruence class modulo n containing x. Then $|\mathbb{Z}_n| = n$ and

$$\mathbb{Z}_n = \{\overline{0}, \overline{1}, \ldots, \overline{n-1}\} = \{\overline{x} \mid x \in \mathbb{Z}\}.$$

We have that for $a \in \mathbb{Z}$,

$$\begin{aligned} a\mathbb{Z}_n &= (a, n)\mathbb{Z}_n \\ &= \left\{\overline{0}, \overline{(a,n)}, 2\overline{(a,n)}, \ldots, \left(\frac{n}{(a,n)} - 1\right)\overline{(a,n)}\right\} \end{aligned}$$

and $|a\mathbb{Z}_n| = \dfrac{n}{(a,n)}$.

The following powerful theorem in number theory will be used to characterize when $((\mathbb{Z}_n, \cdot), \overline{a})$ and $((\mathbb{Z}_n, \cdot), \overline{b})$ are isomorphic for $a, b \in \mathbb{Z}$.

Theorem 2.8 ([10], p. 258)**.** (Dirichlet : Primes in Arithmetic Progression)[1] *If $a, m \in \mathbb{Z}$ with $(a, m) = 1$, then there are infinitely many primes p of the form $p \equiv a (\text{mod } m)$.*

Let X be a nonempty set and

$$G(X) = \{\alpha \in T(X) \mid \alpha \text{ is bijective}\}.$$

We can see that $G(X)$ is the group of units of $T(X)$, $P(X)$ and $I(X)$. By Proposition 2.1, we have $(G(X), \alpha) \cong G(X)$ for all $\alpha \in G(X)$. Hence for all $\alpha, \beta \in G(X), (G(X), \alpha) \cong (G(X), \beta)$.

Next, let

$$M(X) = \{\alpha \in T(X) \mid \alpha \text{ is 1-1}\},$$
$$E(X) = \{\alpha \in T(X) \mid \alpha \text{ is onto}\}.$$

which are subsemigroups of $T(X)$ containing $G(X)$. Notice that $M(X) = G(X)$

[1]The author is very grateful to Associate Professor Dr. Paisan Nakmahachalasint for introducing me this theorem in order to obtain Theorem 4.1 in Chapter IV.

$[E(X) = G(X)]$ if and only if X is finite. In addition, $G(X)$ is the group of units of $M(X)$ and $E(X)$.

Let $\theta \in T(X)$. The *partition* of X *induced by* θ is defined to be

$$\mathcal{P}(\theta) = \{x\theta^{-1} \mid x \in \operatorname{ran}\theta\}.$$

Then

$$X = \dot{\bigcup_{x \in \operatorname{ran}\theta}} x\theta^{-1}.$$

For $\theta_1, \theta_2 \in T(X)$, we say that the partition $\mathcal{P}(\theta_1)$ and the partition $\mathcal{P}(\theta_2)$ are *equivalent* if there exists a bijection $\varphi : \operatorname{ran}\theta_2 \to \operatorname{ran}\theta_1$ such that $|x\theta_2^{-1}| = |(x\varphi)\theta_1^{-1}|$ for all $x \in \operatorname{ran}\theta_2$. If this is the case, we write $\mathcal{P}(\theta_1) \approx \mathcal{P}(\theta_2)$. Notice that $\approx$ is an equivalence relation on the set of the partitions of X induced by $\theta \in T(X)$. By our definitions, we can see that if X is finite, then for $\theta_1, \theta_2 \in T(X)$, θ_1 and θ_2 have the same type if and only if $\mathcal{P}(\theta_1) \approx \mathcal{P}(\theta_2)$. Then Theorem 2.5 can restate as follows:

Theorem 2.9. *For a finite nonempty set X and $\theta_1, \theta_2 \in T(X)$, $(T(X), \theta_1) \cong (T(X), \theta_2)$ if and only if $\mathcal{P}(\theta_1) \approx \mathcal{P}(\theta_2)$.*

We recall some basic knowledge in linear algebra. Let V and W be vector spaces over a field F. Let $L_F(V, W)$ be the set of all linear transformations $\alpha : V \to W$ and $L_F(V)$ stand for $L_F(V, V)$. Then $L_F(V)$ is a semigroup under composition. For $\alpha \in L_F(V, W)$, let $\ker\alpha$ denote the kernel of α. We call $\dim_F \operatorname{ran}\alpha$ the *rank* of α and it is denoted by rank α. For a subset A of V, let $\langle A \rangle$ denote the subspace of V spanned by A.

The following facts in linear algebra will be used in our research. The proofs are omitted.

Remark 2.10.

(1) For $\alpha \in L_F(V, W)$,

$$\dim_F V = \dim_F \ker\alpha + \text{rank } \alpha.$$

(2) If B is a basis of V, $\alpha \in L_F(V,W)$ are such that $\alpha_{|B}$ is 1-1 and $B\alpha$ is a linearly independent subset of W, then α is a monomorphism.

(3) If B is a basis of V, B' is a basis of W and $\alpha \in L_F(V,W)$ is such that $\alpha_{|B} : B \to B'$ is a bijection, then α is an isomorphism from V onto W.

(4) If $\alpha \in L_F(V,W)$, B_1 is a basis of $\ker\alpha$, B_2 is a basis of $\operatorname{ran}\alpha$ and for each $v \in B_2$, let $v' \in v\alpha^{-1}$, then $B_1 \dot{\cup} \{v' \mid v \in B_2\}$ is a basis of V.

(5) If $\alpha \in L_F(V,W)$, B_1 is a basis of $\ker\alpha$ and B is a basis of V containing B_1, then $(B \smallsetminus B_1)\alpha$ is a basis of $\operatorname{ran}\alpha$ and for any distinct $u, v \in B \smallsetminus B_1$, $u\alpha \neq v\alpha$.

(6) If U is a subspace of V, B_1 is a basis of U and B is a basis of V containing B_1, then $\{v + U \mid v \in B \smallsetminus B_1\}$ is a basis of the quotient space V/U and $u + U \neq v + U$ for all distinct $u, v \in B \smallsetminus B_1$. Hence $\dim_F(V/U) = |B \smallsetminus B_1|$.

(7) If B is a basis of V, then

$$|L_F(V,W)| = |\{\alpha \mid \alpha : B \to W\}| = |W|^{|B|}.$$

(8) If V is finite-dimensional, then $V \cong F^{\dim_F V}$ as vector spaces over F.

(9) If W is finite-dimensional, then

$$|L_F(V,W)| = |F^{\dim_F W}|^{\dim_F V} = |F|^{(\dim_F V)(\dim_F W)}.$$

In particular, $|L_F(V,W)| = |F|^{(\dim_F V)(\dim_F W)} < \infty$ if V is also finite-dimensional and F is a finite field.

For a positive integer n and a field F, let $M_n(F)$ be the set of all $n \times n$ matrices over F.

Theorem 2.11 ([7], p. 330-337)**.** *If V is finite-dimensional and* $\dim_F V = n$, *then there exists a semigroup isomorphism $\varphi : L_F(V) \to (M_n(F), \cdot)$ which preserves ranks.*

Let

$$G_F(V) = \{\alpha \in L_F(V) \mid \alpha \text{ is an isomorphism}\}.$$

Then $G_F(V)$ is the group of units of $L_F(V)$. By Proposition 2.1, we have $(G_F(V), \alpha) \cong G_F(V)$ for all $\alpha \in G_F(V)$. Thus for $\alpha, \beta \in G_F(V)$, $(G_F(V), \alpha) \cong (G_F(V), \beta)$.

Next, let $M_F(V)$ and $E_F(V)$ be the set of all 1-1 linear transformations (monomorphisms) of V and the set of all onto linear transformations (epimorphisms) of V, respectively. Then

$$M_F(V) = \{\alpha \in L_F(V) \mid \ker \alpha = \{0\}\},$$
$$E_F(V) = \{\alpha \in L_F(V) \mid \operatorname{ran} \alpha = V\}$$

which are subsemigroups of $L_F(V)$ containing $G_F(V)$. Moreover, it is well-known that $\dim_F V < \infty$ if and only if $M_F(V) = G_F(V)$ $[E_F(V) = G_F(V)]$. In addition, $G_F(V)$ is also the group of units of the semigroups $M_F(V)$ and $E_F(V)$.

CHAPTER III

MULTIPLICATIVE AND ADDITIVE SEMIGROUPS OF INTEGERS

In this chapter, we determine when two variants of the following semigroups of integers are isomorphic:

$$(\mathbb{N},\cdot),\ (\mathbb{N}_k,+)\ \text{ and }\ (k\mathbb{N},+)$$

where $k \in \mathbb{N}$ and recall that

$$\mathbb{N}_k = \{k, k+1, k+2, \ldots\} = \{k+l \mid l \in \mathbb{N} \cup \{0\}\}.$$

Then $(\mathbb{N},+) = (\mathbb{N}_1,+) = (1\mathbb{N},+)$ and we can see that $\mathbb{N}_k = \mathbb{N}\,[k\mathbb{N} = \mathbb{N}]$ if and only if $k = 1$. Note that $(\mathbb{Z},+)$ is a group. Then $((\mathbb{Z},+),a) \cong (\mathbb{Z},+)$ for all $a \in \mathbb{Z}$ (Proposition 2.1). This chapter also includes necessary conditions for two variants of $(\mathbb{Z},\cdot)$ to be isomorphic.

To obtain an isomorphism theorem for the variants of $(\mathbb{N},\cdot)$, the following lemma is needed.

Lemma 3.1. *Let $a, b \in \mathbb{N}$ and $\varphi : \mathbb{N} \to \mathbb{N}$. Then $\varphi : ((\mathbb{N},\cdot),a) \to ((\mathbb{N},\cdot),b)$ is an isomorphism if and only if $\varphi : (\mathbb{N},\cdot) \to (\mathbb{N},\cdot)$ is an isomorphism such that $a\varphi = b$.*

Proof. Let $\varphi : ((\mathbb{N},\cdot),a) \to ((\mathbb{N},\cdot),b)$ be an isomorphism. Then

$$(xay)\varphi = (x\varphi)b(y\varphi) \quad \text{for all } x, y \in \mathbb{N},$$

so

$$(\mathbb{N}a\mathbb{N})\varphi = (\mathbb{N}\varphi)b(\mathbb{N}\varphi) = \mathbb{N}b\mathbb{N}.$$

Since $\mathbb{N}\mathbb{N} = \mathbb{N}$, it follows that $(a\mathbb{N})\varphi = b\mathbb{N}$. But $b \in b\mathbb{N}$, so $(am)\varphi = b$ for some $m \in \mathbb{N}$. We have that

$$a\varphi = (1a1)\varphi = (1\varphi)b(1\varphi) = (1\varphi)^2 b \geq b.$$

If $a\varphi > b$, then $(1\varphi)^2 b > b$, so $1\varphi > 1$. Consequently,

$$b = (am)\varphi = (1am)\varphi = (1\varphi)b(m\varphi) > b,$$

a contradiction. Hence $a\varphi = b$. This implies that $(1\varphi)^2 b = b$, so $1\varphi = 1$. If $x, y \in \mathbb{N}$, then

$$\begin{aligned}((xy)\varphi)b &= ((xy)\varphi)b(1\varphi)\\ &= (xya1)\varphi\\ &= (xay)\varphi\\ &= (x\varphi)b(y\varphi)\\ &= (x\varphi)(y\varphi)b,\end{aligned}$$

so $(xy)\varphi = (x\varphi)(y\varphi)$ since $(\mathbb{N}, \cdot)$ is cancellative. This proves that $\varphi : (\mathbb{N}, \cdot) \to (\mathbb{N}, \cdot)$ is an isomorphism such that $a\varphi = b$.

For the converse, let $\varphi : (\mathbb{N}, \cdot) \to (\mathbb{N}, \cdot)$ be an isomorphism such that $a\varphi = b$. Then for all $x, y \in \mathbb{N}$,

$$(xay)\varphi = (x\varphi)(a\varphi)(y\varphi) = (x\varphi)b(y\varphi).$$

Hence $\varphi : ((\mathbb{N}, \cdot), a) \to ((\mathbb{N}, \cdot), b)$ is an isomorphism. □

Theorem 3.2. *For $a, b \in \mathbb{N}$, $((\mathbb{N}, \cdot), a) \cong ((\mathbb{N}, \cdot), b)$ if and only if either*

(i) $a = b = 1$ or

(ii) $a = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ and $b = q_1^{r_1} q_2^{r_2} \cdots q_k^{r_k}$ for some distinct primes $p_1, p_2, \ldots, p_k$, some distinct primes $q_1, q_2, \ldots, q_k$ and some $r_1, r_2, \ldots, r_k \in \mathbb{N}$.

Proof. Let $\varphi : ((\mathbb{N}, \cdot), a) \to ((\mathbb{N}, \cdot), b)$ be an isomorphism. By Lemma 3.1, $\varphi : (\mathbb{N}, \cdot) \to (\mathbb{N}, \cdot)$ is an isomorphism such that $a\varphi = b$. Since 1 is the identity of $(\mathbb{N}, \cdot)$, $1\varphi = 1$. Therefore if $a = 1$, then $b = a\varphi = 1\varphi = 1$. Assume that $a > 1$ and let $a = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ where $p_1, p_2, \ldots, p_k$ are distinct primes and $r_1, r_2, \ldots, r_k \in \mathbb{N}$. Then

$$b = a\varphi = (p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k})\varphi = (p_1\varphi)^{r_1} (p_2\varphi)^{r_2} \cdots (p_k\varphi)^{r_k}.$$

It remains to show that $p_1\varphi, p_2\varphi, \ldots, p_k\varphi$ are distinct primes. Since φ is 1-1, it suffices to show that if p is a prime, then $p\varphi$ is also a prime. Let p be a prime number and suppose that $p\varphi = mn$ for some $m, n \in \mathbb{N} \smallsetminus \{1\}$. Since φ is a bijection and $1\varphi = 1$, it follows that $m'\varphi = m$ and $n'\varphi = n$ for some $m', n' \in \mathbb{N} \smallsetminus \{1\}$. Then $p\varphi = (m'\varphi)(n'\varphi) = (m'n')\varphi$ which implies that $p = m'n'$, a contradiction. Hence $p\varphi$ is a prime. This shows that if $a > 1$, then (ii) holds.

For the converse, assume that (i) or (ii) holds. It is trivial if (i) holds. Assume that a and b satisfy (ii). Let P be the set of all prime numbers in $\mathbb{N}$. Then $|P \smallsetminus \{p_1, p_2, \ldots, p_k\}| = |P \smallsetminus \{q_1, q_2, \ldots, q_k\}|$. Let $\varphi : P \smallsetminus \{p_1, p_2, \ldots, p_k\} \to P \smallsetminus \{q_1, q_2, \ldots, q_k\}$ be a bijection. Define $\bar{\varphi} : P \to P$ by

$$\bar{\varphi} = \begin{pmatrix} p_i & x \\ q_i & x\varphi \end{pmatrix}_{\substack{i \in \{1,2,\ldots,k\}, \\ x \in P \smallsetminus \{p_1, p_2, \ldots, p_k\}}}.$$

Then $\bar{\varphi}$ is a bijection on P. Let $\theta : \mathbb{N} \to \mathbb{N}$ be defined by

$$1\theta = 1 \quad \text{and} \quad (s_1^{t_1} s_2^{t_2} \cdots s_m^{t_m})\theta = (s_1\bar{\varphi})^{t_1} (s_2\bar{\varphi})^{t_2} \cdots (s_m\bar{\varphi})^{t_m}$$

for any primes $s_1, s_2, \ldots, s_m \in \mathbb{N}$ and $t_1, t_2, \ldots, t_m \in \mathbb{N}$. Then $a\theta = b$. From the definition of $\bar{\varphi}$ and the fact that every element of $\mathbb{N} \smallsetminus \{1\}$ can be written uniquely as a product of primes, we have that θ is an isomorphism from $(\mathbb{N}, \cdot)$ onto itself. From Lemma 3.1, $\theta : ((\mathbb{N}, \cdot), a) \to ((\mathbb{N}, \cdot), b)$ is an isomorphism.

Hence the proof is completed. □

Example 3.3. From Theorem 3.2, we have that

$$((\mathbb{N}, \cdot), 6) \cong ((\mathbb{N}, \cdot), 35) \quad \text{and} \quad ((\mathbb{N}, \cdot), 6) \ncong ((\mathbb{N}, \cdot), 12)$$

since $6 = 2 \cdot 3$, $35 = 5 \cdot 7$ and $12 = 2^2 \cdot 3$.

The following lemma is needed to obtain an isomorphism theorem for the variants of $(\mathbb{Z}, \cdot)$.

Lemma 3.4. *Let* $a, b \in \mathbb{Z} \smallsetminus \{0\}$ *and* $\varphi : \mathbb{Z} \to \mathbb{Z}$. *Then* $\varphi : ((\mathbb{Z}, \cdot), a) \to ((\mathbb{Z}, \cdot), b)$ *is an isomorphism if and only if* φ *satisfies the following three conditions*:

(*i*) φ *is a bijection*;

(*ii*) $a\varphi = b$;

(*iii*) *for all* $x, y \in \mathbb{Z}$, $(xy)\varphi = (x\varphi)(y\varphi)$ *or for all* $x, y \in \mathbb{Z}$, $(xy)\varphi = -(x\varphi)(y\varphi)$.

Proof. Assume that $\varphi : ((\mathbb{Z}, \cdot), a) \to ((\mathbb{Z}, \cdot), b)$ is an isomorphism. Then (i) holds. But for all $x, y \in \mathbb{Z}, (xay)\varphi = (x\varphi)b(y\varphi)$ and $\mathbb{Z}\mathbb{Z} = \mathbb{Z}$, so we have

$$(a\mathbb{Z})\varphi = (\mathbb{Z}a\mathbb{Z})\varphi = (\mathbb{Z}\varphi)b(\mathbb{Z}\varphi) = \mathbb{Z}b\mathbb{Z} = b\mathbb{Z}.$$

Thus $(am)\varphi = b$ for some $m \in \mathbb{Z}$, so $|(am)\varphi| = |b|$. We have that

$$|a\varphi| = |(1a1)\varphi| = |(1\varphi)b(1\varphi)| = (1\varphi)^2|b| \geq |b|.$$

If $|a\varphi| > |b|$, then $(1\varphi)^2|b| > |b|$ which implies that $|1\varphi| > 1$, so

$$|(am)\varphi| = |(1am)\varphi| = |(1\varphi)b(m\varphi)| > |b|,$$

a contradiction. Thus $|a\varphi| = |b|$. Hence $a\varphi = b$ or $a\varphi = -b$. Since $(1\varphi)^2|b| = |a\varphi| = |b|, (1\varphi)^2 = 1$. Thus $a\varphi = (1a1)\varphi = (1\varphi)^2 b = b$ and $1\varphi = \pm 1$. Hence (ii) holds.

Case 1: $1\varphi = 1$. If $x, y \in \mathbb{Z}$, then

$$\begin{aligned} ((xy)\varphi)b &= ((xy)\varphi)b(1\varphi) \\ &= (xya1)\varphi \\ &= (xay)\varphi \\ &= (x\varphi)b(y\varphi) \\ &= (x\varphi)(y\varphi)b \end{aligned}$$

which implies that $(xy)\varphi = (x\varphi)(y\varphi)$.

Case 2: $1\varphi = -1$. If $x, y \in \mathbb{Z}$, then

$$\begin{aligned}((xy)\varphi)b &= ((xy)\varphi)b(-(1\varphi)) \\ &= -(xya1)\varphi \\ &= -(xay)\varphi \\ &= -(x\varphi)b(y\varphi) \\ &= -(x\varphi)(y\varphi)b,\end{aligned}$$

so $(xy)\varphi = -(x\varphi)(y\varphi)$.

Hence (iii) holds.

Conversely, assume that (i), (ii) and (iii) hold. To show that $\varphi : ((\mathbb{Z}, \cdot), a) \to ((\mathbb{Z}, \cdot), b)$ is an isomorphism, from (i) it remains to show that φ is a homomorphism.

Case 1: $(xy)\varphi = (x\varphi)(y\varphi)$ for all $x, y \in \mathbb{Z}$. If $x, y \in \mathbb{Z}$, then

$$(xay)\varphi = (x\varphi)(a\varphi)(y\varphi) = (x\varphi)b(y\varphi)$$

since $a\varphi = b$ by (ii).

Case 2: $(xy)\varphi = -(x\varphi)(y\varphi)$ for all $x, y \in \mathbb{Z}$. If $x, y \in \mathbb{Z}$, then

$$(xay)\varphi = -((xa)\varphi)(y\varphi) = -(-(x\varphi)(a\varphi))(y\varphi) = (x\varphi)b(y\varphi).$$

Therefore the lemma is proved. □

Theorem 3.5. *For $a, b \in \mathbb{Z}$, if $((\mathbb{Z}, \cdot), a) \cong ((\mathbb{Z}, \cdot), b)$, then one of the following conditions holds.*

(i) $a = b = 0$.

(ii) $|a| = |b| = 1$.

(iii) $a = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ *and* $b = q_1^{r_1} q_2^{r_2} \cdots q_k^{r_k}$ *for some distinct primes* $p_1, p_2, \ldots, p_k$ *in* $\mathbb{Z}$, *some distinct primes* $q_1, q_2, \ldots, q_k$ *in* $\mathbb{Z}$, *some* $r_1, r_2, \ldots, r_k \in \mathbb{N}$ *and for* $i, j \in \{1, 2, \ldots, k\}$, *if* $p_i = -p_j$, *then* $q_i = -q_j$.

Proof. Assume that $\varphi : ((\mathbb{Z}, \cdot), a) \to ((\mathbb{Z}, \cdot), b)$ is an isomorphism. Since 0 is the zero of $((\mathbb{Z}, \cdot), a)$ and $((\mathbb{Z}, \cdot), b)$, $0\varphi = 0$. It is clearly seen that $((\mathbb{Z}, \cdot), a)$ is a zero semigroup if and only if $a = 0$. It follows that $a = 0$ if and only if $b = 0$. Assume that $a \neq 0$. Then $b \neq 0$. By Lemma 3.4, φ is a bijection, $a\varphi = b$ and $(xy)\varphi = (x\varphi)(y\varphi)$ for all $x, y \in \mathbb{Z}$ or $(xy)\varphi = -(x\varphi)(y\varphi)$ for all $x, y \in \mathbb{Z}$. Since

$$b = a\varphi = (1a1)\varphi = (1\varphi)^2 b \quad \text{and} \quad b = a\varphi = ((-1)a(-1))\varphi = ((-1)\varphi)^2 b,$$

we have that $1\varphi = \pm 1$ and $(-1)\varphi = \pm 1$. Thus $\{-1, 0, 1\}\varphi = \{-1, 0, 1\}$ and $0\varphi = 0$.

Assume that $|a| = 1$. Then $|b| = |a\varphi| = 1$.

Next, assume that $|a| > 1$. Then $|b| > 1$. Let $a = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ where $p_1, p_2, \ldots, p_k$ are distinct primes in $\mathbb{Z}$ and $r_1, r_2, \ldots, r_k \in \mathbb{N}$. Then

$$b = a\varphi = (p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k})\varphi = \pm (p_1\varphi)^{r_1} (p_2\varphi)^{r_2} \cdots (p_k\varphi)^{r_k}.$$

It remains to show that $p_1\varphi, p_2\varphi, \ldots, p_k\varphi$ are distinct primes in $\mathbb{Z}$. Since φ is 1-1, it suffices to show that if p is a prime in $\mathbb{Z}$, then $p\varphi$ is also a prime in $\mathbb{Z}$. Let p be a prime in $\mathbb{Z}$ and suppose that $p\varphi = mn$ for some $m, n \in \mathbb{Z} \smallsetminus \{-1, 0, 1\}$. Since φ is a bijection and $\{-1, 0, 1\}\varphi = \{-1, 0, 1\}$, it follows that $m'\varphi = m$ and $n'\varphi = n$ for some $m', n' \in \mathbb{Z} \smallsetminus \{-1, 0, 1\}$. Then $p\varphi = (m'\varphi)(n'\varphi) = \pm (m'n')\varphi$.

Case 1: $p\varphi = (m'n')\varphi$. Then $p = m'n'$, a contradiction.

Case 2: $p\varphi = -(m'n')\varphi$. If $1\varphi = 1$, then by the proof of Case 1 of the "only if" part in Lemma 3.4, $(xy)\varphi = (x\varphi)(y\varphi)$ for all $x, y \in \mathbb{Z}$. Since $1\varphi = 1$, $(-1)\varphi = -1$, so

$$\begin{aligned} p\varphi &= (-1)\varphi(m'n')\varphi \\ &= ((-1)(m'n'))\varphi \\ &= (-m'n')\varphi. \end{aligned}$$

It follows that $p = -m'n'$, a contradiction. If $1\varphi = -1$, then by the proof of Case 2 of the "only if" part in Lemma 3.4, we have that $(xy)\varphi = -(x\varphi)(y\varphi)$ for all

$x, y \in \mathbb{Z}$. Since $1\varphi = -1$, $(-1)\varphi = 1$, so

$$\begin{aligned} p\varphi &= -((-1)\varphi(m'n')\varphi) \\ &= ((-1)(m'n'))\varphi \\ &= (-m'n')\varphi \end{aligned}$$

which implies that $p = -m'n'$, a contradiction.

Finally, let $i, j \in \{1, 2, \ldots, k\}$ be such that $p_i = -p_j$. If $1\varphi = 1$, then $(-1)\varphi = -1$ and $(xy)\varphi = (x\varphi)(y\varphi)$ for all $x, y \in \mathbb{Z}$, so

$$\begin{aligned} p_i\varphi &= (-p_j)\varphi \\ &= ((-1)p_j)\varphi \\ &= ((-1)\varphi)(p_j\varphi) \\ &= -(p_j\varphi). \end{aligned}$$

If $1\varphi = -1$, then $(-1)\varphi = 1$ and $(xy)\varphi = -(x\varphi)(y\varphi)$ for all $x, y \in \mathbb{Z}$, so

$$\begin{aligned} p_i\varphi &= (-p_j)\varphi \\ &= ((-1)p_j)\varphi \\ &= -((-1)\varphi)(p_j\varphi) \\ &= -(p_j\varphi). \end{aligned}$$

Hence (iii) holds. □

Example 3.6. From Theorem 3.5, we have that

$$((\mathbb{Z}, \cdot), 6) \ncong ((\mathbb{Z}, \cdot), -12) \ncong ((\mathbb{Z}, \cdot), 25)$$

since $6 = 2 \cdot 3$, $-12 = 2^2 \cdot (-3)$ and $25 = 5^2$.

It is natural to ask whether $((\mathbb{Z}, \cdot), a)$ and $((\mathbb{Z}, \cdot), -a)$ are isomorphic. The answer is positive and the proof is given by making use of Lemma 3.4.

Theorem 3.7. *For $a \in \mathbb{Z}$, $((\mathbb{Z}, \cdot), a) \cong ((\mathbb{Z}, \cdot), -a)$ through the mapping $x \mapsto -x$.*

Proof. Define $\varphi : \mathbb{Z} \to \mathbb{Z}$ by $x\varphi = -x$ for all $x \in \mathbb{Z}$. Then φ is a bijection and $a\varphi = -a$, so φ satisfies (i) and (ii) of Lemma 3.4. If $x, y \in \mathbb{Z}$, then

$$(xy)\varphi = -(xy) = -(-x)(-y) = -(x\varphi)(y\varphi),$$

so φ satisfies (iii) of Lemma 3.4. Therefore by Lemma 3.4, $\varphi : ((\mathbb{Z}, \cdot), a) \to ((\mathbb{Z}, \cdot), -a)$ is an isomorphism. □

Theorem 3.8. *For $k, a, b \in \mathbb{N}$, $((\mathbb{N}_k, +), a) \cong ((\mathbb{N}_k, +), b)$ if and only if $a = b$.*

Proof. Let $\varphi : ((\mathbb{N}_k, +), a) \to ((\mathbb{N}_k, +), b)$ be an isomorphism. Since $\mathbb{N}_k + a + k = \{a+2k, a+2k+1, a+2k+2, \ldots\}$ and for all $i \in \mathbb{N}$, $\mathbb{N}_k + a + (k+i) \subseteq \mathbb{N}_k + a + k$, it follows that

$$\mathbb{N}_k + a + \mathbb{N}_k = \bigcup_{i=0}^{\infty} (\mathbb{N}_k + a + (k+i)) = \mathbb{N}_k + a + k.$$

Therefore we have that

$$\begin{aligned}
(\mathbb{N}_k + a + k)\varphi &= (\mathbb{N}_k + a + \mathbb{N}_k)\varphi \\
&= (\mathbb{N}_k\varphi) + b + (\mathbb{N}_k\varphi) \\
&= \mathbb{N}_k + b + \mathbb{N}_k \qquad\qquad \text{(since } \varphi \text{ is onto)} \\
&= \mathbb{N}_k + b + k.
\end{aligned}$$

Since $\varphi : \mathbb{N}_k \to \mathbb{N}_k$ is a bijection,

$$(\mathbb{N}_k \smallsetminus (\mathbb{N}_k + a + k))\varphi = \mathbb{N}_k \smallsetminus (\mathbb{N}_k + b + k),$$

so $|\mathbb{N}_k \smallsetminus (\mathbb{N}_k + a + k)| = |\mathbb{N}_k \smallsetminus (\mathbb{N}_k + b + k)|$, i.e., $|\{k, k+1, \ldots, k + (k+a-1) = a + 2k - 1\}| = |\{k, k+1, \ldots, k + (k+b-1) = b + 2k - 1\}|$. Hence $k + a = k + b$, so $a = b$. □

As a consequence of Theorem 3.8, we have the following result.

Corollary 3.9. *For $a, b \in \mathbb{N}$, $((\mathbb{N}, +), a) \cong ((\mathbb{N}, +), b)$ if and only if $a = b$.*

We can see that for $k \in \mathbb{N}, (k\mathbb{N}, +)$ is the infinite cyclic semigroup generated by k. Since $(\mathbb{N}, +)$ is the infinite cyclic semigroup generated by 1, it follows that $(k\mathbb{N}, +) \cong (\mathbb{N}, +)$. Therefore from Corollary 3.9, the following result is directly obtained.

Corollary 3.10. *For $k, a, b \in \mathbb{N}$, $((k\mathbb{N}, +), a) \cong ((k\mathbb{N}, +), b)$ if and only if $a = b$.*

CHAPTER IV
THE MULTIPLICATIVE SEMIGROUP $\mathbb{Z}_n$

In this chapter, we deal with isomorphism theorems for the variants of the semigroups $(\mathbb{Z}_n, \cdot)$ and $(k\mathbb{Z}_n, \cdot)$. We characterize when two variants of $(\mathbb{Z}_n, \cdot)$ are isomorphic and give a necessary condition for being isomorphic of two variants of $(k\mathbb{Z}_n, \cdot)$.

Recall that

$$\begin{aligned} \mathbb{Z}_n &= \{\overline{0}, \overline{1}, \ldots, \overline{n-1}\} = \{\overline{x} \mid x \in \mathbb{Z}\}, \\ k\mathbb{Z}_n &= (k,n)\mathbb{Z}_n \\ &= \left\{\overline{0}, \overline{(k,n)}, 2\overline{(k,n)}, \ldots, \left(\frac{n}{(k,n)} - 1\right)\overline{(k,n)}\right\}, \end{aligned}$$

$|\mathbb{Z}_n| = n$ and $|k\mathbb{Z}_n| = \dfrac{n}{(k,n)}$. Note that $l\overline{x} = \overline{l}\overline{x}$ for all $l, x \in \mathbb{Z}$ and for any $l \in \mathbb{Z}$, $l\mathbb{Z}_n = \overline{l}\mathbb{Z}_n$. In addition, we have that

$$\{\overline{x} \mid x \in \mathbb{Z} \text{ and } \overline{x} \text{ is a unit of } (\mathbb{Z}_n, \cdot)\} = \{\overline{x} \mid x \in \mathbb{Z} \text{ and } (x,n) = 1\}.$$

The following main result uses Theorem 2.8 as the main tool.

Theorem 4.1. *For $a, b \in \mathbb{Z}$, $((\mathbb{Z}_n, \cdot), \bar{a}) \cong ((\mathbb{Z}_n, \cdot), \bar{b})$ if and only if $(a,n) = (b,n)$.*

Proof. Let $\varphi : ((\mathbb{Z}_n, \cdot), \bar{a}) \to ((\mathbb{Z}_n, \cdot), \bar{b})$ be an isomorphism. Then for all $x, y \in \mathbb{Z}$,

$$(\bar{x}\bar{a}\bar{y})\varphi = (\bar{x}\varphi)\bar{b}(\bar{y}\varphi).$$

This implies that $(\mathbb{Z}_n\bar{a}\mathbb{Z}_n)\varphi = (\mathbb{Z}_n\varphi)\bar{b}(\mathbb{Z}_n\varphi) = \mathbb{Z}_n\bar{b}\mathbb{Z}_n$. Since $\mathbb{Z}_n\mathbb{Z}_n = \mathbb{Z}_n$, it follows that

$$(\bar{a}\mathbb{Z}_n)\varphi = \bar{b}\mathbb{Z}_n.$$

But since φ is 1-1, we have that $|\bar{a}\mathbb{Z}_n| = |\bar{b}\mathbb{Z}_n|$. Hence $\dfrac{n}{(a,n)} = \dfrac{n}{(b,n)}$ which implies that $(a,n) = (b,n)$.

For the converse, assume that $(a,n) = (b,n)$. Then

$$\bar{a}\mathbb{Z}_n = (a,n)\mathbb{Z}_n = (b,n)\mathbb{Z}_n = \bar{b}\mathbb{Z}_n,$$

so $\bar{a} = \bar{b}\bar{x}$ for some $x \in \mathbb{Z}$. This implies that $n|(a-bx)$. Hence $\dfrac{n}{(a,n)} \bigg| \dfrac{a-bx}{(a,n)}$.

Since $(a,n) = (b,n)$, $(a,n)|b$. Therefore $\dfrac{n}{(a,n)} \bigg| \left(\dfrac{a}{(a,n)} - \dfrac{b}{(a,n)}x\right)$, and so

$$\frac{a}{(a,n)} \equiv \frac{b}{(a,n)}x \left(\bmod \frac{n}{(a,n)}\right).$$

Let $l \in \mathbb{Z}$ be such that $\dfrac{a}{(a,n)} - \dfrac{b}{(a,n)}x = \dfrac{n}{(a,n)}l$. Since $\left(x, \dfrac{n}{(a,n)}\right) \bigg| \dfrac{b}{(a,n)}x$ and $\left(x, \dfrac{n}{(a,n)}\right) \bigg| \dfrac{n}{(a,n)}l$, it follows that $\left(x, \dfrac{n}{(a,n)}\right) \bigg| \dfrac{a}{(a,n)}$. Hence $\left(x, \dfrac{n}{(a,n)}\right) \bigg| \left(\dfrac{a}{(a,n)}, \dfrac{n}{(a,n)}\right)$. But $\left(\dfrac{a}{(a,n)}, \dfrac{n}{(a,n)}\right) = 1$, so $\left(x, \dfrac{n}{(a,n)}\right) = 1$. By Theorem 2.8, there are infinitely many primes p of the form

$$p \equiv x \left(\bmod \frac{n}{(a,n)}\right).$$

Then there exists a prime $q > n$ such that

$$q \equiv x \left(\bmod \frac{n}{(a,n)}\right).$$

Thus

$$\frac{b}{(a,n)}q \equiv \frac{b}{(a,n)}x \left(\bmod \frac{n}{(a,n)}\right)$$

and hence

$$\frac{a}{(a,n)} \equiv \frac{b}{(a,n)}q \left(\bmod \frac{n}{(a,n)}\right).$$

This implies that

$$a \equiv bq (\text{mod } n),$$

so $\bar{a} = \bar{b}\bar{q}$. Since $q > n$ and q is a prime, we have that $(q, n) = 1$. Thus $\bar{q}$ is a unit of $(\mathbb{Z}_n, \cdot)$. By Proposition 2.2, we have that $((\mathbb{Z}_n, \cdot), \bar{a}) \cong ((\mathbb{Z}_n, \cdot), \bar{b})$. □

Example 4.2. From Theorem 4.1, we have that

$$((\mathbb{Z}_{12}, \cdot), \bar{2}) \cong ((\mathbb{Z}_{12}, \cdot), \overline{10}) \ncong ((\mathbb{Z}_{12}, \cdot), \bar{4})$$

since $(2, 12) = 2 = (10, 12)$ and $(4, 12) = 4$.

Theorem 4.3. *For $a, b \in \mathbb{Z}$, if $((k\mathbb{Z}_n, \cdot), k\bar{a}) \cong ((k\mathbb{Z}_n, \cdot), k\bar{b})$, then $(k^3a, n) = (k^3b, n)$.*

Proof. Let $\varphi : ((k\mathbb{Z}_n, \cdot), k\bar{a}) \to ((k\mathbb{Z}_n, \cdot), k\bar{b})$ be an isomorphism. Then for all $x, y \in \mathbb{Z}$,

$$((k\bar{x})(k\bar{a})(k\bar{y}))\varphi = ((k\bar{x})\varphi)(k\bar{b})((k\bar{y})\varphi).$$

Since φ is onto, $((k\mathbb{Z}_n)(k\bar{a})(k\mathbb{Z}_n))\varphi = ((k\mathbb{Z}_n)\varphi)(k\bar{b})((k\mathbb{Z}_n)\varphi) = (k\mathbb{Z}_n)(k\bar{b})(k\mathbb{Z}_n)$. It follows that

$$(k^3\bar{a}\mathbb{Z}_n)\varphi = k^3\bar{b}\mathbb{Z}_n.$$

Since φ is 1-1, we have that $|k^3\bar{a}\mathbb{Z}_n| = |k^3\bar{b}\mathbb{Z}_n|$. Hence $\dfrac{n}{(k^3a, n)} = \dfrac{n}{(k^3b, n)}$ which implies that $(k^3a, n) = (k^3b, n)$. □

Example 4.4. Consider $(2\mathbb{Z}_{12}, \cdot)$. Since $(2^3 \cdot 1, 12) = 4 \neq 12 = (2^3 \cdot 3, 12)$, by Theorem 4.3, we have that $((2\mathbb{Z}_{12}, \cdot), 2 \cdot \bar{1}) \ncong ((2\mathbb{Z}_{12}, \cdot), 2 \cdot \bar{3})$, i.e., $((2\mathbb{Z}_{12}, \cdot), \bar{2}) \ncong ((2\mathbb{Z}_{12}, \cdot), \bar{6})$.

CHAPTER V

SEMIGROUPS OF TRANSFORMATIONS OF SETS

In this chapter, some transformation semigroups on sets are considered. We are motivated to study isomorphism theorems for the variants of $I(X)$ and $T(X)$ by Theorem 2.4 and Theorem 2.5 given by Tsyaputa, respectively, where X is an infinite set. In addition, $M(X)$ and $E(X)$ are also considered. We obtain a necessary condition for two variants of $I(X)$ to be isomorphic. A sufficient condition for this case is provided. We give sufficient conditions for the variants of $M(X), E(X)$ and $T(X)$ in the same manner.

Recall the following notations:

$$\begin{aligned} I(X) &= \text{the symmetric inverse transformation semigroup on } X \\ &\quad \text{(the 1-1 partial transformation semigroup on } X), \\ M(X) &= \{\alpha : X \to X \mid \alpha \text{ is 1-1}\}, \\ E(X) &= \{\alpha : X \to X \mid \alpha \text{ is onto}\}, \\ T(X) &= \text{the full transformation semigroup on } X. \end{aligned}$$

The following facts are also recalled: $G(X)$ is the group of units of all above transformation semigroups; if G is a group, then $(G, a) \cong G$ for all $a \in G$; $M(X) = G(X)$ $[E(X) = G(X)]$ if and only if X is finite.

Throughout this chapter, we assume that X is infinite.

To prove that for $\theta_1, \theta_2 \in I(X)$, $|\operatorname{ran}\theta_1| = |\operatorname{ran}\theta_2|$ if $(I(X), \theta_1) \cong (I(X), \theta_2)$, the following lemma is needed. Note that X need not be required to be infinite in the lemma.

Lemma 5.1. *For* $\theta \in I(X), |\mathrm{E}(I(X), \theta)| = 2^{|\operatorname{ran}\theta|}$.

Proof. We claim that $\mathrm{E}(I(X), \theta) = \{(\theta^{-1})_{|A} \mid A \subseteq \operatorname{ran}\theta\}$. Let $\alpha \in I(X)$ be such that $\alpha = \alpha\theta\alpha$. Then $\operatorname{ran}\alpha \subseteq \operatorname{dom}\theta$. To show that $\operatorname{ran}\alpha\theta \subseteq \operatorname{dom}\alpha$, let $x \in \operatorname{ran}\alpha\theta$.

Then there exists $y \in \operatorname{dom} \alpha\theta$ such that $y\alpha\theta = x$. But $\operatorname{dom} \alpha\theta \subseteq \operatorname{dom} \alpha$, so $y \in \operatorname{dom} \alpha$. Since $\alpha = \alpha\theta\alpha$, we have that $y\alpha = y\alpha\theta\alpha$. Thus $y\alpha = (y\alpha\theta)\alpha = x\alpha$. Since α is 1-1, it follows that $x = y \in \operatorname{dom} \alpha$. Hence we have $\operatorname{ran} \alpha\theta \subseteq \operatorname{dom} \alpha$. If $x \in \operatorname{dom} \alpha$, then

$$x = (x\alpha)\alpha^{-1} = (x\alpha\theta\alpha)\alpha^{-1} = x\alpha\theta 1_{\operatorname{dom} \alpha} = x\alpha\theta \in \operatorname{ran} \theta,$$

so $\operatorname{dom} \alpha \subseteq \operatorname{ran} \theta$ and $\alpha\theta = 1_{\operatorname{dom} \alpha}$. Since $\operatorname{ran} \alpha \subseteq \operatorname{dom} \theta$,

$$\alpha = \alpha 1_{\operatorname{dom} \theta} = \alpha\theta\theta^{-1} = 1_{\operatorname{dom} \alpha}\theta^{-1} = (\theta^{-1})_{|\operatorname{dom} \alpha}.$$

To prove the reverse inclusion, let $A \subseteq \operatorname{ran} \theta$. Then

$$(\theta^{-1})_{|A}\theta(\theta^{-1})_{|A} = 1_A(\theta^{-1})_{|A} = (\theta^{-1})_{|A}.$$

Hence $(\theta^{-1})_{|A} \in \mathrm{E}(I(X), \theta)$.

If $A, B \subseteq \operatorname{ran} \theta$ are such that $A \neq B$, then

$$\operatorname{dom}(\theta^{-1})_{|A} = A \neq B = \operatorname{dom}(\theta^{-1})_{|B}.$$

This implies that

$$|\{(\theta^{-1})_{|A} \mid A \subseteq \operatorname{ran} \theta\}| = |\{A \mid A \subseteq \operatorname{ran} \theta\}| = 2^{|\operatorname{ran} \theta|}.$$

From the claim, we have

$$|\mathrm{E}(I(X), \theta)| = 2^{|\operatorname{ran} \theta|} \quad \text{for all } \theta \in I(X),$$

as desired. □

Theorem 5.2. *For $\theta_1, \theta_2 \in I(X)$, if $(I(X), \theta_1) \cong (I(X), \theta_2)$, then $|\operatorname{ran} \theta_1| = |\operatorname{ran} \theta_2|$.*

Proof. Assume that $(I(X), \theta_1) \cong (I(X), \theta_2)$. Then $|\mathrm{E}(I(X), \theta_1)| = |\mathrm{E}(I(X), \theta_2)|$, so by Lemma 5.1, $2^{|\operatorname{ran} \theta_1|} = 2^{|\operatorname{ran} \theta_2|}$. Hence $|\operatorname{ran} \theta_1| = |\operatorname{ran} \theta_2|$ by the generalized continuum hypothesis. □

Example 5.3. Let $a_1, a_2, \ldots \in X$ be such that $a_i \neq a_j$ if $i \neq j$. Let

$$A_k = \{a_1, a_2, \ldots a_k\}$$

for all $k \in \mathbb{N}$. Then $|\operatorname{ran} 1_{A_k}| = k$, so $|\operatorname{ran} 1_{A_k}| \neq |\operatorname{ran} 1_{A_l}|$ for all distinct $k, l \in \mathbb{N}$. By Theorem 5.2, $(I(X),\, 1_{A_k}) \ncong (I(X),\, 1_{A_l})$ for all distinct $k, l \in \mathbb{N}$. This also shows that there are infinitely many variants of $I(X)$ such that any two of them are not isomorphic.

The following example shows that the converse of Theorem 5.2 is not true in general.

Example 5.4. Let $a \in X$. Then $|X| = |X \smallsetminus \{a\}|$. Let $\theta : X \to X \smallsetminus \{a\}$ be a bijection. Then $\theta \in I(X) \smallsetminus G(X)$, so θ is not a unit of $I(X)$. By Theorem 2.3, $(I(X), \theta) \ncong I(X)$. Then $(I(X), \theta) \ncong (I(X), 1_X)$ but $|\operatorname{ran} \theta| = |X \smallsetminus \{a\}| = |X| = |\operatorname{ran} 1_X|$.

Theorem 5.5. *For* $\theta_1, \theta_2 \in I(X)$, *if* $|\operatorname{ran} \theta_1| = |\operatorname{ran} \theta_2|$, $|X \smallsetminus \operatorname{ran} \theta_1| = |X \smallsetminus \operatorname{ran} \theta_2|$ *and* $|X \smallsetminus \operatorname{dom} \theta_1| = |X \smallsetminus \operatorname{dom} \theta_2|$, *then* $(I(X), \theta_1) \cong (I(X), \theta_2)$.

Proof. Assume that $|\operatorname{ran} \theta_1| = |\operatorname{ran} \theta_2|$, $|X \smallsetminus \operatorname{ran} \theta_1| = |X \smallsetminus \operatorname{ran} \theta_2|$ and $|X \smallsetminus \operatorname{dom} \theta_1| = |X \smallsetminus \operatorname{dom} \theta_2|$. Then there are bijections $\varphi_1 : \operatorname{ran} \theta_2 \to \operatorname{ran} \theta_1$, $\varphi_2 : X \smallsetminus \operatorname{ran} \theta_2 \to X \smallsetminus \operatorname{ran} \theta_1$ and $\psi_1 : X \smallsetminus \operatorname{dom} \theta_1 \to X \smallsetminus \operatorname{dom} \theta_2$. Let $\psi_2 = \theta_1 \varphi_1^{-1} \theta_2^{-1}$. Then $\psi_2 : \operatorname{dom} \theta_1 \to \operatorname{dom} \theta_2$ is a bijection. Define φ and $\psi \in G(X)$ by

$$\varphi = \begin{pmatrix} x & y \\ x\varphi_1 & y\varphi_2 \end{pmatrix}_{\substack{x \in \operatorname{ran} \theta_2 \\ y \in X \smallsetminus \operatorname{ran} \theta_2}} \quad \text{and} \quad \psi = \begin{pmatrix} x & y \\ x\psi_1 & y\psi_2 \end{pmatrix}_{\substack{x \in X \smallsetminus \operatorname{dom} \theta_1 \\ y \in \operatorname{dom} \theta_1}}.$$

It follows that

$$\psi\theta_2 = \psi_2\theta_2 \quad \text{and} \quad \theta_2\varphi = \theta_2\varphi_1.$$

Hence

$$\theta_1 = \theta_1 1_{\operatorname{ran} \theta_1}$$

$$
\begin{aligned}
&= \theta_1\varphi_1^{-1}\varphi_1 \\
&= \theta_1\varphi_1^{-1}1_{\operatorname{ran}\theta_2}\varphi_1 \\
&= \theta_1\varphi_1^{-1}\theta_2^{-1}\theta_2\varphi_1 \\
&= (\theta_1\varphi_1^{-1}\theta_2^{-1})\theta_2\varphi_1 \\
&= \psi_2\theta_2\varphi_1 \\
&= (\psi_2\theta_2)\varphi_1 \\
&= (\psi\theta_2)\varphi_1 \\
&= \psi(\theta_2\varphi_1) \\
&= \psi(\theta_2\varphi) \\
&= \psi\theta_2\varphi.
\end{aligned}
$$

Since φ and ψ are units of $I(X)$, by Proposition 2.2, $(I(X), \theta_1) \cong (I(X), \theta_2)$. □

Example 5.6. Let a, b be distinct elements of X. Then

$$
\begin{gathered}
|\operatorname{ran} 1_{\{a\}}| = 1 = |\operatorname{ran} 1_{\{b\}}|, \\
|X \smallsetminus \operatorname{ran} 1_{\{a\}}| = |X \smallsetminus \{a\}| = |X| = |X \smallsetminus \{b\}| = |X \smallsetminus \operatorname{ran} 1_{\{b\}}|, \\
|X \smallsetminus \operatorname{dom} 1_{\{a\}}| = |X \smallsetminus \{a\}| = |X| = |X \smallsetminus \{b\}| = |X \smallsetminus \operatorname{dom} 1_{\{b\}}|.
\end{gathered}
$$

By Theorem 5.5, $(I(X), 1_{\{a\}}) \cong (I(X), 1_{\{b\}})$. We can show similarly that $(I(X), 1_{X\smallsetminus\{a\}}) \cong (I(X), 1_{X\smallsetminus\{b\}})$. Notice that by Theorem 5.2, $(I(X), 1_{\{a\}}) \ncong (I(X), 1_{X\smallsetminus\{a\}})$ for all $a \in X$. In addition, we have that

$$
|\{(I(X), 1_{\{a\}}) \mid a \in X\}| = |X| = |\{(I(X), 1_{X\smallsetminus\{a\}}) \mid a \in X\}|.
$$

Theorem 5.7. *For* $\theta_1, \theta_2 \in M(X)$, *if* $|X \smallsetminus \operatorname{ran}\theta_1| = |X \smallsetminus \operatorname{ran}\theta_2|$, *then* $(M(X), \theta_1) \cong (M(X), \theta_2)$.

Proof. Assume that $|X \smallsetminus \operatorname{ran}\theta_1| = |X \smallsetminus \operatorname{ran}\theta_2|$. Since θ_1 and θ_2 are 1-1, $|\operatorname{ran}\theta_1| = |X| = |\operatorname{ran}\theta_2|$. Let $\varphi_1 : \operatorname{ran}\theta_2 \to \operatorname{ran}\theta_1$ and $\varphi_2 : X \smallsetminus \operatorname{ran}\theta_2 \to X \smallsetminus \operatorname{ran}\theta_1$ be bijections. Define $\varphi \in G(X)$ by

$$\varphi = \begin{pmatrix} x & y \\ x\varphi_1 & y\varphi_2 \end{pmatrix}_{\substack{x \in \operatorname{ran}\theta_2 \\ y \in X \smallsetminus \operatorname{ran}\theta_2}}.$$

Let $\psi = \theta_1\varphi_1^{-1}\theta_2^{-1}$. We can see that $\psi \in G(X)$ and $\theta_2\varphi = \theta_2\varphi_1$. Then

$$\begin{aligned}
\theta_1 &= \theta_1 1_{\operatorname{ran}\theta_1} \\
&= \theta_1\varphi_1^{-1}\varphi_1 \\
&= \theta_1\varphi_1^{-1}1_{\operatorname{ran}\theta_2}\varphi_1 \\
&= \theta_1\varphi_1^{-1}\theta_2^{-1}\theta_2\varphi_1 \\
&= \psi\theta_2\varphi_1 \\
&= \psi\theta_2\varphi.
\end{aligned}$$

By Proposition 2.2, we have that $(M(X), \theta_1) \cong (M(X), \theta_2)$. □

Example 5.8. Since X is infinite, $|X \times X| = |X|$. Let $\varphi : X \times X \to X$ be a bijection. Since

$$X \times X = \dot{\bigcup_{a \in X}} (\{a\} \times X),$$

it follows that

$$X = \dot{\bigcup_{a \in X}} ((\{a\} \times X)\varphi)$$

and $|(\{a\} \times X)\varphi| = |\{a\} \times X| = |X|$ for all $a \in X$. For each $a \in X$, let $X_a = (\{a\} \times X)\varphi$. Then

$$X = \dot{\bigcup_{a \in X}} X_a \quad \text{and} \quad |X_a| = |X| \quad \text{for all } a \in X.$$

For each $a \in X$, let $\theta_a : X \to X_a$ be a bijection. Then $\theta_a \neq \theta_b$ for all distinct $a, b \in X$. We also have that

$$|X \smallsetminus \operatorname{ran}\theta_a| = \left| \dot{\bigcup_{b \in X \smallsetminus \{a\}}} X_b \right| = |X|.$$

By Theorem 5.7, $(M(X), \theta_a) \cong (M(X), \theta_b)$ for all $a, b \in X$. This shows that there is a set $\mathcal{V}$ of variants of $M(X)$ such that

(1) $|\mathcal{V}| \geq |X|$ and

(2) any two variants in $\mathcal{V}$ are isomorphic.

Theorem 5.9. *For $\theta_1, \theta_2 \in E(X)$, if $\mathcal{P}(\theta_1) \approx \mathcal{P}(\theta_2)$, then $(E(X), \theta_1) \cong (E(X), \theta_2)$.*

Proof. Assume that $\mathcal{P}(\theta_1) \approx \mathcal{P}(\theta_2)$. Then there is a bijection $\varphi : \operatorname{ran}\theta_2 (= X) \to \operatorname{ran}\theta_1 (= X)$ such that $|x\theta_2^{-1}| = |(x\varphi)\theta_1^{-1}|$ for all $x \in \operatorname{ran}\theta_2 (= X)$, so we have $\varphi \in G(X)$ and

$$X = \dot{\bigcup_{x \in \operatorname{ran}\theta_2}} (x\varphi)\theta_1^{-1} = \dot{\bigcup_{x \in X}} (x\varphi)\theta_1^{-1}.$$

For each $x \in X$, let $\psi_x : (x\varphi)\theta_1^{-1} \to x\theta_2^{-1}$ be a bijection. Define $\psi : X \to X$ by

$$\psi = \begin{pmatrix} y \\ y\psi_x \end{pmatrix}_{\substack{x \in X \\ y \in (x\varphi)\theta_1^{-1}}}.$$

It follows that $\psi \in G(X)$. If $x \in X$ and $y \in (x\varphi)\theta_1^{-1}$, then $y\psi_x \in x\theta_2^{-1}$, so

$$y\psi\theta_2\varphi = y\psi_x\theta_2\varphi = ((y\psi_x)\theta_2)\varphi = x\varphi = y\theta_1.$$

This shows that $\psi\theta_2\varphi = \theta_1$. Therefore $(E(X), \theta_1) \cong (E(X), \theta_2)$ by Proposition 2.2. $\square$

Example 5.10. From Example 5.8, X can be written as

$$X = \dot{\bigcup_{a \in X}} X_a \quad \text{and} \quad |X_a| = |X| \quad \text{for all } a \in X.$$

Let $a \in X$. Since $|X \smallsetminus X_a| = |X| = |X \smallsetminus \{a\}|$, there is a bijection $\varphi_a : X \smallsetminus X_a \to X \smallsetminus \{a\}$. Define $\theta_a \in E(X)$ by

$$\theta_a = \begin{pmatrix} X_a & y \\ a & y\varphi_a \end{pmatrix}_{y \in X \smallsetminus X_a}.$$

Then $|a\theta_a^{-1}| = |X_a|$ and $|z\theta_a^{-1}| = 1$ for all $z \in X \smallsetminus \{a\}$. Notice that $\theta_a \neq \theta_b$ for all distinct $a, b \in X$. Claim that $\mathcal{P}(\theta_a) \approx \mathcal{P}(\theta_b)$ for all $a, b \in X$. Let $a, b \in X$. Define $\varphi : X \to X$ by

$$\varphi = \begin{pmatrix} a & b & x \\ b & a & x \end{pmatrix}_{x \in X \smallsetminus \{a,b\}}.$$

Then $\varphi : \operatorname{ran}\theta_b (= X) \to \operatorname{ran}\theta_a (= X)$ is a bijection and

$$|(b\varphi)\theta_a^{-1}| = |a\theta_a^{-1}| = |X_a| = |X| = |X_b| = |b\theta_b^{-1}|.$$

If $x \in X \smallsetminus \{b\}$, then $x\varphi \neq a$, so

$$|(x\varphi)\theta_a^{-1}| = 1 = |x\theta_b^{-1}|.$$

Hence we have the claim. By Theorem 5.9, $(E(X), \theta_a) \cong (E(X), \theta_b)$. This indicates that there is a set $\mathcal{V}$ of variants of $E(X)$ such that

(1) $|\mathcal{V}| \geq |X|$ and

(2) any two variants in $\mathcal{V}$ are isomorphic.

Theorem 5.11. *For $\theta_1, \theta_2 \in T(X)$, if $|X \smallsetminus \operatorname{ran}\theta_1| = |X \smallsetminus \operatorname{ran}\theta_2|$ and $\mathcal{P}(\theta_1) \approx \mathcal{P}(\theta_2)$, then $(T(X), \theta_1) \cong (T(X), \theta_2)$.*

Proof. Assume that $|X \smallsetminus \operatorname{ran}\theta_1| = |X \smallsetminus \operatorname{ran}\theta_2|$ and $\mathcal{P}(\theta_1) \approx \mathcal{P}(\theta_2)$. Let $\varphi_1 : X \smallsetminus \operatorname{ran}\theta_2 \to X \smallsetminus \operatorname{ran}\theta_1$ be a bijection and $\varphi_2 : \operatorname{ran}\theta_2 \to \operatorname{ran}\theta_1$ be a bijection such that $|x\theta_2^{-1}| = |(x\varphi_2)\theta_1^{-1}|$ for all $x \in \operatorname{ran}\theta_2$. Define $\varphi : X \to X$ by

$$\varphi = \begin{pmatrix} x & y \\ x\varphi_1 & y\varphi_2 \end{pmatrix}_{\substack{x \in X \smallsetminus \operatorname{ran}\theta_2 \\ y \in \operatorname{ran}\theta_2}}.$$

Then $\varphi \in G(X)$. For each $x \in \operatorname{ran}\theta_2$, let $\psi_x : (x\varphi_2)\theta_1^{-1} \to x\theta_2^{-1}$ be a bijection. Note that $X = \dot{\bigcup}_{x \in \operatorname{ran}\theta_2} x\theta_2^{-1} = \dot{\bigcup}_{x \in \operatorname{ran}\theta_2} (x\varphi_2)\theta_1^{-1}$. Define $\psi : X \to X$ by

$$\psi = \begin{pmatrix} y \\ y\psi_x \end{pmatrix}_{\substack{x \in \operatorname{ran}\theta_2 \\ y \in (x\varphi_2)\theta_1^{-1}}}.$$

We can see that $\psi \in G(X)$. Claim that $\psi\theta_2\varphi = \theta_1$. Let $x \in \operatorname{ran}\theta_2$ and $y \in (x\varphi_2)\theta_1^{-1}$. Then $y\psi_x \in x\theta_2^{-1}$, so

$$y\psi\theta_2\varphi = y\psi_x\theta_2\varphi = (y\psi_x\theta_2)\varphi = x\varphi = x\varphi_2 = y\theta_1.$$

Hence we have the claim. By Propostion 2.2, $(T(X), \theta_1) \cong (T(X), \theta_2)$, as desired.

□

Example 5.12. From Example 5.8, X can be written as

$$X = \dot{\bigcup_{a \in X}} X_a \quad \text{and} \quad |X_a| = |X| \quad \text{for all } a \in X.$$

For $a \in X$, choose $a' \in X \smallsetminus \{a\}$ and define $\theta_a : X \to X$ by

$$\theta_a = \begin{pmatrix} X_a & X \smallsetminus X_a \\ a & a' \end{pmatrix}.$$

Then $\theta_a \neq \theta_b$ for all distinct $a, b \in X$ since

$$a\theta_a^{-1} = X_a \quad \text{and} \quad a\theta_b^{-1} = \begin{cases} \varnothing & \text{if } a \neq b', \\ X \smallsetminus X_b & \text{if } a = b'. \end{cases}$$

If $a, b \in X$, then

$$|X \smallsetminus \operatorname{ran}\theta_a| = |X \smallsetminus \{a, a'\}| = |X| = |X \smallsetminus \{b, b'\}| = |X \smallsetminus \operatorname{ran}\theta_b|.$$

Define $\varphi : \operatorname{ran}\theta_b \to \operatorname{ran}\theta_a$ by

$$\varphi = \begin{pmatrix} b & b' \\ a & a' \end{pmatrix}.$$

Then

$$|(b\varphi)\theta_a^{-1}| = |a\theta_a^{-1}| = |X_a| = |X| = |X_b| = |b\theta_b^{-1}|$$

and

$$|(b'\varphi)\theta_a^{-1}| = |a'\theta_a^{-1}| = |X \smallsetminus X_a| = |X| = |X \smallsetminus X_b| = |b'\theta_b^{-1}|.$$

This proves that $\mathcal{P}(\theta_a) \approx \mathcal{P}(\theta_b)$. Then by Theorem 5.11, $(T(X), \theta_a) \cong (T(X), \theta_b)$. Hence we have a set $\mathcal{V}$ of variants of $T(X)$ such that

(1) $|\mathcal{V}| \geq |X|$ and

(2) any two variants in $\mathcal{V}$ are isomorphic.

CHAPTER VI

SEMIGROUPS OF LINEAR TRANSFORMATIONS

The main result of this chapter is to determine when two variants of the semigroup $L_F(V)$ are isomorphic where V is a finite-dimensional vector space over a finite field F. This idea relating to finiteness is motivated by Tsyaputa's works (Theorem 2.4, Theorem 2.5 and Theorem 2.6). As a consequence, we characterize when two variants of $(M_n(F), \cdot)$ are isomorphic where F is a finite field. However, we obtain some theorems of sufficiency for this matter when V or F is arbitrary. The semigroups $(M_n(F), \cdot)$ where F is any field, $M_F(V)$ and $E_F(V)$ are also considered in this chapter.

Recall that $L_F(V)$ is the semigroup under composition of all linear transformations $\alpha : V \to V$, $M_F(V)$ and $E_F(V)$ are subsemigroups of $L_F(V)$ defined by

$$\begin{aligned} M_F(V) &= \{\alpha \in L_F(V) \mid \alpha \text{ is 1-1}\} \\ &\quad (= \{\alpha \in L_F(V) \mid \ker \alpha = \{0\}\}), \\ E_F(V) &= \{\alpha \in L_F(V) \mid \alpha \text{ is onto}\} \\ &\quad (= \{\alpha \in L_F(V) \mid \operatorname{ran} \alpha = V\}) \end{aligned}$$

and $M_n(F)$ is the set of all $n \times n$ matrices over F. Also, we recall that $G_F(V)$ is the set of all isomorphisms from V onto itself and $G_F(V)$ is the group of units of $L_F(V), M_F(V)$ and $E_F(V)$.

Throughout, let V be a vector space over a field F and $n \in \mathbb{N}$.

To prove the main result, the following two lemmas are needed.

Lemma 6.1. *For $\theta_1, \theta_2 \in L_F(V)$, if* $\operatorname{rank} \theta_1 = \operatorname{rank} \theta_2$, $\dim_F \ker \theta_1 = \dim_F \ker \theta_2$ *and* $\dim_F(V/\operatorname{ran}\theta_1) = \dim_F(V/\operatorname{ran}\theta_2)$, *then there exist isomorphisms $\varphi, \psi \in L_F(V)$ such that $\psi\theta_2\varphi = \theta_1$.*

Proof. Let B_1 and B_2 be bases of $\ker\theta_1$ and $\ker\theta_2$, respectively, and let $\bar{B}_1$ be a basis of V containing B_1 and $\bar{B}_2$ a basis of V containing B_2. It follows that $(\bar{B}_1 \setminus B_1)\theta_1$ and $(\bar{B}_2 \setminus B_2)\theta_2$ are bases of $\operatorname{ran}\theta_1$ and $\operatorname{ran}\theta_2$, respectively. We also have that $u\theta_1 \neq v\theta_1$ for distinct $u, v \in \bar{B}_1 \setminus B_1$ and $u\theta_2 \neq v\theta_2$ for distinct $u, v \in \bar{B}_2 \setminus B_2$. Then

$$|(\bar{B}_1 \setminus B_1)\theta_1| = |\bar{B}_1 \setminus B_1| \quad \text{and} \quad |(\bar{B}_2 \setminus B_2)\theta_2| = |\bar{B}_2 \setminus B_2|.$$

Next, let $\bar{\bar{B}}_1$ be a basis of V containing $(\bar{B}_1 \setminus B_1)\theta_1$ and $\bar{\bar{B}}_2$ a basis of V containing $(\bar{B}_2 \setminus B_2)\theta_2$. By assumption, $|(\bar{B}_1 \setminus B_1)\theta_1| = \text{rank } \theta_1 = \text{rank } \theta_2 = |(\bar{B}_2 \setminus B_2)\theta_2|$ and $|B_1| = \dim_F \ker\theta_1 = \dim_F \ker\theta_2 = |B_2|$. Then $|\bar{B}_1 \setminus B_1| = |\bar{B}_2 \setminus B_2|$. Let $\psi_1 : B_1 \to B_2$ and $\psi_2 : \bar{B}_1 \setminus B_1 \to \bar{B}_2 \setminus B_2$ be bijections. Define $\psi \in L_F(V)$ on $\bar{B}_1$ by

$$\psi = \begin{pmatrix} u & v \\ u\psi_1 & v\psi_2 \end{pmatrix}_{\substack{u \in B_1 \\ v \in \bar{B}_1 \setminus B_1}}.$$

Then $\psi_{|_{\bar{B}_1}} : \bar{B}_1 \to \bar{B}_2$ is a bijection, so we have that $\psi \in G_F(V)$. Since $\dim_F(V/\operatorname{ran}\theta_1) = \dim_F(V/\operatorname{ran}\theta_2)$, it follows that $|\bar{\bar{B}}_1 \setminus (\bar{B}_1 \setminus B_1)\theta_1| = |\bar{\bar{B}}_2 \setminus (\bar{B}_2 \setminus B_2)\theta_2|$. Let $\pi : \bar{\bar{B}}_2 \setminus (\bar{B}_2 \setminus B_2)\theta_2 \to \bar{\bar{B}}_1 \setminus (\bar{B}_1 \setminus B_1)\theta_1$ be a bijection. Note that $\bar{\bar{B}}_2 = ((\bar{B}_2 \setminus B_2)\theta_2) \,\dot{\cup}\, (\bar{\bar{B}}_2 \setminus (\bar{B}_2 \setminus B_2)\theta_2) = ((\bar{B}_1 \setminus B_1)\psi\theta_2) \,\dot{\cup}\, (\bar{\bar{B}}_2 \setminus (\bar{B}_2 \setminus B_2)\theta_2)$. Define $\varphi \in L_F(V)$ on $\bar{\bar{B}}_2$ by

$$\varphi = \begin{pmatrix} (u\psi)\theta_2 & v \\ u\theta_1 & v\pi \end{pmatrix}_{\substack{u \in \bar{B}_1 \setminus B_1, \\ v \in \bar{\bar{B}}_2 \setminus (\bar{B}_2 \setminus B_2)\theta_2}}.$$

Since $\psi_{|_{\bar{B}_1 \setminus B_1}} = \psi_2 : \bar{B}_1 \setminus B_1 \to \bar{B}_2 \setminus B_2$ is a bijection and $\langle B_2 \rangle \cap \langle \bar{B}_2 \setminus B_2 \rangle = \{0\}$, we have that φ is well-defined. Since $\langle B_1 \rangle \cap \langle \bar{B}_1 \setminus B_1 \rangle = \{0\}$, it follows that for $u, v \in \bar{B}_1 \setminus B_1$, $u\theta_1 = v\theta_1$ if and only if $u = v$. Thus $\varphi_{|_{\bar{\bar{B}}_2}}$ is a bijection from $\bar{\bar{B}}_2$ onto $\bar{\bar{B}}_1$. Hence $\varphi \in G_F(V)$. Claim that $\psi\theta_2\varphi = \theta_1$. If $u \in B_1$, then $u\psi \in B_2$ which is a basis of $\ker\theta_2$, so

$$u\psi\theta_2\varphi = (u\psi\theta_2)\varphi = 0\varphi = 0 = u\theta_1.$$

If $u \in \bar{B}_1 \setminus B_1$, then by the definition of φ, $u\psi\theta_2\varphi = u\theta_1$. It follows that $\psi\theta_2\varphi = \theta_1$ on $\bar{B}_1$. Therefore we have $\psi\theta_2\varphi = \theta_1$, as desired. □

Lemma 6.2. *Assume that V is finite-dimensional. If $\theta_1, \theta_2 \in L_F(V)$ are such that* rank θ_1 = rank θ_2, *then there exist $\varphi, \psi \in G_F(V)$ such that $\psi\theta_2\varphi = \theta_1$.*

Proof. Since $\dim_F \ker\theta_1 + \text{rank } \theta_1 = \dim_F V = \dim_F \ker\theta_2 + \text{rank } \theta_2$, $\dim_F V$ is finite and rank θ_1 = rank θ_2, it follows that $\dim_F \ker\theta_1 = \dim_F \ker\theta_2$. Also, we have $\dim_F(V/\operatorname{ran}\theta_1) = \dim_F V - \text{rank } \theta_1 = \dim_F V - \text{rank } \theta_2 = \dim_F(V/\operatorname{ran}\theta_2)$. Hence by Lemma 6.1, the desired result follows. □

Theorem 6.3. *Assume that V is finite-dimensional and F is a finite field. Then for $\theta_1, \theta_2 \in L_F(V)$, $(L_F(V), \theta_1) \cong (L_F(V), \theta_2)$ if and only if* rank θ_1 = rank θ_2.

Proof. First, assume that $(L_F(V), \theta_1) \cong (L_F(V), \theta_2)$ through an isomorphism φ. Let 0_V be the zero mapping on V. Then 0_V is the zero of both $(L_F(V), \theta_1)$ and $(L_F(V), \theta_2)$, so we have that $0_V\varphi = 0_V$. We claim that $\alpha\theta_1 = \beta\theta_1$ if and only if $(\alpha\varphi)\theta_2 = (\beta\varphi)\theta_2$ for all $\alpha, \beta \in L_F(V)$. Let $\alpha, \beta \in L_F(V)$ and assume that $\alpha\theta_1 = \beta\theta_1$. Then $\alpha\theta_1\lambda = \beta\theta_1\lambda$ for all $\lambda \in L_F(V)$, it follows that $(\alpha\varphi)\theta_2(\lambda\varphi) = (\beta\varphi)\theta_2(\lambda\varphi)$ for all $\lambda \in L_F(V)$. Since $(L_F(V))\varphi = L_F(V)$, we have $(\alpha\varphi)\theta_2 = (\alpha\varphi)\theta_2 1_V = (\beta\varphi)\theta_2 1_V = (\beta\varphi)\theta_2$. But since φ^{-1} is an isomorphism from $(L_F(V), \theta_2)$ onto $(L_F(V), \theta_1)$, if $(\alpha\varphi)\theta_2 = (\beta\varphi)\theta_2$, then from the above proof we have similarly that $(\alpha\varphi)\varphi^{-1}\theta_1 = (\beta\varphi)\varphi^{-1}\theta_1$, i.e., $\alpha\theta_1 = \beta\theta_1$. Therefore we prove that $\alpha\theta_1 = \beta\theta_1$ if and only if $(\alpha\varphi)\theta_2 = (\beta\varphi)\theta_2$. In particular, if $\beta = 0_V$, then $\alpha\theta_1 = 0_V$ if and only if $(\alpha\varphi)\theta_2 = 0_V$. This proves that for every $\alpha \in L_F(V)$, $\alpha\theta_1 = 0_V$ if and only if $(\alpha\varphi)\theta_2 = 0_V$. It follows that $\operatorname{ran}\alpha \subseteq \ker\theta_1$ if and only if $\operatorname{ran}\alpha\varphi \subseteq \ker\theta_2$ for all $\alpha \in L_F(V)$. This proves that $(L_F(V, \ker\theta_1))\varphi = L_F(V, \ker\theta_2)$ since α is an arbitrary element in $L_F(V)$. Consequently, $|L_F(V, \ker\theta_1)| = |L_F(V, \ker\theta_2)|$. By Remark 2.10 (9), $|L_F(V, \ker\theta_1)| = |F|^{(\dim_F V)(\dim_F \ker\theta_1)}$ and $|L_F(V, \ker\theta_2)| = |F|^{(\dim_F V)(\dim_F \ker\theta_2)}$.

It follows that $\dim_F \ker\theta_1 = \dim_F \ker\theta_2$. Hence rank $\theta_1 = \dim_F V - \dim_F \ker\theta_1 = \dim_F V - \dim_F \ker\theta_2 =$ rank θ_2.

The converse follows directly from Proposition 2.2 and Lemma 6.2.

The proof is thereby completed. □

Corollary 6.4. *Assume that* F *is a finite field. Then for* $P_1, P_2 \in M_n(F)$, $((M_n(F),\cdot),P_1) \cong ((M_n(F),\cdot),P_2)$ *if and only if* rank $P_1 =$ rank P_2.

Proof. Let V be a vector space over F of dimension n. Then by Theorem 2.11, there exists a semigroup isomorphism $\varphi : L_F(V) \to M_n(F)$ which preserves ranks. Let $\theta_1, \theta_2 \in L_F(V)$ be such that $\theta_1\varphi = P_1$ and $\theta_2\varphi = P_2$. Then for all $\alpha, \beta \in L_F(V)$,

$$(\alpha\theta_1\beta)\varphi = (\alpha\varphi)P_1(\beta\varphi) \quad \text{and} \quad (\alpha\theta_2\beta)\varphi = (\alpha\varphi)P_2(\beta\varphi).$$

Since $\varphi : L_F(V) \to M_n(F)$ is a bijection, it follows from the above equalities that φ is an isomorphism from $(L_F(V),\theta_1)$ onto $((M_n(F),\cdot),P_1)$ and an isomorphism from $(L_F(V),\theta_2)$ onto $((M_n(F),\cdot),P_2)$, i.e., $(L_F(V),\theta_1) \cong ((M_n(F),\cdot),P_1)$ and $(L_F(V),\theta_2) \cong ((M_n(F),\cdot),P_2)$.

First, assume that $((M_n(F),\cdot),P_1) \cong ((M_n(F),\cdot),P_2)$. This implies that $(L_F(V),\theta_1) \cong (L_F(V),\theta_2)$. By Theorem 6.3, rank $\theta_1 =$ rank θ_2. Since φ preserves ranks, it follows that rank $P_1 =$ rank P_2.

Conversely, assume that rank $P_1 =$ rank P_2. Then rank $\theta_1 =$ rank θ_2 since φ preserves ranks. By Theorem 6.3, $(L_F(V),\theta_1) \cong (L_F(V),\theta_2)$. Consequently, $((M_n(F),\cdot),P_1) \cong ((M_n(F),\cdot),P_2)$. □

From Proposition 2.2 and Lemma 6.1, the following theorem is obtained.

Theorem 6.5. *If* $\theta_1, \theta_2 \in L_F(V)$ *are such that* rank $\theta_1 =$ rank θ_2, $\dim_F \ker\theta_1 = \dim_F \ker\theta_2$ *and* $\dim_F(V/\operatorname{ran}\theta_1) = \dim_F(V/\operatorname{ran}\theta_2)$, *then* $(L_F(V),\theta_1) \cong (L_F(V),\theta_2)$.

Example 6.6. Let V be an infinite-dimensional vector space over F and let B be a basis of V. Then B can be written as

$$B = \dot{\bigcup_{v \in B}} B_v \quad \text{and} \quad |B_v| = |B| \text{ for all } v \in B$$

(see Example 5.8). For each $v \in B$, let $\varphi_v : B \to B_v$ be a bijection and let $\theta_v \in L_F(V)$ be such that $(\theta_v)_{|B} = \varphi_v$. Then for every $v \in B$, θ_v is a monomorphism whose range is $\langle B_v \rangle$. We also have that $\theta_v \neq \theta_w$ if $v \neq w$. Therefore $\ker \theta_v = \{0\}$ and rank $\theta_v = |B_v|$ and

$$\dim_F(V/\operatorname{ran}\theta_v) = \left| \dot{\bigcup_{w \in B \smallsetminus \{v\}}} B_w \right| = |B|.$$

By Theorem 6.5, $(L_F(V), \theta_v) \cong (L_F(V), \theta_w)$ for all $v, w \in B$. Hence there is a set $\mathcal{V}$ of cardinality at least $\dim_F V$ of variants of $L_F(V)$ such that all variants in $\mathcal{V}$ are isomorphic.

The following theorem is directly obtained from Proposition 2.2 and Lemma 6.2.

Theorem 6.7. *Assume that V is finite-dimensional. If $\theta_1, \theta_2 \in L_F(V)$ are such that* rank $\theta_1 =$ rank θ_2, *then* $(L_F(V), \theta_1) \cong (L_F(V), \theta_2)$.

From Theorem 6.7 and the proof of Corollary 6.4, the following result holds.

Corollary 6.8. *If* $P_1, P_2 \in M_n(F)$ *are such that* rank $P_1 =$ rank P_2, *then* $((M_n(F), \cdot), P_1) \cong ((M_n(F), \cdot), P_2)$.

Example 6.9. Let F be a field of characteristic greater than 8. Define $P_1, P_2, P_3 \in M_n(F)$ by

$$P_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 3 & 4 \end{bmatrix} \quad \text{and} \quad P_3 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 4 & 6 & 8 \end{bmatrix}.$$

Then rank $P_1 = 3 = \text{rank } P_2$ and rank $P_3 = 2$. By Corollary 6.8, $((M_4(F), \cdot), P_1) \cong ((M_4(F), \cdot), P_2)$.

If F is a finite field, then by Corollary 6.4, we have that $((M_4(F), \cdot), P_1) \ncong ((M_4(F), \cdot), P_3)$.

Theorem 6.10. *For* $\theta_1, \theta_2 \in M_F(V)$, *if* $\dim_F(V/\operatorname{ran}\theta_1) = \dim_F(V/\operatorname{ran}\theta_2)$, *then* $(M_F(V), \theta_1) \cong (M_F(V), \theta_2)$.

Proof. Let B be a basis of V. Since θ_1 and θ_2 are monomorphisms, $B\theta_1$ and $B\theta_2$ are bases of $\operatorname{ran}\theta_1$ and $\operatorname{ran}\theta_2$, respectively, and $|B\theta_1| = |B| = |B\theta_2|$. Let B_1 be a basis of V containing $B\theta_1$ and B_2 a basis of V containing $B\theta_2$. By assumption, $|B_1 \smallsetminus (B\theta_1)| = \dim_F(V/\operatorname{ran}\theta_1) = \dim_F(V/\operatorname{ran}\theta_2) = |B_2 \smallsetminus (B\theta_2)|$. Let $\varphi_1 : B\theta_2 \to B\theta_1$ and $\varphi_2 : B_2 \smallsetminus (B\theta_2) \to B_1 \smallsetminus (B\theta_1)$ be bijections. Define $\varphi \in L_F(V)$ on B_2 by

$$\varphi = \begin{pmatrix} u & v \\ u\varphi_1 & v\varphi_2 \end{pmatrix}_{\substack{u \in B\theta_2 \\ v \in B_2 \smallsetminus (B\theta_2)}}.$$

Then $\varphi_{|_{B_2}} : B_2 \to B_1$ is a bijection, so we have that $\varphi \in G_F(V)$. Since $v\theta_2\varphi = v\theta_2\varphi_1$ for all $v \in B$, it follows that $\theta_2\varphi = \theta_2\varphi_1$. Define $\psi \in L_F(V)$ on B by $\psi = \theta_1\varphi_1^{-1}\theta_2^{-1}$. We can see that $\psi_{|_B} : B \to B$ is a bijection, so $\psi \in G_F(V)$. If $v \in B$, then

$$\begin{aligned} v\theta_1 &= v\theta_1 1_{B\theta_1} \\ &= v\theta_1\varphi_1^{-1}\varphi_1 \\ &= v\theta_1\varphi_1^{-1}1_{B\theta_2}\varphi_1 \\ &= v\theta_1\varphi_1^{-1}\theta_2^{-1}\theta_2\varphi_1 \\ &= v(\theta_1\varphi_1^{-1}\theta_2^{-1})\theta_2\varphi_1 \\ &= v\psi\theta_2\varphi_1 \\ &= v\psi(\theta_2\varphi_1) \\ &= v\psi\theta_2\varphi. \end{aligned}$$

Hence $\theta_1 = \psi\theta_2\varphi$ on B. Consequently, $\theta_1 = \psi\theta_2\varphi$. By Proposition 2.2, we have that $(M_F(V), \theta_1) \cong (M_F(V), \theta_2)$. □

Example 6.11. From Example 6.6, we can see that $\theta_v \in M_F(V)$. By Theorem 6.10, we have that $(M_F(V), \theta_v) \cong (M_F(V), \theta_w)$ for all $v, w \in B$. Therefore there is a set $\mathcal{V}$ of cardinality at least $\dim_F V$ of variants of $M_F(V)$ such that all variants in $\mathcal{V}$ are isomorphic.

Theorem 6.12. *For* $\theta_1, \theta_2 \in E_F(V)$, *if* $\dim_F \ker\theta_1 = \dim_F \ker\theta_2$, *then* $(E_F(V), \theta_1) \cong (E_F(V), \theta_2)$.

Proof. Let B_1 be a basis of $\ker\theta_1$ and B_2 a basis of $\ker\theta_2$. By assumption, $|B_1| = |B_2|$. Let $\psi_1 : B_1 \to B_2$ be a bijection. Let $\bar{B}_1$ be a basis of V containing B_1 and $\bar{B}_2$ a basis of V containing B_2. Since θ_1 and θ_2 are epimorphisms, $(\bar{B}_1 \smallsetminus B_1)\theta_1$ and $(\bar{B}_2 \smallsetminus B_2)\theta_2$ are bases of V. Let $\varphi \in L_F(V)$ such that $\varphi : (\bar{B}_2 \smallsetminus B_2)\theta_2 \to (\bar{B}_1 \smallsetminus B_1)\theta_1$ be a bijection. Then $\varphi \in G_F(V)$. For $v \in (\bar{B}_1 \smallsetminus B_1)\theta_1$, let $v' \in v\theta_1^{-1}$ and for $v \in (\bar{B}_2 \smallsetminus B_2)\theta_2$, let $v'' \in v\theta_2^{-1}$. Then $B_1 \mathbin{\dot\cup} \{v' \mid v \in (\bar{B}_1 \smallsetminus B_1)\theta_1\}$ and $B_2 \mathbin{\dot\cup} \{v'' \mid v \in (\bar{B}_2 \smallsetminus B_2)\theta_2\} = B_2 \mathbin{\dot\cup} \{(v\varphi^{-1})'' \mid v \in (\bar{B}_1 \smallsetminus B_1)\theta_1\}$ are bases of V. Since $|(\bar{B}_1 \smallsetminus B_1)\theta_1| = |(\bar{B}_2 \smallsetminus B_2)\theta_2|$ and φ is a bijection, it follows that $|\{v' \mid v \in (\bar{B}_1 \smallsetminus B_1)\theta_1\}| = |\{v'' \mid v \in (\bar{B}_2 \smallsetminus B_2)\theta_2\}| = |\{(v\varphi^{-1})'' \mid v \in (\bar{B}_1 \smallsetminus B_1)\theta_1\}|$. We can see that the mapping ψ_2 defined by $v'\psi_2 = (v\varphi^{-1})''$ for all $v \in (\bar{B}_1 \smallsetminus B_1)\theta_1$ is a bijection from $\{v' \mid v \in (\bar{B}_1 \smallsetminus B_1)\theta_1\}$ onto $\{(v\varphi^{-1})'' \mid v \in (\bar{B}_1 \smallsetminus B_1)\theta_1\}$. Define $\psi \in L_F(V)$ on the basis $B_1 \mathbin{\dot\cup} \{v' \mid v \in (\bar{B}_1 \smallsetminus B_1)\theta_1\}$ by

$$\psi = \begin{pmatrix} u & v' \\ u\psi_1 & v'\psi_2 \end{pmatrix}_{\substack{u \in B_1 \\ v \in (\bar{B}_1 \smallsetminus B_1)\theta_1}}.$$

Then the restriction of ψ to $B_1 \mathbin{\dot\cup} \{v' \mid v \in (\bar{B}_1 \smallsetminus B_1)\theta_1\}$ is a bijection from $B_1 \mathbin{\dot\cup} \{v' \mid v \in (\bar{B}_1 \smallsetminus B_1)\theta_1\}$ onto $B_2 \mathbin{\dot\cup} \{(v\varphi^{-1})'' \mid v \in (\bar{B}_1 \smallsetminus B_1)\theta_1\}$, so we have $\psi \in G_F(V)$. Notice that $v'\psi = v'\psi_2 = (v\varphi^{-1})''$ for all $v \in (\bar{B}_1 \smallsetminus B_1)\theta_1$. If $v \in B_1$, then $v\psi = v\psi_1 \in B_2$, so

$$v\theta_1 = 0 = 0\varphi = (v\psi\theta_2)\varphi = v(\psi\theta_2\varphi).$$

If $v \in (\bar{B}_1 \setminus B_1)\theta_1$, then

$$v'\theta_1 = v = (v\varphi^{-1})\varphi = (v\varphi^{-1})''\theta_2\varphi = v'\psi\theta_2\varphi.$$

This proves that $\theta_1 = \psi\theta_2\varphi$ on the basis $B_1 \mathbin{\dot\cup} \{v' \mid v \in (B \setminus B_1)\theta_1\}$. Hence $\theta_1 = \psi\theta_2\varphi$, as desired. Therefore $(E_F(V), \theta_1) \cong (E_F(V), \theta_2)$, by Proposition 2.2. □

Example 6.13. Let v, B and B_v be as in Example 6.6. Since for each $v \in B$, $|B_v| = |B|$, there exists a bijection $\psi_v : B_v \to B$. Define $\theta'_v \in L_F(V)$ on B by

$$\theta'_v = \begin{pmatrix} u & B \setminus B_v \\ u\psi_v & 0 \end{pmatrix}_{u \in B_v}.$$

Then for every $v \in B, \operatorname{ran}\theta'_v = \langle B \cup \{0\}\rangle = V$, so $\theta'_v \in E_F(V)$. We can see that $\ker\theta'_v = \langle B \setminus B_v\rangle$ for all $v \in B$, so $\dim_F \ker\theta'_v = |B \setminus B_v| = \left|\dot{\bigcup}_{w \in B \setminus \{v\}} B_w\right| = |B|$ for all $v \in B$ and $\theta'_v \neq \theta'_w$ if $v \neq w$. By Theorem 6.12, $(E_F(V), \theta'_v) \cong (E_F(V), \theta'_w)$ for all $v, w \in B$. Hence we have that there is a set $\mathcal{V}$ of cardinality at least $\dim_F V$ of variants of $E_F(V)$ such that all variants in $\mathcal{V}$ are isomorphic.

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