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BOUNDS ON NORMAL APPROXIMATION OF LATIN HYPERCUBE AND ORTROGONAL ARRAY SAMPLINGS

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วิทยานิพนธ์ฉบับนี้ประกอบด้วย 3 ส่วน ในส่วนแรกได้ให้ค่าคงตัวของขอบเขตการประมาณค่าอย่างสม่ำเสมอด้วยการแจกแจงปกติของการชักตัวอย่างลาตินไฮเพอร์คิวบ์เมื่อสมมติให้ค่าสัมบูรณ์ของโมเมนต์อันดับที่สามมีค่าจำกัด ภายใต้เงื่อนไขเดียวกันเราได้ให้ค่าคงตัวของขอบเขตการประมาณค่าอย่างสม่ำเสมอในทฤษฎีบทลิมิตกลางเชิงการวัดในส่วนที่สอง ส่วนสุดท้ายมีบทพิสูจน์ของขอบเขตการประมาณค่าแบบไม่สม่ำเสมอด้วยการแจกแจงปกติของการชักตัวอย่างแถวเชิงตั้งฉากภายใต้เงื่อนไขว่าโมเมนต์อันดับที่หกมีค่าจำกัด

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This thesis consists of three parts. In the first part, a constant on a uniform bound on normal approximation for latin hypercube sampling is obtained assuming the finiteness of absolute third moment. Under the same condition, we derive a constant on a uniform bound on a combinatorial central limit theorem in the second part. The last part contains a proof of a non-uniform bound on normal approximation of randomized orthogonal array sampling designs under the finiteness of sixth moment.

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CHAPTER I

INTRODUCTION

This thesis composes of three parts. In the first part, we give a uniform bound on normal approximation of latin hypercube sampling. The second part, we obtain a uniform bound on a combinatorial central limit theorem. In the last part, we give a non-uniform bound on normal approximation of randomized orthogonal array sampling.

1.1 A Uniform Bound on Normal Approximation of Latin Hypercube Sampling

Latin hypercube sampling (LHS) is introduced by McKay, Beckman and Conover in 1978 [17]. For positive integers n and d, $d \geq 2$, a latin hypercube sample of size n (taken from the d-dimensional hypercube $[0,1]^d$) is defined to be $\{X(\eta_1(i), \eta_2(i), ..., \eta_d(i)) : 1 \leq i \leq n\}$, where

1. for all $1 \leq i_1, ..., i_d \leq n,\ 1 \leq j \leq d$,

$$X_j(i_1, ..., i_d) = (i_j - U_{i_1,...,i_d,j})/n,$$

and $$X(i_1, ..., i_d) = (X_1(i_1, ..., i_d), ..., X_d(i_1, ..., i_d));$$

2. $\eta_k = (\eta_k(1), \eta_k(2), \ldots, \eta_k(n)),\ 1 \leq k \leq d$, are random permutations of $\{1, ..., n\}$, each of which is uniformly distributed over all the $n!$ possible permutations;
3. $U_{i_1,...,i_d,j}$ $1 \leq i_1, ..., i_d \leq n,\ 1 \leq j \leq d$, be $[0,1]$ uniform random variables;
4. the $U_{i_1,...,i_d,j}$'s and η_k's all be stochastically independent.

Hence an unbiased estimator for $\mu = E(f \circ X)$, where X is a random vector having a uniform distribution on $[0,1]^d$ and f is a measurable function from $[0,1]^d$ to $\mathbb{R}$, that based on a latin hypercube sampling is

$$\hat{\mu}_n = \frac{1}{n}\sum_{k=1}^{n} f(X(\eta_1(k), \eta_2(k), ..., \eta_d(k))).$$

If $Var(\hat{\mu}_n) > 0$, we define

$$W = \frac{\hat{\mu}_n - \mu}{\sqrt{Var(\hat{\mu}_n)}}.$$

Then $EW = 0$ and $VarW = 1$. To use the Stein's method to approximate the distribution of W with the standard normal distribution, Loh [15] wrote

$$W = \sum_{i=1}^{n} V(\eta_1(i), \dots, \eta_d(i)),$$

where

$$V(i_1, ..., i_d) = \frac{1}{n\sqrt{Var\hat{\mu}_n}}[f \circ X(i_1, ..., i_d) - \sum_{k=1}^{d} \mu_{-k}(i_k) + (d-1)\mu],$$

$$\mu(i_1, ..., i_d) = Ef \circ X(i_1, ..., i_d) \ \text{ and } \ \mu_{-k}(i_k) = \frac{1}{n^{(d-1)}} \sum_{j \neq k} \sum_{i_j=1}^{n} \mu(i_1, ..., i_d).$$

Moreover, he gave the rate of convergence $\frac{C_d}{\sqrt{n}}$ without the value of C_d under the finiteness of the absolute third moment. Theorem 1.1 is his result.

Theorem 1.1. *There exists a positive constant C_d which depends only on d such that for sufficiently large n,*

$$\sup_{z \in \mathbb{R}} |P(W \leq z) - \Phi(z)| \leq C_d \beta_3,$$

where Φ is the standard normal distribution and

$$\beta_3 = \frac{1}{n^{d-1}} \sum_{i_1=1}^{n} ... \sum_{i_d=1}^{n} E|V(i_1, ..., i_d)|^3.$$

Corollary 1.2. *If $E|f \circ X|^3 < \infty$, then*

$$\sup_{z \in \mathbb{R}} |P(W \leq z) - \Phi(z)| \leq \frac{C_d}{\sqrt{n}}.$$

Recently, Rattanawong [31] showed that there exist random permutations $\pi_1, \ldots, \pi_{d-1}$ on $\{1, 2, \ldots, n\}$ which are uniformly distributed over all the $n!$ possible permutations such that

$$W = \sum_{i=1}^{n} Y(i, \pi_1(i), \ldots, \pi_{d-1}(i))$$

and $Y(i_1, \ldots, i_d)$'s and π_k's are stochastically independent. Indeed, for any j in $\{1, \ldots, d\}$, let $\pi_j(\omega) = \eta_{j+1}(\omega)(\eta_1(\omega)^{-1})$ and for each $i_1, ..., i_d \in \{1, ..., n\}$, define

$$Y(i_1, ..., i_d) = \frac{1}{n\sqrt{Var(\hat{\mu}_n)}}[f \circ X(i_1, ..., i_d) + \sum_{k=1}^{d-1} U_k(i_1, ..., i_d) + (-1)^d \mu], \quad (1.1)$$

where

$$\mu(i_1, ..., i_d) = Ef \circ X(i_1, ..., i_d),$$

$$U_k(i_1, ..., i_d) = \frac{(-1)^k}{n^k} \sum_{1 \le j_1 < j_2 < ... < j_k \le d} \sum_{q_{j_1}=1}^{n} ... \sum_{q_{j_k}=1}^{n} \mu(l_1, ..., l_d),$$

and

$$l_p = \begin{cases} q_p & \text{if } \ p = j_1, ..., j_k, \\ i_p & \text{otherwise.} \end{cases}$$

Note that the definition of $Y(i_1, ..., i_d)$'s are different from that of $V(i_1, ..., i_d)$'s given by Loh [15] that the random variable W satisfies the following property: for each $j \in \{1, 2, \ldots, d\}$,

$$\sum_{i_j=1}^{n} EY(i_1, \ldots, i_d) = 0. \quad (1.2)$$

Furthermore, Neammanee and Rattanawong used a concentration inequality approach and assumed the finiteness of fourth moment to give a constant C_d. This is their result.

Theorem 1.3. *Suppose that $E(f \circ X(i_1, ..., i_d))^4 < \infty$, $1 \le i_1, ..., i_d \le n$. Then for $n \ge 6^d + 3$,*

$$\sup_{z \in \mathbb{R}} |P(W \le z) - \Phi(z)| \le \frac{11.946}{\sqrt{n}} + \frac{1.037\sqrt{d}\beta_4^{\frac{1}{4}}}{n^{\frac{3}{8}}} + 8.314 d^{\frac{1}{4}} \beta_4 + 11.765\beta_4 + \frac{5.014 d \beta_4^{\frac{3}{4}}}{n^{\frac{1}{8}}} + \frac{2\sqrt{2\pi}\beta_4^{\frac{3}{4}}}{n^{\frac{1}{8}}},$$

where

$$\beta_4 = \frac{1}{n^{d-\frac{3}{2}}} \sum_{i_1=1}^{n} \cdots \sum_{i_d=1}^{n} E|Y(i_1, ..., i_d)|^4.$$

In this work, we use a concentation inequality approach and the ideas of Neammanee and Rattanawong [19], [21] to obtain the constant C_d by assuming the finiteness of the absolute third moment. Theorem 1.4 is our main result.

Theorem 1.4. *Suppose that* $E|f \circ X(i_1, ..., i_d)|^3 < \infty$, $1 \le i_1, ..., i_d \le n$. *Then for* $n \ge 6^d$,

$$\sup_{z\in\mathbb{R}} |P(W \le z) - \Phi(z)| \le (22.88 + 28.99d)\delta_3 + \frac{3.88 + 2.09d}{\sqrt{n}} + 1.03\delta_3^2$$
$$+ (\frac{C_d}{n^{\frac{1}{24}}})(\frac{C\delta_3^{\frac{3}{4}}}{n^{\frac{1}{8}}} + Cn^{\frac{1}{8}}\delta_3^{\frac{5}{4}}) + O(\frac{1}{n}),$$

where

$$\delta_3 = \frac{1}{n^{d-1}} \sum_{i_1=1}^{n} \cdots \sum_{i_d=1}^{n} E|Y(i_1, ..., i_d)|^3,$$

and the definition of $Y(i_1, ..., i_d)$ *is given in (1.1).*

Corollary 1.5. *Suppose that* $E|f \circ X(i_1, ..., i_d)|^3 < \infty$, $1 \le i_1, ..., i_d \le n$. *If* $n \ge 6^d$ *and* $\delta_3 \sim \frac{1}{\sqrt{n}}$, *then*

$$\sup_{z\in\mathbb{R}} |P(W \le z) - \Phi(z)| \le \frac{26.76 + 31.08d}{\sqrt{n}} + O(\frac{1}{n^{\frac{13}{24}}}).$$

Observe that if $\delta_4 \sim \frac{1}{\sqrt{n}}$, then Theorem 1.3 yields

$$\sup_{z\in\mathbb{R}} |P(W \le z) - \Phi(z)| \le \frac{28.725 + 5.014d + 1.037\sqrt{d} + 8.314\sqrt[4]{d}}{\sqrt{n}}.$$

Although this bound is sharper than the result in Corollary 1.5, we establish a uniform bound under the finiteness of absolute third moment.

1.2 A Uniform Bound on a Combinatorial Central Limit Theorem

Wald and Wolfowitz [38] established the strong sufficient conditions for the asymptotic normality of $\eta_n = \sum_{i=1}^{n} a_i b_{i\alpha(i)}$ where a_i and b_{ij} are two sequences of real numbers and α is a random permutation of $\{1, 2, \ldots, n\}$. The weak conditions were given by Noether [24] and then simplified by Hoeffding [9] who also studied the random variable $\xi_n = \sum_{i=1}^{n} c(i, \alpha(i))$ where $c(i, j)$, $i, j = 1, 2, \ldots, n$, is a sequence of real numbers. In 1957, Matoo [16] proved that a Lindeberg-type condition is sufficient for the asymptotic normality of ξ_n. This condition is also necessary for η_n, which is showed by Hájek [6]. After that, a limit theorem for η_n and ξ_n is given under various conditions see, for examples, Bolthausen [3], Kolchin and Chistyakov [10], Robinson [32].

In the case that $c(i, j)$'s are any random variables, we use the notation $Y(i, j)$'s. A theorem for the asymptotic normality of $V_n = \sum_{i=1}^{n} Y(i, \pi(i))$ where π is a random permutation of $\{1, 2, \ldots, n\}$ is called a combinatorial central limit theorem. If $Y(i, j)$'s and π are all independent, the rate of convergence $O(\frac{1}{\sqrt{n}})$ have been provided by Ho and Chen [8] and Von Bahr [37] under some bounded condition, like $\sup |Y(i, j)| = O(\frac{1}{\sqrt{n}})$. In 2005, Neammanee and Suntornchost [23] applied a concentration inequality of Stein's method to obtain the uniform rate of convergence under the finiteness of third moment and gave the rate $\frac{198}{\sqrt{n}}$. More precisely, for each $i, j = 1, 2, \ldots, n$, let μ_{ij} and σ_{ij}^2 be the mean and variance of $Y(i, j)$, respectively, and

$$\mu_{i.} = \frac{1}{n}\sum_{j=1}^{n} \mu_{ij}, \quad \mu_{.j} = \frac{1}{n}\sum_{i=1}^{n} \mu_{ij}, \quad \mu_{ij} = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \mu_{ij},$$

$$d^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}(\mu_{ij} - \mu_{i.} - \mu_{.j} + \mu_{..})^2 \text{ and } \sigma^2 = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}^2.$$

From Ho and Chen [8], we know that $VarV_n = d^2 + \sigma^2$. Define

$$W = \frac{1}{\sqrt{d^2 + \sigma^2}}\sum_{i=1}^{n}(Y(i, \pi(i)) - \mu_{..}).$$

Then $EW = 0$, $VarW = 1$ and

$$W = \frac{V_n - EV_n}{\sqrt{VarV_n}} = \sum_{i=1}^{n} \overline{Y}(i, \pi(i))$$

where $\overline{Y}(i,j) = \dfrac{1}{d^2+\sigma^2}(Y(i,j) + \mu_{i.} - \mu_{.j} + \mu_{..})$. Their result is Theorem 1.6.

Theorem 1.6. *Suppose that* $E|\overline{Y}(i,j)|^3 < \infty$, $1 \le i, j \le n$. *Then for* $n \ge 32$,

$$\sup_{z\in\mathbb{R}} |P(W \le z) - \Phi(z)| \le 198\beta + \frac{18}{n},$$

where

$$\beta = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} E|\overline{Y}(i,j)|^3.$$

Later, Neammanee and Rattanawong [20] assumed the finiteness of fourth moment to yield the constant to be 27.72. The result is the following theorem.

Theorem 1.7. *Suppose that* $EY^4(i,j) < \infty$, $1 \le i, j \le n$, $VarW = 1$, *and*

$$\sum_{i=1}^{n} EY(i,j_0) = 0 \text{ for a fixed } j_0 \text{ and } \sum_{j=1}^{n} EY(i_0,j) = 0 \text{ for a fixed } i_0.$$

Then for $n \ge 30$,

$$\sup_{z\in\mathbb{R}} |P(W \le z) - \Phi(z)| \le 11\delta_4 + 10.02\frac{\delta_4^{3/4}}{n^{1/8}} + 1.467\frac{\delta_4^{1/2}}{n^{1/4}} + \frac{5.228}{\sqrt{n}},$$

where

$$\delta_4 = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\sum_{j=1}^{n} EY^4(i,j).$$

Furthermore, if $\delta_4 \sim \dfrac{1}{\sqrt{n}}$, *then*

$$\sup_{z\in\mathbb{R}} |P(W \le z) - \Phi(z)| \le \frac{27.72}{\sqrt{n}}.$$

By assuming the finiteness of third moments, we give a shaper constant than that of Neammanee and Suntornchost [23]. Indeed, we obtain the rate to be $\dfrac{78.9}{\sqrt{n}}$. Theorem 1.8 is our main theorem.

Theorem 1.8. *Suppose that* $E|Y(i,j)|^3 < \infty$, $1 \le i,j \le n$, $VarW = 1$, *and*

$$\sum_{i=1}^{n} EY(i,j_0) = 0 \text{ for a fixed } j_0 \text{ and } \sum_{j=1}^{n} EY(i_0,j) = 0 \text{ for a fixed } i_0.$$

For $n \ge 36$,

$$\sup_{z\in\mathbb{R}}|P(W \le z) - \Phi(z)|$$
$$\le 70.85\delta_3 + \frac{8.06 + 10.81\delta_3}{\sqrt{n}} + 1.03\delta_3^2 + (\frac{C}{n^{\frac{1}{24}}})(\frac{C\delta_3^{\frac{3}{4}}}{n^{\frac{1}{8}}} + Cn^{\frac{1}{8}}\delta_3^{\frac{5}{4}}) + O(\frac{1}{n})$$

where

$$\delta_3 = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} E|Y(i,j)|^3.$$

Futhermore, if $\delta_3 \sim \frac{1}{\sqrt{n}}$, *then*

$$\sup_{z\in\mathbb{R}}|P(W \le z) - \Phi(z)| \le \frac{78.9}{\sqrt{n}} + O(\frac{1}{n^{\frac{13}{24}}}).$$

By substituting d by 2 in Corollary 1.5, we obtain the bound to be 88.92 which is larger than the above result.

1.3 A Non-Uniform Bound on Normal Approximation of Randomized Orthogonal Array Sampling

An orthogonal array of strength t with index λ ($\lambda \ge 1$), is an $n \times d$ matrix with elements taken from the set $\{0, 1, ..., q-1\}$ such that for any $n \times t$ submatrix, each of the q^t possible rows appears the same number λ of times where d, n, q and t are positive integers with $t \le d$ and $q \ge 2$. Of course $n = \lambda q^t$. The class of such arrays is denoted by $OA(n, d, q, t)$ (see Raghavarao [30] for more details).

In this work, we consider the class $OA(q^2, 3, q, 2)$. For this class, Loh [14] constructed the samplings $X_1, X_2, ..., X_{q^2}$ on the unit cube $[0,1]^3$ as follows: Let

(a) π_1, π_2, π_3 be random permutations of $\{0, 1, ..., q-1\}$, each uniformly distributed on all the $q!$ possible permutations;

(b) $U_{i_1,i_2,i_3,j}$ be $[0,1]$ uniform random variables where $i_1, i_2, i_3 \in \{0, 1, ..., q-1\}$,

$j \in \{1,2,3\}$; and

(c) $U_{i_1,i_2,i_3,j}$'s and π_k's be all stochastically independent.

An orthogonal array-based sample of size q^2, $\{X_1, X_2, ..., X_{q^2}\}$, is defined to be the set

$$\{X(\pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3})) : 1 \leq i \leq q^2\},$$

where, for each $i_1, i_2, i_3 \in \{0, 1, ..., q-1\}$ and $j \in \{1,2,3\}$,

$$X\big(i_1, i_2, i_3\big) = \big(X_1(i_1, i_2, i_3), X_2(i_1, i_2, i_3), X_3(i_1, i_2, i_3)\big),$$
$$X_j(i_1, i_2, i_3) = (i_j + U_{i_1,i_2,i_3,j})/q,$$

and $a_{i,j}$ is the $(i,j)^{th}$ element of some arbitrary but fixed $A \in OA(q^2, 3, q, 2)$. We use

$$\hat{\mu} = \frac{1}{q^2}\sum_{i=1}^{q^2} f \circ X(\pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3}))$$

as an estimator of $\mu = E(f \circ X)$ where X is a random vector having a uniform distribution on a unit hypercube $[0,1]^3$ and f is a measurable function from $[0,1]^3$ to $\mathbb{R}$. Assume that $Var(\hat{\mu}) > 0$, and define

$$W = \frac{\hat{\mu} - \mu}{\sqrt{Var(\hat{\mu})}}. \tag{1.3}$$

Note that $EW = 0$ and $VarW = EW^2 = 1$. Loh [14] gave a first version of uniform bound on normal approximation of W. He assumed that $E(f \circ X)^r < \infty$ for some even integer $r \geq 4$ and obtained that

$$\sup_{z \in \mathbb{R}} |P(W \leq z) - \Phi(z)| = O(q^{\frac{2-r}{2r-2}}). \tag{1.4}$$

Observe that, the rate of convergence in (1.4) approach $O(\frac{1}{\sqrt{q}})$ as $r \to \infty$. Neammanee and Laipaporn [18] obtained this rate by assuming only the sixth moment exist. Recently, Neammanee, Laipaporn and Sungkamongkol [22] relaxed the assumption by assuming the finite fourth moment and got the rate to be $O(\frac{1}{\sqrt{q}})$.

For a non-uniform bound, Neammanee and Laipaporn [12] took the first step. Their result is

$$|P(W \leq z) - \Phi(z)| \leq \max(\frac{1}{(1+|z|)^{1-\frac{2}{r}}} O(\frac{1}{q^{\frac{r-8}{2r}}}), \frac{1}{(1+|z|)^{\frac{11}{12}}} O(\frac{1}{q^{\frac{1}{6}}})) \tag{1.5}$$

where $E(f \circ X)^{3r} < \infty$ for some even number $r \geq 10$. The proof of (1.5) is very complicated and the result is obtained when they assume at least thirtieth moment. In this work, we give a better bound and a simple proof by using Stein's method. This is our result.

Theorem 1.9. *Assume that $E(f \circ X)^6 < \infty$. Then for every $z \in \mathbb{R}$*

$$(1+|z|)|P(W \leq z) - \Phi(z)| \leq O(\frac{1}{\sqrt{q}}).$$

In this thesis, we organize as follows. In chapter 2, we review some basic concepts in probability theory and the idea of Stein's method. A proof of Theorem 1.4 and Theorem 1.8 are in chapter 3 and 4, respectively. In the last chapter, we prove Theorem 1.9.

CHAPTER II

PRELIMINARIES

In this chapter, we review some basic concepts in probability which will be used in our work and the idea of Stein's method. The proof is omited but can be found in [1], [2], [29], [31] and [35].

2.1 Probability Space and Random Variables

A **probability space** is a measure space $(\Omega, \mathcal{F}, P)$ for which $P(\Omega) = 1$. The measure P is called a **probability measure.** The set Ω will be referred as a **sample space** and its elements are called **points** or **elementary events.** The elements of $\mathcal{F}$ are called **events.** For any event A, the value $P(A)$ is called the **probability of** A**.**

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A function $X : \Omega \to \mathbb{R}$ is called a **random variable** if for every Borel set B in $\mathbb{R}$, $X^{-1}(B)$ belongs to $\mathcal{F}$. We shall use the notation $P(X \in B)$ in place of $P(\{\omega \in \Omega \mid X(\omega) \in B\})$. In the case where $B = (-\infty, a]$ or $[a, b]$, $P(X \in B)$ is denoted by $P(X \leq a)$ or $P(a \leq X \leq b)$, respectively.

Let X be a random variable. A function $F : \mathbb{R} \to [0, 1]$ which is defined by

$$F(x) = P(X \leq x)$$

is called the **distribution function** of X.

Let X be a random variable with the distribution function F. Then X is said to be a **discrete random variable** if the image of X is countable and X is called a **continuous random variable** if F can be written in the form

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

for some nonnegative integrable function f on $\mathbb{R}$. In this case, we say that f is the **probability density function** (or the probability function) of X.

Now we will give some examples of random variables.

We say that X is a **normal** random variable with parameter μ and σ^2, written as $X \sim N(\mu, \sigma^2)$, if its probability function is defined by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{1}{2\sigma^2}(x-\mu)^2\Big).$$

Moreover, if $X \sim N(0,1)$ then X is said to be a **standard normal** random variable.

We say that X is a **uniform** random variable with parameter n if there exist $x_1, x_2, ..., x_n$ such that $P(X = x_i) = \frac{1}{n}$ for any $i = 1, 2, ..., n$ and denoted by $X \sim U(n)$.

If X is a continuous random variable with probability function

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if} \quad a \le x \le b, \\ 0 & \text{otherwise} \end{cases}$$

We say that X is **uniform** on $[a, b]$.

We say that $X = (X_1, X_2, \ldots, X_n)$ is a **continuous random vector** if and only if there are measurable function $f : \mathbb{R}^n \to [0, \infty)$ and **joint distribution function** F of X satisfying

$$F(x_1, x_2, \ldots, x_n) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_n} f(t_1, t_2, \ldots, t_n) dt_1 dt_2 \ldots dt_n$$

for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Then f is called **joint probability function** of X.

Furthermore, we say that π is a **random permutation** if π is a permutation-valued random variable.

2.2 Independence

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{F}_\alpha$ a sub σ-algebra of $\mathcal{F}$ for each $\alpha \in \Lambda$. We say that $\{\mathcal{F}_\alpha | \alpha \in \Lambda\}$ is **independent** if and only if for any subset $J =$

$\{1, 2, ..., k\}$ of Λ,

$$P(\bigcap_{m=1}^{k} A_m) = \prod_{m=1}^{k} P(A_m)$$

where $A_m \in \mathcal{F}_m$ for $m = 1, ..., k$.

Let $\mathcal{E}_\alpha \subseteq \mathcal{F}$ for all $\alpha \in \Lambda$. We say that $\{\mathcal{E}_\alpha \mid \alpha \in \Lambda\}$ is **independent** if and only if $\{\sigma(\mathcal{E}_\alpha) \mid \alpha \in \Lambda\}$ is independent where $\sigma(\mathcal{E}_\alpha)$ is the smallest σ-algebra with $\mathcal{E}_\alpha \subseteq \sigma(\mathcal{E}_\alpha)$.

We say that the set of random variables $\{X_\alpha \mid \alpha \in \Lambda\}$ is **independent** if $\{\sigma(X_\alpha) \mid \alpha \in \Lambda\}$ is independent, where $\sigma(X) = \{X^{-1}(B) \mid B$ is a Borel subset of $\mathbb{R}\}$.

Theorem 2.1. *Random variables $X_1, X_2, ..., X_n$ are* **independent** *if for any Borel sets $B_1, B_2, ..., B_n$ we have*

$$P(\bigcap_{i=1}^{n}\{X_i \in B_i\}) = \prod_{i=1}^{n} P(X_i \in B_i).$$

Proposition 2.2. *If X_{ij} ; $i = 1, 2, ..., n,\ j = 1, 2, ..., m_i$ are independent and $f_i : \mathbb{R}^{m_i} \to \mathbb{R}$ are measurable, then $\{f_i(X_{i1}, X_{i2}, ..., X_{im_i}) \mid i = 1, 2, ..., n\}$ is independent.*

2.3 Expectation, Variance, Conditional Expectation and Conditional

Let X be any random variable on a probability space $(\Omega, \mathcal{F}, P)$.
If $\int_\Omega |X| dP < \infty$, then we define its **expected value** to be

$$E(X) = \int_\Omega X dP.$$

Proposition 2.3.

1. *If X is a discrete random variable, then $E(X) = \sum_{x \in ImX} xP(X = x)$.*

2. If X is a continuous random variable with probability function f, then

$$E(X) = \int_{\mathbb{R}} xf(x)dx.$$

Proposition 2.4. *Let X and Y be random variables such that $E(|X|) < \infty$ and $E(|Y|) < \infty$ and $a, b \in R$. Then we have the followings:*

1. $E(aX + bY) = aE(X) + bE(Y)$.

2. *If $X \leq Y$, then $E(X) \leq E(Y)$.*

3. $|E(X)| \leq E(|X|)$.

4. *If X and Y are independent , then $E(XY) = E(X)E(Y)$.*

Let X be a random variable which $E(|X|^k) < \infty$. Then $E(|X|^k)$ is called the **k-th moment** of X about the origin and call $E[(X - E(X))^k]$ the **k-th moment** of X about the mean.

We call the second moment of X about the mean, the **variance** of X, denoted by $Var(X)$. That is

$$Var(X) = E[X - E(X)]^2.$$

We note that

1. $Var(X) \ = \ E(X^2) - E^2(X)$.

2. If $X \sim N(\mu, \sigma^2)$ then $E(X) = \mu$ and $Var(X) = \sigma^2$.

Proposition 2.5. *If $X_1, ..., X_n$ are independent and $E|X_i| < \infty$ for $i = 1, 2, ..., n$, then*

1. $E(X_1X_2...X_n) = E(X_1)E(X_2)...E(X_n)$,

2. *$Var(a_1X_1 + \cdots + a_nX_n) \ = \ a_1^2Var(X_1) + \cdots + a_n^2Var(X_n)$ for any real number $a_1, ..., a_n$.*

Theorem 2.6. *Let (X_n) be an increasing sequence of randon variables on a probability space $(\Omega, \mathcal{F}, P)$ to $[0, \infty)$ and $\lim\limits_{n\to\infty} X_n = X$ a.s. Then $\lim\limits_{n\to\infty} E(X_n) = EX$.*

The following inequalities are useful in our work.

1. **Hölder's inequality :**

$$E(|XY|) \leq E^{\frac{1}{p}}(|X|^p)E^{\frac{1}{q}}(|Y|^q)$$

where $1 \leq p, q \leq \infty$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and $E(|X|^p) < \infty$,$E(|Y|^q) < \infty$.

2. **Chebyshev's inequality :**

$$P(|X| \geq \varepsilon) \leq \frac{E|X|^p}{\varepsilon^p} \quad \text{for all} \quad \varepsilon, p > 0$$

where $E|X|^p < \infty$.

Let X be a random variable with finite expected value on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{D}$ a sub σ-algebra of $\mathcal{F}$. Define a probability measure $P_{\mathcal{D}} : \mathcal{D} \to [0, 1]$ by

$$P_{\mathcal{D}}(E) = P(E)$$

and a signed measure $\mathcal{Q}_X : \mathcal{D} \to \mathbb{R}$ by

$$\mathcal{Q}_X(E) = \int_E X dP.$$

Then, by Radon-Nikodym theorem we have $\mathcal{Q}_X \ll P_{\mathcal{D}}$ and there exists a unique measurable function $E^{\mathcal{D}}(X)$ on $(\Omega, \mathcal{F}, P)$ such that

$$\int_A E^{\mathcal{D}}(X) dP_{\mathcal{D}} = \mathcal{Q}_X(A) = \int_A X dP \quad \text{for any } A \in \mathcal{D}.$$

We will say that $E^{\mathcal{D}}(X)$ is the **conditional expectation** of X with respect to $\mathcal{D}$.

Moreover, for any random variables X and Y on the same probability space $(\Omega, \mathcal{F}, P)$ such that $E(|X|) < \infty$, we will denote $E^{\sigma(Y)}(X)$, the conditional expectation of X with respect to the σ-algebra generated by Y, by $E^Y(X)$.

Theorem 2.7. *Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ such that $E(|X|) < \infty$, then the followings hold for any sub σ-algebra $\mathcal{D}$ of $\mathcal{F}$.*

1. *If X is a random variable on $(\Omega, \mathcal{D}, P_{\mathcal{D}})$, then $E^{\mathcal{D}}(X) = X$ a.s.$[P_{\mathcal{D}}]$.*

2. $E^{\mathcal{F}}(X) = X$ *a.s.*$[P]$.

3. *If* $\sigma(X)$ *and* $\mathcal{D}$ *are independent, then* $E^{\mathcal{D}}(X) = E(X)$ *a.s.*$[P_{\mathcal{D}}]$.

Theorem 2.8. *Let* X *and* Y *be random variables on the same probability space* $(\Omega, \mathcal{F}, P)$ *such that* $E(|X|)$ *and* $E(|Y|)$ *are finite. Then for any sub* σ*-algebra* $\mathcal{D}$ *of* $\mathcal{F}$ *the followings hold.*

1. *If* $X \le Y$, *then* $E^{\mathcal{D}}(X) \le E^{\mathcal{D}}(Y)$ *a.s.* $[P_{\mathcal{D}}]$.

2. $E^{\mathcal{D}}(aX + bY) = aE^{\mathcal{D}}(X) + bE^{\mathcal{D}}(X)$ *a.s.* $[P_{\mathcal{D}}]$ *for any* $a,\ b \in \mathbb{R}$.

3. $|E^{\mathcal{D}}(X)| \le E^{\mathcal{D}}(|X|)$ *a.s.* $[P_{\mathcal{D}}]$.

Theorem 2.9. *Let* X *and* Y *be random variables on the same probability space* $(\Omega, \mathcal{F}, P)$ *such that* $E(|XY|)$ *and* $E(|Y|)$ *are finite and* $\mathcal{D}_1, \mathcal{D}_2$ *be any sub* σ*-algebras of* $\mathcal{F}$. *If* X *is a random variable with respect to* $\mathcal{D}_1$, *then*

1. $E^{\mathcal{D}_1}(XY) = XE^{\mathcal{D}_1}(Y)$ *a.s.* $[P_{\mathcal{D}_1}]$,

2. $E^{\mathcal{D}_2}(XY) = E^{\mathcal{D}_2}(XE^{\mathcal{D}_1}(Y))$ *a.s.* $[P_{\mathcal{D}_2}]$.

Theorem 2.10. *Let* (X_n) *be an increasing sequence of non-negative random variables on the same probability space* $(\Omega, \mathcal{F}, P)$. *If* $X_n \to X$ *a.s. and* $E|X| \le \infty$, *then* $0 \le \lim\limits_{n\to\infty} E^{\mathcal{D}} X_n = E^{\mathcal{D}} X$ *a.s.* $[P_{\mathcal{D}}]$ *for any sub* σ*-algebra of* $\mathcal{F}$.

Theorem 2.11. *Let* X *be a random variable on the probability space* $(\Omega, \mathcal{F}, P)$ *such that* $E(|X|)$ *is finite and* $\mathcal{D}_1, \mathcal{D}_2$ *be any sub* σ*-algebras of* $\mathcal{F}$. *If* $\sigma\{X, \mathcal{D}_1\}$ *is independent of* $\mathcal{D}_2$, *then*

$$E^{\sigma\{\mathcal{D}_1,\mathcal{D}_2\}}(X) = E^{\mathcal{D}_1}(X) \ a.s.\ [P].$$

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{D}$ a sub σ-algebra of $\mathcal{F}$. For any event A on $\mathcal{F}$, we define the **conditional probability of** A **given** $\mathcal{D}$ by

$$P(A|\mathcal{D}) = E^{\mathcal{D}}\mathbb{I}(A)$$

where $\mathbb{I}(A)$ is the indicator function defined by

$$\mathbb{I}(A)(\omega) = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A. \end{cases}$$

Let X, Y and Z be discrete random variables on a probability space $(\Omega, \mathcal{F}, P)$. If $P(X \le x, Y \le y \mid Z = z) = P(X \le x \mid Z = z)P(Y \le y \mid Z = z)$ for all $x, y, z \in \mathbb{R}$ such that $P(Z = z) > 0$, then we say that X and Y are **conditionally independent** given Z.

Proposition 2.12. *Let X, Y and Z be random variables on a probability space $(\Omega, \mathcal{F}, P)$ such that Z is discrete, X and Y are conditionally independent given Z. Then*

$$EXY = E(E^Z X E^Z Y).$$

2.4 Stein's Method for Normal Approximation

In 1972, Stein [33] introduced a powerful and general method for obtaining an explicit bound for the error in the normal approximation to the distribution of a sum of dependent random variables. The technique used was novel. Stein's technique is free of Fourier methods and relied instead on the elementary differential equation. This method was adapted and applied to the Poisson approximation by Chen in 1975 [4]. Since then, Stein's method has stimulated an area of intensive research in combinatorics, probability and statistics.

There are at least three approaches to use Stein's method when the limit distribution is normal, namely a concentration inequality approach(see, for examples, Ho and Chen [8], and Chen and Shao [5]), an inductive approach (see, for example, Bolthausen [3]), and a coupling approach (see, for example, Stein [35]). In this work, we use the concentration inequality approach.

Let Z be a standard normally distributed random variable and let C_{bd} be the set of continuous and piecewise continuously differentiable finctions $g : \mathbb{R} \to \mathbb{R}$ with $E|g'(Z)| < \infty$.

For $g \in C_{bd}$ and any real valued function I with $E|I(Z)| < \infty$, the equation

$$g'(w) - wg(w) = I(w) - EI(Z) \tag{2.1}$$

is called **Stein's equation**. If we choose I in (2.1) to be the indicator function $\mathbb{I}_z(w) = 1(w \leq z)$ where z is a real number, then (2.1) becomes

$$g'(w) - wg(w) = \mathbb{I}_z(w) - \Phi(z). \tag{2.2}$$

The solution g_z of (2.2) is given by

$$g_z(w) = \begin{cases} \sqrt{2\pi}e^{\frac{w^2}{2}}\Phi(w)[1-\Phi(z)] & \text{, if} \quad w \leq z, \\ \sqrt{2\pi}e^{\frac{w^2}{2}}\Phi(z)[1-\Phi(w)] & \text{, if} \quad w > z. \end{cases}$$

The following properties of g_z are used in this work.

Proposition 2.13. *For all real number w, u, v, we have*

1. $0 < g_z(w) \leq \min(\frac{\sqrt{2\pi}}{4}, \frac{1}{|z|})$, *([5], p.246),*

2. $|g'_z(w)| \leq 1$, *([35], p.23),*

3.

$$\begin{aligned} &g'_z(w+u) - g'_z(w+v) \\ &\qquad \leq \begin{cases} 1 & \textit{if} \;\; w+u < z, w+v > z, \\ (|w| + \frac{\sqrt{2\pi}}{4})(|u| + |v|) & \textit{if } u \geq v, \\ 0 & \textit{otherwise}, \end{cases} \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &g'_z(w+u) - g'_z(w+v) \\ &\qquad \geq \begin{cases} -1 & \textit{if} \;\; w+u > z, w+v < z, \\ -(|w| + \frac{\sqrt{2\pi}}{4})(|u| + |v|) & \textit{if } u \geq v, \\ 0 & \textit{otherwise}, \end{cases} \end{aligned} \tag{2.4}$$

([5], p.247).

From (2.2), we note that

$$E(g_z'(W)) - EWg_z(W) = P(W \leq z) - \Phi(z) \tag{2.5}$$

for any random variable W. Thus, it suffices to find a bound for

$$E(g_z'(W)) - EWg_z(W)$$

instead of

$$P(W \leq z) - \Phi(z).$$

CHAPTER III

UNIFORM BOUND ON NORMAL APPROXIMATION OF LATIN HYPERCUBE SAMPLING

Latin hypercube sampling (LHS) is introduced by McKay, Beckman and Conover in 1978 [17] as a tool to improve the efficiency of different important sampling method. After the original paper appeared, LHS has been used widely in many computer experiments. For example, it is a way to choose the points to calculate the integral

$$\int_{[0,1]^d} f(x)dx,$$

where f is a measurable function from $[0,1]^d$ to $\mathbb{R}$. Computing this integral is equivalent to finding $\mu = E(f(X))$ where X is a random vector having a uniform distribution on a unit hypercube $[0,1]^d$. The Monte Carlo method give the estimation for the integral to be

$$\hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} f(X_i),$$

where $X_1, X_2, \ldots, X_n$ are random samples on $[0,1]^d$.

For positive integers n and d, $d \geq 2$, a latin hypercube sample of size n (taken from the d-dimensional hypercube $[0,1]^d$) is defined to be $\{X(\eta_1(i), \eta_2(i), ..., \eta_d(i)) : 1 \leq i \leq n\}$, where

1. for all $1 \leq i_1, ..., i_d \leq n,\ 1 \leq j \leq d$,

$$X_j(i_1, ..., i_d) = (i_j - U_{i_1,...,i_d,j})/n,$$

$$\text{and} \qquad X(i_1, ..., i_d) = (X_1(i_1, ..., i_d), ..., X_d(i_1, ..., i_d));$$

2. $\eta_k = (\eta_k(1), \eta_k(2), \ldots, \eta_k(n)),\ 1 \leq k \leq d$, are random permutations of $\{1, ..., n\}$, each of which is uniformly distributed over all the $n!$ possible

permutations;

3. $U_{i_1,...,i_d,j}$ $1 \le i_1, ..., i_d \le n,\ 1 \le j \le d$, be $[0,1]$ uniform random variables;

4. the $U_{i_1,...,i_d,j}$'s and η_k's all be stochastically independent.

Hence an unbiased estimator for μ that based on a latin hypercube sampling is

$$\hat{\mu}_n = \frac{1}{n}\sum_{k=1}^{n} f(X(\eta_1(k), \eta_2(k), ..., \eta_d(k))).$$

If $Var(\hat{\mu}_n) > 0$, we define

$$W = \frac{\hat{\mu}_n - \mu}{\sqrt{Var(\hat{\mu}_n)}}.$$

Then

$$EW = 0 \quad \text{and} \quad VarW = 1. \tag{3.1}$$

In this work, we use a concentation inequality approach and ideas in [19] and [21] to obtain a uniform bound on normal approximation of LHS by assuming the finiteness of the absolute third moment. Theorem 3.1 is our main result.

Theorem 3.1. *Suppose that* $E|f \circ X(i_1, ..., i_d)|^3 < \infty$, $1 \le i_1, ..., i_d \le n$. *Then for* $n \ge 6^d$,

$$\sup_{z\in\mathbb{R}} |P(W \le z) - \Phi(z)| \le (22.88 + 28.99d)\delta_3 + \frac{3.88 + 2.09d}{\sqrt{n}} + 1.03\delta_3^2$$
$$+ (\frac{C_d}{n^{\frac{1}{24}}})(\frac{C\delta_3^{\frac{3}{4}}}{n^{\frac{1}{8}}} + Cn^{\frac{1}{8}}\delta_3^{\frac{5}{4}}) + O(\frac{1}{n}),$$

where

$$\delta_3 = \frac{1}{n^{d-1}}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n} E|Y(i_1, ..., i_d)|^3$$

and the definition of $Y(i_1, ..., i_d)$ *is given by (1.1).*

Immediately, we have the following corollary.

Corollary 3.2. *Suppose that* $E|f \circ X(i_1, ..., i_d)|^3 < \infty$, $1 \le i_1, ..., i_d \le n$. *If* $n \ge 6^d$ *and* $\delta_3 \sim \frac{1}{\sqrt{n}}$, *then*

$$\sup_{z\in\mathbb{R}} |P(W \le z) - \Phi(z)| \le \frac{26.76 + 31.08d}{\sqrt{n}} + O(\frac{1}{n^{\frac{13}{24}}}).$$

In case of $d = 2$, Corollary 3.2 yield

$$\sup_{z\in\mathbb{R}} |P(W \le z) - \Phi(z)| \le \frac{88.92}{\sqrt{n}} + O(\frac{1}{n^{\frac{13}{24}}}).$$

3.1 Auxiliary Results

In this section, we will give some lemmas which are used in the next section. Almost all of them, we improve the results from [21] by relaxing their assumption to be the finiteness of the absolute third moment. Throughout this work, we assume that for $1 \leq i_1, \ldots, i_d \leq n$,

$$E|f \circ X(i_1, i_2, \ldots, i_d)|^3 < \infty$$

and let

$$\delta_2 = \frac{1}{n^{d-\frac{1}{2}}} \sum_{i_1=1}^{n} \cdots \sum_{i_d=1}^{n} E|Y(i_1, ..., i_d)|^2 \quad \text{and}$$
$$\delta_3 = \frac{1}{n^{d-1}} \sum_{i_1=1}^{n} \cdots \sum_{i_d=1}^{n} E|Y(i_1, ..., i_d)|^3.$$

Lemma 3.3. *For $n \geq 36$,*

$$\delta_2 \leq \frac{1.02943}{\sqrt{n}}.$$

Proof. By (3.1), we have

$$\begin{aligned}
1 &= EW^2 \\
&= E[\sum_{i=1}^{n} Y(i, \pi_1(i), \ldots, \pi_{d-1}(i))]^2 \\
&= \sum_{i=1}^{n} EY^2(i, \pi_1(i), \ldots, \pi_{d-1}(i)) \\
&\quad + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} EY(i, \pi_1(i), \ldots, \pi_{d-1}(i)) Y(j, \pi_1(j), \ldots, \pi_{d-1}(j)) \qquad (3.2) \\
&= \sqrt{n}\delta_2 + \frac{1}{n^{d-1}(n-1)^{d-1}} \sum_{i_1=1}^{n} \cdots \sum_{i_d=1}^{n} \sum_{\substack{j_1=1 \\ j_1 \neq i_1}}^{n} \cdots \sum_{\substack{j_d=1 \\ j_d \neq i_d}}^{n} EY(i_1, \ldots, i_d) EY(j_1, \ldots, j_d) \\
&= \sqrt{n}\delta_2 + \frac{(-1)^d}{n^{d-1}(n-1)^{d-1}} \sum_{i_1=1}^{n} \cdots \sum_{i_d=1}^{n} [EY(i_1, \ldots, i_d)]^2.
\end{aligned}$$

Hence

$$\begin{aligned}\sqrt{n}\delta_2 &\le 1 + \frac{1}{n^{d-1}(n-1)^{d-1}}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}[EY(i_1,\ldots,i_d)]^2\\ &\le 1 + \frac{1}{n^{d-1}(n-1)^{d-1}}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}EY^2(i_1,\ldots,i_d)\\ &\le 1 + \frac{\sqrt{n}\delta_2}{(n-1)^{d-1}}.\end{aligned}$$

This implies that for $n \ge 36$ and $d \ge 2$,

$$\sqrt{n}\delta_2 \le 1 + \frac{\sqrt{n}\delta_2}{35} = 1 + 0.02858\sqrt{n}\delta_2$$

and hence

$$\delta_2 \le \frac{1}{(1-0.02858)\sqrt{n}} \le \frac{1.02943}{\sqrt{n}}.$$

□

Lemma 3.4. *If $n \ge 36$, then $E[\sum_{i=1}^{n}\sum_{k=1}^{n}Y(i,\pi_1(k),\ldots,\pi_{d-1}(k))]^2 \le 1.02943n$.*

Proof. Observe that

$$\begin{aligned}&E[\sum_{i=1}^{n}\sum_{k=1}^{n}Y(i,\pi_1(k),\ldots,\pi_{d-1}(k))]^2\\ &\quad= \sum_{i=1}^{n}\sum_{k=1}^{n}EY^2(i,\pi_1(k),\ldots,\pi_{d-1}(k))\\ &\qquad+ \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{\substack{l=0\\(i,k)\neq(l,m)}}^{n}\sum_{m=0}^{n}EY(i,\pi_1(k),\ldots,\pi_{d-1}(k))Y(l,\pi_1(m),\ldots,\pi_{d-1}(m))\\ &\quad= \sum_{i=1}^{n}\sum_{k=1}^{n}EY^2(i,\pi_1(k),\ldots,\pi_{d-1}(k))\\ &\qquad+ \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{\substack{l=1\\l\neq i}}^{n}\sum_{\substack{m=1\\m\neq k}}^{n}EY(i,\pi_1(k),\ldots,\pi_{d-1}(k))Y(l,\pi_1(m),\ldots,\pi_{d-1}(m))\\ &\qquad+ \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{\substack{l=1\\l\neq i}}^{n}EY(i,\pi_1(k),\ldots,\pi_{d-1}(k))Y(l,\pi_1(k),\ldots,\pi_{d-1}(k))\\ &\qquad+ \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{\substack{m=1\\m\neq k}}^{n}EY(i,\pi_1(k),\ldots,\pi_{d-1}(k))Y(i,\pi_1(m),\ldots,\pi_{d-1}(m))\end{aligned}$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{n} EY^2(i,\pi_1(k),\ldots,\pi_{d-1}(k))$$
$$+ \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{\substack{m=1\\ m\neq k}}^{n} EY(i,\pi_1(k),\ldots,\pi_{d-1}(k))Y(l,\pi_1(m),\ldots,\pi_{d-1}(m))$$
$$+ \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{\substack{l=1\\ l\neq i}}^{n} EY(i,\pi_1(k),\ldots,\pi_{d-1}(k))Y(l,\pi_1(k),\ldots,\pi_{d-1}(k))$$
$$= \frac{1}{n^{d-2}}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n} EY^2(i_1,\ldots,i_d)$$
$$+ \frac{1}{(n(n-1))^{d-2}}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}\sum_{\substack{l_2=1\\ l_2\neq i_2}}^{n}\cdots\sum_{\substack{l_d=1\\ l_d\neq i_d}}^{n} EY(i_1,\ldots,i_d)\sum_{l_1=1}^{n} EY(l_1,\ldots,l_d)$$
$$+ \frac{1}{n^{d-2}}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}\sum_{\substack{l=1\\ l\neq i_1}}^{n} EY(i_1,\ldots,i_d)EY(l,i_2,\ldots,i_d).$$

By Lemma 3.3 and (1.2), we have

$$E[\sum_{i=1}^{n}\sum_{k=1}^{n}Y(i,\pi_1(k),\ldots,\pi_{d-1}(k))]^2$$
$$= n\sqrt{n}\delta_2 + \frac{1}{n^{d-2}}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n} EY(i_1,\ldots,i_d)\sum_{l=1}^{n} EY(l,i_2,\ldots,i_d)$$
$$- \frac{1}{n^{d-2}}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}[EY(i_1,\ldots,i_d)]^2$$
$$\leq n\sqrt{n}\delta_2$$
$$\leq 1.02943n.$$

□

For each $i_1,\ldots,i_d \in \{1,2,\ldots,n\}$, we define

$$Y_0(i_1,\ldots,i_d) = Y(i_1,\ldots,i_d)\mathbb{I}(|Y(i_1,\ldots,i_d)| > 1), \quad (3.3)$$

$$\widehat{Y}_0(i_1,\ldots,i_d) = Y(i_1,\ldots,i_d)\mathbb{I}(|Y(i_1,\ldots,i_d)| \leq 1), \quad (3.4)$$

and

$$\widehat{Y}(\pi) = \sum_{i=1}^{n}\widehat{Y}_0(i,\pi_1(i),\ldots,\pi_{d-1}(i)), \quad (3.5)$$

where

$$\mathbb{I}(A)(\omega) = \begin{cases} 1 & \text{if } \ \omega \in A, \\ 0 & \text{if } \ \omega \notin A \end{cases}$$

for any nonempty set A. Next, we note that for any integers m, n and r such that $m \geq 0$, and $n, r > 0$,

$$\begin{aligned} E|Y^m(i_1,\dots,i_d)Y_0^n(i_1,\dots,i_d)| &\leq E|Y^m(i_1,\dots,i_d)Y_0^n(i_1,\dots,i_d)||Y_0(i_1,\dots,i_d)|^r \\ &\leq E|Y(i_1,\dots,i_d)|^{m+n+r}. \end{aligned} \tag{3.6}$$

Lemma 3.5.

$$E\widehat{Y}^2(\pi) \leq 1 + \frac{2\sqrt{n}\delta_2}{(n-1)^{d-1}} + \frac{n^{d-1}\delta_3^2}{(n-1)^{d-1}}.$$

Furthermore, if $n \geq 6^d$ *and* $\delta_3 \leq \frac{1}{30}$, *then* $E\widehat{Y}^2(\pi) \leq 1.05998$.

Proof. Note that $\widehat{Y}_0 = Y - Y_0$. By (1.2), (3.2) and (3.6), we have

$$\begin{aligned} E\widehat{Y}^2(\pi) &= E\Big(\sum_{i=1}^{n} \widehat{Y}_0(i,\pi_1(i),\dots,\pi_{d-1}(i))\Big)^2 \\ &= \sum_{i=1}^{n} E\widehat{Y}_0^2(i,\pi_1(i),\dots,\pi_{d-1}(i))) \\ &\quad + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} E\widehat{Y}_0(i,\pi_1(i),\dots,\pi_{d-1}(i))\widehat{Y}_0(j,\pi_1(j),\dots,\pi_{d-1}(j)) \\ &\leq \sum_{i=1}^{n} EY^2(i,\pi_1(i),\dots,\pi_{d-1}(i)) \\ &\quad + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} EY(i,\pi_1(i),\dots,\pi_{d-1}(i))Y(j,\pi_1(j),\dots,\pi_{d-1}(j)) \\ &\quad - 2\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} EY(i,\pi_1(i),\dots,\pi_{d-1}(i))Y_0(j,\pi_1(j),\dots,\pi_{d-1}(j)) \\ &\quad + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} EY_0(i,\pi_1(i),\dots,\pi_{d-1}(i))Y_0(j,\pi_1(j),\dots,\pi_{d-1}(j)) \end{aligned}$$

$$= 1 - \frac{2}{(n(n-1))^{d-1}} \sum_{i_1=1}^{n} \cdots \sum_{i_d=1}^{n} \sum_{\substack{j_1=1 \\ j_1 \neq i_1}}^{n} \cdots \sum_{\substack{j_d=1 \\ j_d \neq i_d}}^{n} EY_0(i_1, \ldots, i_d) EY(j_1, \ldots, j_d)$$

$$+ \frac{1}{(n(n-1))^{d-1}} \sum_{i_1=1}^{n} \cdots \sum_{i_d=1}^{n} \sum_{\substack{j_1=1 \\ j_1 \neq i_1}}^{n} \cdots \sum_{\substack{j_d=1 \\ j_d \neq i_d}}^{n} EY_0(i_1, \ldots, i_d) EY_0(j_1, \ldots, j_d)$$

$$\leq 1 + \frac{2}{(n(n-1))^{d-1}} \sum_{i_1=1}^{n} \cdots \sum_{i_d=1}^{n} EY^2(i_1, \ldots, i_d)$$

$$+ \frac{1}{(n(n-1))^{d-1}} \sum_{i_1=1}^{n} \cdots \sum_{i_d=1}^{n} E|Y(i_1, \ldots, i_d)|^3 \sum_{j_1=1}^{n} \cdots \sum_{j_d=1}^{n} E|Y(j_1, \ldots, j_d)|^3$$

$$\leq 1 + \frac{2\sqrt{n}\delta_2}{(n-1)^{d-1}} + \frac{n^{d-1}\delta_3^2}{(n-1)^{d-1}}.$$

In [19], p.15, Neammanee and Rattanawong showed that for $n \geq 6^d$,

$$(\frac{n}{n-1})^{d-1} \leq 1.03. \tag{3.7}$$

For $\delta_3 \leq \frac{1}{30}$, we use Lemma 3.3 and (3.7) to obtain

$$E\widehat{Y}^2(\pi) \leq 1 + (\frac{2\sqrt{n}}{35})(\frac{1.02943}{\sqrt{n}}) + 1.03(\frac{1}{30})^2 = 1.05998.$$

□

Throughout this thesis, C is always a constant which is greater than zero. It may vary in each situation.

Lemma 3.6. *For $n \geq 6^d$ and $\delta_3 \leq \frac{1}{30}$,*

$$E\widehat{Y}^4(\pi) \leq C(n + n^2\delta_3^2).$$

Proof. We begin this proof by writing

$$E\widehat{Y}^4(\pi) = E(\sum_{i=1}^{n} \widehat{Y}_0(i, \pi_1(i), \ldots, \pi_{d-1}(i)))^4 = Q_1 + Q_2 + Q_3 + Q_4 + Q_5,$$

where

$$Q_1 = \sum_{i=1}^{n} E\widehat{Y}_0^4(i, \pi_1(i), \ldots, \pi_{d-1}(i)),$$

$$Q_2 = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} E\widehat{Y}_0^3(i,\pi_1(i),\dots,\pi_{d-1}(i))\widehat{Y}_0(j,\pi_1(j),\dots,\pi_{d-1}(j)),$$

$$Q_3 = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} E\widehat{Y}_0^2(i,\pi_1(i),\dots,\pi_{d-1}(i))\widehat{Y}_0^2(j,\pi_1(j),\dots,\pi_{d-1}(j)),$$

$$Q_4 = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \sum_{\substack{k=1 \\ k\neq i \\ k\neq j}}^{n} E\widehat{Y}_0^2(i,\pi_1(i),\dots,\pi_{d-1}(i))\widehat{Y}_0(j,\pi_1(j),\dots,\pi_{d-1}(j))$$

$$\times \widehat{Y}_0(k,\pi_1(k),\dots,\pi_{d-1}(k)) \quad \text{and}$$

$$Q_5 = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \sum_{\substack{k=1 \\ k\neq i,j}}^{n} \sum_{\substack{l=1 \\ l\neq i \\ l\neq j \\ l\neq k}}^{n} E\widehat{Y}_0(i,\pi_1(i),\dots,\pi_{d-1}(i))\widehat{Y}_0(j,\pi_1(j),\dots,\pi_{d-1}(j))$$

$$\times \widehat{Y}_0(k,\pi_1(k),\dots,\pi_{d-1}(k))\widehat{Y}_0(l,\pi_1(l),\dots,\pi_{d-1}(l)).$$

Note that for $a_i \geq 0$ and $\alpha_i > 0$, $i \in \{1,2,\dots,n\}$ with $\alpha_1 + \dots + \alpha_n = 1$,

$$a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n} \leq \alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_n a_n. \tag{3.8}$$

Then

$$|Q_2| \leq nQ_1 \text{ and } |Q_3| \leq nQ_1.$$

Hence

$$\begin{aligned} |Q_1 + Q_2 + Q_3| &\leq CnQ_1 \\ &= \frac{Cn}{n^{d-1}} \sum_{i_1=1}^{n} \cdots \sum_{i_d=1}^{n} E\widehat{Y}_0^4(i_1,\dots,i_d) \\ &\leq \frac{Cn}{n^{d-1}} \sum_{i_1=1}^{n} \cdots \sum_{i_d=1}^{n} E|Y(i_1,\dots,i_d)|^3 \\ &= Cn\delta_3. \end{aligned}$$

Next, we observe that Q_4 is bounded by a sum of the term

$$Q_{4,1} = |\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \sum_{\substack{k=1 \\ k\neq i \\ k\neq j}}^{n} E\widehat{Y}_0^2(i,\pi_1(i),\dots,\pi_{d-1}(i))\widehat{Y}_0(j,\pi_1(j),\dots,\pi_{d-1}(j))$$

$$\times Y_0(k,\pi_1(k),\dots,\pi_{d-1}(k))|$$

and 2^d terms each of the form

$$Q_{4,2}(q_1,\ldots,q_d)=\frac{1}{(n(n-1)(n-2))^{d-1}}\Big|\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}\sum_{\substack{j_1=1\\ j_1\neq i_1}}^{n}\cdots\sum_{\substack{j_d=1\\ j_d\neq i_d}}^{n}$$
$$E\widehat{Y}_0^2(i_1,\ldots,i_d)E\widehat{Y}_0(j_1,\ldots,j_d)EY(q_1,\ldots,q_d),\Big|$$

where $q_r=i_r$ or j_r for $r=1,\ldots,d$. By (3.8), we obtain

$$\begin{aligned}
Q_{4,1}&\leq\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\sum_{\substack{k=1\\ k\neq i\\ k\neq j}}^{n}E[|\widehat{Y}_0(i,\pi_1(i),\ldots,\pi_{d-1}(i))|^3]^{\frac{2}{3}}[|\widehat{Y}_0(j,\pi_1(j),\ldots,\pi_{d-1}(j))|^3]^{\frac{1}{3}}\\
&\quad\times|Y_0(k,\pi_1(k),\ldots,\pi_{d-1}(k))|\\
&\leq\frac{2n}{3}\sum_{i=1}^{n}\sum_{\substack{k=1\\ k\neq i}}^{n}E|\widehat{Y}_0(i,\pi_1(i),\ldots,\pi_{d-1}(i))|^3|Y_0(k,\pi_1(k),\ldots,\pi_{d-1}(k))|\\
&\quad+\frac{n}{3}\sum_{j=1}^{n}\sum_{\substack{k=1\\ k\neq j}}^{n}E|\widehat{Y}_0(j,\pi_1(j),\ldots,\pi_{d-1}(j))|^3|Y_0(k,\pi_1(k),\ldots,\pi_{d-1}(k))|\\
&=\frac{n}{(n(n-1))^{d-1}}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}\sum_{\substack{k_1=1\\ k_1\neq i_1}}^{n}\cdots\sum_{\substack{k_d=1\\ k_d\neq i_d}}^{n}E|\widehat{Y}_0(i_1,\ldots,i_d)|^3E|Y_0(k_1,\ldots,k_d)|\\
&\leq\frac{n}{(n(n-1))^{d-1}}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}E|Y(i_1,\ldots,i_d)|^3\sum_{k_1=1}^{n}\cdots\sum_{k_d=1}^{n}E|Y(k_1,\ldots,k_d)|^3\\
&\leq\frac{n^d\delta_3^2}{(n-1)^{d-1}}. \qquad (3.9)
\end{aligned}$$

Fix $q_1,\ldots,q_d$ where $q_r\in\{i_r,j_r\}$ and $r\in\{1,2,\ldots,d\}$. Then (3.8) implies

$$\begin{aligned}
&Q_{4,2}(q_1,\ldots,q_d)\\
&\quad\leq\frac{1}{(n(n-1)(n-2))^{d-1}}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}\sum_{\substack{j_1=1\\ j_1\neq i_1}}^{n}\cdots\sum_{\substack{j_d=1\\ j_d\neq i_d}}^{n}[E|\widehat{Y}_0(i_1,\ldots,i_d)|^3]^{\frac{1}{3}}\\
&\qquad\times[E|\widehat{Y}_0(j_1,\ldots,j_d)|^3]^{\frac{1}{3}}[E|Y(q_1,\ldots,q_d)|^3]^{\frac{1}{3}}\\
&\quad\leq\frac{n^d}{(n(n-1)(n-2))^{d-1}}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}E|Y(i_1,\ldots,i_d)|^3\\
&\quad=\frac{n^d\delta_3}{((n-1)(n-2))^{d-1}}. \qquad (3.10)
\end{aligned}$$

From (3.7), (3.9) and (3.10), we have

$$|Q_4| \le Cn\delta_3^2 + \frac{2^d n^d \delta_3}{((n-1)(n-2))^{d-1}} \le Cn\delta_3^2 + Cn\delta_3.$$

Similar to Q_4, Q_5 is bounded by a sum of the term

$$\begin{aligned} Q_{5,1} = |&\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\sum_{\substack{k=1\\ k\neq i\\ k\neq j}}^{n}\sum_{\substack{l=1\\ l\neq i\\ l\neq j\\ l\neq k}}^{n} E\widehat{Y}_0(i,\pi_1(i),\ldots,\pi_{d-1}(i))\widehat{Y}_0(j,\pi_1(j),\ldots,\pi_{d-1}(j)) \\ &\times \widehat{Y}_0(k,\pi_1(k),\ldots,\pi_{d-1}(k))Y_0(l,\pi_1(l),\ldots,\pi_{d-1}(l))| \end{aligned}$$

and 3^d terms each of the form

$$\begin{aligned} &Q_{5,2}(q_1,\ldots,q_d) \\ &\quad = \frac{1}{(n(n-1)(n-2)(n-3))^{d-1}}|\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}\sum_{\substack{j_1=1\\ j_1\neq i_1}}^{n}\cdots\sum_{\substack{j_d=1\\ j_d\neq i_d}}^{n}\sum_{\substack{k_1=1\\ k_1\neq i_1\\ k_1\neq j_1}}^{n}\cdots\sum_{\substack{k_d=1\\ k_d\neq i_d\\ k_d\neq j_d}}^{n} \\ &\qquad \times E\widehat{Y}_0(i_1,\ldots,i_d)\widehat{Y}_0(j_1,\ldots,j_d)\widehat{Y}_0(k_1,\ldots,k_d)EY(q_1,\ldots,q_d)| \\ &\quad \le \frac{1}{(n(n-1)(n-2)(n-3))^{d-1}}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}\sum_{\substack{j_1=1\\ j_1\neq i_1}}^{n}\cdots\sum_{\substack{j_d=1\\ j_d\neq i_d}}^{n}\sum_{\substack{k_1=1\\ k_1\neq i_1\\ k_1\neq j_1}}^{n}\cdots\sum_{\substack{k_d=1\\ k_d\neq i_d\\ k_d\neq j_d}}^{n} \\ &\qquad \times E|\widehat{Y}_0(i_1,\ldots,i_d)|E|\widehat{Y}_0(j_1,\ldots,j_d)|E|Y(q_1,\ldots,q_d)|, \end{aligned}$$

where $q_r \in \{i_r, j_r, k_r\}$ for $r = 1,\cdots,d$. Similar to $Q_{4,1}$ and $Q_{4,2}(q_1,\ldots,q_d)$, we have

$$Q_{5,1} \le \frac{n^d\delta_3^2}{(n-1)^{d-2}} \text{ and } Q_{5,2}(q_1,\ldots,q_d) \le \frac{n^{2d}\delta_3}{((n-1)(n-2)(n-3))^{d-1}},$$

respectively. Thus

$$|Q_5| \le Cn^2\delta_3^2 + Cn\delta_3.$$

It follows from Q_1 to Q_5 that

$$E\widehat{Y}^4(\tau) \le C(n + n^2\delta_3^2).$$

□

In the rest of this chapter, we use the following system giving by Ho and Chen [8] and Neammanee and Rattanawong [19]. Let $I, K, L_1, \ldots, L_{d-1}, M_1, \ldots, M_{d-1}$ be uniformly distributed random variables on $\{1, 2, \ldots, n\}$ and $\rho_1, \ldots, \rho_{d-1}$ and $\tau_1, \ldots, \tau_{d-1}$ are random permutations of $\{1, 2, \ldots, n\}$. Assume that

$\{I, K, L_1, \ldots, L_{d-1}, M_1, \ldots, M_{d-1}, \rho_1, \ldots, \rho_{d-1}, \tau_1, \ldots, \tau_{d-1}\}$ is independent of $Y(i_1, \ldots, i_d)$'s,

$(I, K), (L_1, M_1), \ldots, (L_{d-1}, M_{d-1})$ are uniformly distributed on $\{(i,k) | i, k = 1, 2, ..., n \text{ and } i \neq k\}$,

$(I, K), (L_1, M_1), \ldots, (L_{d-1}, M_{d-1})$ and $\tau_1, \ldots, \tau_{d-1}$ are mutually independent,

(I, K) and $\rho_1, \ldots, \rho_{d-1}$ are mutually independent, and

$$\rho_i(\alpha) = \begin{cases} \tau_i(\alpha) & \text{if} \quad \alpha \neq I, K, \tau_i^{-1}(L), \tau_i^{-1}(M) \\ L_i & \text{if} \quad \alpha = I, \\ M_i & \text{if} \quad \alpha = K, \\ \tau_i(I) & \text{if} \quad \alpha = \tau_i^{-1}(L) \\ \tau_i(K) & \text{if} \quad \alpha = \tau_i^{-1}(M), \end{cases}$$

where $\rho_i(\rho_i^{-1}(\alpha)) = \rho_i^{-1}(\rho_i(\alpha)) = \alpha$ for $i = 1, 2, \ldots, d-1$. In the case of $d = 2$, Ho [7] gave an example of random vectors $I, K, L, M, \pi, \rho, \tau$ and $Y(i, j), i, j = 1, 2, \ldots, n$ which satisfy the above conditions. It is easy to generalize his example naturally by extending to d-dimensions. Now, we give some notations;

$$\widetilde{Y}(\rho) = \widehat{Y}(\rho) - \widehat{S}_{1,0} - \widehat{S}_{2,0} + \widehat{S}_{3,0} + \widehat{S}_{4,0},$$

where

$$\widehat{Y}(\rho) = \sum_{i=1}^{n} \widehat{Y}_0(i, \rho_1(i), \ldots, \rho_{d-1}(i)),$$

$$\widehat{S}_{1,0} = \widehat{Y}_0(I, \rho_1(I), \ldots, \rho_{d-1}(I)), \quad \widehat{S}_{2,0} = \widehat{Y}_0(K, \rho_1(K), \ldots, \rho_{d-1}(K)),$$

$$\widehat{S}_{3,0} = \widehat{Y}_0(I, \rho_1(K), \ldots, \rho_{d-1}(K)), \quad \widehat{S}_{4,0} = \widehat{Y}_0(K, \rho_1(I), \ldots, \rho_{d-1}(I)).$$

We can show that $\widehat{S}_{1,0}, \widehat{S}_{2,0}, \widehat{S}_{3,0}, \widehat{S}_{4,0}$ have the same distribution (see [31] for more detail).

Lemma 3.7.

$$E|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^2 = \frac{4}{n}+\mathcal{R},$$

where

$$|\mathcal{R}| \le \frac{36\delta_2}{\sqrt{n}(n-1)} + \frac{8\delta_3}{n-1} + \frac{4n^{d-1}\delta_2^2}{(n-1)^d} + \frac{4n^{d-2}\delta_3^2}{(n-1)^{d-1}}.$$

Furthermore, if $n \ge 6^d$ *and* $\delta_3 \le \frac{1}{30}$, *then* $E|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^2 \le \frac{4.27125}{n-1} + O(\frac{1}{n^2})$.

Proof. First, we write

$$E|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^2 = E(-\widehat{S}_{1,0}-\widehat{S}_{2,0}+\widehat{S}_{3,0}+\widehat{S}_{4,0})^2 = \sum_{i=1}^{4} E\widehat{S}_{i,0}^2 + \mathcal{R}_1, \qquad (3.11)$$

where

$$|\mathcal{R}_1| \le 2\sum_{1\le i<j\le 4} |E\widehat{S}_{i,0}\widehat{S}_{j,0}|.$$

From (1.2), we have

$$\begin{aligned}
&E\widehat{S}_{1,0}\widehat{S}_{2,0}\\
&= E\widehat{Y}_0(I,\rho_1(I),\ldots,\rho_{d-1}(I))\widehat{Y}_0(K,\rho_1(K),\ldots,\rho_{d-1}(K))\\
&= \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{\substack{k=1\\k\ne i}}^{n} E\widehat{Y}_0(i,\rho_1(i),\ldots,\rho_{d-1}(i))\widehat{Y}_0(k,\rho_1(k),\ldots,\rho_{d-1}(k))\\
&= \frac{1}{n^d(n-1)^d}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}\sum_{\substack{k_1=1\\k_1\ne i_1}}^{n}\cdots\sum_{\substack{k_d=1\\k_d\ne i_d}}^{n} E\widehat{Y}_0(i_1,\ldots,i_d)\widehat{Y}_0(k_1,\ldots,k_d)\\
&= \frac{1}{n^d(n-1)^d}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}\sum_{\substack{k_1=1\\k_1\ne i_1}}^{n}\cdots\sum_{\substack{k_d=1\\k_d\ne i_d}}^{n} EY(i_1,\ldots,i_d)EY(k_1,\ldots,k_d)\\
&\quad - \frac{2}{n^d(n-1)^d}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}\sum_{\substack{k_1=1\\k_1\ne i_1}}^{n}\cdots\sum_{\substack{k_d=1\\k_d\ne i_d}}^{n} EY_0(i_1,\ldots,i_d)EY(k_1,\ldots,k_d)\\
&\quad + \frac{1}{n^d(n-1)^d}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}\sum_{\substack{k_1=1\\k_1\ne i_1}}^{n}\cdots\sum_{\substack{k_d=1\\k_d\ne i_d}}^{n} EY_0(i_1,\ldots,i_d)EY_0(k_1,\ldots,k_d)
\end{aligned}$$

$$= \frac{(-1)^d}{n^d(n-1)^d}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}[EY(i_1,\ldots,i_d)]^2$$
$$- \frac{2(-1)^d}{n^d(n-1)^d}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}EY_0(i_1,\ldots,i_d)EY(i_1,\ldots,i_d)$$
$$+ \frac{1}{n^d(n-1)^d}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}EY_0(i_1,\ldots,i_d)\sum_{\substack{k_1=1\\k_1\neq i_1}}^{n}\cdots\sum_{\substack{k_d=1\\k_d\neq i_d}}^{n}EY_0(k_1,\ldots,k_d).$$

From this fact and (3.6), we have

$$|E\widehat{S}_{1,0}\widehat{S}_{2,0}|$$
$$\leq \frac{1}{n^d(n-1)^d}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}[EY(i_1,\ldots,i_d)]^2$$
$$+ \frac{2}{n^d(n-1)^d}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}[E|Y(i_1,\ldots,i_d)|]^2$$
$$+ \frac{1}{n^d(n-1)^d}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}EY^2(i_1,\ldots,i_d)\sum_{k_1=1}^{n}\cdots\sum_{k_d=1}^{n}EY^2(k_1,\ldots,k_d)$$
$$\leq \frac{3}{n^d(n-1)^d}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}EY^2(i_1,\ldots,i_d)$$
$$+ \frac{n^{d-1}}{(n-1)^d}\Big[\frac{1}{n^{d-\frac{1}{2}}}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}EY^2(i_1,\ldots,i_d)\Big]^2$$
$$= \frac{3\delta_2}{\sqrt{n}(n-1)^d} + \frac{n^{d-1}\delta_2^2}{(n-1)^d}.$$

By the same argument of $E\widehat{S}_{1,0}\widehat{S}_{2,0}$, we get that

$$|E\widehat{S}_{1,0}\widehat{S}_{2,0}| = |E\widehat{S}_{3,0}\widehat{S}_{4,0}| \leq \frac{3\delta_2}{\sqrt{n}(n-1)^d} + \frac{n^{d-1}\delta_2^2}{(n-1)^d},$$

$$|E\widehat{S}_{1,0}\widehat{S}_{3,0}| = |E\widehat{S}_{2,0}\widehat{S}_{4,0}| \leq \frac{3\delta_2}{\sqrt{n}(n-1)^{d-1}} + \frac{n^{d-2}\delta_3^2}{(n-1)^{d-1}} \text{ and}$$
$$|E\widehat{S}_{1,0}\widehat{S}_{4,0}| = |E\widehat{S}_{2,0}\widehat{S}_{3,0}| \leq \frac{2\delta_2}{\sqrt{n}(n-1)} + \frac{\delta_3}{n-1}.$$

Thus

$$|\mathcal{R}_1| \leq 2\sum_{1\leq i<j\leq 4}|E\widehat{S}_{i,0}\widehat{S}_{j,0}|$$
$$\leq \frac{12\delta_2}{\sqrt{n}(n-1)^d} + \frac{4n^{d-1}\delta_2^2}{(n-1)^d} + \frac{12\delta_2}{\sqrt{n}(n-1)^{d-1}} + \frac{4n^{d-2}\delta_3^2}{(n-1)^{d-1}}$$
$$+ \frac{8\delta_2}{\sqrt{n}(n-1)} + \frac{4\delta_3}{n-1}. \tag{3.12}$$

By (3.2), (3.6) and the fact that $Y(i_1,\ldots,i_d)Y_0(i_1,\ldots,i_d) = Y_0^2(i_1,\ldots,i_d)$, we have

$$
\begin{aligned}
E\widehat{S}_{1,0}^2 &= E(\widehat{Y}_0(I,\rho_1(I),\ldots,\rho_{d-1}(I)))^2 \\
&= \frac{1}{n}\sum_{i=1}^{n} EY^2(i,\rho_1(i),\ldots,\rho_{d-1}(i)) \\
&\quad - \frac{2}{n}\sum_{i=1}^{n} EY(i,\rho_1(i),\ldots,\rho_{d-1}(i))Y_0(i,\rho_1(i),\ldots,\rho_{d-1}(i)) \\
&\quad + \frac{1}{n}\sum_{i=1}^{n} EY_0^2(i,\rho_1(i),\ldots,\rho_{d-1}(i)) \\
&= \frac{1}{n} - \frac{1}{n}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} EY(i,\rho_1(i),\ldots,\rho_{d-1}(i))Y(j,\rho_1(j),\ldots,\rho_{d-1}(j)) \\
&\quad - \frac{2}{n}\sum_{i=1}^{n} EY(i,\rho_1(i),\ldots,\rho_{d-1}(i))Y_0(i,\rho_1(i),\ldots,\rho_{d-1}(i)) \\
&\quad + \frac{1}{n}\sum_{i=1}^{n} EY_0^2(i,\rho_1(i),\ldots,\rho_{d-1}(i)) \\
&= \frac{1}{n} - \frac{(-1)^d}{n^d(n-1)^{d-1}}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}[EY(i_1,\ldots,i_d)]^2 \\
&\quad - \frac{2}{n^d}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n} EY(i_1,\ldots,i_d)Y_0(i_1,\ldots,i_d) \\
&\quad + \frac{1}{n^d}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n} EY_0^2(i_1,\ldots,i_d) \\
&= \frac{1}{n} - \frac{(-1)^d}{n^d(n-1)^{d-1}}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}[EY(i_1,\ldots,i_d)]^2 \\
&\quad - \frac{1}{n^d}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n} EY_0^2(i_1,\ldots,i_d) \\
&= \frac{1}{n} + \mathcal{R}_2,
\end{aligned}
$$

where

$$
\begin{aligned}
|\mathcal{R}_2| &\leq \frac{1}{n^d(n-1)^{d-1}}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n} EY^2(i_1,\ldots,i_d) + \frac{1}{n^d}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n} E|Y(i_1,\ldots,i_d)|^3 \\
&\leq \frac{\delta_2}{\sqrt{n}(n-1)^{d-1}} + \frac{\delta_3}{n-1}. \qquad (3.13)
\end{aligned}
$$

Since $\widehat{S}_{1,0}, \widehat{S}_{2,0}, \widehat{S}_{3,0}$ and $\widehat{S}_{4,0}$ are identically, we have

$$\sum_{i=1}^{4} E\widehat{S}_{i,0}^2 \le \frac{4}{n} + 4|\mathcal{R}_2|. \tag{3.14}$$

From (3.11) to (3.14), we have

$$E|\widetilde{Y}(\rho) - \widehat{Y}(\rho)|^2 = \frac{4}{n} + \mathcal{R},$$

where $\mathcal{R} = \mathcal{R}_1 + 4\mathcal{R}_2$ and

$$\begin{aligned}|\mathcal{R}| &\le |\mathcal{R}_1| + 4|\mathcal{R}_2| \\ &\le \frac{36\delta_2}{\sqrt{n}(n-1)} + \frac{8\delta_3}{n-1} + \frac{4n^{d-1}\delta_2^2}{(n-1)^d} + \frac{4n^{d-2}\delta_3^2}{(n-1)^{d-1}}.\end{aligned}$$

In case of $n \ge 6^d$ and $\delta_3 \le \frac{1}{30}$, we use Lemma 3.3 and (3.7) to get that

$$\begin{aligned}|\mathcal{R}| &\le \frac{36(1.02943)}{n(n-1)} + \frac{8}{30(n-1)} + \frac{4(1.03)}{n-1}(\frac{1.0012}{\sqrt{n}})^2 + \frac{4(1.03)}{900(n-1)} \\ &\le \frac{0.26667}{n-1} + \frac{0.00458}{n-1} + O(\frac{1}{n^2}) \\ &= \frac{0.27125}{n-1} + O(\frac{1}{n^2}).\end{aligned}$$

□

3.2 Proof of Theorem 3.1

Since $\sup_{z\in\mathbb{R}} |P(W \le z) - \Phi(z)| \le 0.55$ ([5], p.246), we can assume $\delta_3 \le \frac{1}{30}$. Assume that $z > 0$. In case of $z \le 0$, we use the fact that $\Phi(z) = 1 - \Phi(-z)$ and then apply the result to $-W$. We observe that

$$\begin{aligned}P(W \ne \widehat{Y}) &= P(\sum_{i=1}^{n} \mathbb{I}(|Y(i, \pi_1(i), \ldots, \pi_{d-1}(i))| > 1) \ge 1) \\ &\le \sum_{i=1}^{n} E\mathbb{I}(|Y(i, \pi_1(i), \ldots, \pi_{d-1}(i))| > 1) \\ &= \frac{1}{n^{d-1}} \sum_{i_1=1}^{n} \cdots \sum_{i_d=1}^{n} P(|Y(i_1, i_2, \ldots, i_d)| > 1) \\ &\le \frac{1}{n^{d-1}} \sum_{i_1=1}^{n} \cdots \sum_{i_d=1}^{n} E|Y(i_1, i_2, \ldots, i_d)|^3 \\ &= \delta_3.\end{aligned}$$

Then

$$|P(W \leq z) - \Phi(z)| \leq P(W \neq \widehat{Y}(\pi)) + |P(\widehat{Y}(\pi) \leq z) - \Phi(z)|$$
$$\leq \delta_3 + |P(\widehat{Y}(\pi) \leq z) - \Phi(z)|. \tag{3.15}$$

Hence, it suffices to bound the second term on the right hand side of (3.15). From Lemma 2.1 in [21], we recall that for a continuous and piecewise continuously differentiable function g, we have

$$E\widehat{Y}(\rho)g(\widehat{Y}(\rho)) = E\int_{-\infty}^{\infty} g^{'}(\widehat{Y}(\rho)+t)K(t)dt + \Delta g(\widehat{Y}(\rho)), \tag{3.16}$$

where

$$\Delta g(\widehat{Y}(\rho)) = \frac{1}{n}Eg(\widehat{Y}(\rho))\sum_{i=1}^{n}\sum_{k=1}^{n}\widehat{Y}_0(i,\rho_1(k),\ldots,\rho_{d-1}(k))$$

and

$$K(t) = \frac{n-1}{4}(\widetilde{Y}(\rho)-\widehat{Y}(\rho))(\mathbb{I}(0\leq t\leq \widetilde{Y}(\rho)-\widehat{Y}(\rho)) - \mathbb{I}(\widetilde{Y}(\rho)-\widehat{Y}(\rho)\leq t<0)).$$

By using (2.5), (3.16) and applying the same argument in [21], we get that

$$|P(\widehat{Y}(\pi)\leq z)-\Phi(z)| \leq |T_1|+|T_2|+|T_3|+|T_4|,$$

where

$$T_1 = Eg_z'(\widehat{Y}(\tau))\int_{-\infty}^{\infty}K(t)dt - E\int_{-\infty}^{\infty}g_z'(\widehat{Y}(\rho)+t)K(t)dt,$$
$$T_2 = Eg_z'(\widehat{Y}(\tau))E\int_{-\infty}^{\infty}K(t)dt - Eg_z'(\widehat{Y}(\tau))\int_{-\infty}^{\infty}K(t)dt,$$
$$T_3 = Eg_z'(\widehat{Y}(\tau)) - Eg_z'(\widehat{Y}(\tau))E\int_{-\infty}^{\infty}K(t)dt,$$
$$T_4 = \frac{1}{n}Eg_z(\widehat{Y}(\rho))\sum_{i=1}^{n}\sum_{k=1}^{n}\widehat{Y}_0(i,\rho_1(k),\ldots,\rho_{d-1}(k)),$$

and g_z is the solution of the Stein's equation (2.2). Firstly, we will bound T_4 by using Lemma 3.4 and Proposition 2.13(1).

Then

$$
\begin{aligned}
|T_4| &= |\frac{1}{n}Eg_z(\widehat{Y}(\rho))\sum_{i=1}^{n}\sum_{k=1}^{n}Y(i,\rho_1(k),\dots,\rho_{d-1}(k)) \\
&\quad - \frac{1}{n}Eg_z(\widehat{Y}(\rho))\sum_{i=1}^{n}\sum_{k=1}^{n}Y_0(i,\rho_1(k),\dots,\rho_{d-1}(k))| \\
&\leq \frac{1}{n}E|g_z(\widehat{Y}(\rho))||\sum_{i=1}^{n}\sum_{k=1}^{n}Y(i,\rho_1(k),\dots,\rho_{d-1}(k))| \\
&\quad + \frac{1}{n}E|g_z(\widehat{Y}(\rho))||\sum_{i=1}^{n}\sum_{k=1}^{n}Y_0(i,\rho_1(k),\dots,\rho_{d-1}(k))| \\
&\leq \frac{1}{n}\{Eg_z^2(\widehat{Y}(\rho))\}^{\frac{1}{2}}\{E[\sum_{i=1}^{n}\sum_{k=1}^{n}Y(i,\rho_1(k),\dots,\rho_{d-1}(k))]^2\}^{\frac{1}{2}} \\
&\quad + \frac{\sqrt{2\pi}}{4n}\sum_{i=1}^{n}\sum_{k=1}^{n}E|Y_0(i,\rho_1(k),\dots,\rho_{d-1}(k))| \\
&\leq \frac{\sqrt{2\pi}}{4n}[\sqrt{1.02943n} + \frac{1}{n^{d-2}}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}E|Y(i_1,\dots,i_d)|^3] \\
&= \frac{\sqrt{2\pi}}{4n}[\sqrt{1.02943n} + n\delta_3] \\
&\leq \frac{0.63582}{\sqrt{n}} + 0.62666\delta_3. \qquad (3.17)
\end{aligned}
$$

Secondly, we apply Lemma 3.7 and Proposition 2.13(2) to obtain

$$
\begin{aligned}
|T_3| &\leq E|g_z'(\widehat{Y}(\tau))||1 - E\int_{-\infty}^{\infty}K(t)dt| \\
&\leq |1 - \frac{n-1}{4}E|\widetilde{Y}(\rho) - \widehat{Y}(\rho)|^2| \\
&= |\frac{1}{n} - \frac{(n-1)}{4}\mathcal{R}| \\
&\leq \frac{1}{n} + \frac{9\delta_2}{\sqrt{n}} + 2\delta_3 + \frac{n^{d-1}\delta_2^2}{(n-1)^{d-1}} + \frac{n^{d-2}\delta_3^2}{(n-1)^{d-2}} \\
&\leq \frac{1}{n} + \frac{C}{n} + 2\delta_3 + \frac{C}{n} + 1.03\delta_3^2 \\
&= 2\delta_3 + 1.03\delta_3^2 + O(\frac{1}{n}). \qquad (3.18)
\end{aligned}
$$

Next, we will bound T_2 by using the construction of Ho and Chen [8] and Neammanee and Rattanawong [21]. Let

$$A = \{\tau_i(I) \neq L_i, \tau_i(K) \neq M_i, \tau_i(I) \neq M_i, \tau_i(K) \neq L_i, i = 1,\dots,d-1\}$$

and

$$\widehat{G} = \widehat{Y}_0(I, M_1, \ldots, M_{d-1}) + \widehat{Y}_0(K, L_1, \ldots, L_{d-1}) - \widehat{Y}_0(I, L_1, \ldots, L_{d-1})$$
$$- \widehat{Y}_0(K, M_1, \ldots, M_{d-1}).$$

Since $\tau_1, \ldots, \tau_{d-1}$ and $\widehat{G}$ are independent, we have $E\widehat{G}^2 = E^{\tau_1,\ldots,\tau_{d-1}}\widehat{G}^2$. Moreover, $\widehat{Y}(\tau)$ and $\widehat{G}^2\mathbb{I}(A)$ are conditionally independent given $\tau_1, \ldots, \tau_{d-1}$ (see Lemma 3.6 in [31] for more detail). Now, we define $\mathcal{B}$ to be the σ-algebra generated by

$$\{I, K, L_1, \ldots, L_{d-1}, M_1, \ldots, M_{d-1}, Y(i_1, \ldots, i_d) : 1 \le i_1, i_2, \ldots, i_d \le n\}.$$

By the same argument of Neammanee and Rattanawong [19], pp.24-26, we have

$$E^{\mathcal{B}}\mathbb{I}(A^c) = 1 - [1 + \frac{6-4n}{n(n-1)}]^{d-1} \le \frac{4}{n}\sum_{r=1}^{d-1}\binom{d-1}{r} \tag{3.19}$$

and

$$|T_2| \le \frac{n-1}{2}E[\widehat{G}^2 E^{\mathcal{B}}\mathbb{I}(A^c)].$$

By Lemma 3.7, we have

$$\begin{aligned}|T_2| &\le (\frac{n-1}{2})[\frac{4}{n}\sum_{r=1}^{d-1}\binom{d-1}{r}]E\widehat{G}^2\\ &\le 2(2^{d-1}-1)E|\widetilde{Y}(\rho) - \widehat{Y}(\rho)|^2\\ &\le O(\frac{1}{n}).\end{aligned} \tag{3.20}$$

Finally, we will bound T_1. Denote $\Delta\widehat{Y} = \widehat{Y}(\rho) - \widehat{Y}(\tau)$. Then by (2.3), we have

$$\begin{aligned}T_1 &= E\int_{-\infty}^{\infty}[g_z'(\widehat{Y}(\tau)) - g_z'(\widehat{Y}(\rho)+t)]K(t)dt\\ &= E\int_{-\infty}^{\infty}[g_z'(\widehat{Y}(\tau)) - g_z'(\widehat{Y}(\tau)+\Delta\widehat{Y}+t)]K(t)dt\\ &\le E\int_{\substack{\widehat{Y}(\tau)<z\\ \widehat{Y}(\tau)+\Delta\widehat{Y}+t>z}} K(t)dt + E\int_{\Delta\widehat{Y}+t\le 0}(|\widehat{Y}(\tau)| + \frac{\sqrt{2\pi}}{4})(|\Delta\widehat{Y}| + |t|)K(t)dt\\ &= B_1 + B_2,\end{aligned} \tag{3.21}$$

where

$$B_1 = E\int_{\substack{\widehat{Y}(\tau)<z\\ \widehat{Y}(\tau)+\Delta\widehat{Y}+t>z}} K(t)dt \quad \text{and}$$

$$B_2 = E\int_{\Delta\widehat{Y}+t\le 0}(|\widehat{Y}(\tau)| + \frac{\sqrt{2\pi}}{4})(|\Delta\widehat{Y}| + |t|)K(t)dt.$$

Neammanee and Rattanawong showed in [19], p.21, that

$$B_1 \le E \int_{|t| \le |\widetilde{Y}(\rho) - \widehat{Y}(\rho)|} \mathbb{I}(z - |\widetilde{Y}(\rho) - \widehat{Y}(\rho)| < \widehat{Y}(\rho) < z + \Delta\widehat{Y}) K(t) dt. \tag{3.22}$$

For each $\delta \ge 0$ and $a, b \in \mathbb{R}$ which is $a < b$, we define a function f_δ by

$$f_\delta(t) = \begin{cases} -\frac{1}{2}(b-a) - \delta & \text{if } \ t < a - \delta, \\ -\frac{1}{2}(b+a) + t & \text{if } \ a - \delta \le t \le b + \delta, \\ \frac{1}{2}(b-a) + \delta & \text{if } \ b + \delta < t. \end{cases}$$

Observe that

$$|f_\delta(t)| \le \frac{1}{2}(b-a) + \delta \ \text{ for every } \ t \in \mathbb{R} \tag{3.23}$$

and

$$\begin{aligned} E \int_{|t| \le \delta} \mathbb{I}(a \le \widehat{Y}(\rho) \le b) K(t) dt &= E \int_{\substack{a \le \widehat{Y}(\rho) \le b \\ |t| \le \delta}} K(t) dt \\ &= E \int_{\substack{a \le \widehat{Y}(\rho) \le b \\ |t| \le \delta}} f'_\delta(\widehat{Y}(\rho) + t) K(t) dt \\ &\le E \int_{-\infty}^{\infty} f'_\delta(\widehat{Y}(\rho) + t) K(t) dt \\ &= E\widehat{Y}(\rho) f_\delta(\widehat{Y}(\rho)) - \Delta f_\delta(\widehat{Y}(\rho)), \end{aligned} \tag{3.24}$$

where we apply f_δ with (3.16) in the last equality. Hence,

$$B_1 \le B_{1,1} + B_{1,2}, \tag{3.25}$$

where

$$B_{1,1} = E|\widehat{Y}(\rho)||f_{|\widetilde{Y}(\rho) - \widehat{Y}(\rho)|}(\widehat{Y}(\rho))| \ \text{ and}$$
$$B_{1,2} = |\Delta f_{|\widetilde{Y}(\rho) - \widehat{Y}(\rho)|}(\widehat{Y}(\rho))|.$$

We note that $E|\Delta\widehat{Y}|^k$ is bounded by a sum of $(4d)^k$ terms each of the form $E|\widehat{Y}_0(I, \rho_1(I), \ldots, \rho_{d-1}(I))|^k$. This implies

$$E|\Delta\widehat{Y}|^k \le \frac{(4d)^k \delta_k}{n^{\frac{k-1}{2}}} \tag{3.26}$$

for $k \in \{2,3\}$. Then Lemma 3.3, 3.5 and (3.26) yield

$$\begin{aligned} E|\widehat{Y}(\rho)||\Delta\widehat{Y}| &\le \{E|\widehat{Y}(\rho)|^2 E|\Delta\widehat{Y}|^2\}^{\frac{1}{2}} \\ &\le \{\frac{(1.05998)(1.02943)(4d)^2}{n}\}^{\frac{1}{2}} \\ &= \frac{4.17838d}{\sqrt{n}} \end{aligned} \tag{3.27}$$

By Lemma 3.5 and 3.7, we get that

$$\begin{aligned} E|\widehat{Y}(\rho)||\widetilde{Y}(\rho)-\widehat{Y}(\rho)| &\le \{E|\widehat{Y}(\rho)|^2 E|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^2\}^{\frac{1}{2}} \\ &\le \sqrt{1.05998}(\frac{4.27125}{n-1}+O(\frac{1}{n^2}))^{\frac{1}{2}} \\ &\le \frac{3.23695}{\sqrt{n}}+O(\frac{1}{n}). \end{aligned} \tag{3.28}$$

By (3.23), (3.27) and (3.28), we have

$$\begin{aligned} B_{1,1} &\le E|\widehat{Y}(\rho)|(\frac{1}{2}|\Delta\widehat{Y}|+\frac{3}{2}|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|) \\ &\le \frac{2.08919d}{\sqrt{n}}+\frac{3.23695}{\sqrt{n}}+O(\frac{1}{n}). \end{aligned} \tag{3.29}$$

Note also that $|\Delta\widehat{Y}|^k \le (4d)^k$ and $|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^k \le 4^k$. Similar to T_4, we obtain

$$\begin{aligned} B_{1,2} &= |\Delta f_{|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|}(\widehat{Y}(\rho))| \\ &= \frac{1}{n}|Ef_{|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|}(\widehat{Y}(\rho))\sum_{i=1}^{n}\sum_{k=1}^{n}\widehat{Y}_0(i,\rho_1(k),\ldots,\rho_{d-1}(k))| \\ &\le \frac{1}{n}\{Ef^2_{|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|}(\widehat{Y}(\rho))\}^{\frac{1}{2}}\{E[\sum_{i=1}^{n}\sum_{k=1}^{n}Y(i,\rho_1(k),\ldots,\rho_{d-1}(k))]^2\}^{\frac{1}{2}} \\ &\quad + \frac{1}{n}E[\frac{1}{2}|\Delta\widehat{Y}|+\frac{3}{2}|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|]|\sum_{i=1}^{n}\sum_{k=1}^{n}Y_0(i,\rho_1(k),\ldots,\rho_{d-1}(k))| \\ &\le \frac{\sqrt{1.02943n}}{n}(Ef^2_{|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|}(\widehat{Y}(\rho)))^{\frac{1}{2}} \\ &\quad + \frac{1}{n}E[\frac{4d}{2}+\frac{3\cdot 4}{2}]|\sum_{i=1}^{n}\sum_{k=1}^{n}Y_0(i,\rho_1(k),\ldots,\rho_{d-1}(k))| \\ &\le \frac{1.01461}{\sqrt{n}}(Ef^2_{|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|}(\widehat{Y}(\rho)))^{\frac{1}{2}}+(\frac{2d+6}{n})(\frac{1}{n^{d-2}})\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}E|Y(i_1,\ldots,i_d)|^3 \\ &= \frac{1.01461}{\sqrt{n}}(Ef^2_{|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|}(\widehat{Y}(\rho)))^{\frac{1}{2}}+(2d+6)\delta_3. \end{aligned} \tag{3.30}$$

Moreover, Lemma 3.7, (3.23) and (3.26) yield

$$\begin{aligned}\{Ef^2_{|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|}(\widehat{Y}(\rho))\}^{\frac{1}{2}} &\le \{E(\frac{1}{2}|\Delta\widehat{Y}|+\frac{3}{2}|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|)^2\}^{\frac{1}{2}}\\
&\le \{2E[\frac{1}{2}|\Delta\widehat{Y}|]^2+2E[\frac{3}{2}|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|]^2\}^{\frac{1}{2}}\\
&\le \frac{1}{\sqrt{2}}\{E|\Delta\widehat{Y}|^2+9E|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^2\}^{\frac{1}{2}}\\
&\le \frac{1}{\sqrt{2}}\{\frac{16d^2\delta_2}{\sqrt{n}}+\frac{9(4.27125)}{n-1}+O(\frac{1}{n^2})\}^{\frac{1}{2}}\\
&\le \frac{\sqrt{16(1.02943)d^2}}{\sqrt{2}\sqrt{n}}+\frac{\sqrt{9(4.27125)}}{\sqrt{2}\sqrt{n}}\sqrt{1+\frac{1}{35}}+O(\frac{1}{n})\\
&\le \frac{2.86975d}{\sqrt{n}}+\frac{4.44633}{\sqrt{n}}+O(\frac{1}{n}). \qquad (3.31)\end{aligned}$$

Thus, (3.30) and (3.31) imply

$$B_{1,2} \le (2d+6)\delta_3+O(\frac{1}{n}). \qquad (3.32)$$

It follows from (3.29) and (3.32) that

$$B_1 \le \frac{2.08919d}{\sqrt{n}}+\frac{3.23695}{\sqrt{n}}+(2d+6)\delta_3+O(\frac{1}{n}). \qquad (3.33)$$

Next consider B_2,

$$\begin{aligned}B_2 &= E\int_{\Delta\widehat{Y}+t\le 0}(|\widehat{Y}(\tau)|+\frac{\sqrt{2\pi}}{4})(|\Delta\widehat{Y}|+|t|)K(t)dt\\
&\le E\int_{\mathbb{R}}|\widehat{Y}(\tau)||\Delta\widehat{Y}|K(t)dt+E\int_{\mathbb{R}}|\widehat{Y}(\tau)||t|K(t)dt+\frac{\sqrt{2\pi}}{4}E\int_{\mathbb{R}}|\Delta\widehat{Y}|K(t)dt\\
&\quad+\frac{\sqrt{2\pi}}{4}E\int_{\mathbb{R}}|t|K(t)dt\\
&= B_{2,1}+B_{2,2}+B_{2,3}+B_{2,4}, \qquad (3.34)\end{aligned}$$

where

$$B_{2,1}=E\int_{\mathbb{R}}|\widehat{Y}(\tau)||\Delta\widehat{Y}|K(t)dt, \qquad B_{2,2}=E\int_{\mathbb{R}}|\widehat{Y}(\tau)||t|K(t)dt$$
$$B_{2,3}=\frac{\sqrt{2\pi}}{4}E\int_{\mathbb{R}}|\Delta\widehat{Y}|K(t)dt, \qquad B_{2,4}=\frac{\sqrt{2\pi}}{4}E\int_{\mathbb{R}}|t|K(t)dt.$$

We start to bound $B_{2,1}$ by writing this equation

$$B_{2,1}=\frac{n-1}{4}E|\widehat{Y}(\tau)||\Delta\widehat{Y}||\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^2\mathbb{I}(A)+\frac{n-1}{4}E|\widehat{Y}(\tau)||\Delta\widehat{Y}||\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^2\mathbb{I}(A^c),$$

where A is defined in the proof of T_2. Moreover, $|\widehat{Y}(\tau)|^{\frac{3}{2}}$ and $|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^3\mathbb{I}(A)$ are identically independent given $\tau_1,\ldots,\tau_{d-1}$ (see Lemma 3.6 in [31] for more detail). By theorem 2.7 and proposition 2.12, we have

$$\begin{aligned}E|\widehat{Y}(\tau)|^{\frac{3}{2}}|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^3\mathbb{I}(A) &= E[(E^{\tau_1,\ldots,\tau_{d-1}}|\widehat{Y}(\tau)|^{\frac{3}{2}})(E^{\tau_1,\ldots,\tau_{d-1}}|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^3\mathbb{I}(A))]\\ &= E[(E^{\tau_1,\ldots,\tau_{d-1}}|\widehat{Y}(\tau)|^{\frac{3}{2}})(E|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^3\mathbb{I}(A))]\\ &= [E|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^3\mathbb{I}(A)][EE^{\tau_1,\ldots,\tau_{d-1}}|\widehat{Y}(\tau)|^{\frac{3}{2}}]\\ &= E|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^3\mathbb{I}(A)E|\widehat{Y}(\tau)|^{\frac{3}{2}}. \qquad (3.35)\end{aligned}$$

Note that

$$E|\widehat{S}_{1,0}|^3 = E|\widehat{Y}_0(I,\rho_1(I),\ldots,\rho_{d-1}(I))|^3 \le \frac{1}{n^d}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}E|Y(i_1,\ldots,i_d)|^3 = \frac{\delta_3}{n}$$

which implies

$$E|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^3 \le 4^2\sum_{i=1}^{4}E|\widehat{S}_{1,0}|^3 \le \frac{64\delta_3}{n}. \qquad (3.36)$$

Note also that

$$\begin{aligned}&\frac{n-1}{4}E|\widehat{Y}(\tau)||\Delta\widehat{Y}||\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^2\mathbb{I}(A)\\ &\quad\le \frac{n-1}{4}\{E|\Delta\widehat{Y}|^3\}^{\frac{1}{3}}\{E|\widehat{Y}(\tau)|^{\frac{3}{2}}|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^3\mathbb{I}(A)\}^{\frac{2}{3}}\\ &\quad= \frac{n-1}{4}\{E|\Delta\widehat{Y}|^3\}^{\frac{1}{3}}\{E|\widehat{Y}(\tau)|^{\frac{3}{2}}E|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^3\mathbb{I}(A)\}^{\frac{2}{3}}\\ &\quad\le \frac{n-1}{4}\{E|\Delta\widehat{Y}|^3\}^{\frac{1}{3}}\{[E|\widehat{Y}(\tau)|^2]^{\frac{3}{4}}E|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^3\mathbb{I}(A)\}^{\frac{2}{3}}\\ &\quad\le \frac{n-1}{4}\{E|\Delta\widehat{Y}|^3\}^{\frac{1}{3}}\{E|\widehat{Y}(\rho)|^2\}^{\frac{1}{2}}\{E|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^3\}^{\frac{2}{3}} \qquad (3.37)\end{aligned}$$

and

$$\begin{aligned}&\frac{n-1}{4}E|\widehat{Y}(\tau)||\Delta\widehat{Y}||\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^2\mathbb{I}(A^c)\\ &\quad\le \frac{n-1}{4}\{E|\widehat{Y}(\tau)|^3|\Delta\widehat{Y}|^3\}^{\frac{1}{3}}\{E|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^3\mathbb{I}(A^c)\}^{\frac{2}{3}}\\ &\quad\le \frac{n-1}{4}[\{E|\widehat{Y}(\tau)|^4\}^{\frac{3}{4}}\{E|\Delta\widehat{Y}|^{12}\}^{\frac{1}{4}}]^{\frac{1}{3}}\{E|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^3E^{\mathcal{B}}\mathbb{I}(A^c)\}^{\frac{2}{3}}. \qquad (3.38)\end{aligned}$$

By Lemma 3.5, 3.6, (3.19), (3.26) and (3.35) to (3.38), we have

$$
\begin{aligned}
B_{2,1} \le\ & \frac{d(n-1)\delta_3^{\frac{1}{3}}}{n^{\frac{1}{3}}} \frac{(1.05998)^{\frac{1}{2}} 4^2 \delta_3^{\frac{2}{3}}}{n^{\frac{2}{3}}} \\
& + \frac{n-1}{4} \{E|\widehat{Y}(\tau)|^4\}^{\frac{1}{4}} \{E|\Delta\widehat{Y}|^{12}\}^{\frac{1}{12}} \{\frac{4}{n} \sum_{r=1}^{d-1} \binom{d-1}{r} E|\widetilde{Y}(\rho) - \widehat{Y}(\rho)|^3\}^{\frac{2}{3}} \\
\le\ & \frac{16.95968 d(n-1)\delta_3}{n} \\
& + \frac{(4d)^{\frac{9}{12}}(n-1)}{4} \{E|\widehat{Y}(\tau)|^4\}^{\frac{1}{4}} \{E|\Delta\widehat{Y}|^3\}^{\frac{1}{12}} \{\frac{4(2^{d-1}-1)}{n} E|\widetilde{Y}(\rho) - \widehat{Y}(\rho)|^3\}^{\frac{2}{3}} \\
\le\ & 16.95968 d\delta_3 \\
& + \frac{(4d)^{\frac{9}{12}}(n-1)}{4} [C(n + n^2\delta_3^2)]^{\frac{1}{4}} \frac{(4d)^{\frac{3}{12}} \delta_3^{\frac{1}{12}}}{n^{\frac{1}{12}}} (\frac{4}{n})^{\frac{2}{3}} (2^{d-1}-1)^{\frac{2}{3}} \frac{(4^3)^{\frac{2}{3}} \delta_3^{\frac{2}{3}}}{n^{\frac{2}{3}}} \\
\le\ & 16.95968 d\delta_3 + \frac{C_d \delta_3^{\frac{3}{4}}}{n^{\frac{5}{12}}} [C(n + n^2\delta_3^2)]^{\frac{1}{4}} \\
\le\ & 16.95968 d\delta_3 + \frac{C_d}{n^{\frac{1}{24}}} [\frac{C\delta_3^3}{\sqrt{n}} + C\sqrt{n}\delta_3^5]^{\frac{1}{4}} \\
\le\ & 16.95968 d\delta_3 + \frac{C_d}{n^{\frac{1}{24}}} [\frac{C\delta_3^{\frac{3}{4}}}{n^{\frac{1}{8}}} + C n^{\frac{1}{8}} \delta_3^{\frac{5}{4}}]. \qquad (3.39)
\end{aligned}
$$

It is easy to see that

$$
B_{2,2} = \frac{n-1}{8} E|\widehat{Y}(\tau)||\widetilde{Y}(\rho) - \widehat{Y}(\rho)|^3 \mathbb{I}(A) + \frac{n-1}{8} E|\widehat{Y}(\tau)||\widetilde{Y}(\rho) - \widehat{Y}(\rho)|^3 \mathbb{I}(A^c).
$$

Similar to $B_{2,1}$, we have

$$
\begin{aligned}
\frac{n-1}{8} E|\widehat{Y}(\tau)||\widetilde{Y}(\rho) - \widehat{Y}(\rho)|^3 \mathbb{I}(A) &= \frac{n-1}{8} E|\widehat{Y}(\tau)| E|\widetilde{Y}(\rho) - \widehat{Y}(\rho)|^3 \mathbb{I}(A) \\
&\le \frac{n-1}{8} \{E|\widehat{Y}(\tau)|^2\}^{\frac{1}{2}} E|\widetilde{Y}(\rho) - \widehat{Y}(\rho)|^3
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{n-1}{8} E|\widehat{Y}(\tau)||\widetilde{Y}(\rho) - \widehat{Y}(\rho)|^3 \mathbb{I}(A^c) \\
&\quad = \frac{n-1}{8} E|\widehat{Y}(\tau)||\widetilde{Y}(\rho) - \widehat{Y}(\rho)||\widetilde{Y}(\rho) - \widehat{Y}(\rho)|^2 \mathbb{I}(A^c) \\
&\quad \le \frac{n-1}{8} \{E|\widehat{Y}(\tau)|^3 |\widetilde{Y}(\rho) - \widehat{Y}(\rho)|^3\}^{\frac{1}{3}} \{E|\widetilde{Y}(\rho) - \widehat{Y}(\rho)|^3 \mathbb{I}(A^c)\}^{\frac{2}{3}} \\
&\quad \le \frac{n-1}{8} [\{E|\widehat{Y}(\tau)|^4\}^{\frac{3}{4}} \{E|\widetilde{Y}(\rho) - \widehat{Y}(\rho)|^{12}\}^{\frac{1}{4}}]^{\frac{1}{3}} \{E|\widetilde{Y}(\rho) - \widehat{Y}(\rho)|^3 E^{\mathcal{B}} \mathbb{I}(A^c)\}^{\frac{2}{3}}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&B_{2,2}\\
&\le \frac{4^3\delta_3\sqrt{1.05998}(n-1)}{8n}\\
&\quad + \frac{n-1}{8}\{E|\widehat{Y}(\tau)|^4\}^{\frac{1}{4}}\{E|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^{12}\}^{\frac{1}{12}}\{\frac{4}{n}\sum_{r=1}^{d-1}\binom{d-1}{r}E|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^3\}^{\frac{2}{3}}\\
&= \frac{8.23643(n-1)\delta_3}{n}\\
&\quad + \frac{(n-1)(4^9)^{\frac{1}{12}}}{8}\{E|\widehat{Y}(\tau)|^4\}^{\frac{1}{4}}\{E|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^3\}^{\frac{1}{12}}\{\frac{4(2^{d-1}-1)}{n}E|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^3\}^{\frac{2}{3}}\\
&\le 8.23643\delta_3 + \frac{4^{\frac{3}{4}}(n-1)}{8}[C(n+n^2\delta_3^2)]^{\frac{1}{4}}[\frac{4}{n}(2^{d-1}-1)]^{\frac{2}{3}}\{E|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^3\}^{\frac{3}{4}}\\
&\le 8.23643\delta_3 + \frac{4^{\frac{3}{4}}(n-1)}{8}[C(n+n^2\delta_3^2)]^{\frac{1}{4}}[\frac{4}{n}(2^{d-1}-1)]^{\frac{2}{3}}\frac{(4^3)^{\frac{3}{4}}\delta_3^{\frac{3}{4}}}{n^{\frac{3}{4}}}\\
&\le 8.23643\delta_3 + \frac{C_d\delta_3^{\frac{3}{4}}}{n^{\frac{5}{12}}}[C(n+n^2\delta_3^2)]^{\frac{1}{4}}\\
&\le 8.23643\delta_3 + \frac{C_d}{n^{\frac{1}{24}}}[\frac{C\delta_3^3}{\sqrt{n}}+C\sqrt{n}\delta_3^5]^{\frac{1}{4}}\\
&\le 8.23643\delta_3 + \frac{C_d}{n^{\frac{1}{24}}}[\frac{C\delta_3^{\frac{3}{4}}}{n^{\frac{1}{8}}}+Cn^{\frac{1}{8}}\delta_3^{\frac{5}{4}}]. \qquad (3.40)
\end{aligned}
$$

Next, we will bound $B_{2,3}$. Then

$$
\begin{aligned}
B_{2,3} &= \frac{\sqrt{2\pi}(n-1)}{16}E|\Delta\widehat{Y}||\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^2\\
&\le \frac{\sqrt{2\pi}(n-1)}{16}\{E|\Delta\widehat{Y}|^3\}^{\frac{1}{3}}\{E|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^3\}^{\frac{2}{3}}\\
&= \frac{\sqrt{2\pi}(n-1)}{16}(\frac{(4d)^3\delta_3}{n})^{\frac{1}{3}}(\frac{4^3\delta_3}{n})^{\frac{2}{3}}\\
&= \frac{4^3d\sqrt{2\pi}(n-1)\delta_3}{16n}\\
&= 10.02652d\delta_3. \qquad (3.41)
\end{aligned}
$$

To finish this proof, we need to bound $B_{2,4}$. Then

$$B_{2,4} = \frac{\sqrt{2\pi}(n-1)}{32}E|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^3 = (\frac{n-1}{8})(\frac{\sqrt{2\pi}}{4})(\frac{4^3\delta_3}{n}) \le 5.01326\delta_3. \qquad (3.42)$$

By (3.34) and (3.39) to (3.42), we have

$$B_2 \le 13.24969\delta_3 + 26.9862d\delta_3 + (\frac{C_d}{n^{\frac{1}{24}}})[\frac{C\delta_3^{\frac{3}{4}}}{n^{\frac{1}{8}}}+Cn^{\frac{1}{8}}\delta_3^{\frac{5}{4}}]. \qquad (3.43)$$

By (3.33) and (3.43),we get that

$$T_1 \le \frac{3.23695}{\sqrt{n}} + \frac{2.08919d}{\sqrt{n}} + 19.24969\delta_3 + 28.9862d\delta_3$$
$$+ (\frac{C_d}{n^{\frac{1}{24}}})[\frac{C\delta_3^{\frac{3}{4}}}{n^{\frac{1}{8}}} + Cn^{\frac{1}{8}}\delta_3^{\frac{5}{4}}] + O(\frac{1}{n}). \quad (3.44)$$

By using (2.4) instead of (2.3) and applying the same argument of (3.44), we have

$$T_1 \ge \frac{-3.23695}{\sqrt{n}} - \frac{2.08919d}{\sqrt{n}} - 19.24969\delta_3 - 28.9862d\delta_3$$
$$- (\frac{C_d}{n^{\frac{1}{24}}})[\frac{C\delta_3^{\frac{3}{4}}}{n^{\frac{1}{8}}} + Cn^{\frac{1}{8}}\delta_3^{\frac{5}{4}}] - O(\frac{1}{n}). \quad (3.45)$$

Hence

$$|T_1| \le \frac{3.23695}{\sqrt{n}} + \frac{2.08919d}{\sqrt{n}} + 19.24969\delta_3 + 28.9862d\delta_3$$
$$+ (\frac{C_d}{n^{\frac{1}{24}}})[\frac{C\delta_3^{\frac{3}{4}}}{n^{\frac{1}{8}}} + Cn^{\frac{1}{8}}\delta_3^{\frac{5}{4}}] + O(\frac{1}{n}). \quad (3.46)$$

Therefore, (3.17), (3.18), (3.20) and (3.46) yield

$$|P(W \le z) - \Phi(z)|$$
$$\le 22.87635\delta_3 + \frac{3.87277}{\sqrt{n}} + 28.9862d\delta_3 + \frac{2.08919d}{\sqrt{n}} + 1.03\delta_3^2$$
$$+ (\frac{C_d}{n^{\frac{1}{24}}})[\frac{C\delta_3^{\frac{3}{4}}}{n^{\frac{1}{8}}} + Cn^{\frac{1}{8}}\delta_3^{\frac{5}{4}}] + O(\frac{1}{n}).$$

CHAPTER IV

AN IMPROVEMENT OF A UNIFORM BOUND ON A COMBINATORIAL CENTRAL LIMIT THEOREM

In this chapter, we concern with the random variable

$$W = \sum_{i=1}^{n} Y(i, \pi(i)),$$

where $Y(i,j)$'s and π are all independent. A theorem for the asymptotic normality of W is called a combinatorial central limit theorem. By assuming the finiteness of absolute third moment, Neammanee and Suntornchost [23] applied a concentration inequality of Stein's method to obtain the uniform rate of convergence $\frac{198}{\sqrt{n}}$. Our work improve the result of Neammanee and Rattanawong [21]. We give the constant of approximation to be 78.9 which is shaper than the result in [23].

Throughout this chapter, we assume that $VarW = 1$,

$$E|Y(i_1, i_2)|^3 < \infty, \ 1 \le i_1, i_2 \le n,$$

$$\sum_{i=1}^{n} EY(i, j_0) = 0 \text{ for a fixed } j_0 \text{ and } \sum_{j=1}^{n} EY(i_0, j) = 0 \text{ for a fixed } i_0. \tag{4.1}$$

The main theorem of this chapter is Theorem 4.1.

Theorem 4.1. *For $n \ge 36$,*

$$\begin{aligned}\sup_{z\in\mathbb{R}} &|P(W \le z) - \Phi(z)| \\ &\le 70.85\delta_3 + \frac{8.06 + 10.81\delta_3}{\sqrt{n}} + 1.03\delta_3^2 + \left(\frac{C}{n^{\frac{1}{24}}}\right)\left(\frac{C\delta_3^{\frac{3}{4}}}{n^{\frac{1}{8}}} + Cn^{\frac{1}{8}}\delta_3^{\frac{5}{4}}\right) + O\left(\frac{1}{n}\right)\end{aligned}$$

where $\delta_3 = \frac{1}{n}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n} E|Y(i_1, i_2)|^3$. *Futhermore, if* $\delta_3 \sim \frac{1}{\sqrt{n}}$, *then*

$$\sup_{z\in\mathbb{R}} |P(W \le z) - \Phi(z)| \le \frac{78.9}{\sqrt{n}} + O\left(\frac{1}{n^{\frac{13}{24}}}\right).$$

Observe that this bound is sharper than the result of Corollary 3.2.

4.1 Auxiliary Results

If we choose $d = 2$ in (3.3)-(3.5), we have

$$Y_0(i_1,i_2) = Y(i_1,i_2)\mathbb{I}(|Y(i_1,i_2)| > 1),$$

$$\widehat{Y}_0(i_1,i_2) = Y(i_1,i_2)\mathbb{I}(|Y(i_1,i_2)| \le 1),$$

and $$\widehat{Y}(\pi) = \sum_{i=1}^{n} \widehat{Y}_0(i,\pi(i))$$

The following lemma improves the result in Lemma 2.1 of Neammanee and Rattanawong [21].

Lemma 4.2. *For $n \ge 36$, $E[\sum_{i=1}^{n}\sum_{k=1}^{n}\widehat{Y}_0(i,k)]^2 \le n + n^2\delta_3^2$.*

Proof. Note that

$$E[\sum_{i=1}^{n}\sum_{k=1}^{n}\widehat{Y}_0(i,k)]^2 = \sum_{i=1}^{n}\sum_{k=1}^{n} E\widehat{Y}_0^2(i,k) + \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{\substack{l=1 \\ (l,m)\ne(i,k)}}^{n}\sum_{m=1}^{n} E\widehat{Y}_0(i,k)\widehat{Y}_0(l,m).$$

From (4.1) and the fact that $\widehat{Y}_0 = Y - Y_0$, we have

$$\begin{aligned}
&\sum_{i=0}^{n}\sum_{k=0}^{n}\sum_{\substack{l=0 \\ (l,m)\ne(i,k)}}^{n}\sum_{m=0}^{n} E\widehat{Y}_0(i,k)\widehat{Y}_0(l,m) \\
&= \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{\substack{l_1=1 \\ l_1\ne i_1}}^{n}\sum_{\substack{l_2=1 \\ l_2\ne i_2}}^{n} E\widehat{Y}_0(i_1,i_2)E\widehat{Y}_0(l_1,l_2) \\
&\quad + \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{\substack{l_1=1 \\ l\ne i_1}}^{n} E\widehat{Y}_0(i_1,i_2)E\widehat{Y}_0(l_1,i_2) + \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{\substack{l_2=1 \\ l_2\ne i_2}}^{n} E\widehat{Y}_0(i_1,i_2)E\widehat{Y}_0(i_1,l_2) \\
&= \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{l_1=1}^{n}\sum_{l_2=1}^{n} E\widehat{Y}_0(i_1,i_2)E\widehat{Y}_0(l_1,l_2) \\
&\quad + \sum_{i_1=1}^{n}\sum_{i_2=1}^{n} E\widehat{Y}_0(i_1,i_2)E\widehat{Y}_0(i_1,i_2) - \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{l_2=1}^{n} E\widehat{Y}_0(i_1,i_2)E\widehat{Y}_0(i_1,l_2) \\
&\quad - \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{l_1=1}^{n} E\widehat{Y}_0(i_1,i_2)E\widehat{Y}_0(l_1,i_2) + \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{\substack{l_1=1 \\ l_1\ne i_1}}^{n} E\widehat{Y}_0(i_1,i_2)E\widehat{Y}_0(l_1,i_2)
\end{aligned}$$

$$+\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{\substack{l_2=1\\l_2\neq i_2}}^{n}E\widehat{Y}_0(i_1,i_2)E\widehat{Y}_0(i_1,l_2)$$

$$=\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{l_1=1}^{n}\sum_{l_2=1}^{n}E\widehat{Y}_0(i_1,i_2)E\widehat{Y}_0(l_1,l_2)$$

$$+\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}E\widehat{Y}_0(i_1,i_2)E\widehat{Y}_0(i_1,i_2)-\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{l_2=1}^{n}E\widehat{Y}_0(i_1,i_2)E\widehat{Y}_0(i_1,l_2)$$

$$-\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{l_1=1}^{n}E\widehat{Y}_0(i_1,i_2)E\widehat{Y}_0(l_1,i_2)+\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{l_1=1}^{n}E\widehat{Y}_0(i_1,i_2)E\widehat{Y}_0(l_1,i_2)$$

$$-\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}E\widehat{Y}_0(i_1,i_2)E\widehat{Y}_0(i_1,i_2)+\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{l_2=1}^{n}E\widehat{Y}_0(i_1,i_2)E\widehat{Y}_0(i_1,l_2)$$

$$-\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}E\widehat{Y}_0(i_1,i_2)E\widehat{Y}_0(i_1,i_2)$$

$$\leq(\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}E\widehat{Y}_0(i_1,i_2))^2$$

$$=(\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}EY_0(i_1,i_2))^2.$$

Then (3.6) implies that

$$\sum_{i=0}^{n}\sum_{k=0}^{n}\sum_{\substack{l=0\\(l,m)\neq(i,k)}}^{n}\sum_{m=0}^{n}E\widehat{Y}_0(i,k)\widehat{Y}_0(l,m)\leq(\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}E|Y(i_1,i_2)|^3)^2=(n\delta_3)^2. \tag{4.2}$$

Moreover, we use (4.1) to obtain this useful equation

$$\begin{aligned}1&=EW^2\\&=E(\sum_{i=1}^{n}Y(i,\pi(i)))^2\\&=\sum_{i=1}^{n}EY^2(i,\pi(i))+\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}EY(i,\pi(i))Y(j,\pi(j))\\&=\frac{1}{n}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}EY^2(i_1,i_2)+\frac{1}{n(n-1)}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{j_1=1\neq i_1}^{n}\sum_{\substack{j_2=1\\j_2\neq i_2}}^{n}EY(i_1,i_2)EY(j_1,j_2)\\&=\frac{1}{n}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}EY^2(i_1,i_2)+\frac{1}{n(n-1)}\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}[EY(i_1,i_2)]^2.\end{aligned}\tag{4.3}$$

From this fact and (4.2), we have the lemma. □

4.2 Proof of Theorem 4.1

We apply the proof of Theorem 3.1 by substituting d by 2. Then

$$\begin{aligned}|P(W \le z) - \Phi(z)| &\le P(W \ne \widehat{Y}(\pi)) + |P(\widehat{Y}(\pi) \le z) - \Phi(z)| \\ &\le \delta_3 + |P(\widehat{Y}(\pi) \le z) - \Phi(z)| \\ &\le \delta_3 + |T_1| + |T_2| + |T_3| + |T_4|\end{aligned}$$

where

$$|T_4| \le \frac{0.63582}{\sqrt{n}} + 0.62666\delta_3, \ (3.17), \tag{4.4}$$

$$|T_3| \le 2\delta_3 + 1.03\delta_3^2 + O(\frac{1}{n}), \ (3.18), \tag{4.5}$$

$$|T_2| \le O(\frac{1}{n}), \ (3.20), \tag{4.6}$$

$$T_1 \le B_{1,1} + B_{1,2} + B_2, \ ((3.21) \text{ and } (3.25)), \tag{4.7}$$

$$B_{1,1} \le \frac{7.41533}{\sqrt{n}} + O(\frac{1}{n}), \ (3.29), \tag{4.8}$$

and $$B_2 \le 67.22209\delta_3 + (\frac{C_d}{n^{\frac{1}{24}}})[\frac{C\delta_3^{\frac{3}{4}}}{n^{\frac{1}{8}}} + Cn^{\frac{1}{8}}\delta_3^{\frac{5}{4}}], \ (3.43). \tag{4.9}$$

Thus it suffices to bound $B_{1,2}$. By Lemma 4.2 and (3.31), we have

$$\begin{aligned}B_{1,2} &= |\Delta f_{|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|}(\widehat{Y}(\rho))| \\ &= |\frac{1}{n} E f_{|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|}(\widehat{Y}(\rho)) \sum_{i=1}^{n}\sum_{k=1}^{n} \widehat{Y}_0(i, \rho_1(k))| \\ &\le \frac{1}{n}\{E f^2_{|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|}(\widehat{Y}(\rho))\}^{\frac{1}{2}} \{E[\sum_{i=1}^{n}\sum_{k=1}^{n} \widehat{Y}_0(i, \rho_1(k))]^2\}^{\frac{1}{2}} \\ &\le \frac{1}{n}(n + n^2\delta_3^2)^{\frac{1}{2}}(\frac{2(2.86975)}{\sqrt{n}} + \frac{4.44633}{\sqrt{n}} + O(\frac{1}{n})) \\ &\le \frac{10.81\delta_3}{\sqrt{n}} + O(\frac{1}{n}).\end{aligned} \tag{4.10}$$

By (4.7)–(4.9) and (4.10), we get that

$$T_1 \le \frac{7.41533}{\sqrt{n}} + 67.22209\delta_3 + \frac{10.81\delta_3}{\sqrt{n}} + (\frac{C}{n^{\frac{1}{24}}})[\frac{C\delta_3^{\frac{3}{4}}}{n^{\frac{1}{8}}} + Cn^{\frac{1}{8}}\delta_3^{\frac{5}{4}}] + O(\frac{1}{n}). \tag{4.11}$$

By the same idea of (3.45), we get that

$$T_1 \ge \frac{-7.41533}{\sqrt{n}} - 67.22209\delta_3 - \frac{10.81\delta_3}{\sqrt{n}} - (\frac{C}{n^{\frac{1}{24}}})[\frac{C\delta_3^{\frac{3}{4}}}{n^{\frac{1}{8}}} + Cn^{\frac{1}{8}}\delta_3^{\frac{5}{4}}] - O(\frac{1}{n})$$

Hence

$$|T_1| \leq \frac{7.41533}{\sqrt{n}} + 67.22209\delta_3 + \frac{10.81\delta_3}{\sqrt{n}} + (\frac{C}{n^{\frac{1}{24}}})[\frac{C\delta_3^{\frac{3}{4}}}{n^{\frac{1}{8}}} + Cn^{\frac{1}{8}}\delta_3^{\frac{5}{4}}] + O(\frac{1}{n}). \quad (4.12)$$

Therefore, we conclude from (4.4)-(4.6) and (4.12) that

$$\begin{aligned}
&|P(W \leq z) - \Phi(z)| \\
&\quad \leq \delta_3 + \frac{7.41533}{\sqrt{n}} + 67.22209\delta_3 + \frac{10.81\delta_3}{\sqrt{n}} + (\frac{C}{n^{\frac{1}{24}}})[\frac{C\delta_3^{\frac{3}{4}}}{n^{\frac{1}{8}}} + Cn^{\frac{1}{8}}\delta_3^{\frac{5}{4}}] \\
&\qquad + O(\frac{1}{n}) + 2\delta_3 + 1.03\delta_3^2 + O(\frac{1}{n}) + \frac{0.63582}{\sqrt{n}} + 0.62666\delta_3 \\
&\quad = 70.84875\delta_3 + \frac{8.05115}{\sqrt{n}} + \frac{10.81\delta_3}{\sqrt{n}} + 1.03\delta_3^2 + (\frac{C}{n^{\frac{1}{24}}})(\frac{C\delta_3^{\frac{3}{4}}}{n^{\frac{1}{8}}} + Cn^{\frac{1}{8}}\delta_3^{\frac{5}{4}}) + O(\frac{1}{n}).
\end{aligned}$$

CHAPTER V
AN IMPROVEMENT OF A NON-UNIFORM BOUND ON NORMAL APPROXIMATION OF RANDOMIZED ORTHOGONAL ARRAY SAMPLING

This chapter is organized as follows. We introduce orthogonal arrays and orthogonal array sampling designs in section 5.1. Auxiliary results and the proof of main theorem are given in section 5.2 and section 5.3, respectively.

5.1 Orthogonal Arrays and Orthogonal Array Sampling Designs

An orthogonal array of strength t with index λ ($\lambda \geq 1$), is an $n \times d$ matrix with elements taken from the set $\{0, 1, ..., q-1\}$ such that for any $n \times t$ submatrix, each of the q^t possible rows appears the same number λ of times where d, n, q and t are positive integers with $t \leq d$ and $q \geq 2$. Of course $n = \lambda q^t$. The class of such arrays is denoted by $OA(n, d, q, t)$ (see Raghavarao [30] for more details).

In this work, we consider the class $OA(q^2, 3, q, 2)$. For example, let $q = 2$ and

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix},\ C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then A and B are orthogonal arrays in the class $OA(q^2, 3, q, 2)$ but C and D are not. Observe that if A is an orthogonal array in $OA(q^2, 3, q, 2)$, then every element in $\{0, 1, \ldots, q-1\}$ appear q times in each column of A. Let $A = [a_{ij}]$ be an orthogonal array in $OA(q^2, 3, q, 2)$ and π_1, π_2, π_3 be independent random permutations. In this work, we define $\rho_\pi : \{0, 1, ..., q-1\}^2 \to \{0, 1, ..., q-1\}$ to be a random function such that

$$\big(i_1, i_2, \rho_\pi(i_1, i_2)\big) = \big(\pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3})\big)$$

for some $i \in \{1, ..., q^2\}$. Some properties of ρ_π are given in the following examples.

Example 5.1. *If i_2, k_2, l_2 are all distinct, then*

$$P(\rho_\pi(i_1, i_2) = i_3, \rho_\pi(i_1, k_2) = k_3, \rho_\pi(i_1, l_2) = l_3) = \begin{cases} \dfrac{1}{q(q-1)(q-2)} & \textit{if } \ i_3, k_3, l_3 \ \textit{are all distinct}, \\ 0 & \textit{otherwise}. \end{cases} \tag{5.1}$$

Proof. From the independent of π_1, π_2, π_3, we have

$$\begin{aligned} &P(\rho_\pi(i_1, i_2) = i_3, \rho_\pi(i_1, k_2) = k_3, \rho_\pi(i_1, l_2) = l_3) \\ &= \sum_{n=1}^{q^2}\sum_{m=1}^{q^2}\sum_{s=1}^{q^2} P(\pi_1(a_{n1}) = i_1, \pi_2(a_{n2}) = i_2, \pi_3(a_{n3}) = i_3, \pi_1(a_{m1}) = i_1, \\ &\qquad \pi_2(a_{m2}) = k_2, \pi_3(a_{m3}) = k_3, \pi_1(a_{s1}) = i_1, \pi_2(a_{s2}) = l_2, \pi_3(a_{s3}) = l_3) \\ &= \sum_{n=1}^{q^2}\sum_{m=1}^{q^2}\sum_{s=1}^{q^2} P(\pi_1(a_{n1}) = i_1, \pi_1(a_{m1}) = i_1, \pi_1(a_{s1}) = i_1) P(\pi_2(a_{n2}) = i_2, \\ &\qquad \pi_2(a_{m2}) = k_2, \pi_2(a_{s2}) = l_2) P(\pi_3(a_{n3}) = i_3, \pi_3(a_{m3}) = k_3, \pi_3(a_{s3}) = l_3). \end{aligned} \tag{5.2}$$

Case 1 i_3, k_3, l_3 are all distinct. Observe that

$$P(\pi_1(a_{n1}) = i_1, \pi_1(a_{m1}) = i_1, \pi_1(a_{s1}) = i_1) = \begin{cases} \dfrac{1}{q} & \text{if } a_{n1} = a_{m1} = a_{s1}, \\ 0 & \text{otherwise}, \end{cases}$$

$$P(\pi_2(a_{n2}) = i_2, \pi_2(a_{m2}) = k_2, \pi_2(a_{s2}) = l_2) = \begin{cases} \dfrac{1}{q(q-1)(q-2)} & \text{if } a_{n2}, a_{m2}, a_{s2} \text{ are all distinct}, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$P(\pi_3(a_{n3}) = i_3, \pi_3(a_{m3}) = k_3, \pi_3(a_{s3}) = l_3)$$
$$= \begin{cases} \dfrac{1}{q(q-1)(q-2)} & \text{if } a_{n3}, a_{m3}, a_{s3} \text{ are all distinct,} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$P(\pi_1(a_{n1}) = i_1, \pi_1(a_{m1}) = i_1, \pi_1(a_{s1}) = i_1)P(\pi_2(a_{n2}) = i_2, \pi_2(a_{m2}) = k_2,$$
$$\pi_2(a_{s2}) = l_2)P(\pi_3(a_{n3}) = i_3, \pi_3(a_{m3}) = k_3, \pi_3(a_{s3}) = l_3)$$
$$= \begin{cases} \dfrac{1}{q^3(q-1)^2(q-2)^2} & \text{if } a_{n1} = a_{m1} = a_{s1}, \\ & \quad a_{n2}, a_{m2}, a_{s2} \text{ are all distinct,} \\ & \quad a_{n3}, a_{m3}, a_{s3} \text{ are all distinct,} \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

Note that if we fix $n \in \{1, 2, \ldots, q^2\}$, there are only $q-1$ of m's such that

$$a_{m1} = a_{n1},\ a_{m2} \neq a_{n2} \text{ and } a_{m3} \neq a_{n3}$$

and if we fix $n, m \in \{1, 2, \ldots, q^2\}$, then there are only $q-2$ of s's such that

$$a_{s1} = a_{m1} = a_{n1},\ a_{s2}, a_{m2}, a_{n2} \text{ are all distinct,}$$
$$\text{and } a_{s3}, a_{m3}, a_{n3} \text{ are all distinct.}$$

Thus there are $q^2(q-1)(q-2)$ terms such that $a_{n1} = a_{m1} = a_{s1}$ and a_{n2}, a_{m2}, a_{s2} are all distinct and a_{n3}, a_{m3}, a_{s3} are all distinct. From this fact, (5.2) and (5.3), we have

$$P(\rho_\pi(i_1, i_2) = i_3, \rho_\pi(i_1, k_2) = k_3, \rho_\pi(i_1, l_2) = l_3)$$
$$= \frac{q^2(q-1)(q-2)}{q^3(q-1)^2(q-2)^2}$$
$$= \frac{1}{q(q-1)(q-2)}.$$

Case 2 i_3, k_3, l_3 are not all distinct. WLOG, we let $i_3 = k_3 \neq l_3$. Suppose that

$$P(\rho_\pi(i_1, i_2) = i_3, \rho_\pi(i_1, k_2) = i_3, \rho_\pi(i_1, l_2) = l_3) \neq 0.$$

By (5.2), there exist $m, n, s \in \{1, 2, \ldots, q^2\}$ such that

$$P(\pi_1(a_{n1}) = i_1, \pi_2(a_{n2}) = i_2, \pi_3(a_{n3}) = i_3, \pi_1(a_{m1}) = i_1, \pi_2(a_{m2}) = k_2,$$
$$\pi_3(a_{m3}) = i_3, \pi_1(a_{s1}) = i_1, \pi_2(a_{s2}) = l_2, \pi_3(a_{s3}) = l_3) \neq 0.$$

Since π_1 and π_3 are random permutations, $\pi_1(a_{n1}) = \pi_1(a_{m1})$ and $\pi_3(a_{n3}) = \pi_3(a_{m3})$, we have $a_{n1} = a_{m1}$ and $a_{n3} = a_{m3}$. By the fact that A is an orthogonal array, we obtain $n = m$. Since π_2 is a random permutation and $\pi_2(a_{n2}) \neq \pi_2(a_{m2})$, we get that $a_{n2} \neq a_{m2}$ and hence $n \neq m$. This is a contradiction. Therefore, $P(\rho_\pi(i_1, i_2) = i_3, \rho_\pi(i_1, k_2) = i_3, \rho_\pi(i_1, l_2) = l_3) = 0.$ □

Example 5.2. *If i_2, j_2, k_2, l_2 are all distinct, then*

$$P(\rho_\pi(i_1, i_2) = i_3, \rho_\pi(i_1, j_2) = j_3, \rho_\pi(i_1, k_2) = k_3, \rho_\pi(i_1, l_2) = l_3) = \begin{cases} \dfrac{1}{q(q-1)(q-2)(q-3)} & \text{if } i_3, j_3, k_3, l_3 \text{ are all distinct,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Similar to (5.2), we get that

$$\begin{aligned} &P(\rho_\pi(i_1, i_2) = i_3, \rho_\pi(i_1, j_2) = j_3, \rho_\pi(i_1, k_2) = k_3, \rho_\pi(i_1, l_2) = l_3) \\ &= \sum_{n=1}^{q^2}\sum_{m=1}^{q^2}\sum_{s=1}^{q^2}\sum_{t=1}^{q^2} P(\pi_1(a_{n1}) = i_1, \pi_1(a_{m1}) = i_1, \pi_1(a_{s1}) = i_1, \pi_1(a_{t1}) = i_1) \\ &\quad P(\pi_2(a_{n2}) = i_2, \pi_2(a_{m2}) = j_2, \pi_2(a_{s2}) = k_2, \pi_2(a_{t2}) = j_2) P(\pi_3(a_{n3}) = i_3, \\ &\quad \pi_3(a_{m3}) = j_3, \pi_3(a_{s3}) = k_3, \pi_3(a_{t3}) = l_3). \end{aligned} \tag{5.4}$$

Case 1 i_3, j_3, k_3, l_3 are all distinct. Observe that

$$P(\pi_1(a_{n1}) = i_1, \pi_1(a_{m1}) = i_1, \pi_1(a_{s1}) = i_1, \pi_1(a_{t1}) = i_1) = \begin{cases} \dfrac{1}{q} & \text{if } a_{n1} = a_{m1} = a_{s1} = a_{t1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$P(\pi_2(a_{n2}) = i_2, \pi_2(a_{m2}) = j_2, \pi_2(a_{s2}) = k_2, \pi_2(a_{t2}) = l_2)$$
$$= \begin{cases} \dfrac{1}{q(q-1)(q-2)(q-3)} & \text{if } a_{n2}, a_{m2}, a_{s2}, a_{t2} \text{ are all distinct,} \\ 0 & \text{otherwise,} \end{cases}$$

$$P(\pi_3(a_{n3}) = i_3, \pi_3(a_{m3}) = j_3, \pi_3(a_{s3}) = k_3, \pi_3(a_{t3}) = l_3)$$
$$= \begin{cases} \dfrac{1}{q(q-1)(q-2)(q-3)} & \text{if } a_{n3}, a_{m3}, a_{s3}, a_{t3} \text{ are all distinct,} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$P(\pi_1(a_{n1}) = i_1, \pi_1(a_{m1}) = i_1, \pi_1(a_{s1}) = i_1, \pi_1(a_{t1}) = i_1)P(\pi_2(a_{n2}) = i_2,$$
$$\pi_2(a_{m2}) = j_2, \pi_2(a_{s2}) = k_2, \pi_2(a_{t2}) = l_2)P(\pi_3(a_{n3}) = i_3, \pi_3(a_{m3}) = j_3,$$
$$\pi_3(a_{s3}) = k_3, \pi_3(a_{t3}) = l_3)$$
$$= \begin{cases} \dfrac{1}{q^3(q-1)^2(q-2)^2(q-3)^2} & \text{if } a_{n1} = a_{m1} = a_{s1} = a_{t1}, \\ & a_{n2}, a_{m2}, a_{s2}, a_{t2} \text{ are all distinct,} \\ & a_{n3}, a_{m3}, a_{s3}, a_{t3} \text{ are all distinct,} \\ 0 & \text{otherwise.} \end{cases} \tag{5.5}$$

Note that if we fix $n \in \{1, 2, \ldots, q^2\}$, there are only $q-1$ of m's such that

$$a_{m1} = a_{n1}, \; a_{m2} \neq a_{n2} \text{ and } \; a_{m3} \neq a_{n3},$$

if we fix $n, m \in \{1, 2, \ldots, q^2\}$, then there are only $q-2$ of s's such that

$$a_{s1} = a_{m1} = a_{n1}, \; a_{s2}, a_{m2}, a_{n2} \text{ are all distinct,}$$
$$\text{and } a_{s3}, a_{m3}, a_{n3} \text{ are all distinct}$$

and if we fix $n, m, s \in \{1, 2, \ldots, q^2\}$, then there are only $q-3$ of t's such that

$$a_{t1} = a_{s1} = a_{m1} = a_{n1}, \; a_{t2}, a_{s2}, a_{m2}, a_{n2} \text{ are all distinct,}$$
$$\text{and } a_{t3}, a_{s3}, a_{m3}, a_{n3} \text{ are all distinct}$$

Thus there are $q^2(q-1)(q-2)(q-3)$ terms such that $a_{n1} = a_{m1} = a_{s1} = a_{t1}$ and $a_{n2}, a_{m2}, a_{s2}, a_{t2}$ are all distinct and $a_{n3}, a_{m3}, a_{s3}, a_{t3}$ are all distinct. From this fact, (5.4) and (5.5), we have

$$\begin{aligned} &P(\rho_\pi(i_1,i_2) = i_3, \rho_\pi(i_1,j_2) = j_3, \rho_\pi(i_1,k_2) = k_3, \rho_\pi(i_1,l_2) = l_3) \\ &\quad = \frac{q^2(q-1)(q-2)(q-3)}{q^3(q-1)^2(q-2)^2(q-3)^2} \\ &\quad = \frac{1}{q(q-1)(q-2)(q-3)}. \end{aligned}$$

Case 2 i_3, k_3, l_3 are not all distinct. We can prove this case by the same argument of Case 2 in Example 5.1. □

Example 5.3. *If $i_1 \neq k_1$ and $i_2 = k_2 \neq j_2 = l_2$, then*

$$\begin{aligned} &P(\rho_\pi(i_1,i_2) = i_3, \rho_\pi(i_1,j_2) = j_3, \rho_\pi(k_1,i_2) = k_3, \rho_\pi(k_1,j_2) = l_3) \\ &\quad = \begin{cases} \frac{1}{q} & \text{if } i_3 = l_3 \neq j_3 = k_3, \\ \frac{1}{q(q-1)} & \text{if } i_3 \neq j_3 = k_3 \neq l_3 \text{ and } i_3 \neq l_3, \\ \frac{1}{q(q-1)} & \text{if } i_3 = l_3 \neq j_3 \neq k_3 \text{ and } i_3 \neq k_3, \\ \frac{1}{q(q-1)^2} & \text{if } i_3, j_3, k_3, l_3 \text{ are all distinct}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. We can prove this Lemma by using the same idea of Example 5.1. □

For the class $OA(q^2, 3, q, 2)$, Loh [14] constructed the samplings $X_1, X_2, ..., X_{q^2}$ on the unit cube $[0,1]^3$ as follows: Let

(a) π_1, π_2, π_3 be random permutations of $\{0, 1, ..., q-1\}$, each uniformly distributed on all the $q!$ possible permutations;

(b) $U_{i_1,i_2,i_3,j}$ be $[0,1]$ uniform random variables where $i_1, i_2, i_3 \in \{0, 1, ..., q-1\}$, $j \in \{1, 2, 3\}$; and

(c) $U_{i_1,i_2,i_3,j}$'s and π_k's be all stochastically independent.

An orthogonal array-based sample of size q^2, $\{X_1, X_2, ..., X_{q^2}\}$, is defined to be the set

$$\{X(\pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3})) : 1 \leq i \leq q^2\},$$

where, for each $i_1, i_2, i_3 \in \{0, 1, ..., q-1\}$ and $j \in \{1, 2, 3\}$,

$$X\big(i_1, i_2, i_3\big) = \big(X_1(i_1, i_2, i_3), X_2(i_1, i_2, i_3), X_3(i_1, i_2, i_3)\big),$$
$$X_j(i_1, i_2, i_3) = (i_j + U_{i_1, i_2, i_3, j})/q,$$

and $a_{i,j}$ is the $(i, j)^{th}$ element of some arbitary but fixed $A \in OA(q^2, 3, q, 2)$.
We use

$$\hat{\mu} = \frac{1}{q^2} \sum_{i=1}^{q^2} f \circ X(\pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3}))$$

as an estimator of $\mu = E(f \circ X)$ where X is a random vector having a uniform distribution on a unit hypercube $[0, 1]^3$ and f is a measureable function from $[0, 1]^3$ to $\mathbb{R}$. Assume that $Var(\hat{\mu}) > 0$, and define

$$W = \frac{\hat{\mu} - \mu}{\sqrt{Var(\hat{\mu})}}. \tag{5.6}$$

Note that $EW = 0$ and $VarW = EW^2 = 1$. In 1996, Loh ([14], pp.1213) rewrite W in (5.6) as

$$W = \sum_{i_1=0}^{q-1} \sum_{i_2=0}^{q-1} Y\big(i_1, i_2, \rho_\pi(i_1, i_2)\big), \tag{5.7}$$

where

$$\mu(i_1, i_2, i_3) = Ef \circ X(i_1, i_2, i_3),$$
$$\mu_j(i_j) = \frac{1}{q^2} \sum_{\substack{i_k=0 \\ k \neq j}}^{q-1} [\mu(i_1, i_2, i_3) - \mu],$$
$$\mu_{k,l}(i_k, i_l) = \frac{1}{q} \sum_{\substack{i_j=0 \\ j \neq k,l}}^{q-1} [\mu(i_1, i_2, i_3) - \mu - \mu_k(i_k) - \mu_l(i_l)],$$
$$Y(i_1, i_2, i_3) = \frac{1}{q^2 \sigma_{oas}} \Big[f \circ X(i_1, i_2, i_3) - \mu - \sum_{j=1}^{3} \mu_j(i_j) - \sum_{1 \leq k < l \leq 3} \mu_{k,l}(i_k, i_l)\Big],$$
$$\tilde{\mu}(i_1, i_2, i_3) = EY(i_1, i_2, i_3).$$

Moreover, W in (5.7) satisfies the following property:

$$\sum_{i_j=0}^{q-1} \tilde{\mu}(i_1, i_2, i_3) = 0 \quad \text{for each} \quad j \in \{1, 2, 3\}, ([14], \text{ pp.1212}), \tag{5.8}$$

In this chapter, we use (5.7) to obtain Theorem 5.4.

Theorem 5.4. *Assume that* $E(f\circ X)^6<\infty$. *Then for every* $z\in\mathbb{R}$

$$(1+|z|)|P(W\le z)-\Phi(z)|=O(\frac{1}{\sqrt{q}})\quad as\quad q\to\infty.$$

5.2 Auxiliary Results

Note that W in (5.7) has the following useful properties:

$$\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}EY^r(i,j,k)=O(q^{3-r})\text{ for }r=2,4,6,([14],\ \text{pp.1217}),\tag{5.9}$$

$$E\{\sum_{i_1=0}^{q-1}(W^{(i_1)})^2\}^2=1+O(\frac{1}{q}),([14],\ \text{pp.1219}),\tag{5.10}$$

where

$$W^{(i_1)}=\sum_{i_2=1}^{n}Y(i_1,i_2,\rho_\pi(i_1,i_2)).$$

Let I and K be uniformly distributed random variables on $\{0,1,...,q-1\}$, (I,K) uniformly distributed on $\big\{(i,k)|i,k=0,1,...,q-1,i\neq k\big\}$ and assume that they are independent of all π_1,π_2,π_3 and $U_{i_1,i_2,i_3,j}$'s defined priviously.

Define

$$S_1=\sum_{i_2=0}^{q-1}Y\big(I,i_2,\rho_\pi(I,i_2)\big),\qquad S_2=\sum_{i_2=0}^{q-1}Y\big(K,i_2,\rho_\pi(K,i_2)\big),$$

$$S_3=\sum_{i_2=0}^{q-1}Y\big(I,i_2,\rho_\pi(K,i_2)\big),\qquad S_4=\sum_{i_2=0}^{q-1}Y\big(K,i_2,\rho_\pi(I,i_2)\big).$$

The following properties of S_i's are used in our work (see [14], pp.1219).

$$S_1,S_2,S_3,S_4\ \text{ are identically distributed,}\tag{5.11}$$

$$ES_i^4=O(\frac{1}{q^2})\text{ for }i\in\{1,2,3,4\}.\tag{5.12}$$

For the rest of this chapter, we use the following notations; for $n\in\mathbb{N}$ and $i_1,\dots,i_n,j_1,\dots,j_n\in\{0,\dots,q-1\}$,

$$\sum_{i_1} = \sum_{i_1=0}^{q-1}, \quad \sum_{\substack{i_1 \\ i_1 \neq j_1}} = \sum_{\substack{i_1=0 \\ i_1 \neq j_1}}^{q-1}, \quad \sum_{i_1,\ldots,i_n} = \sum_{i_1=0}^{q-1} \cdots \sum_{i_n=0}^{q-1}, \quad \text{and}$$

$$\sum_{\substack{i_1,\ldots,i_n \\ (i_1,\ldots,i_n) \neq (j_1,\ldots,j_n)}} = \sum_{\substack{i_1=0 \\ (i_1,\ldots,i_n) \neq (j_1,\ldots,j_n)}}^{q-1} \cdots \sum_{i_n=0}^{q-1} .$$

In Lemma 5.5–5.9, we prove some properties of S_i's which will be used in the proof of theorem 5.4. Let $\mathcal{B}$ be the σ-algebra generated by π_1, π_2, π_3 and $U_{i_1,i_2,i_3,j}$'s, $j = 1, 2, 3, 4$.

Lemma 5.5. *If $E(f \circ X)^4 < \infty$, then $E(E^{\mathcal{B}} S_1 S_2)^2 = E(E^{\mathcal{B}} S_3 S_4)^2 = O(\frac{1}{q^4})$.*

Proof. By theorem 2.7(3) and 2.9(1), we obtain

$$\begin{aligned}
&E^{\mathcal{B}}[\sum_{i_2} Y(I, i_2, \rho_\pi(I, i_2))][\sum_{j_2} Y(K, j_2, \rho_\pi(K, j_2))] \\
&= \sum_{i} \sum_{\substack{k \\ k \neq i}} E^{\mathcal{B}}[\sum_{i_2} Y(i, i_2, \rho_\pi(i, i_2))][\sum_{j_2} Y(k, j_2, \rho_\pi(k, j_2))] \chi_{B_{i,k}} \\
&= \sum_{i} \sum_{\substack{k \\ k \neq i}} [\sum_{i_2} Y(i, i_2, \rho_\pi(i, i_2))][\sum_{j_2} Y(k, j_2, \rho_\pi(k, j_2))] E^{\mathcal{B}} \chi_{B_{i,k}} \\
&= \sum_{i} \sum_{\substack{k \\ k \neq i}} [\sum_{i_2} Y(i, i_2, \rho_\pi(i, i_2))][\sum_{j_2} Y(k, j_2, \rho_\pi(k, j_2))] E \chi_{B_{i,k}} \\
&= \frac{1}{q(q-1)} \sum_{i} \sum_{\substack{k \\ k \neq i}} [\sum_{i_2} Y(i, i_2, \rho_\pi(i, i_2))][\sum_{j_2} Y(k, j_2, \rho_\pi(k, j_2))]
\end{aligned}$$

where $B_{i,k} = \{I = i, K = k\}, i \neq k$. Using the fact that $EW^4 = O(1)$, ([22], pp.717) and (5.10), we have

$$\begin{aligned}
&E(E^{\mathcal{B}} S_1 S_2)^2 \\
&= E\{E^{\mathcal{B}}[\sum_{i} Y(I, i, \rho_\pi(I, i))][\sum_{j} Y(K, j, \rho_\pi(K, j))]\}^2 \\
&= \frac{1}{q^2(q-1)^2} E\{\sum_{i_1} \sum_{\substack{j_1 \\ j_1 \neq i_1}} [\sum_{i_2} Y(i_1, i_2, \rho_\pi(i_1, i_2))][\sum_{j_2} Y(j_1, j_2, \rho_\pi(j_1, j_2))]\}^2
\end{aligned}$$

$$= \frac{1}{q^2(q-1)^2} E\{\sum_{i_1}\sum_{j_1}[\sum_{i_2} Y(i_1,i_2,\rho_\pi(i_1,i_2))][\sum_{j_2} Y(j_1,j_2,\rho_\pi(j_1,j_2))]$$

$$- \sum_{i_1}[\sum_{i_2} Y(i_1,i_2,\rho_\pi(i_1,i_2))][\sum_{j_2} Y(i_1,j_2,\rho_\pi(i_1,j_2))]\}^2 \quad (5.13)$$

$$= \frac{1}{q^2(q-1)^2} E\{[\sum_{i_1}\sum_{i_2} Y(i_1,i_2,\rho_\pi(i_1,i_2))]^2 - \sum_{i_1}[\sum_{i_2} Y(i_1,i_2,\rho_\pi(i_1,i_2))]^2\}^2$$

$$= \frac{1}{q^2(q-1)^2} E\{W^2 - \sum_{i_1}(W^{(i_1)})^2\}^2$$

$$\leq \frac{2}{q^2(q-1)^2}\{EW^4 + E[\sum_{i_1}(W^{(i_1)})^2]^2\}$$

$$= O(\frac{1}{q^4}).$$

Let $S^{(q)}$ be the set of all permutations on $\{0,1,\ldots,q-1\}$. Note that

$$P((E^{\mathcal{B}}S_1S_2)^2 < a)$$

$$= P(\{E^{\mathcal{B}}[\sum_{i_2} Y(I,i_2,\rho(I,i_2))][\sum_{j_2} Y(K,j_2,\rho(K,j_2))]\}^2 < a)$$

$$= P(\frac{1}{q^2(q-1)^2}\{\sum_{i_1}\sum_{\substack{j_1\\ j_1\neq i_1}}[\sum_{i_2} Y(i_1,i_2,\rho(i_1,i_2))][\sum_{j_2} Y(j_1,j_2,\rho(j_1,j_2))]\}^2 < a)$$

$$= \sum_{\alpha,\beta\in S^{(q)}} P(\frac{1}{q^2(q-1)^2}\{\sum_{i_1}\sum_{\substack{j_1\\ j_1\neq i_1}}[\sum_{i_2} Y(i_1,i_2,\alpha(i_2))][\sum_{j_2} Y(j_1,j_2,\beta(j_2))]\}^2 < a,$$

$$\rho(i_1,\cdot)=\alpha, \rho(j_1,\cdot)=\beta)$$

$$= \sum_{\alpha,\beta\in S^{(q)}} P(\frac{1}{q^2(q-1)^2}\{\sum_{i_1}\sum_{\substack{j_1\\ j_1\neq i_1}}[\sum_{i_2} Y(i_1,i_2,\alpha(i_2))][\sum_{j_2} Y(j_1,j_2,\beta(j_2))]\}^2 < a,$$

$$\rho(j_1,\cdot)=\alpha, \rho(i_1,\cdot)=\beta)$$

$$= P(\frac{1}{q^2(q-1)^2}\{\sum_{i_1}\sum_{\substack{j_1\\ j_1\neq i_1}}[\sum_{i_2} Y(i_1,i_2,\rho(j_1,i_2))][\sum_{j_2} Y(j_1,j_2,\rho(i_1,j_2))]\}^2 < a)$$

$$= P(\{E^{\mathcal{B}}[\sum_{i_2} Y(I,i_2,\rho(K,i_2))][\sum_{j_2} Y(K,j_2,\rho(I,j_2))]\}^2 < a)$$

$$= P((E^{\mathcal{B}}S_3S_4)^2 < a).$$

Thus $(E^{\mathcal{B}}S_1S_2)^2$ and $(E^{\mathcal{B}}S_3S_4)^2$ have the same distribution. This implies that $E(E^{\mathcal{B}}S_3S_4)^2 = O(\frac{1}{q^4})$. This complete the proof. □

Lemma 5.6. *If* $E(f \circ X)^4 < \infty$*, then* $E(E^{\mathcal{B}}S_1S_3)^2 = E(E^{\mathcal{B}}S_2S_4)^2 = O(\frac{1}{q^3})$.

Proof. Note that

$$\begin{aligned} ES_1^4 &= E(\sum_{i_2} Y(I, i_2, \rho(I, i_2)))^4 \\ &= \frac{1}{q}\sum_i E(\sum_{i_2} Y(i, i_2, \rho(i, i_2)))^4 \\ &= \frac{1}{q}\sum_i E(W^{(i)})^4. \end{aligned}$$

By this fact and (5.12), we get that

$$\sum_i E(W^{(i)})^4 = qES_1^4 = O(\frac{1}{q}). \tag{5.14}$$

Similar to (5.13), we have

$$\begin{aligned} &E(E^{\mathcal{B}}S_1S_3)^2 \\ &= \frac{1}{q^2(q-1)^2} E\{\sum_{i_1}\sum_k [\sum_{i_2} Y(i_1, i_2, \rho_\pi(i_1, i_2))][\sum_j Y(i_1, j, \rho_\pi(k, j))] \\ &\quad - \sum_{i_1}[\sum_{i_2} Y(i_1, i_2, \rho_\pi(i_1, i_2))][\sum_j Y(i_1, j, \rho_\pi(i_1, j))]\}^2 \\ &= \frac{1}{q^2(q-1)^2} E\{\sum_{i_1}\sum_{i_2} Y(i_1, i_2, \rho_\pi(i_1, i_2)) \sum_j\sum_k Y(i_1, j, k) \\ &\quad - \sum_{i_1}[\sum_{i_2} Y(i_1, i_2, \rho_\pi(i_1, i_2))]^2\}^2 \\ &\le \frac{C}{q^4}\{E[\sum_i W^{(i)} \sum_j\sum_k Y(i, j, k)]^2 + E[\sum_i (W^{(i)})^2]^2\} \\ &\le \frac{C}{q^4}\{q\sum_i E(W^{(i)})^2(\sum_j\sum_k Y(i, j, k))^2 + E[\sum_i (W^{(i)})^2]^2\} \\ &= \frac{C}{q^3} E\sum_i [q(W^{(i)})^4]^{\frac{1}{2}}[\frac{1}{q}(\sum_j\sum_k Y(i, j, k))^4]^{\frac{1}{2}} + \frac{C}{q^4}E[\sum_i (W^{(i)})^2]^2 \\ &\le \frac{C}{q^2}E\sum_i (W^{(i)})^4 + \frac{C}{q^4}E\sum_i(\sum_j\sum_k Y(i, j, k))^4 + \frac{C}{q^4}E[\sum_i (W^{(i)})^2]^2 \\ &\le \frac{C}{q^4}E\sum_i(\sum_j\sum_k Y(i, j, k))^4 + O(\frac{1}{q^3}) \end{aligned} \tag{5.15}$$

where we use (3.8) in the third inequality and for the last one, we use (5.10) and (5.14). Next, we will bound the first term on the right hand side of (5.15). Note that

$$E\sum_i(\sum_j\sum_k Y(i,j,k))^4 = A_1+A_2+A_3+A_4+A_5,$$

where

$$A_1 = \sum_{i,j,k} EY^4(i,j,k),$$

$$A_2 = \sum_{i,j_1,k_1}\sum_{\substack{j_2,k_2\\(j_2,k_2)\neq(j_1,k_1)}} EY^3(i,j_1,k_1)EY(i,j_2,k_2),$$

$$A_3 = \sum_{i,j_1,k_1}\sum_{\substack{j_2,k_2\\(j_2,k_2)\neq(j_1,k_1)}} EY^2(i,j_1,k_1)EY^2(i,j_2,k_2),$$

$$A_4 = \sum_{i,j_1,k_1}\sum_{\substack{j_2,k_2\\(j_2,k_2)\neq(j_1,k_1)}}\sum_{\substack{j_3,k_3\\(j_3,k_3)\neq(j_1,k_1)\\(j_3,k_3)\neq(j_2,k_2)}} EY^2(i,j_1,k_1)EY(i,j_2,k_2)EY(i,j_3,k_3), \text{ and}$$

$$A_5 = \sum_{i,j_1,k_1}\sum_{\substack{j_2,k_2\\(j_2,k_2)\neq(j_1,k_1)}}\sum_{\substack{j_3,k_3\\(j_3,k_3)\neq(j_1,k_1)\\(j_3,k_3)\neq(j_2,k_2)}}\sum_{\substack{j_4,k_4\\(j_4,k_4)\neq(j_1,k_1)\\(j_4,k_4)\neq(j_2,k_2)\\(j_4,k_4)\neq(j_3,k_3)}} EY(i,j_1,k_1)EY(i,j_2,k_2)$$

$$\times EY(i,j_3,k_3)EY(i,j_4,k_4).$$

It is easy to show by (3.8) that

$$A_2 \le Cq^2\sum_{i,j,k} EY^4(i,j,k) \text{ and } A_3 \le Cq^2\sum_{i,j,k} EY^4(i,j,k).$$

Note by (3.8) and (5.8) that

$$\begin{aligned}A_4 = &\sum_{i,j_1,k_1}\sum_{j_2,k_2}\sum_{j_3,k_3} EY^2(i,j_1,k_1)EY(i,j_2,k_2)EY(i,j_3,k_3)\\ &-\sum_{i,j_1,k_1} EY^2(i,j_1,k_1)[EY(i,j_1,k_1)]^2\\ &-\sum_{i,j_1,k_1}\sum_{\substack{j_2,k_2\\(j_2,k_2)\neq(j_1,k_1)}} EY^2(i,j_1,k_1)[EY(i,j_2,k_2)]^2\\ &-2\sum_{i,j_1,k_1}\sum_{\substack{j_2,k_2\\(j_2,k_2)\neq(j_1,k_1)}} EY^2(i,j_1,k_1)EY(i,j_1,k_1)EY(i,j_2,k_2)\end{aligned}$$

$$\leq 2\sum_{i,j_1,k_1}\sum_{j_2,k_2}\{E|Y(i,j_1,k_1)|^4\}^{\frac{1}{2}}\{E|Y(i,j_1,k_1)|^4\}^{\frac{1}{4}}\{E|Y(i,j_2,k_2)|^4\}^{\frac{1}{4}}$$
$$\leq Cq^2\sum_{i,j,k}EY^4(i,j,k),$$

and

$$\begin{aligned}
A_5 = &\sum_{i,j_1,k_1}\sum_{j_2,k_2}\sum_{j_3,k_3}\sum_{j_4,k_4} EY(i,j_1,k_1)EY(i,j_2,k_2)EY(i,j_3,k_3)EY(i,j_4,k_4) \\
&- \sum_{i,j_1,k_1} EY^4(i,j_1,k_1) - 4\sum_{i,j_1,k_1}\sum_{\substack{j_2,k_2 \\ (j_2,k_2)\neq(j_1,k_1)}} [EY(i,j_1,k_1)]^3 EY(i,j_2,k_2) \\
&- 6\sum_{i,j_1,k_1}\sum_{\substack{j_2,k_2 \\ (j_2,k_2)\neq(j_1,k_1)}} [EY(i,j_1,k_1)]^2[EY(i,j_2,k_2)]^2 \\
\leq\ & 4\sum_{i,j_1,k_1}\sum_{j_2,k_2}\{E|Y(i,j_1,k_1)|^4\}^{\frac{3}{4}}\{E|Y(i,j_2,k_2)|^4\}^{\frac{1}{4}} \\
\leq\ & Cq^2\sum_{i,j,k}EY^4(i,j,k).
\end{aligned}$$

Equation (5.9) give $A_i \leq O(q)$ for all $i \in \{1,2,3,4,5\}$. Hence

$$E\sum_i(\sum_j\sum_k Y(i,j,k))^4 = O(q). \tag{5.16}$$

Combine (5.15) and (5.16), we have $E(E^{\mathcal{B}}S_1S_3)^2 = O(\frac{1}{q^3})$. By the symmetry of S_1S_3 and S_2S_4, we obtain

$$E(E^{\mathcal{B}}S_2S_4)^2 = E(E^{\mathcal{B}}S_1S_3)^2 = O(\frac{1}{q^3}).$$

The proof is completed. □

Lemma 5.7. *If* $E(f\circ X)^4 < \infty$, *then* $E(E^{\mathcal{B}}S_1S_4)^2 = E(E^{\mathcal{B}}S_2S_3)^2 = O(\frac{1}{q^3})$.

Proof. Observe that

$$\begin{aligned}
&E(E^{\mathcal{B}}S_1S_4)^2 \\
&= E\{E^{\mathcal{B}}[\sum_i Y(I,i,\rho_\pi(I,i))][\sum_j Y(K,j,\rho_\pi(I,j))]\}^2 \\
&= \frac{1}{q^2(q-1)^2}E\{\sum_{i_1}\sum_{\substack{j_1 \\ j_1\neq i_1}}[\sum_{i_2}Y(i_1,i_2,\rho_\pi(i_1,i_2))][\sum_{j_2}Y(j_1,j_2,\rho_\pi(i_1,j_2))]\}^2 \\
&= \frac{1}{q^2(q-1)^2}(A_1+A_2),
\end{aligned} \tag{5.17}$$

where

$$A_1 = E\sum_{i_1}\sum_{\substack{j_1\\ j_1\neq i_1}}[\sum_{i_2} Y(i_1,i_2,\rho_\pi(i_1,i_2))]^2[\sum_{j_2} Y(j_1,j_2,\rho_\pi(i_1,j_2))]^2, \text{ and}$$

$$\begin{aligned}A_2 = \sum_{i_1}\sum_{\substack{j_1\\ j_1\neq i_1}}\sum_{k_1}\sum_{\substack{l_1\\ l_1\neq k_1\\ (k_1,l_1)\neq(i_1,j_1)}} & E[\sum_{i_2} Y(i_1,i_2,\rho_\pi(i_1,i_2))][\sum_{j_2} Y(j_1,j_2,\rho_\pi(i_1,j_2))]\\ & \times [\sum_{k_2} Y(k_1,k_2,\rho_\pi(k_1,k_2))][\sum_{l_2} Y(l_1,l_2,\rho_\pi(k_1,l_2))].\end{aligned}$$

Note that

$$qES_1^4 = qE(\sum_{i_2} Y(I,i_2,\rho(I,i_2)))^4 = \sum_{i_1} E(\sum_{i_2} Y(i_1,i_2,\rho(i_1,i_2)))^4$$

and

$$qES_3^4 = qE(\sum_{j_2} Y(I,j_2,\rho(K,j_2)))^4 = \frac{1}{q-1}\sum_{i_1}\sum_{\substack{j_1\\ j_1\neq i_1}} E(\sum_{j_2} Y(j_1,j_2,\rho(i_1,j_2)))^4.$$

By this fact, (3.8) and (5.12), we obtain

$$\begin{aligned}A_1 &\le CE\{\sum_{i_1}\sum_{\substack{j_1\\ j_1\neq i_1}}[\sum_{i_2} Y(i_1,i_2,\rho_\pi(i_1,i_2))]^4 + \sum_{i_1}\sum_{\substack{j_1\\ j_1\neq i_1}}[\sum_{j_2} Y(j_1,j_2,\rho_\pi(i_1,j_2))]^4\}\\ &\le Cq^2(ES_1^4 + ES_3^4)\\ &= O(1). \qquad (5.18)\end{aligned}$$

In order to bound A_2, we consider A_2 into 2 cases, i.e., $i_1 = k_1$ and $i_1 \neq k_1$. So we have

$$A_2 = A_{2,1} + A_{2,2}$$

where

$$\begin{aligned}A_{2,1} = \sum_{i_1}\sum_{\substack{j_1\\ j_1\neq i_1}}\sum_{\substack{l_1\\ l_1\neq i_1\\ l_1\neq j_1}}\sum_{i_2}\sum_{j_2}\sum_{k_2}\sum_{l_2} & EY(i_1,i_2,\rho_\pi(i_1,i_2))Y(j_1,j_2,\rho_\pi(i_1,j_2))\\ & \times Y(i_1,k_2,\rho_\pi(i_1,k_2))Y(l_1,l_2,\rho_\pi(i_1,l_2)) \quad \text{and}\end{aligned}$$

$$\begin{aligned}A_{2,2} = \sum_{i_1}\sum_{\substack{j_1\\ j_1\neq i_1}}\sum_{\substack{k_1\\ k_1\neq i_1}}\sum_{\substack{l_1\\ l_1\neq k_1}}\sum_{i_2}\sum_{j_2}\sum_{k_2}\sum_{l_2} & EY(i_1,i_2,\rho_\pi(i_1,i_2))Y(j_1,j_2,\rho_\pi(i_1,j_2))\\ & \times Y(k_1,k_2,\rho_\pi(k_1,k_2))Y(l_1,l_2,\rho_\pi(k_1,l_2)).\end{aligned}$$

If $i_2 = j_2 = k_2 = l_2$, then we use (3.8) to obtain

$$
\begin{aligned}
A_{2,1} &= \sum_{i_1,i_2} \sum_{\substack{j_1 \\ j_1\neq i_1}} \sum_{\substack{l_1 \\ l_1\neq i_1 \\ l_1\neq j_1}} EY^2(i_1,i_2,\rho_\pi(i_1,i_2))Y(j_1,i_2,\rho_\pi(i_1,i_2))Y(l_1,i_2,\rho_\pi(i_1,i_2)) \\
&\leq \frac{1}{2} \sum_{i_1,j_1,l_1,i_2} EY^4(i_1,i_2,\rho_\pi(i_1,i_2)) + \frac{1}{4} \sum_{i_1,j_1,l_1,i_2} EY^4(j_1,i_2,\rho_\pi(i_1,i_2)) \\
&\quad + \frac{1}{4} \sum_{i_1,j_1,l_1,i_2} EY^4(l_1,i_2,\rho_\pi(i_1,i_2)) \\
&= \frac{q^2}{q} \sum_{i_1,i_2,i_3} EY^4(i_1,i_2,i_3) \\
&= O(1). \qquad (5.19)
\end{aligned}
$$

Suppose that i_2, j_2, k_2, l_2 are not equal. We consider 4 possibilities as follow.

Case 1 three of i_2, j_2, k_2, l_2 are equal.

Case 2 i_2, j_2, k_2, l_2 have 2 equal-pairs.

Case 3 i_2, j_2, k_2, l_2 have 1 equal-pair.

Case 4 i_2, j_2, k_2, l_2 are all distinct.

To bound $A_{2,1}$ in Case 1 and 2, we use the same argument of (5.19) and hence $A_{2,1} \leq O(q)$. To prove Case 3, we will bound $A_{2,1}$ in case of $i_2 = j_2 \neq k_2 \neq l_2$ and $i_2 \neq l_2$. For the other cases, we can prove by the similar way. In this case,

$$
\begin{aligned}
A_{2,1} &= \sum_{i_1,i_2} \sum_{\substack{j_1 \\ j_1\neq i_1}} \sum_{\substack{l_1 \\ l_1\neq i_1 \\ l_1\neq j_1}} \sum_{\substack{k_2 \\ k_2\neq i_2}} \sum_{\substack{l_2 \\ l_2\neq i_2 \\ l_2\neq k_2}} EY(i_1,i_2,\rho_\pi(i_1,i_2))Y(j_1,i_2,\rho_\pi(i_1,i_2)) \\
&\quad \times Y(i_1,k_2,\rho_\pi(i_1,k_2))Y(l_1,l_2,\rho_\pi(i_1,l_2)) \\
&= \sum_{i_1,i_2,i_3,k_3,l_3} \sum_{\substack{j_1 \\ j_1\neq i_1}} \sum_{\substack{k_2 \\ k_2\neq i_2}} \sum_{\substack{l_1 \\ l_1\neq i_1 \\ l_1\neq j_1}} \sum_{\substack{l_2 \\ l_1\neq i_2 \\ l_1\neq k_2}} EY(i_1,i_2,i_3)Y(j_1,i_2,i_3)Y(i_1,k_2,k_3) \\
&\quad \times Y(l_1,l_2,l_3)P(\rho_\pi(i_1,i_2)=i_3, \rho_\pi(i_1,k_2)=k_3, \rho_\pi(i_1,l_2)=l_3).
\end{aligned}
$$

By the fact that

$$P(\rho_\pi(i_1,i_2)=i_3,\rho_\pi(i_1,k_2)=k_3,\rho_\pi(i_1,l_2)=l_3) = \begin{cases} \dfrac{1}{q(q-1)(q-2)} & \text{if } i_3,k_3,l_3 \text{ are all distinct,} \\ 0 & \text{otherwise,} \end{cases}$$

(see Example 5.1). We get that

$$\begin{aligned} A_{2,1} &= \frac{1}{q(q-1)(q-2)} \sum_{i_1,i_2,i_3} \sum_{\substack{j_1 \\ j_1\neq i_1}} \sum_{\substack{k_2 \\ k_2\neq i_2}} \sum_{\substack{k_3 \\ k_3\neq i_3}} \sum_{\substack{l_1 \\ l_1\neq i_1 \\ l_1\neq j_1}} \sum_{\substack{l_2 \\ l_2\neq i_2 \\ l_2\neq k_2}} \sum_{\substack{l_3 \\ l_3\neq i_3 \\ l_3\neq k_3}} EY(i_1,i_2,i_3) \\ &\quad \times EY(j_1,i_2,i_3)EY(i_1,k_2,k_3)EY(l_1,l_2,l_3) \\ &= \frac{-1}{q(q-1)(q-2)} \sum_{i_1,i_2,i_3} \sum_{\substack{j_1 \\ j_1\neq i_1}} \sum_{\substack{k_2 \\ k_2\neq i_2}} \sum_{\substack{k_3 \\ k_3\neq i_3}} \sum_{\substack{l_1 \\ l_1\neq i_1 \\ l_1\neq j_1}} \sum_{\substack{l_2 \\ l_2\neq i_2 \\ l_2\neq k_2}} EY(i_1,i_2,i_3) \\ &\quad \times EY(j_1,i_2,i_3)EY(i_1,k_2,k_3)EY(l_1,l_2,i_3) \\ &\quad - \frac{1}{q(q-1)(q-2)} \sum_{i_1,i_2,i_3} \sum_{\substack{j_1 \\ j_1\neq i_1}} \sum_{\substack{k_2 \\ k_2\neq i_2}} \sum_{\substack{k_3 \\ k_3\neq i_3}} \sum_{\substack{l_1 \\ l_1\neq i_1 \\ l_1\neq j_1}} \sum_{\substack{l_2 \\ l_2\neq i_2 \\ l_2\neq k_2}} EY(i_1,i_2,i_3) \\ &\quad \times EY(j_1,i_2,i_3)EY(i_1,k_2,k_3)EY(l_1,l_2,k_3) \\ &\leq \frac{C}{q^3} \sum_{i_1,i_2,i_3} \sum_{j_1,k_2,k_3} \sum_{l_1,l_2} \{EY^4(i_1,i_2,i_3)\}^{\frac{1}{4}} \{EY^4(j_1,i_2,i_3)\}^{\frac{1}{4}} \{EY^4(i_1,k_2,k_3)\}^{\frac{1}{4}} \\ &\quad \times \{EY^4(l_1,l_2,i_3)\}^{\frac{1}{4}} \\ &\leq \frac{Cq^5}{q^3} \sum_{i_1,i_2,i_3} EY^4(i_1,i_2,i_3) \\ &= O(q). \end{aligned} \tag{5.20}$$

For Case 4, we use the fact that

$$P(\rho_\pi(i_1,i_2)=i_3,\rho_\pi(i_1,j_2)=j_3,\rho_\pi(i_1,k_2)=k_3,\rho_\pi(i_1,l_2)=l_3) = \begin{cases} \dfrac{1}{q(q-1)(q-2)(q-3)} & \text{if } i_3,j_3,k_3,l_3 \text{ are all distinct} \\ 0 & \text{otherwise.} \end{cases}$$

(see Example 5.2). Then

$$A_{2,1} = \frac{1}{q(q-1)(q-2)(q-3)} \sum_{i_1,i_2,i_3} \sum_{\substack{j_1 \\ j_1 \neq i_1}} \sum_{\substack{j_2 \\ j_2 \neq i_2}} \sum_{\substack{j_3 \\ j_3 \neq i_3}} \sum_{\substack{k_2 \\ k_2 \neq i_2 \\ k_2 \neq j_2}} \sum_{\substack{k_3 \\ k_3 \neq i_3 \\ k_3 \neq j_3}} \sum_{\substack{l_1 \\ l_1 \neq i_1 \\ l_1 \neq j_1}} \sum_{\substack{l_2 \\ l_2 \neq i_2 \\ l_2 \neq j_2 \\ l_2 \neq k_2}} \sum_{\substack{l_3 \\ l_3 \neq i_3 \\ l_3 \neq j_3 \\ l_3 \neq k_3}}$$

$$EY(i_1, i_2, i_3)EY(j_1, j_2, j_3)EY(i_1, k_2, l_3)EY(l_1, l_2, l_3)$$

$$= \frac{(-1)^2}{q(q-1)(q-2)(q-3)}$$

$$\sum_{i_1,i_2,i_3} \sum_{\substack{j_1=0 \\ j_1 \neq i_1}}^{q-1} \sum_{\substack{l_1=0 \\ l_1 \neq k_1 \\ l_1 \neq j_1}}^{q-1} \sum_{\substack{j_2=0 \\ j_2 \neq i_2}}^{q-1} \sum_{\substack{k_2=0 \\ k_2 \neq i_2 \\ k_2 \neq j_2}}^{q-1} \sum_{\substack{j_3=0 \\ j_3 \neq i_3}}^{q-1} \sum_{\substack{k_3=0 \\ k_3 \neq i_3 \\ k_3 \neq j_3}}^{q-1} \sum_{l_2 \in \{i_2, j_2, k_2\}} \sum_{l_3 \in \{i_3, j_3, k_3\}}$$

$$EY(i_1, i_2, i_3)EY(j_1, j_2, j_3)EY(i_1, k_2, k_3)EY(l_1, l_2, l_3)$$

$$\leq \frac{Cq^6}{q^4} \sum_{i_1,i_2,i_3} EY^4(i_1, i_2, i_3)$$

$$= O(q).$$

Thus, (5.19) and Case 1 – Case 4 imply

$$A_{2,1} = O(q). \tag{5.21}$$

Next, we will bound $A_{2,2}$. If $i_2 = j_2 = k_2 = l_2$, then we use the same argument of (5.19) to show that

$$A_{2,2} = O(q). \tag{5.22}$$

In order to prove Case 1–Case 4, we will show only Case 2 and the other case are similarly. If i_2, j_2, k_2, l_2 have 2 equal-pairs, then $i_2 = j_2 \neq k_2 = l_2$ or $i_2 = k_2 \neq j_2 = l_2$ or $i_2 = l_2 \neq j_2 = k_2$. In this work, we will bound $A_{2,2}$ in case of $i_2 = j_2 \neq k_2 = l_2$ and the other cases can be proved in the same way. Then

$$A_{2,2} = \sum_{i_1,i_2} \sum_{\substack{j_1 \\ j_1 \neq i_1}} \sum_{\substack{k_1 \\ k_1 \neq i_1}} \sum_{\substack{l_1 \\ l_1 \neq k_1}} \sum_{\substack{j_2 \\ j_2 \neq i_2}} EY(i_1, i_2, \rho_\pi(i_1, i_2))Y(j_1, j_2, \rho_\pi(i_1, j_2))$$

$$\times Y(k_1, i_2, \rho_\pi(k_1, i_2))Y(l_1, j_2, \rho_\pi(k_1, j_2))$$

$$= \sum_{i_1,i_2,i_3} \sum_{\substack{j_1 \\ j_1 \neq i_1}} \sum_{\substack{j_2 \\ j_2 \neq i_2}} \sum_{\substack{k_1 \\ k_1 \neq i_1}} \sum_{\substack{l_1 \\ l_1 \neq k_1}} \sum_{j_3,k_3,l_3} EY(i_1, i_2, i_3)Y(j_1, j_2, j_3)Y(k_1, i_2, k_3)Y(l_1, j_2, l_3)$$

$$\times P(\rho_\pi(i_1, i_2) = i_3, \rho_\pi(i_1, j_2) = j_3, \rho_\pi(k_1, i_2) = k_3, \rho_\pi(k_1, j_2) = l_3).$$

By the fact that

$$P(\rho_\pi(i_1,i_2)=i_3,\rho_\pi(i_1,j_2)=j_3,\rho_\pi(k_1,i_2)=k_3,\rho_\pi(k_1,j_2)=l_3)$$
$$=\begin{cases}\frac{1}{q} & \text{if } i_3=l_3\neq j_3=k_3,\\ \frac{1}{q(q-1)} & \text{if } i_3\neq j_3=k_3\neq l_3 \text{ and } i_3\neq l_3,\\ \frac{1}{q(q-1)} & \text{if } i_3=l_3\neq j_3\neq k_3 \text{ and } i_3\neq k_3,\\ 0 & \text{otherwise,}\end{cases}$$

(see Example 5.3) and the same argument of (5.20), we get that $A_{2,2}\leq O(q)$. By (5.22) and Case 1–Case 4, we get that

$$A_{2,2}=O(q). \tag{5.23}$$

Thus (5.21) and (5.23) imply

$$A_2=O(q). \tag{5.24}$$

Hence, (5.18) and (5.24) yield

$$E(E^{\mathcal{B}}S_1S_4)^2=\frac{1}{q^2(q-1)^2}(A_1+A_2)=O(\frac{1}{q^3}).$$

By the symmetry of S_1S_4 and S_2S_3, we get that

$$E(E^{\mathcal{B}}S_1S_4)^2=E(E^{\mathcal{B}}S_2S_3)^2=O(\frac{1}{q^3}).$$

□

For each i,j and $k\in\{1,2,...,q\}$, we let

$$Y_0(i,j,k)=Y(i,j,k)\mathbb{I}\big(\big|Y(i,j,k)\big|>1\big),$$
$$\widehat{Y}_0(i,j,k)=Y(i,j,k)\mathbb{I}\big(\big|Y(i,j,k)\big|\leq 1\big),$$
$$S_{1,0}=\sum_{j=0}^{q-1}Y_0(I,j,\rho_\pi(I,j)),\ \ S_{2,0}=\sum_{j=0}^{q-1}Y_0(K,j,\rho_\pi(K,j)),$$
$$S_{3,0}=\sum_{j=0}^{q-1}Y_0(I,j,\rho_\pi(K,j)),\ \ S_{4,0}=\sum_{j=0}^{q-1}Y_0(K,j,\rho_\pi(I,j)),$$

$$\widehat{S}_{1,0} = \sum_{j=0}^{q-1} \widehat{Y}_0(I, j, \rho_\pi(I, j)), \quad \widehat{S}_{2,0} = \sum_{j=0}^{q-1} \widehat{Y}_0(K, j, \rho_\pi(K, j))$$

and
$$\widehat{S}_{3,0} = \sum_{j=0}^{q-1} \widehat{Y}_0(I, j, \rho_\pi(K, j)), \quad \widehat{S}_{4,0} = \sum_{j=0}^{q-1} \widehat{Y}_0(K, j, \rho_\pi(I, j))$$

where $\mathbb{I}$ is the indicator function.

Lemma 5.8. *If $E(f \circ X)^6 < \infty$, then $E(E^{\mathcal{B}} S_{i,0}^2)^2 = O(\frac{1}{q^4})$ for $i = 1, 2, 3, 4$.*

Proof. Observe that

$$\begin{aligned}
E(E^{\mathcal{B}} S_{1,0}^2)^2 &= E[E^{\mathcal{B}}(\sum_{i_2=0}^{q-1} Y_0(I, i_2, \rho(I, i_2)))^2]^2 \\
&= \frac{1}{q^2} E[\sum_{i_1=0}^{q-1}(\sum_{i_2=0}^{q-1} Y_0(i_1, i_2, \rho(i_1, i_2)))^2]^2 \\
&\leq \frac{q}{q^2} E[\sum_{i_1=0}^{q-1}(\sum_{i_2=0}^{q-1} Y_0(i_1, i_2, \rho_\pi(i_1, i_2)))^4] \\
&= \frac{1}{q}(A_1 + A_2 + A_3 + A_4 + A_5),
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= \sum_{i_1, i_2} EY_0^4(i_1, i_2, \rho_\pi(i_1, i_2)), \\
A_2 &= \sum_{i_1, i_2} \sum_{\substack{j_2 \\ j_2 \neq i_2}} EY_0^3(i_1, i_2, \rho_\pi(i_1, i_2)) Y_0(i_1, j_2, \rho_\pi(i_1, j_2)), \\
A_3 &= \sum_{i_1, i_2} \sum_{\substack{j_2 \\ j_2 \neq i_2}} EY_0^2(i_1, i_2, \rho_\pi(i_1, i_2)) Y_0^2(i_1, j_2, \rho_\pi(i_1, j_2)), \\
A_4 &= \sum_{i_1, i_2} \sum_{\substack{j_2 \\ j_2 \neq i_2}} \sum_{\substack{k_2 \\ k_2 \neq i_2 \\ k_2 \neq j_2}} EY_0^2(i_1, i_2, \rho_\pi(i_1, i_2)) Y_0(i_1, j_2, \rho_\pi(i_1, j_2)) \\
&\quad \times Y_0(i_1, k_2, \rho_\pi(i_1, k_2)), \text{ and} \\
A_5 &= \sum_{i_1, i_2} \sum_{\substack{j_2 \\ j_2 \neq i_2}} \sum_{\substack{k_2 \\ k_2 \neq i_2 \\ k_2 \neq j_2}} \sum_{\substack{l_2 \\ l_2 \neq i_2 \\ l_2 \neq j_2 \\ l_2 \neq k_2}} EY_0(i_1, i_2, \rho_\pi(i_1, i_2)) Y_0(i_1, j_2, \rho_\pi(i_1, j_2)) \\
&\quad \times Y_0(i_1, k_2, \rho_\pi(i_1, k_2)) Y_0(i_1, l_2, \rho_\pi(i_1, l_2)).
\end{aligned}$$

From (5.9), we note that

$$\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} E|Y(i,j,k)^m Y_0(i,j,k)^n| \le \sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} E|Y(i,j,k)|^{m+n+t}$$

$$= O(q^{3-m-n-t}) \qquad (5.25)$$

for any integers m, n and t where $m \ge 0, n, t > 0$ and $m+n+t$ is an even number. Then

$$A_1 = \frac{1}{q}\sum_{i_1,i_2,i_3} EY_0^4(i_1,i_2,i_3) \le \frac{1}{q}\sum_{i_1,i_2,i_3} EY^6(i_1,i_2,i_3) = O(\frac{1}{q^4})$$

and

$$\begin{aligned} A_2 &= \frac{1}{q(q-1)}\sum_{i_1,i_2,i_3}\sum_{\substack{j_2\\ j_2\neq i_2}}\sum_{\substack{j_3\\ j_3\neq i_2}} EY_0^3(i_1,i_2,i_3)EY_0(i_1,j_2,j_3) \\ &\le \frac{1}{q(q-1)}\sum_{i_1,i_2,i_3} E|Y(i_1,i_2,i_3)|^4 \sum_{j_1,j_2,j_3} E|Y(j_1,j_2,j_3)|^4 \\ &= O(\frac{1}{q^4}). \end{aligned}$$

By the same technique of A_2, we have

$$A_3 = O(\frac{1}{q^4}),$$

$$\begin{aligned} A_4 &= \frac{1}{q(q-1)(q-2)}\sum_{i_1,i_2,i_3}\sum_{\substack{j_2\\ j_2\neq i_2}}\sum_{\substack{j_3\\ j_3\neq i_3}}\sum_{\substack{k_2\\ k_2\neq i_2\\ k_2\neq j_2}}\sum_{\substack{k_3\\ k_3\neq i_3\\ k_3\neq j_3}} EY_0^2(i_1,i_2,i_3)EY_0(i_1,j_2,j_3) \\ &\quad\times EY_0(i_1,k_2,k_3) \\ &= O(\frac{1}{q^6}), \end{aligned}$$

and

$$\begin{aligned} A_5 &= \frac{1}{q(q-1)(q-2)(q-3)}\sum_{i_1,i_2,i_3}\sum_{\substack{j_2\\ j_2\neq i_2}}\sum_{\substack{j_3\\ j_3\neq i_3}}\sum_{\substack{k_2\\ k_2\neq i_2\\ k_2\neq j_2}}\sum_{\substack{k_3\\ k_3\neq i_3\\ k_3\neq j_3}}\sum_{\substack{l_2\\ l_2\neq i_2\\ l_2\neq j_2\\ l_2\neq k_2}}\sum_{\substack{l_3\\ l_3\neq i_3\\ l_3\neq j_3\\ l_3\neq k_3}} EY_0(i_1,i_2,i_3) \\ &\quad\times EY_0(i_1,j_2,j_3)EY_0(i_1,k_2,k_3)EY_0(i_1,l_2,l_3) \\ &= O(\frac{1}{q^8}). \end{aligned}$$

Hence

$$E(E^{\mathcal{B}}S_{1,0}^2)^2 = O(\frac{1}{q^4})$$

and

$$E(E^{\mathcal{B}}S_{2,0}^2)^2 = E(E^{\mathcal{B}}S_{1,0}^2)^2 = O(\frac{1}{q^4}). \tag{5.26}$$

To bound $E(E^{\mathcal{B}}S_{3,0}^2)^2$, we note that

$$\begin{aligned}
E[\sum_{i_1,i_2,i_3} & Y_0^2(i_1,i_2,i_3)]^2 \\
&= \sum_{i_1,i_2,i_3} EY_0^4(i_1,i_2,i_3) + \sum_{i_1,i_2,i_3} \sum_{\substack{j_1,j_2,j_3 \\ (j_1,j_2,j_3)\neq(i_1,i_2,i_3)}} EY_0^2(i_1,i_2,i_3)Y_0^2(j_1,j_2,j_3)] \\
&\leq \sum_{i_1,i_2,i_3} EY^6(i_1,i_2,i_3) + \sum_{i_1,i_2,i_3} EY^4(i_1,i_2,i_3) \sum_{j_1,j_2,j_3} EY^4(j_1,j_2,j_3) \\
&= O(\frac{1}{q^3}) + O(\frac{1}{q^2}) \\
&= O(\frac{1}{q^2}).
\end{aligned}$$

Then

$$\begin{aligned}
E(E^{\mathcal{B}}S_{3,0}^2)^2 &= E[E^{\mathcal{B}}(\sum_{i_2=0}^{q-1} Y_0(I,i_2,\rho(K,i_2)))^2]^2 \\
&= \frac{1}{q^2(q-1)^2} E[\sum_{i_1=0}^{q-1} \sum_{\substack{k=0 \\ k\neq i_1}}^{q-1} E^{\mathcal{B}}(\sum_{i_2=0}^{q-1} Y_0(i_1,i_2,\rho(k,i_2)))^2]^2 \\
&\leq \frac{1}{q^2(q-1)^2} E[q \sum_{i_1,i_2,k} Y_0^2(i_1,i_2,\rho(k,i_2))]^2 \\
&= \frac{1}{(q-1)^2} E[\sum_{i_1,i_2,i_3} Y_0^2(i_1,i_2,i_3)]^2 \\
&= O(\frac{1}{q^4})
\end{aligned}$$

and it is easy to see that

$$E(E^{\mathcal{B}}S_{4,0}^2)^2 = E(E^{\mathcal{B}}S_{3,0}^2)^2 = O(\frac{1}{q^4}). \tag{5.27}$$

By (5.26) and (5.27), we establish the lemma. □

Lemma 5.9. *If $E(f \circ X)^6 < \infty$, then*

$$E[E^{\mathcal{B}}\{1 - \frac{q-1}{4}(\widehat{S}_{1,0} + \widehat{S}_{2,0} - \widehat{S}_{3,0} - \widehat{S}_{4,0})^2\}]^2 = O(\frac{1}{q}).$$

Proof. First, we note that

$$\begin{aligned}
&E[E^{\mathcal{B}}\{1 - \frac{q-1}{4}(\widehat{S}_{1,0} + \widehat{S}_{2,0} - \widehat{S}_{3,0} - \widehat{S}_{4,0})^2\}]^2 \\
&\leq \frac{1}{16}E[\sum_{i=1}^{4}|E^{\mathcal{B}}(1-(q-1)\widehat{S}_{i,0}^2)| + 2(q-1)\sum_{1\leq i<j\leq 4}|E^{\mathcal{B}}\widehat{S}_{i,0}\widehat{S}_{j,0}|]^2 \\
&\leq C\sum_{i=1}^{4}E|E^{\mathcal{B}}(1-(q-1)\widehat{S}_{i,0}^2)|^2 + Cq^2\sum_{1\leq i<j\leq 4}E|E^{\mathcal{B}}\widehat{S}_{i,0}\widehat{S}_{j,0}|^2 \\
&= C\sum_{i=1}^{4}E|E^{\mathcal{B}}(1-(q-1)(S_i - S_{i,0})^2)|^2 + Cq^2\sum_{1\leq i<j\leq 4}E|E^{\mathcal{B}}(S_i - S_{i,0})(S_j - S_{j,0})|^2 \\
&= C\sum_{i=1}^{4}E|E^{\mathcal{B}}(1-(q-1)(S_i^2 - 2S_iS_{i,0} + S_{i,0}^2))|^2 \\
&\quad + Cq^2\sum_{1\leq i<j\leq 4}E|E^{\mathcal{B}}(S_iS_j - S_jS_{i,0} - S_iS_{j,0} + S_{i,0}S_{j,0})|^2 \\
&= C\sum_{i=1}^{4}E|E^{\mathcal{B}}[(1-qS_i^2) + 2qS_iS_{i,0} - qS_{i,0}^2 + S_i^2 - 2S_iS_{i,0} + S_{i,0}^2]|^2 \\
&\quad + Cq^2\sum_{1\leq i<j\leq 4}E|E^{\mathcal{B}}(S_iS_j - S_jS_{i,0} - S_iS_{j,0} + S_{i,0}S_{j,0})|^2 \\
&\leq Cq^2\sum_{1\leq i<j\leq 4}E|E^{\mathcal{B}}S_iS_j|^2 + Cq^2\sum_{i=1}^{4}E|E^{\mathcal{B}}S_{i,0}^2|^2 + C\sum_{i=1}^{4}E|E^{\mathcal{B}}(1-qS_i^2)|^2 \\
&\quad + C\sum_{i=1}^{4}E|E^{\mathcal{B}}S_i^2|^2 + Cq^2\sum_{i=1}^{4}\sum_{j=1}^{4}E|E^{\mathcal{B}}S_iS_{j,0}|^2 + Cq^2\sum_{1\leq i<j\leq 4}E|E^{\mathcal{B}}S_{i,0}S_{j,0}|^2.
\end{aligned} \tag{5.28}$$

The first and the second terms of (5.28) can be bounded by $O(\frac{1}{q})$ from Lemma 5.5 to 5.8. Moreover, Loh ([14], pp.1218-1220) showed that the third term of (5.28) is bounded by $O(\frac{1}{q})$. Next, we bound the fourth term. By (5.10), we have

$$\begin{aligned}
E(E^{\mathcal{B}}S_1^2)^2 &= E[E^{\mathcal{B}}(\sum_{i_2=0}^{q-1} Y(I, i_2, \rho_\pi(I, i_2)))^2]^2 \\
&= \frac{1}{q^2} E[\sum_{i_1=0}^{q-1}(\sum_{i_2=0}^{q-1} Y(i_1, i_2, \rho_\pi(i_1, i_2)))^2]^2 \\
&= \frac{1}{q^2} + O(\frac{1}{q^3})
\end{aligned}$$

and so

$$E(E^{\mathcal{B}}S_2^2)^2 = E(E^{\mathcal{B}}S_1^2)^2 = \frac{1}{q^2} + O(\frac{1}{q^3}).$$

By the fact that

$$\begin{aligned}
E^{\mathcal{B}}S_3^2 &= \frac{1}{q}[E^{\mathcal{B}}(S_{3;1} - 1) + 1 + E^{\mathcal{B}}S_{3;2}], \\
E(S_{3;1} - 1)^2 &= E(S_{3;2}^2) = O(\frac{1}{q})
\end{aligned}$$

([14], pp. 1219-1220) where

$$S_{3;1} = \frac{1}{q-1} \sum_{i_1=0}^{q-1} \sum_{\substack{j_1=0 \\ j_1 \neq i_1}}^{q-1} \sum_{i_2=0}^{q-1} Y^2(i_1, i_2, \rho_\pi(j_1, i_2)),$$

and $S_{3;2} = \frac{1}{q-1} \sum_{i_1=0}^{q-1} \sum_{\substack{j_1=0 \\ j_1 \neq i_1}}^{q-1} \sum_{i_2=0}^{q-1} \sum_{\substack{j_2=0 \\ j_2 \neq i_2}}^{q-1} \tilde{\mu}(i_1, i_2, \rho_\pi(j_1, i_2))\tilde{\mu}(i_1, j_2, \rho_\pi(j_1, j_2)),$

we have

$$\begin{aligned}
E(E^{\mathcal{B}}S_3^2)^2 &= \frac{1}{q^2} E[E^{\mathcal{B}}(S_{3;1} - 1) + 1 + E^{\mathcal{B}}S_{3;2}]^2 \\
&\leq \frac{3}{q^2} E[S_{3;1} - 1]^2 + \frac{3}{q^2} + \frac{3}{q^2} E[S_{3;2}]^2 \\
&= O(\frac{1}{q^2}).
\end{aligned}$$

By the symmetry of S_3 and S_4,

$$E(E^{\mathcal{B}}S_4^2)^2 = E(E^{\mathcal{B}}S_3^2)^2 = O(\frac{1}{q^2}).$$

Hence, the fourth term of (5.28) is bounded by $O(\frac{1}{q^2})$. Since, for each $i, j \in \{1, 2, 3, 4\}$,

$$\begin{aligned} E(E^{\mathcal{B}} S_i S_{j,0})^2 &= E[E^{\mathcal{B}}\{\frac{1}{q^{\frac{1}{2}}} S_i^2\}^{\frac{1}{2}}\{q^{\frac{1}{2}} S_{j,0}^2\}^{\frac{1}{2}}]^2 \\ &\leq E[\frac{1}{2q^{\frac{1}{2}}} E^{\mathcal{B}} S_i^2 + \frac{q^{\frac{1}{2}}}{2} E^{\mathcal{B}} S_{j,0}^2]^2 \\ &= \frac{1}{2q} E(E^{\mathcal{B}} S_i^2)^2 + \frac{q}{2} E(E^{\mathcal{B}} S_{j,0}^2)^2 \\ &= O(\frac{1}{q^3}), \end{aligned}$$

and similarly,

$$E|E^{\mathcal{B}} S_{i,0} S_{j,0}|^2 \leq \frac{1}{2} E|E^{\mathcal{B}} S_{i,0}^2|^2 + \frac{1}{2} E|E^{\mathcal{B}} S_{j,0}^2|^2 = O(\frac{1}{q^4}),$$

the fifth and the sixth terms of (5.28) are bounded by $O(\frac{1}{q})$. Therefore, we have the lemma. □

5.3 Proof of the Main Theorem

To bound $|P(W \leq z) - \Phi(z)|$, it suffices to consider $z > 0$ as we have used the fact that $\Phi(z) = 1 - \Phi(-z)$ and apply the result to $-W$ when $z \leq 0$. So, from now on, we assume $z > 0$.

In the view of [12], pp. 2360, and the fact that

$$1 = \frac{1+z}{1+z} \leq \frac{2}{1+z} \quad \text{for} \quad 0 < z \leq 1,$$

we have

$$|P(W \leq z) - \Phi(z)| = O\left(\frac{1}{\sqrt{q}}\right) \leq \frac{1}{1+z} O\left(\frac{1}{\sqrt{q}}\right) \quad \text{for} \quad 0 < z \leq 1.$$

Let $z > 1$ and

$$\widehat{Y} = \sum_{i=0}^{q-1} \sum_{j=0}^{q-1} \widehat{Y}_0(i, j, \rho_\pi(i, j)) \text{ and } \widetilde{Y} = \widehat{Y} - \widehat{S}_{1,0} - \widehat{S}_{2,0} + \widehat{S}_{3,0} + \widehat{S}_{4,0}.$$

Note from [12], pp.2360-2361, that

$$|P(W \leq z) - \Phi(z)| \leq P(W \neq \widehat{Y}) + |P(\widehat{Y} \leq z) - \Phi(z)| \tag{5.29}$$

and

$$P(W \neq \widehat{Y}) = \frac{1}{(1+z)^4} O(\frac{1}{q^2}) \leq \frac{1}{1+z} O(\frac{1}{\sqrt{q}}). \tag{5.30}$$

It remains to bound the second term of (5.29). Let g_z be the Stein solution of the Stein's equation (2.2). From (2.2) and the fact that

$$E\widehat{Y}g\big(\widehat{Y}\big) = E\int_{-\infty}^{\infty} g'(\widehat{Y}+t)K(t)dt + \widetilde{\Delta}g\big(\widehat{Y}\big) \tag{5.31}$$

for a continuous and piecewise continuously differrentiable function g, where

$$\widetilde{\Delta}g\big(\widehat{Y}\big) = \frac{1}{q}Eg\big(\widehat{Y}\big)\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\widehat{Y}_0(i,j,k), \quad \text{and}$$

$$K(t) = \frac{q-1}{4}(\widetilde{Y}-\widehat{Y})(\mathbb{I}(0 \leq t \leq \widetilde{Y}-Y) - \mathbb{I}(\widetilde{Y}-Y \leq t < 0)),$$

([11], pp.16),we obtain that

$$\begin{aligned}
|P(\widehat{Y} \leq z) - \Phi(z)| &= |Eg_z'(\widehat{Y}) - E\widehat{Y}g_z(\widehat{Y})| \\
&= |Eg_z'(\widehat{Y}) - E\int_{-\infty}^{\infty} g_z'(\widehat{Y}+t)K(t)dt - \tilde{\Delta}g_z(\widehat{Y})| \\
&\leq T_1 + T_2 + T_3,
\end{aligned} \tag{5.32}$$

where

$$\begin{aligned}
T_1 &= |E\int_{-\infty}^{\infty}\{g_z'(\widehat{Y}) - g_z'(\widehat{Y}+t)\}K(t)dt|, \\
T_2 &= |Eg_z'(\widehat{Y})E^{\mathcal{B}}\{1 - \int_{-\infty}^{\infty}K(t)dt\}|, \text{ and} \\
T_3 &= |\tilde{\Delta}g_z(\widehat{Y})|.
\end{aligned}$$

By the same argument as Chen and Shao ([5], pp.248), we can show that

$$E|g_z'(\widehat{Y})| \leq \frac{C}{(1+z)^2}. \tag{5.33}$$

By Proposition 2.13(2), Lemma 5.9 and (5.33) , we have

$$\begin{aligned}
T_2 &\leq \{E|g_z'(\widehat{Y})|^2\}^{\frac{1}{2}}\{E(E^{\mathcal{B}}\{1 - \int_{-\infty}^{\infty}K(t)dt\})^2\}^{\frac{1}{2}} \\
&\leq \{E|g_z'(\widehat{Y})|^2\}^{\frac{1}{2}}\{E[E^{\mathcal{B}}\{1 - \frac{q-1}{4}(\widehat{S}_{1,0} + \widehat{S}_{2,0} - \widehat{S}_{3,0} - \widehat{S}_{4,0})^2\}]^2\}^{\frac{1}{2}} \\
&\leq \{E|g_z'(\widehat{Y})|\}^{\frac{1}{2}}O(\frac{1}{\sqrt{q}}) \\
&\leq \frac{1}{1+z}O(\frac{1}{\sqrt{q}}).
\end{aligned} \tag{5.34}$$

Neammanee and Laipaporn ([12], pp.2362) showed that

$$T_3 \leq \frac{1}{1+z}O(\frac{1}{\sqrt{q}}), \tag{5.35}$$

and

$$T_1 \leq T_{11} + T_{12} + T_{13}$$

where

$$T_{11} \leq \frac{2}{1+z}E\int_{|t|\leq|\widetilde{Y}-\widehat{Y}|} f'_{|\widetilde{Y}-\widehat{Y}|}(\widehat{Y}+t)\mathbb{I}(z-2|\widetilde{Y}-\widehat{Y}|<\widehat{Y}<z+2|\widetilde{Y}-\widehat{Y}|)K(t)dt,$$
$$T_{12} \leq \frac{1}{(1+z)^2}O(\frac{1}{\sqrt{q}}) + C(1+z)E\int_{-\infty}^{\infty}\int_t^0 K(t)\mathbb{I}(\widehat{Y}+u \geq \frac{z}{2})dudt,$$
$$T_{13} \leq q\{P(|\widetilde{Y}-\widehat{Y}| \geq \frac{1+z}{2})\}^{\frac{1}{2}}\{E|\widetilde{Y}-\widehat{Y}|^4\}^{\frac{1}{2}}$$

and $f_\delta : \mathbb{R} \to \mathbb{R}$ is defined by

$$f_\delta(t) = \begin{cases} 0 & \text{if} \quad t < z-2\delta, \\ (1+t+\delta)(t-z+2\delta) & \text{if} \quad z-2\delta \leq t \leq z+2\delta, \\ 4\delta(1+t+\delta) & \text{if} \quad t > z+2\delta \end{cases}$$

for $\delta > 0$. By (5.31), we have

$$T_{11} \leq \frac{C}{1+z}(E|\widehat{Y}f_{|\widetilde{Y}-\widehat{Y}|}(\widehat{Y})| + |\widetilde{\Delta}f_{|\widetilde{Y}-\widehat{Y}|}(\widehat{Y})|).$$

Note from [12] pp.2364-2365 that

$$|\widetilde{\Delta}f_{|\widetilde{Y}-\widehat{Y}|}(\widehat{Y})| = O(\frac{1}{q})$$

and

$$E|\widehat{Y}f_{|\widetilde{Y}-\widehat{Y}|}(\widehat{Y})| = O(\frac{1}{\sqrt{q}}) + O(\frac{1}{\sqrt{q}})\{E\widehat{Y}^2|\widehat{Y}|^{\frac{2}{r-1}}\}^{\frac{r-1}{r}} \tag{5.36}$$

for some positive integer $r > 1$. Recently, Laipaporn and Sungkamongkol ([13], pp.81) showed that

$$E\widehat{Y}^4 = O(1).$$

Then, if we choose $r = 2$ in (5.36), we have

$$\begin{aligned} T_{11} &\leq \frac{C}{1+z}[O(\frac{1}{\sqrt{q}}) + O(\frac{1}{\sqrt{q}})\{E\widehat{Y}^2|\widehat{Y}|^2\}^{\frac{1}{2}} + O(\frac{1}{q})] \\ &= \frac{C}{1+z}[O(\frac{1}{\sqrt{q}}) + O(\frac{1}{\sqrt{q}})\{E\widehat{Y}^4\}^{\frac{1}{2}} + O(\frac{1}{q})] \\ &\leq \frac{1}{1+z}O(\frac{1}{\sqrt{q}}). \end{aligned} \tag{5.37}$$

To bound T_{12}, we note that

$$E|\widetilde{Y} - \widehat{Y}|^r = O(\frac{1}{q^{\frac{r}{2}}}) \text{ for } r = 2, 4, 6, ([11], \text{ pp.20}) \tag{5.38}$$

and

$$E\int_{-\infty}^{\infty}\int_{t}^{0} K(t)\mathbb{I}(\widehat{Y} + u \geq \frac{z}{2})dudt \leq qE|\widetilde{Y} - \widehat{Y}|^3\mathbb{I}(\widehat{Y} + |\widetilde{Y} - \widehat{Y}| \geq \frac{z}{2})$$

([12], pp.2357). Thus

$$\begin{aligned} T_{12} &\leq \frac{1}{1+z}O(\frac{1}{\sqrt{q}}) + Cq(1+z)\{E|\widetilde{Y} - \widehat{Y}|^6\}^{\frac{1}{2}}\{P(\widehat{Y} + |\widetilde{Y} - \widehat{Y}| \geq \frac{z}{2})\}^{\frac{1}{2}} \\ &\leq \frac{1}{1+z}O(\frac{1}{\sqrt{q}}) + Cq(1+z)\{E|\widetilde{Y} - \widehat{Y}|^6\}^{\frac{1}{2}}\{\frac{E\widehat{Y}^4 + E|\widetilde{Y} - \widehat{Y}|^4}{z^4}\}^{\frac{1}{2}} \\ &\leq \frac{1}{1+z}O(\frac{1}{\sqrt{q}}). \end{aligned} \tag{5.39}$$

Using Chebyshev's inequality and (5.38), we obtain

$$T_{13} \leq Cq\{\frac{E|\widetilde{Y} - \widehat{Y}|^2}{(1+z)^2}\}^{\frac{1}{2}}\{O(\frac{1}{q^2})\}^{\frac{1}{2}} = \frac{1}{(1+z)}O(\frac{1}{\sqrt{q}}). \tag{5.40}$$

Finally, (5.37), (5.39) and (5.40) imply that

$$T_1 \leq \frac{1}{(1+z)}O(\frac{1}{\sqrt{q}}). \tag{5.41}$$

Equations (5.29), (5.30), (5.32), (5.34), (5.35) and (5.41) complete the proof.

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