กึ่งกรุปย่อยการแปลงเชิงเส้นของ $L_R(V,W)$ ซึ่งให้โครงสร้างของกึ่งไฮเพอร์ริงที่มีศูนย์

นายสำคัญ ฮ่อบรรทัด

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LINEAR TRANSFORMATION SUBSEMIGROUPS OF $L_R(V, W)$ ADMITTING THE STRUCTURE OF A SEMIHYPERRING WITH ZERO

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กึ่งไฮเพอร์ริงที่มีศูนย์ คือ ระบบ $(A,+,*)$ โดยที่ $(A,+)$ เป็นกึ่งไฮเพอร์กรุป $(A,*)$ เป็นกึ่งกรุป $*$ แจกแจงบน $+$ และมี $0 \in A$ (เรียกว่า *ศูนย์*) ที่ทำให้ $x+0=0+x=\{x\}$ และ $x*0=0*x=0$ สำหรับทุก $x \in A$ สำหรับกึ่งกรุป S กำหนดให้ S^0 คือ S ถ้า S มีศูนย์และ S มีสมาชิกมากกว่าหนึ่งตัว มิเช่นนั้นกำหนดให้ S^0 คือกึ่งกรุป S ที่ผนวกด้วยศูนย์ ดังนั้น S^0 เป็นกึ่งกรุปที่มีศูนย์ เรากล่าวว่ากึ่งกรุป *S ให้โครงสร้างของกึ่งไฮเพอร์ริงที่มีศูนย์* ถ้ามีการดำเนินการไฮเพอร์ $+$ บน S^0 ที่ทำให้ $(S^0,+,*)$ เป็นกึ่งไฮเพอร์ริงที่มีศูนย์ 0 โดยที่ $*$ เป็นการดำเนินการบน S^0 และ 0 เป็นศูนย์ของ S^0

กำหนดให้ V เป็นปริภูมิเวกเตอร์บนริงการหาร R, W เป็นปริภูมิย่อยของ V และ $L_R(V,W)$ เป็นกึ่งกรุปของการแปลงเชิงเส้นจาก V ไปยัง W ภายใต้การประกอบ สำหรับแต่ละ $\alpha \in L_R(V,W)$ กำหนดให้ $F(\alpha)$ ประกอบด้วยสมาชิกใน V ที่ α ตรึงสมาชิกนั้น กำหนดให้ $OM_R(V,W)$, $OE_R(V,W)$, $G_R(V,W)$, $AI_R(V,\underline{W})$ และ $AI_R(\underline{V},W)$ เป็นดังต่อไปนี้

$$OM_R(V,W)=\{\alpha \in L_R(V,W) \mid \dim_R \operatorname{Ker}\alpha = \infty\}$$
$$OE_R(V,W)=\{\alpha \in L_R(V,W) \mid \dim_R (W/\operatorname{Im}\alpha) = \infty\}$$
$$G_R(V,W)=\{\alpha \in L_R(V,W) \mid \alpha|_W \text{ เป็นสมสัณฐาน }\}$$
$$AI_R(V,\underline{W})=\{\alpha \in L_R(V,W) \mid \dim_R (W/F(\alpha)) < \infty\}$$
$$AI_R(\underline{V},W)=\{\alpha \in L_R(V,W) \mid \dim_R (V/F(\alpha)) < \infty\}$$

นอกจากนี้ กำหนดให้ H, S และ T เป็นกึ่งกรุปของ $G_R(V,W)$, $AI_R(V,\underline{W})$ และ $AI_R(\underline{V},W)$ ตามลำดับ เราแสดงว่า $OM_R(V,W)$, $OE_R(V,W)$, $OM_R(V,W)\cup H$, $OE_R(V,W)\cup H$, $OM_R(V,W)\cup S$, $OE_R(V,W)\cup S$, $OM_R(V,W)\cup T$ และ $OE_R(V,W)\cup T$ เป็นกึ่งกรุป ยิ่งไปกว่านั้น เรากำหนดว่ากึ่งกรุปเหล่านั้นให้โครงสร้างของกึ่งไฮเพอร์ริงที่มีศูนย์หรือไม่

ภาควิชา...........คณิตศาสตร์..........	ลายมือชื่อนิสิต สำคัญ ฮ่อบรรทัด
สาขาวิชา.........คณิตศาสตร์..........	ลายมือชื่ออาจารย์ที่ปรึกษาวิทยานิพนธ์หลัก [signature]
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A *semihyperring with zero* is a triple $(A,+,*)$ such that $(A,+)$ is a semihypergroup, $(A,*)$ is a semigroup, $*$ is distributive over $+$ and there exists $0 \in A$ (called a *zero*) such that $x+0=0+x=\{x\}$ and $x*0=0*x=0$ for all $x \in A$. For a semigroup S, let S^0 be S if S has a zero and S contains more than one element; otherwise, let S^0 be the semigroup S with a zero adjoined. Then S^0 is a semigroup with zero. We say that a semigroup S *admits the structure of a semihyperring with zero* if there exists a hyperoperation $+$ on S^0 such that $(S^0,+,*)$ is a semihyperring with zero 0 where $*$ is the operation on S^0 and 0 is the zero of S^0.

Let V be a vector space over a division ring R, W a subspace of V and $L_R(V,W)$ the semigroup of all linear transformations from V into W under composition. For each $\alpha \in L_R(V,W)$, let $F(\alpha)$ consist of all elements in V fixed by α. Let $OM_R(V,W)$, $OE_R(V,W)$, $G_R(V,W)$, $AI_R(V,\underline{W})$ and $AI_R(\underline{V},W)$ be as follows:

$$OM_R(V,W)=\{\alpha \in L_R(V,W) \mid \dim_R \operatorname{Ker}\alpha=\infty\},$$

$$OE_R(V,W)=\{\alpha \in L_R(V,W) \mid \dim_R(W/\operatorname{Im}\alpha)=\infty\},$$

$$G_R(V,W)=\{\alpha \in L_R(V,W) \mid \alpha|_W \text{ is an isomorphism}\},$$

$$AI_R(V,\underline{W})=\{\alpha \in L_R(V,W) \mid \dim_R(W/F(\alpha))<\infty\},$$

$$AI_R(\underline{V},W)=\{\alpha \in L_R(V,W) \mid \dim_R(V/F(\alpha))<\infty\}.$$

Moreover, let H, S and T be subsemigroups of $G_R(V,W)$, $AI_R(V,\underline{W})$ and $AI_R(\underline{V},W)$, respectively. We show that $OM_R(V,W)$, $OE_R(V,W)$, $OM_R(V,W)\cup H$, $OE_R(V,W)\cup H$, $OM_R(V,W)\cup S$, $OE_R(V,W)\cup S$, $OM_R(V,W)\cup T$ and $OE_R(V,W)\cup T$ are semigroups. Furthermore, we determine whether they admit the structure of a semihyperring with zero.

Department : ....Mathematics....	Student's Signature : ....
Field of study : ....Mathematics...	Principal Advisor's Signature ....
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CONTENTS

FREQUENTLY USED NOTATION

Let V be a vector space over a division ring R, W a subspace of V and α a linear transformation from V into W.

$\dim_R U$	the dimension of a vector space U over R
$\operatorname{Ker}\alpha$	the kernel of α
$\operatorname{Im}\alpha$	the image of α
$v\alpha$	the image of $v \in V$ under α
$\langle A \rangle$	the subspace of V spaned by a subset A of V
$L_R(V)$	the set of all linear transformations on V
$L_R(V, W)$	the set of all linear transformations from V into W
$F(\alpha)$	$= \{v \in V \mid v\alpha = v\}$
$OM_R(V, W)$	$= \{\alpha \in L_R(V, W) \mid \dim_R \operatorname{Ker}\alpha$ is infinite$\}$
$OE_R(V, W)$	$= \{\alpha \in L_R(V, W) \mid \dim_R(W/\operatorname{Im}\alpha)$ is infinite$\}$
$G_R(V, W)$	$= \{\alpha \in L_R(V, W) \mid \alpha$ is an isomorphism$\}$
$AI_R(V, \underline{W})$	$= \{\alpha \in L_R(V, W) \mid \dim_R(W/F(\alpha))$ is finite$\}$
$AI_R(\underline{V}, W)$	$= \{\alpha \in L_R(V, W) \mid \dim_R(V/F(\alpha))$ is finite$\}$
H	a subsemigroup of $G_R(V, W)$
S	a subsemigroup of $AI_R(V, \underline{W})$
T	a subsemigroup of $AI_R(\underline{V}, W)$
$\lvert B \rvert$	the cardinality of a set B
$\alpha = \begin{pmatrix} B_1 & v \\ 0 & v \end{pmatrix}_{v \in B \setminus B_1}$	the linear transformation defined on a basis B of V containing a set B_1 such that $v\alpha = \begin{cases} 0 & \text{if } v \in B_1, \\ v & \text{if } v \in B \setminus B_1 \end{cases}$

CHAPTER I
INTRODUCTION

There are two sections in this chapter. In the first section, we shall give some history of hyperstructure theory and some research in hyperstructure theory that have been studied. Moreover, the main purpose of this thesis will be addressed. In the other section, we shall give basic definitions and some examples of semi-hypergroups, semihyperrings, hypergroups and Krasner hyperrings. Furthermore, the meaning of a semigroup admitting some certain algebraic structures will be provided. At the end of this section, we also gather some results which will be used later on in the rest of this thesis.

1.1 Motivation

Hyperstructure theory was first known in 1934 by Marty. Later, it was interested by several investigator. Now, it is important topic which has conference on Hyperstructures only (International Conference on algebraic Hyperstructures and Its Applications). Basic definition of Marty is hypergroups as a generalization of groups. However, we know that the definition of semihypergroups, hypergroups, semihyperring and Krasner hyperrings are a generalization of semigroups, groups, semiring and rings, respectively. Moreover, any Krasner hyperrings is semihyperring.

The multiplicative structure of a semihyperring with zero, hyperring and ring is a semigroup with zero. It is reasonable to study which semigroups joining zero are isomorphic to the multiplicative structure of some semihyperrings, hyperrings and rings. A semigroup $(S, \cdot)$ with zero *admits the structure of a semihyperring with zero* if and only if there exists a hyperoperation $\circ$ on S such that $(S, \circ, \cdot)$ is a semihyperring with zero. A semigroup with zero admitting a hyperring or a ring structure are defined analogously.

Semigroups admitting a ring structure have long been studied, e.g., [1], [2], [3] and [10]. If we consider linear transformation semigroups, in particular, we found that M. Siripitukdet and Y. Kemprasit [1] studied when these semigroups admit a ring structure; Y. Kemprasit and Y. Punkla [4], Y. Punkla [5] and N. Rompurk [6] investigated when these semigroups admit a hyperring structure; S. Chaopraknoi and Y. Kemprasit [7] analyzed when these semigroups admit the structure of a semihyperring with zero. The work on linear transformation semigroups inspired us to investigate some specific linear transformation semigroups. The semigroups we considered are adopted from S. Chaopraknoi's Ph.D. Thesis [8]. She studied linear transformations from a vector space into itself. Here, we generalize to linear transformations from a vector space into its subspace.

The main purpose of this research is to study various types of linear transformations which form semigroups and to explore whether or when they admit the structure of a semihyperring with zero; furthermore, to extend the result to the case of admitting hyperring and ring structures.

This thesis is divided into three chapters. In Chapter I, we shall give precise definitions, notations, basic results which will be used throughout in Chapter II and Chapter III.

We show, in Chapter II, that the target subsets, which will be given later in page 8, are indeed subsemigroups of $L_R(V, W)$ containing zero.

In Chapter III, we investigate whether the aimed semigroups admit the structure of a semihyperring with zero. Also, the condition for admitting the structure of a semihyperring with zero is provided.

1.2 Preliminaries

For a semigroup $(S, \cdot)$, the semigroup $(S^0, *)$ is defined to be $(S, \cdot)$ if S has a zero and S contains more than one element; otherwise, let S^0 be the semigroup S with a zero 0 adjoined, that is, $S^0 = S \cup \{0\}$ where $0 \notin S$ and the operation $*$ is defined by $0 * x = x * 0 = 0$ for all $x \in S \cup \{0\}$ and $x * y = x \cdot y$ for all

$x, y \in S$. Note that if a semigroup S has only one element, then S^0 is a semigroup (which is not a group) of two elements and $(S^0, *) \cong (\mathbb{Z}_2, \cdot)$. Also, If G is a group, then $G^0 = G \cup \{0\}$. For a set X, let $P(X)$ denote the power set of X and $P^*(X) = P(X) \smallsetminus \{\varnothing\}$ and $|X|$ be the cardinality of X.

A *hyperoperation* on a nonempty set H is a mapping from $H \times H$ into $P^*(H)$. A *hypergroupoid* is a system $(H, \circ)$ consisting of a nonempty set H and a hyperoperation $\circ$ on H.

Let $(H, \circ)$ be a hypergroupoid. For nonempty subsets A and B of H and $x \in H$, let $A \circ x = A \circ \{x\}$, $x \circ A = \{x\} \circ A$ and

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b.$$

We call $(H, \circ)$ *commutative* if and only if $x \circ y = y \circ x$ for all $x, y \in H$. An element e of H is called an *identity* of $(H, \circ)$ if $x \in (x \circ e) \cap (e \circ x)$ for all $x \in H$. An element e of H is called a *scalar identity* of $(H, \circ)$ if $(x \circ e) \cap (e \circ x) = \{x\}$ for all $x \in H$. Then H has at most one scalar identity.

A *semihypergroup* is a hypergroupoid $(H, \circ)$ such that $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in H$.

Example 1.2.1. Let H be a nonempty set. Define a hyperoperation $\circ$ on H by

$$x \circ y = H \qquad \text{for all } x, y \in H.$$

Then $(H, \circ)$ is a semihypergroup.

A tripple $(A, +, \cdot)$ is called a *semihyperring* [*semiring*] if

(i) $(A, +)$ is a semihypergroup [semigroup],

(ii) $(A, \cdot)$ is a semigroup and

(iii) $+$ is distributive over $\cdot$, i.e., $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$ for all x, y, $z \in A$; this property is called the *distributive law*.

A semihyperring [semiring] $(A, +, \cdot)$ is said to be *additively commutative* if $x + y = y + x$ for all $x, y \in A$. An element 0 of a semihyperring [semiring] $(A, +, \cdot)$ is called a *zero* of $(A,+,\cdot)$ if $x+0=0+x=\{x\}$ $[x+0=0+x=x]$ and $x\cdot 0=0\cdot x=0$ for all $x\in A$. A semihyperring [semiring] with zero is a semihyperring [semiring] containing a zero element. By the definition, a semiring and a semiring with zero is a semihyperring and a semihyperring with zero, respectively.

A *hypergroup* is a semihypergroup $(H,\circ)$ such that $H\circ x=x\circ H=H$ for all $x\in H$. For x,y in a hypergroup $(H,\circ)$, x is called an *inverse* of y if there exists an identity e of $(H,\circ)$ such that $e\in(x\circ y)\cap(y\circ x)$. A hypergroup H is called *regular* if every element of H has an inverse in H. A regular hypergroup $(H,\circ)$ is said to be *reversible* if for $x,y,z\in H, x\in y\circ z$ implies $z\in u\circ x$ and $y\in x\circ v$ for some inverse u of y and some inverse v of z.

A *canonical hypergroup* is a hypergroup $(H,\circ)$ such that

(i) $(H,\circ)$ is commutative,

(ii) $(H,\circ)$ has a scalar identity,

(iii) every element of H has a unique inverse in H and

(iv) $(H,\circ)$ is reversible.

We can see that the semihypergroup in Example 1.2.1 is a hypergroup, which is called the *total hypergroup*, but it is not a canonical hypergroup because inverses of each element in H may not be unique.

Example 1.2.2. Let $H=\{0,x\}$ where x and 0 are distinct. Define a hyperoperation $\cdot$ on H by

$\cdot$	0	x
0	$\{0\}$	$\{x\}$
x	$\{x\}$	H

Then $(H,\cdot)$ is a canonical hypergroup.

A *Krasner hyperring* is a system $(A,+,\cdot)$ where

(i) $(A,+)$ is a canonical hypergroup,

(ii) $(A, \cdot)$ is a semigroup with zero 0 where 0 is the scalar identity of $(A, +)$ and

(iii) $+$ is distributive over $\cdot$.

Notice that every Krasner hyperring is a semihyperring with zero. Thus semihyperrings with zero are a generalization of Krasner hyperrings. In this research, by a hyperring we mean a Krasner hyperring.

Example 1.2.3. [7] Let G be a group. Define a hyperoperation $+$ on G^0 by

$$\begin{aligned} x + 0 = 0 + x = \{x\} \quad & \text{for all } x \in G^0, \\ x + x = G^0 \smallsetminus \{x\} \quad & \text{for all } x \in G, \\ x + y = \{x, y\} \quad & \text{for all distinct elements } x, y \in G. \end{aligned}$$

Then $(G^0, +, \cdot)$ is a hyperring where $\cdot$ is the operation on G^0. Note that the zero of the hyperring $(G^0, +, \cdot)$ is 0 and the inverse in $(G^0, +)$ of $x \in G$ is x itself. Also, $(G^0, +, \cdot)$ is not a ring if $|G| > 1$.

Example 1.2.4. [7] Let A be a set whose cardinality is at least 3 and 0 an element of A. Define a hyperoperation $+$ and an operation $\cdot$ on A by

$$\begin{aligned} x + 0 = 0 + x = \{x\} \quad & \text{for all } x \in A, \\ x + y = A \quad & \text{for all } x, y \in A \smallsetminus \{0\}, \\ x \cdot y = 0 \quad & \text{for all } x, y \in A. \end{aligned}$$

Then $(A, +, \cdot)$ is clearly a semihyperring with zero 0 but not a hyperring.

A semigroup S is said to *admit the structure of a semihyperring with zero* if there exists a hyperoperation $+$ on S^0 such that $(S^0, +, \cdot)$ is a semihyperring with zero where $\cdot$ is the operation on S^0. A semigroup S *admitting a hyperring [ring] structure* is given analogously. Observe that if S admits a ring [hyperring] structure, then S admits the structure of a semihyperring with zero. Consequently, if S does not admit the structure of a semihyperring with zero, then S does not admit a ring [hyperring] structure.

Let V be a vector space over a division ring R, W a subspace of V and $L_R(V,W)$ the semigroup of all linear transformations from V into W under composition. In particular, $L_R(V)$ is the set of all linear transformations on V. The image of $v \in V$ under $\alpha \in L_R(V,W)$ is written by $v\alpha$. For $\alpha \in L_R(V,W)$, let $\operatorname{Ker}\alpha$ and $\operatorname{Im}\alpha$ denote the kernel and the image of α, respectively. For $A \subseteq V$, let $\langle A \rangle$ stand for the subspace of V spanned by A. Moreover, $\dim_R U$ denotes the dimension of a vector space U over R. Since every linear transformation can be defined on its basis, for convenience, we write a linear transformation by using a blanket notation. For example,

$$\alpha = \begin{pmatrix} B_1 & v \\ 0 & v \end{pmatrix}_{v \in B \smallsetminus B_1}$$

means that α is a linear transformation from a vector space having B as a basis with $B_1 \subseteq B$ and

$$v\alpha = \begin{cases} 0 & \text{if } v \in B_1, \\ v & \text{if } v \in B \smallsetminus B_1, \end{cases}$$

(if $B = \varnothing$, then $v\alpha = v$ for all $v \in B$) and

$$\beta = \begin{pmatrix} u & w & v \\ w & 0 & v \end{pmatrix}_{v \in B \smallsetminus \{u,\, w\}}$$

means that β is a linear transformation from a vector space having B as a basis, u and w are distinct elements of B and

$$v\beta = \begin{cases} w & \text{if } v = u, \\ 0 & \text{if } v = w, \\ v & \text{if } v \in B \smallsetminus \{u,\, w\}. \end{cases}$$

The following propositions are simple facts of vector spaces and linear transformations which are major tools of our work. The proofs are routine and elementary so they will be omitted.

Proposition 1.2.5. *Let B be a basis of a vector space V. If u and w are distinct elements of B, then $\{u+w\} \cup (B \smallsetminus \{w\})$ is also a basis of V.*

Proposition 1.2.6. *Let B be a basis of a vector space V, $A \subseteq B$ and $\varphi : B \smallsetminus A \to V$ a one-to-one function such that $(B \smallsetminus A)\varphi$ is a linearly independent subset of V. If $\alpha \in L_R(V)$ is defined by*

$$\alpha = \begin{pmatrix} A & v \\ 0 & v\varphi \end{pmatrix}_{v \in B \smallsetminus A},$$

then $\operatorname{Ker} \alpha = \langle A \rangle$ *and* $\operatorname{Im} \alpha = \langle (B \smallsetminus A)\varphi \rangle$.

Proposition 1.2.7. *Let B be a basis of a vector space V and $A \subseteq B$. Then*
(i) $\{v + \langle A \rangle \mid v \in B \smallsetminus A\}$ *is a basis of the quotient space* $V/\langle A \rangle$ *and*
(ii) $\dim_R(V/\langle A \rangle) = |B \smallsetminus A|$.

Some of linear transformation subsemigroups of $L_R(V)$ studied in [8] are the followings:

$$OM_R(V) = \{\alpha \in L_R(V) \mid \dim_R \operatorname{Ker} \alpha \text{ is an infinite}\},$$
$$OE_R(V) = \{\alpha \in L_R(V) \mid \dim_R(V/\operatorname{Im} \alpha) \text{ is an infinite}\},$$
$$G_R(V) = \{\alpha \in L_R(V) \mid \alpha \text{ is an isomorphism}\},$$
$$AI_R(V) = \{\alpha \in L_R(V) \mid \dim_R(V/F(\alpha)) \text{ is finite}\},$$

where $F(\alpha) = \{v \in V \mid v\alpha = v\}$ for all $\alpha \in L_R(V)$. It is proved that $G_R(V)$ admits a ring structure if and only if $\dim_R V \leq 1$; if $\dim_R V$ is infinite, then $OM_R(V)$ and $OE_R(V)$ do not admit the structure of a semihyperring with zero; and if $\dim_R V$ is finite, then$AI_R(V)$ admits a ring structure.

We are interested in $L_R(V, W)$ instead of $L_R(V)$ and some linear transformation subsemigroups of $L_R(V, W)$ where W is a subspace of a vector space V over a division ring R. The natural question arises: "does generalized linear transformation subsemigroups of $L_R(V, W)$ (defined analogously to $OM_R(V)$, $OE_R(V)$, $G_R(V)$ and $AI_R(V)$) admit the structure of a semihyperring with zero?".

In this thesis, let V be a vector space over a division ring R and W a subspace of V. Moreover, let

$$\begin{aligned}
OM_R(V,W) &= \{\alpha \in L_R(V,W) \mid \dim_R \operatorname{Ker}\alpha \text{ is infinite}\}\,,\\
OE_R(V,W) &= \{\alpha \in L_R(V,W) \mid \dim_R (W/\operatorname{Im}\alpha) \text{ is infinite}\}\,,\\
G_R(V,W) &= \{\alpha \in L_R(V,W) \mid \alpha \text{ is an isomorphism}\}\,,\\
AI_R(V,\underline{W}) &= \{\alpha \in L_R(V,W) \mid \dim_R (W/F(\alpha)) \text{ is finite}\}\,,\\
AI_R(\underline{V},W) &= \{\alpha \in L_R(V,W) \mid \dim_R (V/F(\alpha)) \text{ is finite}\}\,,
\end{aligned}$$

where $F(\alpha) = \{v \in V \mid v\alpha = v\}$, the set of all elements of V fixed by α, is a subspace of V for all $\alpha \in L_R(V,W)$. Clearly, $AI_R(\underline{V},W) \subseteq AI_R(V,\underline{W})$. We investigate the following target subsets of $L_R(V,W)$:

$$\begin{array}{llll}
OM_R(V,W), & OM_R(V,W)\cup H, & OM_R(V,W)\cup S, & OM_R(V,W)\cup T,\\
OE_R(V,W), & OE_R(V,W)\cup H, & OE_R(V,W)\cup S, & OE_R(V,W)\cup T
\end{array}$$

where H, S and T are subsemigroups of $G_R(V,W)$, $AI_R(V,\underline{W})$ and $AI_R(\underline{V},W)$, respectively.

Assume that $\dim_R V$ is finite. Then

$$\begin{aligned}
OM_R(V,W) = OE_R(V,W) &= \varnothing,\\
G_R(V,W) &= G_R(V),\\
AI_R(V,\underline{W}) = AI_R(\underline{V},W) &= L_R(V,W).
\end{aligned}$$

Thus $OM_R(V,W)$ and $OE_R(V,W)$ are not semigroups, $G_R(V,W)$ admits a ring structure if and only if $\dim_R V \le 1$ and both $AI_R(V,\underline{W})$ and $AI_R(\underline{V},W)$ admit the structure of a semihyperring with zero because they admit a ring structure under the usual addition. As a result, throughout the rest of this thesis, we consider only when $\dim_R V$ is infinite.

Assume that $\dim_R W$ is finite. Then

$$\begin{aligned}
OM_R(V,W) = AI_R(V,\underline{W}) &= L_R(V,W) \quad \text{and}\\
OE_R(V,W) = G_R(V,W) = AI_R(\underline{V},W) &= \varnothing.
\end{aligned}$$

Hence $OM_R(V,W)$ and $AI_R(V,\underline{W})$ admit the structure of a semihyperring with zero but $OE_R(V,W)$, $G_R(V,W)$ and $AI_R(\underline{V},W)$ are not semigroups. Thus we consider only when $\dim_R W$ is infinite for the remaining of this thesis.

The simple question "are $OM_R(V,W)$, $OE_R(V,W)$, $G_R(V,W)$, $AI_R(V,\underline{W})$ and $AI_R(\underline{V},W)$ subsemigroups of $L_R(V,W)$?" need to be taken into account. It is obvious that $G_R(V,W)$ is a subsemigroup of $L_R(V,W)$. Moreover, $OM_R(V,W)$ and $OE_R(V,W)$ are subsemigroups of $L_R(V,W)$ as follows.

Proposition 1.2.8. *$OM_R(V,W)$ and $OE_R(V,W)$ are subsemigroups of $L_R(V,W)$ containing zero.*

Proof. Note that $\operatorname{Ker}\alpha \subseteq \operatorname{Ker}\alpha\beta$ and $\operatorname{Im}\alpha\beta \subseteq \operatorname{Im}\beta$ for each $\alpha,\ \beta \in L_R(V,W)$. Then $OM_R(V,W)$ and $OE_R(V,W)$ are both subsemigroups of $L_R(V,W)$. Since $\dim_R V$ and $\dim_R W$ are infinite, the zero map belongs to both $OM_R(V,W)$ and $OE_R(V,W)$. In fact, the zero map is, actually, the zero of the semigroups $OM_R(V,W)$ and $OE_R(V,W)$. $\square$

Finally, we present that both $AI_R(V,\underline{W})$ and $AI_R(\underline{V},W)$ are subsemigroups of $L_R(V,W)$.

Proposition 1.2.9. *$AI_R(V,\underline{W})$ and $AI_R(\underline{V},W)$ are subsemigroups of $L_R(V,W)$ not containing zero.*

Proof. We show only that $AI_R(V,\underline{W})$ is a subsemigroup of $L_R(V,W)$ not containing zero because the proof for the case $AI_R(\underline{V},W)$ is obtained similarly.

Let $\alpha,\beta \in AI_R(V,\underline{W})$. Then $\dim_R(W/F(\alpha))$ and $\dim_R(W/F(\beta))$ are finite. We claim that $\dim_R(W/F(\alpha\beta))$ is finite. Since $F(\alpha)\cap F(\beta) \subseteq F(\alpha\beta)$, it suffices to show only that $\dim_R\Big(W/\big(F(\alpha)\cap F(\beta)\big)\Big)$ is finite.

Let B_1 be a basis of $F(\alpha)\cap F(\beta)$, $B_2 \subseteq F(\alpha)\smallsetminus B_1$ and $B_3 \subseteq F(\beta)\smallsetminus B_1$ be such that $B_1\cup B_2$ and $B_1\cup B_3$ are bases of $F(\alpha)$ and $F(\beta)$, respectively. We will show that $B_1\cup B_2\cup B_3$ is linearly independent over R. Let $u_1,\ u_2,\ldots,\ u_k \in B_1\cup B_2$, $v_1,\ v_2,\ldots,\ v_l \in B_3$ be all distinct and $a_1,\ a_2,\ldots,\ a_k,\ b_1,\ b_2,\ldots,\ b_l \in R$ be such that

$$\sum_{i=1}^{k} a_i u_i + \sum_{j=1}^{l} b_j v_j = 0.$$

Then $\sum_{i=1}^{k} a_i u_i = -\sum_{j=1}^{l} b_j v_j \in F(\alpha) \cap F(\beta) = \langle B_1 \rangle$. Hence

$$\sum_{j=1}^{l} b_j v_j \in \langle B_1 \rangle \cap \langle B_3 \rangle = \{0\}.$$

Since B_3 is linearly independent, $b_j = 0$ for all j, so $\sum_{i=1}^{k} a_i u_i = 0$. This implies that $a_i = 0$ for all i because of the linearly independence of $B_1 \cup B_2$. Hence $B_1 \cup B_2 \cup B_3$ is linearly independent over R. Let $B_4 \subseteq W \smallsetminus (B_1 \cup B_2 \cup B_3)$ be such that $B_1 \cup B_2 \cup B_3 \cup B_4$ is a basis of V. Hence $\{v + F(\alpha) \mid v \in B_3 \cup B_4\}$, $\{v + F(\beta) \mid v \in B_2 \cup B_4\}$ and $\{v + (F(\alpha) \cap F(\beta)) \mid v \in B_2 \cup B_3 \cup B_4\}$ are bases of $W/F(\alpha)$, $W/F(\beta)$ and $W/(F(\alpha) \cap F(\beta))$, respectively. This implies that $\dim_R\Big(W/\big(F(\alpha) \cap F(\beta)\big)\Big)$ is finite as desired. Therefore, $\alpha\beta \in AI_R(V, \underline{W})$. □

Finally, we end this chapter by giving an example of a subsemigroup of $L_R(V, W)$ which does not admit the structure of a semihyperring with zero as follows:

Example 1.2.10. Let B and C be bases of V and W, respectively, such that $C \subseteq B$. Also, let v_1 and v_2 be fixed distinct elements in C. Define linear transformations α and β in $L_R(V, W)$ by

$$\alpha = \begin{pmatrix} v_1 & B \smallsetminus \{v_1\} \\ v_2 & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} v_2 & B \smallsetminus \{v_2\} \\ v_1 & 0 \end{pmatrix}.$$

Clearly, $\alpha^2 = \beta^2 = 0$ and

$$\alpha\beta = \begin{pmatrix} v_1 & B \smallsetminus \{v_1\} \\ v_1 & 0 \end{pmatrix} \quad \text{and} \quad \beta\alpha = \begin{pmatrix} v_2 & B \smallsetminus \{v_2\} \\ v_2 & 0 \end{pmatrix}.$$

Let S be the semigroup generated by α and β. It is obvious that

$$S = \{0,\ \alpha,\ \beta,\ \alpha\beta,\ \beta\alpha\}.$$

Next, suppose that there exists a hyperoperation $\oplus$ such that $(S, \oplus, \cdot)$ is a semihyperring with zero where $\cdot$ is the operation on S. By the distributive law,

$$(\alpha \oplus \beta)\beta = \alpha\beta \oplus \beta^2 = \alpha\beta \oplus 0 = \{\alpha\beta\}$$
$$(\alpha \oplus \beta)\alpha = \alpha^2 \oplus \beta\alpha = 0 \oplus \beta\alpha = \{\beta\alpha\}.$$

Let $\lambda \in \alpha \oplus \beta$. Then we have $\lambda\beta = \alpha\beta$ and $\lambda\alpha = \beta\alpha$. Consider $v_1\lambda\beta = v_1\alpha\beta = v_1$ so $v_1\lambda = v_2 + \sum_{i=1}^{n} a_i w_i$ for some $w_i \in C$ and for some $a_i \in R$. Since $\lambda \in S$, we obtain that $\lambda = \alpha$ only. But $v_2\lambda\alpha = v_2\beta\alpha = v_2$ so $v_2\lambda = v_1 + \sum_{i=1}^{n} a_i w_i$ for some $w_i \in C$ and for some $a_i \in R$. This shows that $\lambda = \beta$ which is impossible.

CHAPTER II
CERTAIN SUBSEMIGROUPS OF $L_R(V,W)$

We know from Chapter I that $OM_R(V,W), OE_R(V,W), G_R(V,W), AI_R(V,\underline{W})$ and $AI_R(\underline{V},W)$ are subsemigroups of $L_R(V,W)$. Let U be a subsemigroup of $L_R(V,W)$. Naturally, ones may ask whether $OM_R(V,W)\cup U$ and $OE_R(V,W)\cup U$ are subsemigroups of $L_R(V,W)$. The following examples show that this is not generally true.

Example Let $\dim_R W = \dim_R V$. Then there is a subsemigroup U of $L_R(V,W)$ such that $OM_R(V,W)\cup U$ is not a semigroup.

To see this, let B and C be bases of V and W, respectively, such that $C\subseteq B$. Since C is infinite, there are disjoint subsets C_1 and C_2 of C such that $C_1\cup C_2 = C$ and $|C_1| = |C_2| = |C|$. Hence $|B| = |C_1|$ and then there are bijections $\varphi : B\to C_1$ and $\phi : C_1 \to C_2$. Define $\alpha,\ \gamma \in L_R(V,W)$ by

$$\alpha = \begin{pmatrix} v \\ v\varphi \end{pmatrix}_{v\in B} \quad \text{and} \quad \gamma = \begin{pmatrix} v & B\smallsetminus C_1 \\ v\phi & 0 \end{pmatrix}_{v\in C_1}.$$

It is obvious that α is a bijection and $\operatorname{Im}\alpha = \langle C_1\rangle$. Let U be the subsemigroup of $L_R(V,W)$ generated by α. Clearly, $\operatorname{Im}\beta = \langle C_1\rangle$ for all $\beta\in U$. Moreover, $\gamma\in OM_R(V,W)$ because $\operatorname{Ker}\gamma = \langle B\smallsetminus C_1\rangle$. Consider $\alpha\gamma = \binom{v}{v\varphi\phi}_{v\in B}$. Hence $\operatorname{Ker}\alpha\gamma = \{0\}$ and $\operatorname{Im}\alpha\gamma = \langle C_2\rangle$. Thus $\alpha\gamma\notin OM_R(V,W)$ and $\alpha\gamma\notin U$. Therefore, $OM_R(V,W)\cup U$ is not a semigroup.

Example Let $\dim_R W = \dim_R V$. Then there is a subsemigroup U of $L_R(V,W)$ such that $OE_R(V,W)\cup U$ is not a semigroup.

To show this, let B and C be bases of V and W, respectively, such that $C\subseteq B$. Since C is infinite, there are disjoint subsets C_1 and C_2 of C such that $C_1\cup C_2 = C$

and $|C_1| = |C_2| = |C|$. Hence there are bijections $\varphi : C_2 \to C$ and $\phi : C_1 \to C_2$. Define $\alpha,\ \gamma \in L_R(V,W)$ by

$$\alpha = \begin{pmatrix} v & B \setminus C_2 \\ v\varphi & 0 \end{pmatrix}_{v \in C_2} \quad \text{and} \quad \gamma = \begin{pmatrix} v & B \setminus C_1 \\ v\phi & 0 \end{pmatrix}_{v \in C_1}.$$

Let U be the subsemigroup of $L_R(V,W)$ generated by α. It is obvious that domain of each element in U is $\langle C_2 \rangle$. It is clear that $\gamma \in OE_R(V,W)$ because $\dim_R\big(W/\operatorname{Im}\gamma\big) = |C_1|$. Consider

$$\gamma\alpha = \begin{pmatrix} v & B \setminus C_1 \\ v\phi\varphi & 0 \end{pmatrix}_{v \in C_1}.$$

Hence $\operatorname{Im}\gamma\alpha = W$ and domain of $\gamma\alpha$ is $\langle C_1 \rangle$. Thus $\gamma\alpha \notin OE_R(V,W)$ and $\gamma\alpha \notin U$. Therefore $OE_R(V,W) \cup U$ is not a semigroup.

Proposition 2.1.7 and Proposition 2.2.7 tell that there are subsemigroups U_1 and U_2 of $L_R(V,W)$ such that $OM_R(V,W) \cup U_1$ and $OE_R(V,W) \cup U_2$ are semigroups, respectively. In view of those, the main purpose of this chapter is to show that the following subsets of $L_R(V,W)$ are subsemigroups of $L_R(V,W)$:

(1) subsets containing $OM_R(V,W)$, namely, $OM_R(V,W) \cup H$, $OM_R(V,W) \cup S$ and $OM_R(V,W) \cup T$;

(2) subsets containing $OE_R(V,W)$, namely, $OE_R(V,W) \cup H$, $OE_R(V,W) \cup S$ and $OE_R(V,W) \cup T$,

where H, S and T are subsemigroups of $G_R(V,W)$, $AI_R(V,\underline{W})$ and $AI_R(\underline{V},W)$, respectively.

2.1 Certain Semigroups Containing $OM_R(V,W)$

We illustrate first that $OM_R(V,W)$ is, in fact, a right ideal of $L_R(V,W)$. Then $OM_R(V,W) \cup H$ is shown to be a semigroup.

Lemma 2.1.1. *$OM_R(V,W)$ is a right ideal of $L_R(V,W)$.*

Proof. Proposition 1.2.8 provides that $OM_R(V,W)$ is a subsemigroup of $L_R(V,W)$. To show that $OM_R(V,W)$ is a right ideal of $L_R(V,W)$, let $\alpha \in OM_R(V,W)$ and

$\beta \in L_R(V,W)$. Then $\dim_R \operatorname{Ker}\alpha$ is infinite. Note also that $\operatorname{Ker}\alpha \subseteq \operatorname{Ker}\alpha\beta$. This leads to the conclusion that $\dim_R \operatorname{Ker}\alpha\beta$ is infinite. Thus $\alpha\beta \in OM_R(V,W)$. $\square$

Next example shows that $OM_R(V,W)$ is not a left ideal of $L_R(V,W)$.

Example 2.1.2. Let $\dim_R W = \dim_R V$, B and C be bases of V and W, respectively, such that $C \subseteq B$. Since C is infinite, there are disjoint subsets C_1 and C_2 of C such that $|C_1| = |C_2| = |C| = |B|$. Thus there is a bijection $\phi : B \to C_1$. Define $\alpha, \beta \in L_R(V,W)$ by

$$\alpha = \begin{pmatrix} v & B \setminus C_1 \\ v & 0 \end{pmatrix}_{v \in C_1} \quad \text{and} \quad \beta = \phi.$$

Clearly, $\alpha \in OM_R(V,W)$ but $\beta \notin OM_R(V,W)$. It is obvious that $\beta\alpha = \beta$. This shows that $OM_R(V,W)$ is not a left ideal of $L_R(V,W)$.

We have shown that $OM_R(V,W)$ is only a right ideal but not a left ideal of $L_R(V,W)$. To present that $OM_R(V,W) \cup H$ is a semigroup, we prove the following lemma.

Lemma 2.1.3. *$G_R(V,W)OM_R(V,W) \subseteq OM_R(V,W)$.*

Proof. Let $\alpha \in G_R(V,W)$ and $\beta \in OM_R(V,W)$. We claim that $(\operatorname{Ker}\alpha\beta)\alpha = \operatorname{Ker}\beta$. Clearly, $v\alpha\beta = 0$ for all $v \in \operatorname{Ker}\alpha\beta$ whence $(\operatorname{Ker}\alpha\beta)\alpha \subseteq \operatorname{Ker}\beta$. Let $v \in \operatorname{Ker}\beta$. Note that α^{-1} exists since $\alpha \in G_R(V,W)$. Then $0 = v\beta = (v\alpha^{-1})\alpha\beta$ so that $v\alpha^{-1} \in \operatorname{Ker}\alpha\beta$. Thus

$$v = (v\alpha^{-1})\alpha \in (\operatorname{Ker}\alpha\beta)\alpha.$$

This shows that $\operatorname{Ker}\beta \subseteq (\operatorname{Ker}\alpha\beta)\alpha$. Therefore $(\operatorname{Ker}\alpha\beta)\alpha = \operatorname{Ker}\beta$ as claimed. Since α is an isomorphism and $\dim_R \operatorname{Ker}\beta$ is infinite, $\dim_R \operatorname{Ker}\alpha\beta$ is also infinite. Hence $\alpha\beta \in OM_R(V,W)$. $\square$

Proposition 2.1.4. *$OM_R(V,W) \cup H$ is a subsemigroup of $L_R(V,W)$.*

Proof. The result follows from Lemma 2.1.1, Lemma 2.1.3 and the fact that $OM_R(V,W)$ and H are subsemigroups of $L_R(V,W)$. $\square$

Next, in the same manner, we show that $OM_R(V,W)\cup S$ is a semigroup by proving that $AI_R(V,\underline{W})OM_R(V,W) \subseteq OM_R(V,W)$. However, the following lemma is needed.

Lemma 2.1.5. *Let $\alpha \in AI_R(V,\underline{W})$, B, C and E be bases of V, W and* $\operatorname{Ker}\alpha$*, respectively, such that B contains C and E. If $B \smallsetminus C$ is infinite and E is finite, then there are $w \in B \smallsetminus (C \cup E)$ and $v \in V \smallsetminus \langle E \cup \{w\}\rangle$ such that $w\alpha = v\alpha$.*

Proof. Assume that $B \smallsetminus C$ is infinite and E is finite. Let $E = \{v_1', v_2', \ldots, v_k'\}$. Clearly, $B \smallsetminus (C \cup E)$ is infinite. Suppose that

$$\text{for every } w \in B \smallsetminus (C \cup E) \text{ for every } v \in V \smallsetminus \langle E \cup \{w\}\rangle,\ w\alpha \neq v\alpha. \qquad (1)$$

Hence

$$w_1\alpha \neq w_2\alpha \quad \text{for every distinct } w_1, w_2 \in B \smallsetminus (C \cup E). \qquad (2)$$

We seperate the proof into five steps.

Step 1. $\big\{w\alpha \,|\, w \in B \smallsetminus (C \cup E)\big\}$ is an infinite linearly independent subset of W.

Step 2. For $w \in \langle B \smallsetminus (C \cup E)\rangle$, if $w\alpha \in F(\alpha)$, then $w = 0$.

Step 3. For every $w \in B \smallsetminus (C \cup E)$, $w\alpha \notin F(\alpha)$.

Step 4. $\big\{w\alpha + F(\alpha) \,|\, w \in B \smallsetminus (C \cup E)\big\}$ is a linearly independent subset of $W/F(\alpha)$.

Step 5. For all v, $w \in B \smallsetminus (C \cup E)$, if $v\alpha \neq w\alpha$, then $v\alpha + F(\alpha) \neq w\alpha + F(\alpha)$.

We conclude from these steps that $\{w\alpha + F(\alpha) \,|\, w \in B \smallsetminus (C \cup E)\}$ is an infinite linearly independent subset of $W/F(\alpha)$. Hence $\dim_R\big(W/F(\alpha)\big)$ is infinite contradicting the fact that $\alpha \in AI_R(V,\underline{W})$. Therefore, the result is obtained. It remain to prove Step 1-Step 5.

Step 1. Since $B \smallsetminus (C \cup E)$ is infinite and (2), we obtain that the set $\{w\alpha \,|\, w \in B \smallsetminus (C \cup E)\}$ is infinite. Next, we show that $\{w\alpha \,|\, w \in B \smallsetminus (C \cup E)\}$ is linearly independent. Let $w_1, w_2, \ldots, w_n \in B \smallsetminus (C \cup E)$ be all distinct and $a_1, a_2, \ldots, a_n \in R$ be such that.

$$a_1w_1\alpha + a_2w_2\alpha + \cdots + a_nw_n\alpha = 0.$$

Then

$$(a_1w_1 + a_2w_2 + \cdots + a_nw_n)\alpha = 0$$

so that $a_1w_1 + a_2w_2 + \cdots + a_nw_n \in \operatorname{Ker}\alpha$. Thus

$$a_1w_1 + a_2w_2 + \cdots + a_nw_n \in \langle E\rangle \cap \langle B \smallsetminus (C \cup E)\rangle = \{0\}.$$

As a result, $a_1w_1 + a_2w_2 + \cdots + a_nw_n = 0$ and then $a_1 = a_2 = \cdots = a_n = 0$. Hence $\{w\alpha \,|\, w \in B \smallsetminus (C \cup E)\}$ is a linearly independent set as claimed.

Step 2. Let $w \in \langle B \smallsetminus (C \cup E)\rangle$. Assume that $w\alpha \in F(\alpha)$, i.e., $(w\alpha)\alpha = w\alpha$. Then $w\alpha - w \in \operatorname{Ker}\alpha = \langle E\rangle$. Thus $w\alpha - w = \sum_{i=1}^{n} a_iv_i'$. Hence $w = w\alpha - \sum_{i=1}^{n} a_iv_i' \in \langle C \cup E\rangle$. Therefore $w \in \langle B \smallsetminus (C \cup E)\rangle \cap \langle C \cup E\rangle$. Thus $w = 0$.

Step 3. Let $w \in B \smallsetminus (C \cup E)$. Suppose that $w\alpha \in F(\alpha)$. By Step 2, $w = 0$ leading to a contradiction. Hence $w\alpha \notin F(\alpha)$ for all $w \in B \smallsetminus (C \cup E)$.

Step 4. Let $w_1, w_2, \ldots, w_n \in B \smallsetminus (C \cup E)$ be distinct and $a_1, a_2, \ldots, a_n \in R$ be such that

$$\sum_{i=1}^{n} a_i\big(w_i\alpha + F(\alpha)\big) = F(\alpha).$$

Hence $\sum_{i=1}^{n} a_iw_i\alpha \in F(\alpha)$. Thus $(\sum_{i=1}^{n} a_iw_i\alpha)\alpha = \sum_{i=1}^{n} a_iw_i\alpha$ so $\big(\sum_{i=1}^{n} a_iw_i\alpha - \sum_{i=1}^{n} a_iw_i\big)\alpha = 0$, i.e., $\sum_{i=1}^{n} a_iw_i\alpha - \sum_{i=1}^{n} a_iw_i \in \operatorname{Ker}\alpha$. It follows that

$$\sum_{i=1}^{n} a_iw_i\alpha - \sum_{i=1}^{n} a_iw_i = \sum_{j=1}^{k} b_jv_j'.$$

Thus

$$\sum_{i=1}^{n} a_iw_i = \sum_{i=1}^{n} a_iw_i\alpha - \sum_{j=1}^{k} b_jv_j' \in \langle C \cup E\rangle.$$

This implies that $\sum_{i=1}^{n} a_iw_i \in \big\langle B \smallsetminus (C \cup E)\big\rangle \cap \langle C \cup E\rangle = \{0\}$ so $a_1 = a_2 = \cdots = a_n = 0$. Hence $\big\{w\alpha + F(\alpha) \,|\, w \in B \smallsetminus (C \cup E)\big\}$ is a linearly independent subset of $W/F(\alpha)$ as desired.

Step 5. Let $v, w \in B \smallsetminus (C \cup E)$ be such that $v\alpha \neq w\alpha$. Suppose that $v\alpha + F(\alpha) =$

$w\alpha + F(\alpha)$. We obtain that $v\alpha - w\alpha \in F(\alpha)$. Hence $(v\alpha - w\alpha)\alpha = v\alpha - w\alpha$. Thus $(v\alpha - w\alpha)\alpha + w\alpha = v\alpha$. Therefore

$$(v\alpha - w\alpha + w)\alpha = v\alpha. \tag{3}$$

If $v\alpha - w\alpha + w \in \langle E \cup \{v\}\rangle$, then $v\alpha - w\alpha + w = bv + \sum_{i=1}^{k} a_i v_i'$ where $b,\ a_1,\ a_2,\ \ldots,\ a_k \in R$. It is clear that $bv - w = v\alpha - w\alpha - \sum_{i=1}^{k} a_i v_i' \in \langle C \cup E\rangle$. Therefore

$$bv - w \in \langle B \smallsetminus (C \cup E)\rangle \cap \langle C \cup E\rangle = \{0\}$$

so that $bv = w$ which is impossible. Hence $v\alpha - w\alpha + w \notin \langle E \cup \{v\}\rangle$. From (1), $(v\alpha - w\alpha + w)\alpha \neq v\alpha$ contradicting (3). □

Lemma 2.1.6. $AI_R(V, \underline{W})OM_R(V, W) \subseteq OM_R(V, W)$.

Proof. Let $\alpha \in AI_R(V, \underline{W})$ and $\beta \in OM_R(V, W)$. Let B_1 be a basis of $F(\alpha) \cap \operatorname{Ker}\beta$, $B_2 \subseteq \operatorname{Ker}\beta \smallsetminus B_1$ such that $B_1 \cup B_2$ a basis of $\operatorname{Ker}\beta \cap W$, $B_3 \subseteq \operatorname{Ker}\beta \smallsetminus B_1 \cup B_2$ such that $B_1 \cup B_2 \cup B_3$ a basis of $\operatorname{Ker}\beta$. Then $B_1 \cup B_2 \cup B_3$ is infinite because $\beta \in OM_R(V, W)$. Next, we claim that $\{v + F(\alpha) \mid v \in B_2\}$ is a finite linearly independent subset of quotient space $W/F(\alpha)$. Let $v_1, v_2, \ldots, v_n$ be distinct elements of B_2 and let $a_1, a_2, \ldots, a_n \in R$ be such that $\sum_{i=1}^{n} a_i\big(v_i + F(\alpha)\big) = F(\alpha)$. Then $\sum_{i=1}^{n} a_i v_i \in F(\alpha) \cap \operatorname{Ker}\beta$. But B_1 is a basis of $F(\alpha) \cap \operatorname{Ker}\beta$ and $B_1 \cup B_2$ is linearly independent over R, so $a_i = 0$ for all i. This shows that $\{v + F(\alpha) | v \in B_2\}$ is a linearly independent subset of the quotient space $W/F(\alpha)$ and $u + F(\alpha) \neq w + F(\alpha)$ for all distinct $u, w \in B_2$. Since $\dim_R\big(W/F(\alpha)\big) < \infty$, we obtain that $\{v + F(\alpha) | v \in B_2\}$ is finite. But $|\,\{v + F(\alpha) | v \in B_2\}\,| = |B_2|$, thus B_2 is finite. Let $B_4 \subseteq W \smallsetminus B_1 \cup B_2$ be such that $B_1 \cup B_2 \cup B_4$ is a basis of W and let $B_1 \cup B_2 \cup B_4 = C$. Moreover, let $B_5 \subseteq V \smallsetminus C \cup B_3$ be such that $C \cup B_3 \cup B_5$ is a basis of V and let $B = C \cup B_3 \cup B_5$.

Case 1. $B \smallsetminus C$ is finite. Since $B_3 \subseteq B \smallsetminus C$, we have $|B_3| \leq |B \smallsetminus C|$. Thus B_3 is finite. Hence $B_2 \cup B_3$ is finite. This implies that B_1 is infinite because $B_1 \cup B_2 \cup B_3$

is infinite. Since $B_1 \subseteq F(\alpha) \cap \operatorname{Ker} \beta$, we have $B_1\alpha\beta = B_1\beta = \{0\}$ so $B_1 \subseteq \operatorname{Ker} \alpha\beta$. Hence $\dim_R \operatorname{Ker} \alpha\beta$ is infinite. Thus $\alpha\beta \in OM_R(V, W)$.

Case 2. $B \setminus C$ is infinite. We claim that $\dim_R \operatorname{Ker} \alpha$ is infinite. Suppose that $\dim_R \operatorname{Ker} \alpha$ is finite. Let E be a basis of $\operatorname{Ker} \alpha$. Lemma 2.1.5 provides that there are $w \in B \setminus (C \cup E)$ and

$v \in V \setminus \langle E \cup \{w\} \rangle$ such that $w\alpha = v\alpha$. Since $v \in V = \langle B \rangle$, there are $v_1, v_2, \ldots, v_m \in B$ and $b_1, b_2, \ldots, b_m \in R$ such that $v = b_1v_1 + b_2v_2 + \cdots + b_mv_m$. Assume, without loss of generality, that

$$v = b_1v_1 + b_2v_2 + \cdots + b_lv_l + b_{l+1}v_{l+1} + \cdots + b_mv_m$$

where $v_{l+1}, v_{l+2}, \ldots, v_m \in E$. We see that

$$\begin{aligned} w\alpha &= v\alpha \\ &= (b_1v_1 + b_2v_2 + \cdots + b_lv_l + b_{l+1}v_{l+1} + \cdots + b_mv_m)\alpha \\ &= (b_1v_1 + b_2v_2 + \cdots + b_lv_l)\alpha \end{aligned}$$

Hence $(w - b_1v_1 - b_2v_2 - \cdots - b_lv_l)\alpha = 0$ so $w - b_1v_1 - b_2v_2 - \cdots - b_lv_l \in \operatorname{Ker} \alpha$. Thus

$$w - b_1v_1 - b_2v_2 - \cdots - b_lv_l = c_1v'_1 + c_2v'_2 + \cdots + c_kv'_k.$$

Therefore

$$w = b_1v_1 + b_2v_2 + \cdots + b_lv_l + c_1v'_1 + c_2v'_2 + \cdots + c_kv'_k.$$

Subcase 2.1 $w \neq v_j$ for all $j \in \{1, 2, \ldots, l\}$. Hence w is written in a linear combination of $B \setminus \{w\}$ which is a contradiction.

Subcase 2.2 $w = v_j$ for some $j \in \{1, 2, \ldots, l\}$. Assume, without loss of generality, that $w = v_1$. Hence

$$w = b_1w + b_2v_2 + \cdots + b_lv_l + c_1v'_1 + c_2v'_2 + \cdots + c_kv'_k.$$

Thus $0 = (b_1 - 1)w + b_2v_2 + \cdots + b_lv_l + c_1v'_1 + c_2v'_2 + \cdots + c_kv'_k$. This implies that

$$b_1 - 1 = b_2 = \cdots = b_l = c_1 = \cdots = c_k = 0,$$

so $b_1 = 1$ and

$$\begin{aligned} v &= b_1v_1 + b_2v_2 + \cdots + b_lv_l + b_{l+1}v_{l+1} + \cdots + b_mv_m \\ &= v_1 + b_{l+1}v_{l+1} + \cdots + b_mv_m \\ &= w + b_{l+1}v_{l+1} + \cdots + b_mv_m \\ &\in \langle E \cup \{w\} \rangle . \end{aligned}$$

This is a contradiction.

Hence $\dim_R \operatorname{Ker} \alpha$ is infinite. Consequently, $\dim_R \operatorname{Ker} \alpha\beta$ is infinite because of $\operatorname{Ker} \alpha \subseteq \operatorname{Ker} \alpha\beta$. Therefore $\alpha\beta \in OM_R(V, W)$. □

Proposition 2.1.7. *$OM_R(V, W) \cup S$ is a subsemigroup of $L_R(V, W)$.*

Proof. The result follows from applying Lemma 2.1.1 and Lemma 2.1.6 and the fact that $OM_R(V, W)$ and S are subsemigroups of $L_R(V, W)$. □

Proposition 2.1.8. *$OM_R(V, W) \cup T$ is a subsemigroup of $L_R(V, W)$.*

Proof. The result follows from the fact that $AI_R(\underline{V}, W) \subseteq AI_R(V, \underline{W})$ and Proposition 2.1.7. □

2.2 Certain Semigroups Containing $OE_R(V, W)$

Likewise, we notice that $OE_R(V, W)$ is a left ideal but not a right ideal of $L_R(V, W)$.

Lemma 2.2.1. *$OE_R(V, W)$ is a left ideal of $L_R(V, W)$.*

Proof. Proposition 1.2.8 show that $OE_R(V, W)$ is a subsemigroup of $L_R(V, W)$. Next, let $\alpha \in L_R(V, W)$ and $\beta \in OE_R(V, W)$. Then $\dim_R(W/\operatorname{Im} \alpha\beta)$ is infinite because
$\dim_R(W/\operatorname{Im} \beta)$ is infinite and $\operatorname{Im} \alpha\beta \subseteq \operatorname{Im} \beta$. Thus $\alpha\beta \in OE_R(V, W)$. □

The following example assures that $OE_R(V, W)$ is not a right ideal of $L_R(V, W)$.

Example 2.2.2. Let B and C be bases of vector space V and W, respectively, such that $C \subseteq B$. Since C is infinite set. there are subsets C_1 and C_2 of C such that $|C_1| = |C_2| = |C|$ and $C_1 \cap C_2 = \varnothing$. There is a bijection $\phi : C_1 \to C$. Defined

$$\alpha = \begin{pmatrix} v & B \setminus C_1 \\ v & 0 \end{pmatrix}_{v \in C_1} \quad \text{and} \quad \beta = \begin{pmatrix} v & B \setminus C_1 \\ v\phi & 0 \end{pmatrix}_{v \in C_1}.$$

Clearly that $\alpha \in OE_R(V,W)$ because $\dim_R\big(W/\operatorname{Im}\alpha\big) = |C_2|$ and $\beta \in L_R(V,W)$ but $\beta \notin OE_R(V,W)$. It is obvious that $\alpha\beta = \beta$. This show that $OM_R(V,W)$ is not a left ideal of $L_R(V,W)$.

Lemma 2.2.3. *$OE_R(V,W)G_R(V,W) \subseteq OE_R(V,W)$.*

Proof. Let $\alpha \in OE_R(V,W)$ and $\beta \in G_R(V,W)$. We claim that $W/\operatorname{Im}\alpha \cong W/\operatorname{Im}\alpha\beta$. Thus $\dim_R(W/\operatorname{Im}\alpha\beta) = \dim_R(W/\operatorname{Im}\alpha)$ which is infinite. Hence $\alpha\beta \in OE_R(V,W)$. Therefore, it remains to show that $W/\operatorname{Im}\alpha \cong W/\operatorname{Im}\alpha\beta$. Define $\varphi : W/\operatorname{Im}\alpha \to W/\operatorname{Im}\alpha\beta$ by

$$(w + \operatorname{Im}\alpha)\varphi = w\beta + \operatorname{Im}\alpha\beta \quad \text{for every } w \in W.$$

Then φ is well-defined. Moreover, φ is a bijection. Since β is a homomorphism, φ is an monomorphism. Thus β is an isomorphism, i.e.,

$$W/\operatorname{Im}\alpha \cong W/\operatorname{Im}\alpha\beta.$$

□

Next, we obtain that $OE_R(V,W) \cup H$ is a semigroup.

Proposition 2.2.4. *$OE_R(V,W) \cup H$ is a subsemigroup of $L_R(V,W)$.*

Proof. The result follows from Lemma 2.2.1, Lemma 2.2.3 and the fact that $OE_R(V,W)$ and H are subsemigroups of $L_R(V,W)$ □

Similar to $OE_R(V,W) \cup S$, we show that $OE_R(V,W)AI_R(V,\underline{W}) \subseteq OE_R(V,W)$ and then conclude that $OE_R(V,W) \cup S$ is a semigroup. Nevertheless, we first prove the following lemma.

Lemma 2.2.5. *For every* $\alpha \in AI_R(V, \underline{W})$, $\dim_R \operatorname{Ker} \alpha|_W$ *is finite.*

Proof. Let $\alpha \in AI_R(V, \underline{W})$ and B be a basis of $\operatorname{Ker} \alpha|_W$. We claim that $\{v + F(\alpha) \mid v \in B\}$ is linearly independent over R. Let $v_1, v_2, \ldots, v_n \in B$ be all distinct and $a_1, a_2, \ldots, a_n \in R$ be such that

$$\sum_{i=1}^{n} a_i\big(v_i + F(\alpha)\big) = F(\alpha).$$

Then $\sum_{i=1}^{n} a_i v_i = F(\alpha)$ which implies that $(\sum_{i=1}^{n} a_i v_i)\alpha = \sum_{i=1}^{n} a_i v_i$. Since $v_1, v_2, \ldots, v_n \in \operatorname{Ker} \alpha|_W$, we have $\sum_{i=1}^{n} a_i v_i = 0$. Then $a_i = 0$ for all i because $v_1, v_2, \ldots, v_n$ are linearly independent over R. This proves that $\{v + F(\alpha) | v \in B\}$ is a linearly independent subset of $W/F(\alpha)$ as claimed. Moreover $v + F(\alpha) \neq w + F(\alpha)$ for all distinct $v, w \in B$. Since $\dim_R\big(W/F(\alpha)\big)$ is finite, $\{v + F(\alpha) | v \in B\}$ is finite. Therefore $\dim_R \operatorname{Ker} \alpha|_W| = |B| = |\, \{v + F(\alpha) | v \in B\} \,|$ is finite. □

Lemma 2.2.6. $OE_R(V, W)AI_R(V, \underline{W}) \subseteq OE_R(V, W)$.

Proof. Let $\alpha \in OE_R(V, W)$ and $\beta \in AI_R(V, \underline{W})$. Observe that $\varphi : W/\operatorname{Im} \alpha \to \operatorname{Im} \beta|_W / \operatorname{Im} \alpha\beta$ defined by

$$(w + \operatorname{Im} \alpha)\varphi = w\beta + \operatorname{Im} \alpha\beta \quad \text{for all } w \in W$$

is an epimorphism. Hence

$$(W/\operatorname{Im} \alpha)/\operatorname{Ker} \varphi \cong \operatorname{Im} \beta|_W / \operatorname{Im} \alpha\beta. \tag{1}$$

Then

$$\dim_R(W/\operatorname{Im} \alpha) = \dim_R\big((W/\operatorname{Im} \alpha)/\operatorname{Ker} \varphi\big) + \dim_R \operatorname{Ker} \varphi.$$

We claim that $\dim_R \operatorname{Ker} \alpha$ is finite. Thus $\dim_R\big((W/\operatorname{Im} \alpha)/\operatorname{Ker} \varphi\big)$ must be infinite since $\dim_R(W/\operatorname{Im} \alpha)$ is infinite but $\dim_R \operatorname{Ker} \alpha$ is finite. Together this fact and (1) we ob tain that $\dim_R \operatorname{Im} \beta|_W / \operatorname{Im} \alpha\beta$ is infinite. We see that

$$\dim_R(\operatorname{Im} \beta|_W / \operatorname{Im} \alpha\beta) \leq \dim_R(\operatorname{Im} \beta / \operatorname{Im} \alpha\beta) \leq \dim_R(W/\operatorname{Im} \alpha\beta).$$

Consequently, $\dim_R(W/\operatorname{Im}\alpha)$ is infinite so $\alpha \in OE_R(V,W)$ To complete the proof, it remains showing that $\dim_R \operatorname{Ker}\varphi$ is finite. Let $C \subseteq W$ be such that $\{v+\operatorname{Im}\alpha | v \in C\}$ is a basis of $\operatorname{Ker}\varphi$ and $v+\operatorname{Im}\alpha \neq w+\operatorname{Im}\alpha$ for all distinct $v, w \in C$. We know that $v\beta + \operatorname{Im}\alpha\beta = (v+\operatorname{Im}\alpha)\varphi = \operatorname{Im}\alpha\beta$ for all $v \in C$. Thus $v\beta \in \operatorname{Im}\alpha\beta = (\operatorname{Im}\alpha)\beta$ for all $v \in C$. Hence for each $v \in C$, there exists an element $w_v \in \operatorname{Im}\alpha$ such that $v\beta = w_v\beta$. Fix such w_v for each $v \in C$. Consequently,

$$\{v - w_v | v \in C\} \subseteq \operatorname{Ker}\beta|_W.$$

If distinct elements $v_1, v_2, \ldots, v_n \in B$ and $a_1, a_2, \ldots, a_n \in R$ are such that $\sum_{i=1}^{n} a_i(v_i - w_{v_i}) = 0$, then

$$\sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{n} a_i w_{v_i} \in \operatorname{Im}\alpha$$

and hence $\sum_{i=1}^{n} a_i(v_i + \operatorname{Im}\alpha) = \operatorname{Im}\alpha$. Thus $a_i = 0$ for all i. This show that $\{v - w_v | v \in C\}$ is linearly independent over R and $v - w_v \neq u - w_u$ for all distinct $u, v \in C$ because $v + \operatorname{Im}\alpha \neq w + \operatorname{Im}\alpha$ all distinct $u, v \in C$. It follows that

$$|C| = |\{v + \operatorname{Im}\alpha | v \in C\}| = |\{v - w_v | v \in C\}| \leq \dim_R \operatorname{Ker}\beta|_W.$$

Since $\dim_R \operatorname{Ker}\beta|_W$ is finite from Lemma 2.2.5, we conclude that C is finite. Therefore $\dim_R \operatorname{Ker}\varphi$ is finite as disired. □

Proposition 2.2.7. *$OE_R(V,W) \cup S$ is a subsemigroup of $L_R(V,W)$.*

Proof. Apply Lemma 2.2.1, Lemma 2.2.6 and the fact that $OE_R(V,W)$ and S are subsemigroups of $L_R(V,W)$ to obtain the result. □

Proposition 2.2.8. *$OE_R(V,W) \cup T$ is a subsemigroup of $L_R(V,W)$.*

Proof. The result follows immediately from the fact that $AI_R(\underline{V},W) \subseteq AI_R(V,\underline{W})$ and Proposition 2.2.7. □

CHAPTER III
ADMITTING THE STRUCTURE OF A SEMIHYPERRING WITH ZERO OF SEMIGROUPS

Chapter I and Chapter II illustraste that

$$OM_R(V,W),\quad OM_R(V,W)\cup H,\quad OM_R(V,W)\cup S,\quad OM_R(V,W)\cup T,$$
$$OE_R(V,W),\quad OE_R(V,W)\cup H,\quad OE_R(V,W)\cup S,\quad OE_R(V,W)\cup T$$

are subsemigroups of $L_R(V,W)$ where H, S and T are subsemigroups of $G_R(V,W)$, $AI_R(V,\underline{W})$ and $AI_R(\underline{V},W)$, respectively. In this chapter, we investigate whether or when each of them admits the structure of a semihyperring with zero.

3.1 Semigroups Containing $OM_R(V,W)$

It can be shown that $OM_R(V,W)$ and $L_R(V,W)$ are identical if W is a proper subspace of V with $\dim_R W < \dim_R V$.

Lemma 3.1.1. *If* $\dim_R W < \dim_R V$, *then* $OM_R(V,W) = L_R(V,W)$.

Proof. Assume that $\dim_R W < \dim_R V$. Note that $OM_R(V,W) \subseteq L_R(V,W)$. It suffices to show that $L_R(V,W) \subseteq OM_R(V,W)$. Let $\alpha \in L_R(V,W)$. Suppose that $\alpha \notin OM_R(V,W)$. Then $\dim_R \operatorname{Ker}\alpha$ is finite. Since $\dim_R V = \dim_R \operatorname{Ker}\alpha + \dim_R \operatorname{Im}\alpha$ and $\dim_R V$ is infinite, we obtain that $\dim_R \operatorname{Im}\alpha$ is infinite and

$$\dim_R V = \dim_R \operatorname{Ker}\alpha + \dim_R \operatorname{Im}\alpha = \dim_R \operatorname{Im}\alpha \le \dim_R W$$

which absurd. Hence $\alpha \in OM_R(V,W)$.

Therefore, $L_R(V,W) = OM_R(V,W)$. □

Proposition 3.1.2. *If* $\dim_R W < \dim_R V$, *then* $OM_R(V,W)$, $OM_R(V,W)\cup H$, $OM_R(V,W)\cup S$ *and* $OM_R(V,W)\cup T$ *admit the structure of semihyperring with zero.*

Proof. The result is obtained immediately from Lemma 3.1.1. □

In order the prove the main theorem, we need the follow lemma.

Lemma 3.1.3. *Let W be a proper subspace of V such that $\dim_R W = \dim_R V$, $\mathcal{P}_R(V,W)$ a subsemigroup of $L_R(V,W)$ containing $OM_R(V,W)$ and $\oplus$ a hyperoperation on $\mathcal{P}_R(V,W)$ such that $(\mathcal{P}_R(V,W),\oplus,\cdot)$ is a semihyperring. Then there are α, $\beta \in OM_R(V,W)$ and $\lambda \in L_R(V,W)$ such that $\lambda \in \alpha\oplus\beta$ but $\lambda \notin OM_R(V,W) \cup G_R(V,W) \cup AI_R(V,\underline{W})$. Moreover, $\lambda \notin AI_R(\underline{V},W)$.*

Proof. Let C and B be bases of W and V, respectively, such that $C \subseteq B$. Since W is a proper subspace of V, it follows that $B \smallsetminus C \neq \varnothing$. Let $u \in B \smallsetminus C$ be a fixed element. Moreover, let $D = B \smallsetminus (C \cup \{u\})$ and D_1, D_2 be disjoint subsets of D such that $D_1 \cup D_2 = D$. Since $|B| = |C|$ and C is infinite, there are disjoint subsets C_1 and C_2 of C such that $C_1 \cup C_2 = C$ and $|C_1| = |C_2| = |C| = |B|$. Since $C_1 \subseteq C_1 \cup D_1 \subseteq B$, we have $|C_2| = |C_1| = |C_1 \cup D_1|$. Similarly, $|C_1| = |C_2 \cup D_2|$. [Note that $B = C_1 \cup D_1 \cup C_2 \cup D_2 \cup \{u\}$] As a result, there are bijections $\varphi : C_1 \cup D_1 \to C_2$ and $\gamma : C_2 \cup D_2 \to C_1$. Define α, $\beta \in L_R(V,W)$ by

$$\alpha = \begin{pmatrix} v & C_2 \cup D_2 \cup \{u\} \\ v\varphi & 0 \end{pmatrix}_{v\in C_1\cup D_1} \quad \text{and} \quad \beta = \begin{pmatrix} C_1 \cup D_1 \cup \{u\} & v \\ 0 & v\gamma \end{pmatrix}_{v\in C_2\cup D_2}.$$

Observe that α and β are well-defined because $C_1 \cup D_1 \cup C_2 \cup D_2 \cup \{u\}$ is a partition of B. Hence $\operatorname{Ker}\alpha = \langle C_2 \cup D_2 \cup \{u\}\rangle$ and $\operatorname{Ker}\beta = \langle C_1 \cup D_1 \cup \{u\}\rangle$. Thus α, $\beta \in OM_R(V,W) \subseteq \mathcal{P}_R(V,W)$. Moreover, $\alpha^2 = \beta^2 = 0$ and by the distributive law,

$$\begin{aligned} \alpha(\alpha\oplus\beta) &= \alpha^2 \oplus \alpha\beta = 0 \oplus \alpha\beta = \{\alpha\beta\} = \alpha\beta \oplus 0 = \alpha\beta \oplus \beta^2 = (\alpha\oplus\beta)\beta \\ \beta(\alpha\oplus\beta) &= \beta\alpha \oplus \beta^2 = \beta\alpha \oplus 0 = \{\beta\alpha\} = 0 \oplus \beta\alpha = \alpha^2 \oplus \beta\alpha = (\alpha\oplus\beta)\alpha. \end{aligned} \tag{1}$$

Since $\alpha\oplus\beta \neq \varnothing$, let $\lambda \in \alpha\oplus\beta$. Then $\alpha\lambda = \alpha\beta = \lambda\beta$ and $\beta\lambda = \beta\alpha = \lambda\alpha$.

Now, we would like to determine the linear transformation λ. Let $v \in B$. Then $v \in C_1 \cup D_1$ or $v \in C_2 \cup D_2$ or $v = u$. Clearly, $v\lambda \in W = \langle C_1 \cup C_2\rangle$ and thus

$$v\lambda = a_1w_1 + a_2w_2 + \cdots + a_nw_n + b_1w_1' + b_2w_2' + \cdots + b_mw_m'$$

for some distinct elements $w_1, w_2, \ldots, w_n \in C_1$, $w'_1, w'_2, \ldots, w'_m \in C_2$ and for some $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m \in R$. First, assume that $v \in C_1 \cup D_1$. Then

$$\begin{aligned} 0 = 0\alpha = (v\beta)\alpha &\\ &= v(\beta\alpha) \\ &= v(\lambda\alpha) \\ &= (v\lambda)\alpha \\ &= (a_1w_1 + a_2w_2 + \cdots + a_nw_n + b_1w'_1 + b_2w'_2 + \cdots + b_mw'_m)\alpha \\ &= \sum_{i=1}^{n} a_i(w_i\alpha) + \sum_{j=1}^{m} b_j(w'_j\alpha) \\ &= \sum_{i=1}^{n} a_i(w_i\varphi). \end{aligned}$$

Since φ is one-to-one and w_i's are all distinct, $w_i\varphi$'s are all distinct elements in C_2. Hence $a_i = 0$ for all i because of the linearly independence of C_2. It follows that $v\lambda \in \langle C_2 \rangle$. We see futher that $v\lambda\beta = v\alpha\beta = (v\alpha)\beta = (v\varphi)\beta$. Since $\beta|_{C_2}$ is one-to-one, $\beta|_{\langle C_2 \rangle}$ is also one-to-one. Thus $v\lambda = v\varphi$. This shows that $\lambda|_{C_1 \cup D_1} = \varphi$.

Similarly, if $v \in C_2 \cup D_2$, then $\lambda|_{C_2 \cup D_2} = \psi$. Hence

$$\lambda = \begin{pmatrix} v & w \\ v\varphi & w\gamma \end{pmatrix}_{v \in C_1 \cup D_1, w \in C_2 \cup D_2}.$$

Consequentely, $\lambda \notin OM_R(V,W) \cup G_R(V,W)$ because $\dim_R \operatorname{Ker} \lambda = 0$ and $\lambda|_W$ is not onto since $\operatorname{Im} \lambda = \langle (C_1 \cup C_2) \smallsetminus \{u\} \rangle$. Moreover, $\lambda \notin AI_R(V, \underline{W})$ because $F(\lambda)$ is the zero space so that $\dim_R\big(W/F(\lambda)\big) = \dim_R W$ which is infinite. Furthermore, $\lambda \notin AI_R(\underline{V}, W)$ because $AI_R(\underline{V}, W) \subseteq AI_R(V, \underline{W})$. □

We are ready to give the condition for our target semigroups containing $OM_R(V,W)$ to admit the structure of a semihyperring with zero.

Theorem 3.1.4. *Let $\mathcal{P}_R(V,W)$ be one of $OM_R(V,W)$, $OM_R(V,W) \cup H$, $OM_R(V,W) \cup S$ or $OM_R(V,W) \cup T$. Then $\mathcal{P}_R(V,W)$ does not admit the structure of a semihyperring with zero if and only if $\dim_R V = \dim_R W$.*

Proof. First, we assume that $\dim_R W < \dim_R V$. By Proposition 3.1.2, $\mathcal{P}_R(V,W)$ admits the structure of a semihyperring with zero.

Conversely, assume that $\dim_R V = \dim_R W$. There are two cases to be considered.

Case 1. $W = V$. Then $\mathcal{P}_R(V,W) = \mathcal{P}_R(V,V)$. Then $\mathcal{P}_R(V,V)$ does not admit the structure of a semihyperring with zero from [1].

Case 2. $W \neq V$. Suppose that there exists a hyperoperation $\oplus$ such that $(\mathcal{P}_R(V,W)^0, \oplus, \cdot)$ is a semihyperring with zero where $\cdot$ is the operation on $\mathcal{P}_R(V,W)$. Note that $\mathcal{P}_R(V,W)^0 = \mathcal{P}_R(V,W)$ because $0 \in OM_R(V,W)$ where 0 is the zero map. Lemma 3.1.3 provides that there are $\alpha,\ \beta \in OM_R(V,W) \subseteq \mathcal{P}_R(V,W)$ and $\lambda \in L_R(V,W)$ such that $\lambda \in \alpha \oplus \beta \subseteq \mathcal{P}_R(V,W)$ but $\lambda \notin \mathcal{P}_R(V,W)$. This leads to a contradiction. Hence $\mathcal{P}_R(V,W)$ does not admit the structure of a semihyperring with zero. □

The following corollary is the immediate result from Theorem 3.1.4.

Corollary 3.1.5. *Let $\mathcal{P}_R(V,W)$ be one of $OM_R(V,W)$, $OM_R(V,W) \cup H$, $OM_R(V,W) \cup S$ or $OM_R(V,W) \cup T$. Then $P_R(V,W)$ does not admit a hyperring [ring] structure if and only if* $\dim_R V = \dim_R W$.

3.2 Semigroups Containing $OE_R(V,W)$

Unlike semigroups containing $OM_R(V,W)$, we find that our desired subsemigroups of $L_R(V,W)$ containing $OE_R(V,W)$ does not admit the structure of a semihyperring with zero. Nonetheless, we need the following lemma to prove the main result.

Lemma 3.2.1. *Let W be a proper subspace of V, $\mathcal{Q}_R(V,W)$ a subsemigroup of $L_R(V,W)$ containing $OE_R(V,W)$ and $\oplus$ a hyperoperation on $\mathcal{Q}_R(V,W)$ such that $(\mathcal{Q}_R(V,W), \oplus, \cdot)$ is a semihyperring. Then there are $\alpha,\ \beta \in OE_R(V,W)$ and $\lambda \in L_R(V,W)$ such that $\lambda \in \alpha \oplus \beta$ but $\lambda \notin OE_R(V,W) \cup G_R(V,W)$.*

Proof. The proof is done by choosing α, β and λ defined in the proof of Lemma 3.1.3 $\square$

Next theorem provides that the semigroups $OE_R(V,W)$ and $OE_R(V,W)\cup H$ cannot admit the structure of a semihyperring with zero.

Theorem 3.2.2. *Let $\mathcal{Q}_R(V,W)$ be either $OE_R(V,W)$ or $OE_R(V,W)\cup H$. Then $\mathcal{Q}_R(V,W)$ does not admit the structure of a semihyperring with zero.*

Proof. We separate the proof into two cases.
Case 1. $W = V$. We observe that $\mathcal{Q}_R(V,W) = \mathcal{Q}_R(V,V)$. Then $\mathcal{Q}_R(V,V)$ does not admit the structure of a semihyperring with zero from [1].
Case 2. $W \neq V$. Suppose that there exists a hyperoperation $\oplus$ such that $(\mathcal{Q}_R(V,W)^0, \oplus, \cdot)$ is a semihyperring with zero where $\cdot$ is the operation on $\mathcal{Q}_R(V,W)$. Note that $\mathcal{Q}_R(V,W)^0 = \mathcal{Q}_R(V,W)$ because $0 \in OE_R(V,W)$ where 0 is the zero map. By Lemma 3.2.1, there are α, $\beta \in OE_R(V,W) \subseteq \mathcal{Q}_R(V,W)$ and $\lambda \in L_R(V,W)$ such that $\lambda \in \alpha\oplus\beta \subseteq \mathcal{Q}_R(V,W)$ but $\lambda \notin \mathcal{Q}_R(V,W)$. This is absurd. Hence $\mathcal{Q}(V,W)$ does not admit the structure of a semihyperring with zero. $\square$

The following corollary is the direct result from Theorem 3.2.2.

Corollary 3.2.3. *Let $\mathcal{Q}_R(V,W)$ be either $OE_R(V,W)$ or $OE_R(V,W)\cup H$. Then $\mathcal{Q}_R(V,W)$ does not admit a hyperring[ring] structure.*

Now, we show that the semigroup $OE_R(V,W)\cup S$ with zero never admit the structure of a semihyperring with zero either.

Theorem 3.2.4. *$OE_R(V,W)\cup S$ does not admit the structure of a semihyperring with zero.*

Proof. We divide the argument into two cases.

Case 1. $W = V$. We note that $OE_R(V,W) = OE_R(V,V)$ and $AI_R(V,\underline{W}) = AI_R(V,V)$. Then $OE_R(V,V) \cup S$ does not admit the structure of a semihyperring with zero from [1].

Case 2. $W \neq V$. Let C and B be bases of W and V, respectively, such that $C \subseteq B$. To show that $OE_R(V,W) \cup S$ does not admit the structure of a semihyperring with zero, suppose that there exists a hyperoperation $\oplus$ such that $(OE_R(V,W) \cup S, \oplus, \cdot)$ is a semihyperring with zero where $\cdot$ is the operation on $OE_R(V,W) \cup S$. Since $\dim_R W$ is infinite, there are distinct subsets C_1 and C_2 of C such that $C_1 \cup C_2 = C$ and $|C_1| = |C_2| = |C|$. Then there is a bijection $\varphi : C_1 \to C_2$. Let $D = B \setminus C$ which is not empty. Then $B = C_1 \cup C_2 \cup D$ Define α, $\beta \in L_R(V,W)$ by

$$\alpha = \begin{pmatrix} v & C_2 \cup D \\ v\varphi & 0 \end{pmatrix}_{v \in C_1} \quad \text{and} \quad \beta = \begin{pmatrix} C_1 \cup D & v \\ 0 & v\varphi^{-1} \end{pmatrix}_{v \in C_2}. \tag{1}$$

Observe that α and β are well-defined because $C_1 \cup C_2 \cup D$ is a partition of B. Clearly, $\dim_R(W/\operatorname{Im}\alpha) = |C \setminus C_2| = |C_1|$ and $\dim_R(W/\operatorname{Im}\beta) = |C \setminus C_1| = |C_2|$. Hence α, $\beta \in OE_R(V,W) \subseteq OE_R(V,W) \cup S$. From (1), we see that $\alpha^2 = \beta^2 = 0$. The distributive law provides that

$$\begin{aligned} \alpha(\alpha \oplus \beta) &= \alpha^2 \oplus \alpha\beta = 0 \oplus \alpha\beta = \{\alpha\beta\} = \alpha\beta \oplus 0 = \alpha\beta \oplus \beta^2 = (\alpha \oplus \beta)\beta \\ \beta(\alpha \oplus \beta) &= \beta\alpha \oplus \beta^2 = \beta\alpha \oplus 0 = \{\beta\alpha\} = 0 \oplus \beta\alpha = \alpha^2 \oplus \beta\alpha = (\alpha \oplus \beta)\alpha. \end{aligned} \tag{2}$$

Since $\alpha \oplus \beta \neq \varnothing$, let $\lambda \in \alpha \oplus \beta$, we have $\alpha\lambda = \alpha\beta = \lambda\beta$ and $\beta\lambda = \beta\alpha = \lambda\alpha$.

Now, we determine the linear transformation λ. Let $v \in B = C_1 \cup C_2 \cup D$. Then $v\lambda \in W$ so

$$v\lambda = a_1w_1 + a_2w_2 + \cdots + a_nw_n + b_1w_1' + b_2w_2' + \cdots + b_mw_m'$$

for some distinct elements $w_1, w_2, \ldots, w_n \in C_1$, $w_1', w_2', \ldots, w_m' \in C_2$ and for some $a_1, a_2 \ldots, a_n, b_1, b_2, \ldots, b_m \in R$. First, assume that $v \in C_1 \cup D$. Then

$$0 = v\beta\alpha = v\lambda\alpha = (v\lambda)\alpha = \sum_{i=1}^{n} a_i(w_i\alpha) + \sum_{j=1}^{m} b_j(w_j'\alpha) = \sum_{i=1}^{n} a_i(w_i\alpha) = \sum_{i=1}^{n} a_i(w_i\varphi).$$

Since φ is one-to-one and w_i are all distinct, $w_i\varphi$ are all distinct elements in C_2. Hence $a_i = 0$ for all i. We have shown that if $v \in C_1 \cup D$, then $v\lambda \in \langle C_2 \rangle$. Similarly, if $v \in C_2 \cup D$, then $v\lambda \in \langle C_1\rangle$. Thus, if $v \in D$, then $v\lambda \in \langle C_1\rangle \cap \langle C_2\rangle = \{0\}$ so that $v\lambda = \{0\}$. Furthermore,

$$\begin{aligned} v\lambda\beta &= v\alpha\beta = (v\alpha)\beta = (v\varphi)\beta && \text{for all } v \in C_1, \\ v\lambda\alpha &= v\beta\alpha = (v\beta)\alpha = (v\varphi^{-1})\alpha && \text{for all } v \in C_2. \end{aligned}$$

Since $\beta|_{C_2}$ and $\alpha|_{C_1}$ are one-to-one, $\beta|_{\langle C_2\rangle}$ and $\alpha|_{\langle C_1\rangle}$ are one-to-one. Consequently,

$$v\lambda = \begin{cases} v\varphi, & \text{if } v \in C_1, \\ v\varphi^{-1}, & \text{if } v \in C_2. \end{cases}$$

Hence

$$\lambda = \begin{pmatrix} v & w & D \\ v\varphi & w\varphi^{-1} & 0 \end{pmatrix}_{v\in C_1,\, w\in C_2}.$$

Consequently, $\lambda \notin OE_R(V,W)$ because $\dim_R(W/\operatorname{Im}\lambda) = |C \setminus C| = 0$. Moreover, $\lambda \notin AI_R(V,\underline{W})$ because $F(\lambda)$ is the zero space so that $\dim_R\big(W/F(\lambda)\big) = \dim_R W$ which is infinite. Then $\lambda \notin S$.

So far, we have proved that $\lambda \in \alpha\oplus\beta \subseteq OE_R(V,W)\cup S$ and $\lambda \notin OE_R(V,W)\cup S$ which is impossible. □

The following corollary is the immediate result from Theorem 3.2.4.

Corollary 3.2.5. *$OE_R(V,W)\cup S$ does not admit a hyperring[ring] structure.*

The fact that $AI_R(\underline{V},W) \subseteq AI_R(V,\underline{W})$ yields the following corollaries.

Corollary 3.2.6. *$OE_R(V,W)\cup T$ does not admit the structure of a semihyperring with zero.*

Corollary 3.2.7. *$OE_R(V,W)\cup T$ does not admit a hyperring[ring] structure.*

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