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LEFT REGULAR AND RIGHT REGULAR ELEMENTS
OF SOME SEMIGROUPS

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เราเรียกสมาชิก x ของกึ่งกรุป S ว่า *สมาชิกปรกติซ้าย* [*ขวา*] เมื่อ $x = yx^2$ $[x = x^2 y]$ สำหรับบาง y ใน S หรือสมมูลกับ $x \mathcal{L} x^2$ $[x \mathcal{R} x^2]$ *แวเรียนต์*ของกึ่งกรุป S *โดย* a ใน S คือกึ่งกรุป $(S,*)$ โดยที่ $x * y = xay$ สำหรับทุก x, y ใน S ในการวิจัยนี้เราให้ลักษณะเฉพาะของสมาชิกปรกติซ้ายและสมาชิกปรกติขวาของกึ่งกรุปของการแปลงของเซต และกึ่งกรุปของการแปลงเชิงเส้นบางชนิด ยิ่งไปกว่านั้นเราบอกสมาชิกปรกติซ้ายและสมาชิกปรกติขวาของแวเรียนต์ใดๆของกึ่งกรุปเหล่านี้

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NISSARA SIRASUNTORN : LEFT REGULAR AND RIGHT REGULAR ELEMENTS OF SOME SEMIGROUPS. ADVISOR : PROF. YUPAPORN KEMPRASIT, Ph.D., 62 pp.

We call an element x of a semigroup S a *left* [*right*] *regular element* if $x = yx^2$ [$x = x^2y$] for some $y \in S$, or equivalently, $x\mathcal{L}x^2$ [$x\mathcal{R}x^2$]. The *variant* of a semigroup S *induced by* $a \in S$ is the semigroup $(S, *)$ where $x * y = xay$ for all $x, y \in S$. In this research, we characterize the left regular and right regular elements of some semigroups of transformations of sets and linear transformations. Moreover, the left regular and right regular elements of any variant of these semigroups are determined.

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CONTENTS

CHAPTER I
INTRODUCTION

Green's relations are five equivalence relations that characterize the elements of a semigroup in terms of the principal ideals they generate. These fundamental equivalence relations, definable in any semigroup, were first introduced and studied by Green [7]. The concept of Green's relations is a crucial notion in semigroup theory. It has shed a great deal of light on the structure of semigroups in general. It is interesting to see that we can consider left [right] regularity in terms of the Green's relation $\mathcal{L}\,[\mathcal{R}]$. Recall that an element x of a semigroup S is called a *left [right] regular element* of S if $x = yx^2\,[x = x^2y]$ for some $y \in S$, that is, $x\mathcal{L}x^2\,[x\mathcal{R}x^2]$. Denote by $\mathrm{LReg}(S)\,[\mathrm{RReg}(S)]$ the set of all left [right] regular elements of S. Note that if S is commutative, then $\mathrm{LReg}(S) = \mathrm{RReg}(S) = \mathrm{Reg}(S)$ where $\mathrm{Reg}(S)$ is the set of all regular elements of S, that is, $\mathrm{Reg}(S) = \{x \in S \mid x = xyx \text{ for some } y \in S\}$. We have generally that $\mathrm{LReg}(S) \cap \mathrm{RReg}(S) \subseteq \mathrm{Reg}(S)$. As we know, regularity is an important notion and it is very extensively studied in semigroup theory.

Left [Right] regularity of semigroups has long been studied. In 1954, Clifford [4] proved that S is a band of groups if and only if S is both left and right regular and $Syx = Syx^2$ and $xyS = x^2yS$ for all $x, y \in S$. Kiss [12] generalized left [right] regular elements of semigroups in 1972. It was shown by Anjaneyulu [1] in 1981 that in a duo semigroup S, the set of all left regular elements and the set of all right regular elements coincide. In 1998, left regular partially ordered semigroups and left regular partially ordered Γ-semigroups were studied by Lee and Jung [14] and by Kwon and Lee [13], respectively. In 2005, Mitrović [18] gave a characterization determining when every regular element of a semigroup S is left regular, that is, he characterized when $\mathrm{Reg}(S) \subseteq \mathrm{LReg}(S)$ holds.

Variants of abstract semigroups were studied by Hickey [8] in 1983 and he also provided many results relating to variants of semigroups in many papers.

Semigroups of transformations play an important role in studying semigroups. It is well-known that any semigroup can be realized as a semigroup of transformations, analogous to the Cayley's theorem. This is reasonable to consider those semigroups and their variants and connect them with left and right regularity in which we are interested.

The purpose of this research is to characterize the left regular and right regular elements of some semigroups of transformations of sets and linear transformations and their variants. This research is organized into five chapters as follows:

Chapter II provides basic definitions and known results for later usage in this research.

In Chapter III, we give characterizations of the left regular and right regular elements of the following semigroups of transformations of an infinite set X:

$$\begin{aligned}
M(X) &= \{\alpha \in T(X) \mid \alpha \text{ is 1-1}\},\\
M(X) \smallsetminus G(X) &\ (= \{\alpha \in T(X) \mid \alpha \text{ is 1-1 but not onto}\}),\\
E(X) &= \{\alpha \in T(X) \mid \alpha \text{ is onto}\},\\
E(X) \smallsetminus G(X) &\ (= \{\alpha \in T(X) \mid \alpha \text{ is onto but not 1-1}\}),\\
BL(X,q) &= \{\alpha \in T(X) \mid \alpha \text{ is 1-1 and } |X \smallsetminus \operatorname{ran}\alpha| = q\}\\
&\qquad \text{where } q \text{ is the cardinal number greater than or equal to } \aleph_0,\\
DBL(X,q) &= \{\alpha \in T(X) \mid \alpha \text{ is onto and } |x\alpha^{-1}| = q \text{ for all } x \in X\},\\
KN(X,q) &= \{\alpha \in T(X) \mid \alpha \text{ is 1-1 and } |X \smallsetminus \operatorname{ran}\alpha| \geq q\},\\
Trf(X) &= \{\alpha \in T(X) \mid \operatorname{ran}\alpha \text{ is finite}\},\\
Prf(X) &= \{\alpha \in P(X) \mid \operatorname{ran}\alpha \text{ is finite}\},\\
Irf(X) &= \{\alpha \in I(X) \mid \operatorname{ran}\alpha \text{ is finite}\}
\end{aligned}$$

where $T(X)$, $P(X)$, $I(X)$ and $G(X)$ are the full transformation semigroup, the partial transformation semigroup, the symmetric inverse semigroup (the 1-1 partial transformation semigroup) and the symmetric group on X, respectively. Note that $BL(X,q)$ is called the *Baer-Levi semigroup of type* $(|X|, q)$, which was constructed in [2] and $DBL(X,q)$ is called the *dual Baer-Levi semigroup of type* $(|X|,q)$, which was given in [3].

Let $L_F(V)$ be the semigroup under composition of all linear transformations from a vector space V over a field F into itself. In Chapter IV, we consider the following subsemigroups of $L_F(V)$ analogous to those in Chapter III:

$$\begin{aligned}
M_F(V) &= \{\alpha \in L_F(V) \mid \alpha \text{ is 1-1}\},\\
M_F(V) \smallsetminus G_F(V) &\ (= \{\alpha \in L_F(V) \mid \alpha \text{ is 1-1 but not onto}\}),\\
E_F(V) &= \{\alpha \in L_F(V) \mid \alpha \text{ is onto}\},\\
E_F(V) \smallsetminus G_F(V) &\ (= \{\alpha \in L_F(V) \mid \alpha \text{ is onto but not 1-1}\}),\\
BL_F(V,q) &= \{\alpha \in L_F(V) \mid \alpha \text{ is 1-1 and } \dim_F(V/\operatorname{ran}\alpha) = q\}\\
&\qquad \text{where } q \text{ is the cardinal number greater than or equal to } \aleph_0,\\
DBL_F(V,q) &= \{\alpha \in L_F(V) \mid \alpha \text{ is onto and } \dim_F \ker\alpha = q\},\\
KN_F(V,q) &= \{\alpha \in L_F(V) \mid \alpha \text{ is 1-1 and } \dim_F(V/\operatorname{ran}\alpha) \geq q\},\\
Lrf_F(V) &= \{\alpha \in L_F(V) \mid \dim_F \operatorname{ran}\alpha \text{ is finite}\}.
\end{aligned}$$

In [16], $BL_F(V,q)$ is called the *linear Baer-Levi semigroup on* V *of type* q. To be analogous to $DBL(X,q)$, we may refer to $DBL_F(V,q)$ as the *dual linear Baer-Levi semigroup on* V *of type* q. The results for the left regular and right regular elements of these semigroups are obtained accordingly to those in Chapter III.

In Chapter V, the left regular and right regular elements of the variants of the full transformation semigroup $T(X)$, the partial transformation semigroup $P(X)$ and the symmetric inverse semigroup $I(X)$ on a nonempty set X are determined. In addition, the variants of those semigroups in Chapter III are studied in the same manner.

The variants of the semigroup $L_F(V)$ are considered in Chapter VI. Their left regular and right regular elements are determined. Moreover, the left regular and right regular elements of the variants of those semigroups in Chapter IV are characterized. The results are obtained suitably to those of the variants of semigroups given in Chapter V.

CHAPTER II

PRELIMINARIES

In this chapter, we review some basic materials which will be used in our later discussion.

The cardinality of a set X is denoted by $|X|$. The value of a mapping α at x in the domain of α shall be written as $x\alpha$. The notation $\dot{\bigcup}$ stands for a disjoint union.

If a semigroup S has an identity, set $S^1 = S$. If S does not have an identity, let S^1 be the semigroup S with an identity adjoined, usually denoted by the symbol 1. An element x of a semigroup S with identity 1 is called a *unit* of S if $xy = yx = 1$ for some $y \in S$. We have that such y is unique and it is denoted by x^{-1}. Then the set of all units of S forms a subgroup of S and it is the greatest subgroup of S containing 1. It is usually called the *group of units* of S.

The Green's relations $\mathcal{L}$ and $\mathcal{R}$ on a semigroup S are the equivalence relations on S defined by

$$\begin{aligned} x\mathcal{L}y \quad &\Leftrightarrow \quad S^1x = S^1y \\ &\quad\quad \text{or equivalently, } x = sy \text{ and } y = tx \\ &\quad\quad \text{for some } s, t \in S^1, \\ x\mathcal{R}y \quad &\Leftrightarrow \quad xS^1 = yS^1 \\ &\quad\quad \text{or equivalently, } x = ys \text{ and } y = xt \\ &\quad\quad \text{for some } s, t \in S^1. \end{aligned}$$

From these definitions, we have that $\mathcal{L}$ and $\mathcal{R}$ are right and left compatible, respectively, i.e., for all x, y, z, if $x\mathcal{L}y$ then $xz\mathcal{L}yz$ and if $x\mathcal{R}y$ then $zx\mathcal{R}zy$.

An element x of a semigroup S is called an *idempotent* of S if $x^2 = x$.

We call an element x of a semigroup S *regular* if $x = xyx$ for some $y \in S$.

An element x of a semigroup S is called *left* [*right*] *regular* if $x = yx^2$ [$x = x^2y$] for some $y \in S$. Then an idempotent of S is regular, left regular and right regular. It is clear that if S has an identity, then every unit of S is regular, left regular and right regular. If $x = xyx$, then xy, yx are idempotents. Thus we have that if S contains a regular element, then S contains an idempotent. If $x = xyx$, then $x = x(yxy)x$, so it implies that every ideal of a regular semigroup is regular. We can see that in a commutative semigroup S, the regular elements, the left regular elements and the right regular elements of S are identical. In terms of the Green's relations $\mathcal{L}$ and $\mathcal{R}$ on S, we have that

(1) x is a left regular element of S if and only if $x\mathcal{L}x^2$;

(2) x is a right regular element of S if and only if $x\mathcal{R}x^2$.

A semigroup S is called a *regular semigroup* if every element of S is regular. *Left* [*Right*] *regular semigroups* are defined similarly. For regularity, left regularity and right regularity of semigroups, one does not imply the others. Some examples can be seen later. However, if a semigroup S is both left and right regular, then S is regular. More generally, if an element x of S is both left and right regular, then x is regular. To show this, we first introduce some notations relating to Green's relations. For any $x \in S$, we let L_x be the equivalence class of $\mathcal{L}$ containing x and R_x the equivalence class of $\mathcal{R}$ containing x. It follows from Theorem 2.16 of [5] that if there are $a, b \in L_x \cap R_x$ such that $ab \in L_x \cap R_x$, then $L_x \cap R_x$ is a subgroup of S, i.e., $L_x \cap R_x$ is a subsemigroup of S which forms a group under the operation on S. We assume that $x \in S$ is both left and right regular. Then $x\mathcal{L}x^2$ and $x\mathcal{R}x^2$. This implies that $x^2 \in L_x \cap R_x$. From the above fact, $L_x \cap R_x$ is a subgroup of S. Then $L_x \cap R_x$ is a regular subsemigroup of S. But $x \in L_x \cap R_x$, so x is a regular element of S.

For a semigroup S, let $\mathrm{LReg}(S)$ and $\mathrm{RReg}(S)$ denote the set of all left regular elements of S and the set of all right regular elements of S, respectively. From the previous mention, $\mathrm{LReg}(S) \cap \mathrm{RReg}(S) \subseteq \mathrm{Reg}(S)$ where $\mathrm{Reg}(S)$ is the set of all regular elements of S.

A nonempty subset A of a semigroup S is called a *left* [*right*] *ideal* of S if

$SA \subseteq A\,[AS \subseteq A]$. We call S *left* [*right*] *simple* if S is the only left [right] ideal of S. Characterizations of left simple semigroups and right simple semigroups are given as follows:

Theorem 2.1 ([19], p. 7)**.** *For a semigroup S, the following statements hold.*

(i) *S is left simple if and only if $Sx = S$ for all $x \in S$.*

(ii) *S is right simple if and only if $xS = S$ for all $x \in S$.*

If S is a semigroup and $a \in S$, then the semigroup $(S, *)$ defined by $x * y = xay$ for all $x, y \in S$ is called the *variant* of S *induced by a* and let $(S, *)$ be denoted by (S, a).

For a nonempty set A, let 1_A be the identity mapping on A.

Let X be a nonempty set. The full transformation semigroup, the partial transformation semigroup and the symmetric inverse semigroup (the 1-1 partial transformation semigroup) on X are denoted by $T(X), P(X)$ and $I(X)$, respectively. Notice that $T(X)$ and $I(X)$ are subsemigroups of $P(X)$. Let $G(X)$ be the symmetric group on X. We have that $G(X)$ is the group of units of $P(X)$, $T(X)$ and $I(X)$. The domain and the range (image) of α in $P(X)$ are denoted by $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$, respectively. Recall that for $\alpha, \beta \in P(X)$,

$$\begin{aligned}
&\operatorname{dom}(\alpha\beta) = (\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\alpha^{-1} \subseteq \operatorname{dom}\alpha,\\
&\operatorname{ran}(\alpha\beta) = (\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\beta \subseteq \operatorname{ran}\beta \quad \text{and}\\
&\text{for } x \in X, x \in \operatorname{dom}(\alpha\beta) \Leftrightarrow x \in \operatorname{dom}\alpha \text{ and } x\alpha \in \operatorname{dom}\beta.
\end{aligned}$$

It is well-known that $P(X)$, $T(X)$ and $I(X)$ are regular semigroups, and moreover, $I(X)$ is an inverse semigroup ([9], p. 4). Recall that a semigroup S is called an *inverse semigroup* if for each $x \in S$, there exists a unique $x^{-1} \in S$ such that $x = xx^{-1}x$ and $x^{-1} = x^{-1}xx^{-1}$. We have that the inverse function α^{-1} of $\alpha \in I(X)$ is the unique element of $I(X)$ such that $\alpha = \alpha\alpha^{-1}\alpha$ and $\alpha^{-1} = \alpha^{-1}\alpha\alpha^{-1}$. Note that 1_X is the identity of $P(X)$, $T(X)$ and $I(X)$. The empty transformation 0 is the zero of $P(X)$ and $I(X)$. For each $\alpha \in P(X)$, the equivalence relation π_α on

$\operatorname{dom}\alpha$ defined by $\pi_\alpha = \alpha \circ \alpha^{-1}$ is called the *partition* of $\operatorname{dom}\alpha$ *corresponding to* α (see [5], p. 51). Then

$$\pi_\alpha = \{(x, y) \in \operatorname{dom}\alpha \times \operatorname{dom}\alpha \mid x\alpha = y\alpha\}.$$

Note that for $\alpha, \beta \in P(X)$, if $\pi_\alpha = \pi_\beta$, then $\operatorname{dom}\alpha = \operatorname{dom}\beta$.

Next, let $M(X)$ and $E(X)$ be the subsemigroups of $T(X)$ defined as follows:

$$M(X) = \{\alpha \in T(X) \mid \alpha \text{ is 1-1}\},$$
$$E(X) = \{\alpha \in T(X) \mid \alpha \text{ is onto}\}.$$

We have that $G(X)$ is the group of units of both $M(X)$ and $E(X)$ and $M(X) = G(X)$ $[E(X) = G(X)]$ if and only if X is finite. If X is an infinite set, then $M(X) \smallsetminus G(X) \neq \varnothing$ and $E(X) \smallsetminus G(X) \neq \varnothing$. It is not difficult to see $M(X) \smallsetminus G(X)$ and $E(X) \smallsetminus G(X)$ are ideals of $M(X)$ and $E(X)$, respectively.

The other important semigroups of transformations of sets are the *Baer-Levi semigroups* and their duals. They were respectively defined by Baer and Levi [2] and Chen [3] as follows:

$$BL(X, q) = \{\alpha \in T(X) \mid \alpha \text{ is 1-1 and } |X \smallsetminus \operatorname{ran}\alpha| = q\},$$
$$DBL(X, q) = \{\alpha \in T(X) \mid \alpha \text{ is onto and } |x\alpha^{-1}| = q \text{ for all } x \in X\}$$

where $|X| \geq q \geq \aleph_0$. These semigroups have the following properties.

Theorem 2.2 ([6], p. 82). *If $|X| \geq q \geq \aleph_0$, then $BL(X, q)$ is a right simple and right cancellative semigroup without idempotents.*

Theorem 2.3 ([3]). *If $|X| \geq q \geq \aleph_0$, then $DBL(X, q)$ is a left simple and left cancellative semigroup without idempotents.*

For convenience, we use a bracket notation to represent a mapping. For instance,

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ stands for the mapping α with $\operatorname{dom}\alpha = \{a, b\}$, $\operatorname{ran}\alpha = \{c, d\}$,

$a\alpha = c$ and $b\alpha = d$,

$\begin{pmatrix} A & x \\ a & x' \end{pmatrix}_{x \in X \smallsetminus A}$ stands for the mapping β with $\operatorname{dom}\beta = X$,

$\operatorname{ran}\beta = \{a\} \cup \{x' \mid x \in X \smallsetminus A\}$ and $x\beta = \begin{cases} a & \text{if } x \in A, \\ x' & \text{if } x \in X \smallsetminus A. \end{cases}$

By a bracket notation, a mapping α can be written as $\alpha = \begin{pmatrix} x\alpha^{-1} \\ x \end{pmatrix}_{x \in \operatorname{ran}\alpha}$.

Let V be a vector space over a field F. The semigroup under composition of all linear transformations $\alpha : V \to V$ is denoted by $L_F(V)$. We define the subsemigroups $M_F(V)$ and $E_F(V)$ similarly as follows:

$$\begin{aligned} M_F(V) &= \{\alpha \in L_F(V) \mid \alpha \text{ is 1-1}\} \\ &\quad (= \{\alpha \in L_F(V) \mid \ker\alpha = \{0\}\}), \\ E_F(V) &= \{\alpha \in L_F(V) \mid \alpha \text{ is onto}\} \\ &\quad (= \{\alpha \in L_F(V) \mid \operatorname{ran}\alpha = V\}). \end{aligned}$$

Let $G_F(V)$ be the set of all isomorphisms from V onto itself. We also have that $G_F(V)$ is the group of units of $L_F(V)$, $M_F(V)$ and $E_F(V)$ and $M_F(V) = G_F(V)\,[E_F(V) = G_F(V)]$ if and only if V is finite-dimensional. If V is infinite-dimensional, then $M_F(V) \smallsetminus G_F(V) \neq \varnothing$ and $E_F(V) \smallsetminus G_F(V) \neq \varnothing$, and they are ideals of $M_F(V)$ and $E_F(V)$, respectively.

The Green's relations $\mathcal{L}$ and $\mathcal{R}$ on $T(X)$, $P(X)$ and $L_F(V)$ are well-known as follows:

Theorem 2.4 ([5], p. 52)**.** *In* $T(X)$,

(i) $\alpha\mathcal{L}\beta$ *if and only if* $\operatorname{ran}\alpha = \operatorname{ran}\beta$;

(ii) $\alpha\mathcal{R}\beta$ *if and only if* $\pi_\alpha = \pi_\beta$.

Theorem 2.5 ([10], p. 63)**.** *In* $P(X)$,

(i) $\alpha\mathcal{L}\beta$ *if and only if* $\operatorname{ran}\alpha = \operatorname{ran}\beta$;

(ii) $\alpha\mathcal{R}\beta$ *if and only if* $\pi_\alpha = \pi_\beta$.

Theorem 2.6 ([5], p. 57 and [10], p. 63)**.** *In* $L_F(V)$,

(i) $\alpha\mathcal{L}\beta$ *if and only if* $\operatorname{ran}\alpha = \operatorname{ran}\beta$;

(ii) $\alpha\mathcal{R}\beta$ *if and only if* $\ker\alpha = \ker\beta$.

Observe that for $\alpha \in I(X)$, $\alpha \circ \alpha^{-1} = \{(x, x) \mid x \in \operatorname{dom}\alpha\}$. It follows that for $\alpha, \beta \in I(X)$, $\pi_\alpha = \pi_\beta$ if and only if $\operatorname{dom}\alpha = \operatorname{dom}\beta$. From this fact together with Theorem 2.5 and its proof, we obtain the following theorem for $I(X)$.

Theorem 2.7. *In* $I(X)$,

(i) $\alpha\mathcal{L}\beta$ *if and only if* $\operatorname{ran}\alpha = \operatorname{ran}\beta$;

(ii) $\alpha\mathcal{R}\beta$ *if and only if* $\operatorname{dom}\alpha = \operatorname{dom}\beta$.

For any vector space V over a field F with $\dim_F V \geq q \geq \aleph_0$, we let

$$BL_F(V, q) = \{\alpha \in L_F(V) \mid \alpha \text{ is 1-1 and } \dim_F(V/\operatorname{ran}\alpha) = q\}.$$

It was shown in [15] that for any $\alpha, \beta \in M_F(V)$,

$$\dim_F(V/\operatorname{ran}\alpha\beta) = \dim_F(V/\operatorname{ran}\alpha) + \dim_F(V/\operatorname{ran}\beta).$$

Then $BL_F(V, q)$ is a semigroup which is called the *linear Baer-Levi semigroup on* V *of type* q ([16]). We define the *dual linear Baer-Levi semigroup* $DBL_F(V, q)$ *on* V *of type* q similarly to the dual Baer-Levi semigroup $DBL(X, q)$ defined previously as follows:

$$DBL_F(V, q) = \{\alpha \in L_F(V) \mid \alpha \text{ is onto and } \dim_F \ker\alpha = q\}.$$

Note that $|v\alpha^{-1}| = |\ker\alpha|$ for all $\alpha \in L_F(V)$ and $v \in \operatorname{ran}\alpha$. We have that $DBL_F(V,q)$ is a semigroup by the fact that for any $\alpha, \beta \in E_F(V)$,

$$\dim_F \ker\alpha\beta = \dim_F \ker\alpha + \dim_F \ker\beta,$$

which can be seen by the following proof. Let $\alpha, \beta \in E_F(V)$. We will show that $(\ker\alpha\beta)\alpha = \ker\beta$. If $v \in \ker\alpha\beta$, then $(v\alpha)\beta = v\alpha\beta = 0$, so $v\alpha \in \ker\beta$. Next, let $v \in \ker\beta$. Since α is onto, $v = w\alpha$ for some $w \in V$. Thus $w\alpha\beta = (w\alpha)\beta = v\beta = 0$, so $w \in \ker\alpha\beta$. Hence $v = w\alpha \in (\ker\alpha\beta)\alpha$. This proves that $(\ker\alpha\beta)\alpha = \ker\beta$. Then $\alpha_{|\ker\alpha\beta} : \ker\alpha\beta \to \ker\beta$ is an onto linear transformation. Thus $\dim_F \ker\alpha\beta = \dim_F \ker\left(\alpha_{|\ker\alpha\beta}\right) + \dim_F \ker\beta$. We can see that $\ker\left(\alpha_{|\ker\alpha\beta}\right) = \ker\alpha$. Consequently, $\dim_F \ker\alpha\beta = \dim_F \ker\alpha + \dim_F \ker\beta$, as required.

In [16], the authors gave the next theorem for $BL_F(V,q)$ which has the same result as $BL(X,q)$.

Theorem 2.8 ([16])**.** *If $\dim_F V \geq q \geq \aleph_0$, then $BL_F(V,q)$ is a right simple and right cancellative semigroup without idempotents.*

Mendes-Gançalves [15] introduced the following semigroup.

$$KN_F(V,q) = \{\alpha \in L_F(V) \mid \alpha \text{ is 1-1 and } \dim_F(V/\operatorname{ran}\alpha) \geq q\}$$

where $\dim_F V \geq q \geq \aleph_0$. This semigroup generalizes the semigroup

$$\{\alpha \in L_F(V) \mid \alpha \text{ is 1-1 and } \dim_F(V/\operatorname{ran}\alpha) \text{ is infinite}\}$$

which was introduced by Kemprasit and Namnak [11]. Notice that this semigroup is $KN_F(V,\aleph_0)$. In [15], the authors proved that the prime ideals of $M_F(V)$ are exactly the semigroups $KN_F(V,q)$. Note that a proper ideal I of a semigroup S is called *prime* in [15] if for all a, b in S, $ab \in I$ implies that $a \in I$ or $b \in I$. To be analogous with $KN_F(V,q)$, we define $KN(X,q)$ as follows:

$$KN(X,q) = \{\alpha \in T(X) \mid \alpha \text{ is 1-1 and } |X \smallsetminus \operatorname{ran}\alpha| \geq q\}$$

where $|X| \geq q \geq \aleph_0$. Since $\operatorname{ran}\alpha\beta \subseteq \operatorname{ran}\beta$ for all $\alpha, \beta \in T(X)$, it follows that $KN(X,q)$ is a semigroup.

Finally, we define the following semigroups:

$$Trf(X) = \{\alpha \in T(X) \mid \operatorname{ran}\alpha \text{ is finite}\},$$
$$Prf(X) = \{\alpha \in P(X) \mid \operatorname{ran}\alpha \text{ is finite}\},$$
$$Irf(X) = \{\alpha \in I(X) \mid \operatorname{ran}\alpha \text{ is finite}\},$$
$$Lrf_F(V) = \{\alpha \in L_F(V) \mid \dim_F \operatorname{ran}\alpha \text{ is finite}\}.$$

Notice if X is finite, then $Trf(X) = T(X)$, $Prf(X) = P(X)$ and $Irf(X) = I(X)$. We also have that if V is finite-dimensional, then $Lrf_F(V) = L_F(V)$.

We give some basic knowledge of linear algebra in the following remark. Their proofs are omitted.

Remark 2.9. Let V be a vector space.

(1) If A_1, A_2 are disjoint subsets of V such that $A_1 \cup A_2$ is a linearly independent subset of V, then $\langle A_1\rangle \cap \langle A_2\rangle = \{0\}$.

(2) If A_1 and A_2 are (disjoint) linearly independent subsets of V such that $\langle A_1\rangle \cap \langle A_2\rangle = \{0\}$, then $A_1 \cup A_2$ is a linearly independent subset of V.

(3) If W is a subspace of V, then $\dim_F V = \dim_F(V/W) + \dim_F W$.

(4) For all subspaces U and W of V with $W \subseteq U$, we have
$$(V/W)/(U/W) \cong V/U.$$

(5) If U is a subspace of V, B_1 is a basis of U and B is a basis of V containing B_1, then $v_1 + U \neq v_2 + U$ for all distinct $v_1, v_2 \in B \smallsetminus B_1$ and the set $\{v + U \mid v \in B \smallsetminus B_1\}$ is a basis of the quotient space $V/U\ (= \{v + U \mid v \in V\})$. Hence $\dim_F(V/U) = |B \smallsetminus B_1|$.

Next, let V' be a vector space and $\alpha : V \to V'$ a linear transformation.

(6) If A is a linearly independent subset of V and α is 1-1, then $A\alpha$ is a linearly independent subset of V'. In particular, if B is a basis of V and α is 1-1, then $B\alpha$ is a basis of $\operatorname{ran}\alpha$.

(7) If B is a basis of V, $A \subseteq B$, $A\alpha = \{0\}$, $\alpha_{|_{B \smallsetminus A}}$ is 1-1 and $(B \smallsetminus A)\alpha$ is a linearly independent subset of V', then $\ker\alpha = \langle A \rangle$.

(8) If B is a basis of V, A is a linearly independent subset of V' and $\alpha_{|_B} : B \to A$ is a bijection, then α is a 1-1 linear transformation from V into V'. In particular, if A is also a basis of V', then α is an isomorphism from V onto V'.

(9) Let B_1 be a basis of $\ker\alpha$ and B a basis of V containing B_1. Then for all $u, v \in B \smallsetminus B_1$, if $u \neq v$ then $u\alpha \neq v\alpha$ and $(B \smallsetminus B_1)\alpha$ is a basis of $\operatorname{ran}\alpha$. Hence $\dim_F \operatorname{ran}\alpha = |(B \smallsetminus B_1)\alpha| = |B \smallsetminus B_1|$.

(10) If B_1 is a basis of $\ker\alpha$, B_2 is a basis of $\operatorname{ran}\alpha$ and for each $v \in B_2$, choose $v' \in v\alpha^{-1}$, then

$$v\alpha^{-1} = v' + \ker\alpha \text{ for all } v \in B_2$$

and

$$B_1 \,\dot{\cup}\, \{v' \mid v \in B_2\} \text{ is a basis of } V.$$

(11) If $\alpha : V \to V'$ is 1-1 and W is a subspace of V, then we have that the mapping $v + W \mapsto v\alpha + W\alpha$ is an isomorphism from V/W onto $V\alpha/W\alpha$. Hence $\dim_F(V/W) = \dim_F(V\alpha/W\alpha)$.

CHAPTER III

SEMIGROUPS OF TRANSFORMATIONS OF SETS

This chapter gives characterizations of the left regular and right regular elements of the following semigroups of transformations of X where X is infinite:

$$M(X), M(X) \smallsetminus G(X), E(X), E(X) \smallsetminus G(X),$$
$$BL(X,q), DBL(X,q), KN(X,q) \text{ where } |X| \geq q \geq \aleph_0,$$
$$Trf(X), Prf(X) \text{ and } Irf(X).$$

First of all, we show that the left regular elements and the units of $M(X)$ are identical. We shall introduce the Green's relation $\mathcal{L}$ on $M(X)$ as a lemma.

Lemma 3.1. *For any* $\alpha, \beta \in M(X)$,

$$\alpha\mathcal{L}\beta \textit{ in } M(X) \Leftrightarrow \operatorname{ran}\alpha = \operatorname{ran}\beta.$$

Proof. Assume that $\alpha, \beta \in M(X)$ and $\alpha\mathcal{L}\beta$ in $M(X)$. Then $\alpha = \gamma\beta$ and $\beta = \lambda\alpha$ for some $\gamma, \lambda \in M(X)$. It follows that $\operatorname{ran}\alpha = \operatorname{ran}\gamma\beta \subseteq \operatorname{ran}\beta = \operatorname{ran}\lambda\alpha \subseteq \operatorname{ran}\alpha$, so $\operatorname{ran}\alpha = \operatorname{ran}\beta$.

Conversely, assume that $\operatorname{ran}\alpha = \operatorname{ran}\beta$. Note that $\alpha^{-1} : \operatorname{ran}\alpha\,(= \operatorname{ran}\beta) \to X$ and $\beta^{-1} : \operatorname{ran}\beta\,(= \operatorname{ran}\alpha) \to X$ are bijections. Then $\alpha\beta^{-1}, \beta\alpha^{-1} : X \to X$ are bijections, i.e., $\alpha\beta^{-1}, \beta\alpha^{-1} \in G(X) \subseteq M(X)$. Since $(\alpha\beta^{-1})\beta = \alpha(\beta^{-1}\beta) = \alpha 1_{\operatorname{ran}\beta} = \alpha 1_{\operatorname{ran}\alpha} = \alpha$ and $(\beta\alpha^{-1})\alpha = \beta(\alpha^{-1}\alpha) = \beta 1_{\operatorname{ran}\alpha} = \beta 1_{\operatorname{ran}\beta} = \beta$, it follows that $\alpha\mathcal{L}\beta$ in $M(X)$. □

Theorem 3.2. $\operatorname{LReg}(M(X)) = G(X)$.

Proof. Since $G(X)$ is the group of units of $M(X)$, we have $G(X) \subseteq \text{LReg}(M(X))$.

For the reverse inclusion, let $\alpha \in \text{LReg}(M(X))$. Then $\alpha\mathcal{L}\alpha^2$ in $M(X)$. By Lemma 3.1, $\operatorname{ran}\alpha = \operatorname{ran}\alpha^2$. Thus $X\alpha = (X\alpha)\alpha$. Since α is 1-1, it follows that $X = X\alpha$, which implies that $\alpha \in G(X)$. Hence the result follows. $\square$

Next to determine $\text{RReg}(M(X))$, we first provide the Green's relation $\mathcal{R}$ on $M(X)$.

Lemma 3.3. *For any* $\alpha, \beta \in M(X)$,

$$\alpha\mathcal{R}\beta \textit{ in } M(X) \Leftrightarrow |X \smallsetminus \operatorname{ran}\alpha| = |X \smallsetminus \operatorname{ran}\beta|.$$

Proof. Let $\alpha, \beta \in M(X)$ and assume that $\alpha\mathcal{R}\beta$ in $M(X)$. Then $\alpha = \beta\gamma$ and $\beta = \alpha\lambda$ for some $\gamma, \lambda \in M(X)$. Consequently, $(\operatorname{ran}\beta)\gamma = \operatorname{ran}\alpha$ and $(\operatorname{ran}\alpha)\lambda = \operatorname{ran}\beta$. Since γ and λ are 1-1, we have that $(X \smallsetminus \operatorname{ran}\beta)\gamma \subseteq X \smallsetminus \operatorname{ran}\alpha$ and $(X \smallsetminus \operatorname{ran}\alpha)\lambda \subseteq X \smallsetminus \operatorname{ran}\beta$. These imply that $|X \smallsetminus \operatorname{ran}\beta| \leq |X \smallsetminus \operatorname{ran}\alpha|$ and $|X \smallsetminus \operatorname{ran}\alpha| \leq |X \smallsetminus \operatorname{ran}\beta|$. Hence $|X \smallsetminus \operatorname{ran}\alpha| = |X \smallsetminus \operatorname{ran}\beta|$.

For the converse, assume that $|X \smallsetminus \operatorname{ran}\alpha| = |X \smallsetminus \operatorname{ran}\beta|$. Let $\varphi : X \smallsetminus \operatorname{ran}\beta \to X \smallsetminus \operatorname{ran}\alpha$ be a bijection. Define $\gamma, \lambda : X \to X$ by

$$\gamma = \begin{pmatrix} x\beta & y \\ x\alpha & y\varphi \end{pmatrix}_{\substack{x \in X \\ y \in X \smallsetminus \operatorname{ran}\beta}} \quad \text{and} \quad \lambda = \begin{pmatrix} x\alpha & y \\ x\beta & y\varphi^{-1} \end{pmatrix}_{\substack{x \in X \\ y \in X \smallsetminus \operatorname{ran}\alpha}}.$$

Since α and β are 1-1, we have that γ and λ are well-defined and 1-1. It follows that $\gamma, \lambda \in G(X)$, $\beta\gamma = \alpha$ and $\alpha\lambda = \beta$. Hence $\alpha\mathcal{R}\beta$ in $M(X)$, as desired. $\square$

Note that Lemma 3.3 is found later that it is a special case of Lemma 4.1 in [20].

Theorem 3.4. $\text{RReg}(M(X)) = \{\alpha \in M(X) \mid \operatorname{ran}\alpha = X \textit{ or } X \smallsetminus \operatorname{ran}\alpha \textit{ is infinite}\}$.

Proof. Since $\mathrm{RReg}(M(X)) = \{\alpha \in M(X) \mid \alpha \mathcal{R} \alpha^2 \text{ in } M(X)\}$, by Lemma 3.3, we have

$$\mathrm{RReg}(M(X)) = \{\alpha \in M(X) \mid |X \smallsetminus \operatorname{ran}\alpha| = |X \smallsetminus \operatorname{ran}\alpha^2|\}.$$

Let $\alpha \in M(X)$ be such that $|X \smallsetminus \operatorname{ran}\alpha| = |X \smallsetminus \operatorname{ran}\alpha^2|$ and assume that $X \smallsetminus \operatorname{ran}\alpha$ is finite. Since $\operatorname{ran}\alpha^2 \subseteq \operatorname{ran}\alpha$, we have $X \smallsetminus \operatorname{ran}\alpha \subseteq X \smallsetminus \operatorname{ran}\alpha^2$. Consequently, $X \smallsetminus \operatorname{ran}\alpha = X \smallsetminus \operatorname{ran}\alpha^2$, which implies that $\operatorname{ran}\alpha = \operatorname{ran}\alpha^2$. Hence $X\alpha = (X\alpha)\alpha$. But since α is 1-1, $X = X\alpha$, i.e., $\operatorname{ran}\alpha = X$.

For the reverse inclusion, let $\alpha \in M(X)$ be such that $\operatorname{ran}\alpha = X$ or $X \smallsetminus \operatorname{ran}\alpha$ is infinite. If $\operatorname{ran}\alpha = X$, then $\operatorname{ran}\alpha^2 = X$, so $|X \smallsetminus \operatorname{ran}\alpha| = 0 = |X \smallsetminus \operatorname{ran}\alpha^2|$. Next, suppose that $X \smallsetminus \operatorname{ran}\alpha$ is infinite. Since $\operatorname{ran}\alpha^2 \subseteq \operatorname{ran}\alpha$ and α is 1-1, it follows that

$$\begin{aligned}
|X \smallsetminus \operatorname{ran}\alpha^2| &= |X \smallsetminus \operatorname{ran}\alpha| + |\operatorname{ran}\alpha \smallsetminus \operatorname{ran}\alpha^2| \\
&= |X \smallsetminus \operatorname{ran}\alpha| + |X\alpha \smallsetminus X\alpha^2| \\
&= |X \smallsetminus \operatorname{ran}\alpha| + |(X \smallsetminus X\alpha)\alpha| \\
&= |X \smallsetminus \operatorname{ran}\alpha| + |X \smallsetminus X\alpha| \\
&= 2|X \smallsetminus \operatorname{ran}\alpha| \\
&= |X \smallsetminus \operatorname{ran}\alpha|.
\end{aligned}$$

The theorem is thereby proved. □

The following result is a consequence of Theorem 3.2, Lemma 3.3 and Theorem 3.4.

Corollary 3.5.

(i) $\mathrm{LReg}(M(X) \smallsetminus G(X)) = \varnothing$.

(ii) $\mathrm{RReg}(M(X) \smallsetminus G(X)) = \{\alpha \in M(X) \mid X \smallsetminus \operatorname{ran}\alpha$ *is infinite*$\}$.

Proof. (i) We will prove that $\mathrm{LReg}(M(X) \smallsetminus G(X)) = \varnothing$, suppose not. Let $\alpha \in \mathrm{LReg}(M(X) \smallsetminus G(X))$. Thus $\alpha \in \mathrm{LReg}(M(X))$. But since $\mathrm{LReg}(M(X)) = G(X)$ by Theorem 3.2, it follows that $\alpha \in G(X)$, which is a contradiction.

(ii) Let $\alpha \in \text{RReg}(M(X) \setminus G(X))$. Then $\alpha \in \text{RReg}(M(X))$. By Theorem 3.4, $\operatorname{ran} \alpha = X$ or $X \setminus \operatorname{ran} \alpha$ is infinite. But $\alpha \in M(X) \setminus G(X)$, so $X \setminus \operatorname{ran} \alpha$ is infinite.

For the converse, let $\alpha \in M(X)$ be such that $X \setminus \operatorname{ran} \alpha$ is infinite. By Theorem 3.4, $\alpha \in \text{RReg}(M(X))$, so $\alpha = \alpha^2 \beta$ for some $\beta \in M(X)$. We also have that $|X \setminus \operatorname{ran} \alpha| = |X \setminus \operatorname{ran} \alpha^2|$ by Lemma 3.3. Let $a \in X \setminus \operatorname{ran} \alpha$ be fixed. It follows that $|X \setminus (\operatorname{ran} \alpha \cup \{a\})| = |X \setminus \operatorname{ran} \alpha| = |X \setminus \operatorname{ran} \alpha^2|$. Thus there exists a bijection $\lambda : X \setminus \operatorname{ran} \alpha^2 \to X \setminus (\operatorname{ran} \alpha \cup \{a\})$. Define the mapping γ on X by

$$\gamma = \begin{pmatrix} x & y \\ x\beta & y\lambda \end{pmatrix}_{\substack{x \in \operatorname{ran} \alpha^2 \\ y \in X \setminus \operatorname{ran} \alpha^2}}.$$

Note that $(\operatorname{ran} \alpha^2)\beta = X\alpha^2\beta = X\alpha = \operatorname{ran} \alpha$ and $(X \setminus \operatorname{ran} \alpha^2)\lambda = X \setminus (\operatorname{ran} \alpha \cup \{a\})$. It follows that $(\operatorname{ran} \alpha^2)\beta \cap (X \setminus \operatorname{ran} \alpha^2)\lambda = \varnothing$. But β and λ are 1-1, so we have $\gamma \in M(X)$. Since $\alpha = \alpha^2\beta$, by the definition of γ, we have for any $x \in X$, $x(\alpha^2\gamma) = (x\alpha^2)\gamma = (x\alpha^2)\beta = x(\alpha^2\beta) = x\alpha$. This means that $\alpha = \alpha^2\gamma$. It follows that

$$\begin{aligned} X\gamma &= (\operatorname{ran} \alpha^2)\gamma \cup (X \setminus \operatorname{ran} \alpha^2)\gamma \\ &= \operatorname{ran} \alpha^2\gamma \cup (X \setminus \operatorname{ran} \alpha^2)\lambda \\ &= \operatorname{ran} \alpha \cup (X \setminus (\operatorname{ran} \alpha \cup \{a\})) \\ &= X \setminus \{a\}. \end{aligned}$$

Thus γ is not onto, so $\gamma \in M(X) \setminus G(X)$. Hence $\alpha \in \text{RReg}(M(X) \setminus G(X))$.

Therefore the proof is completed. □

Next, the left regular and right regular elements of $E(X)$ are considered. The following lemma is needed. Note that it is found later that it is a special case of Lemma 5.1 in [20].

Lemma 3.6. *For any $\alpha, \beta \in E(X)$,*

$$\alpha \mathcal{L} \beta \text{ in } E(X) \Leftrightarrow |x\alpha^{-1}| = |x\beta^{-1}| \text{ for all } x \in X.$$

Proof. Let $\alpha, \beta \in E(X)$ be such that $\alpha\mathcal{L}\beta$ in $E(X)$. Then $\alpha = \gamma\beta$ and $\beta = \lambda\alpha$ for some $\gamma, \lambda \in E(X)$. Thus for all $x \in X$ and for all $y \in x\alpha\alpha^{-1}$, $y\gamma\beta = y\alpha = x\alpha$, so $y\gamma \in (x\alpha)\beta^{-1}$. This proves that $(x\alpha\alpha^{-1})\gamma \subseteq (x\alpha)\beta^{-1}$ for all $x \in X$. But α is onto, so $(x\alpha^{-1})\gamma \subseteq x\beta^{-1}$ for all $x \in X$. Since $X = \dot{\bigcup\limits_{x\in X}} x\alpha^{-1} = \dot{\bigcup\limits_{x\in X}} x\beta^{-1}$ and γ is onto, it follows that $(x\alpha^{-1})\gamma = x\beta^{-1}$ for all $x \in X$. This implies that $|x\alpha^{-1}| \geq |x\beta^{-1}|$ for all $x \in X$. By the assumption that $\beta = \lambda\alpha$, we can prove similarly that $|x\beta^{-1}| \geq |x\alpha^{-1}|$ for all $x \in X$. Hence $|x\alpha^{-1}| = |x\beta^{-1}|$ for all $x \in X$.

Conversely, assume that $|x\alpha^{-1}| = |x\beta^{-1}|$ for all $x \in X$. For each $x \in X$, let $\gamma_x : x\alpha^{-1} \to x\beta^{-1}$ be a bijection. Define $\gamma : X \to X$ by

$$\gamma = \begin{pmatrix} y \\ y\gamma_x \end{pmatrix}_{\substack{x\in X \\ y\in x\alpha^{-1}}}.$$

Since $X = \dot{\bigcup\limits_{x\in X}} x\alpha^{-1} = \dot{\bigcup\limits_{x\in X}} x\beta^{-1}$, we have that γ is onto. To show that $\gamma\beta = \alpha$, let $y \in X$. Then $y \in x\alpha^{-1}$ for some $x \in X$, so $y\gamma = y\gamma_x \in x\beta^{-1}$. This implies that $y\gamma\beta = x = y\alpha$. We can show similarly that $\lambda\alpha = \beta$ where $\lambda_x : x\beta^{-1} \to x\alpha^{-1}$ is a bijection for all $x \in X$ and

$$\lambda = \begin{pmatrix} y \\ y\lambda_x \end{pmatrix}_{\substack{x\in X \\ y\in x\beta^{-1}}}.$$

Hence $\alpha\mathcal{L}\beta$ in $E(X)$.

This completes the proof of the lemma. □

The following theorem is an immediate consequence of Lemma 3.6.

Theorem 3.7. $\mathrm{LReg}(E(X)) = \{\alpha \in E(X) \mid |x\alpha^{-1}| = |x(\alpha^2)^{-1}|$ *for all* $x \in X\}$.

Theorem 3.8. $\mathrm{RReg}(E(X)) = G(X)$.

Proof. Since $G(X)$ is the group of units of $E(X)$, we have $G(X) \subseteq \mathrm{RReg}(E(X))$.

For the reverse inclusion, let $\alpha \in \mathrm{RReg}(E(X))$. That is, $\alpha \mathcal{R} \alpha^2$ in $E(X)$. Then $\alpha = \alpha^2\beta$ for some $\beta \in E(X)$. Hence $1_X = \alpha\beta$ since α is onto. Thus α is 1-1, so $\alpha \in G(X)$. $\square$

Theorem 3.7 and Theorem 3.8 yield the following two corollaries, respectively.

Corollary 3.9. *For any $\alpha \in E(X) \smallsetminus G(X)$, $\alpha \in \mathrm{LReg}(E(X) \smallsetminus G(X))$ if and only if α satisfies the following two properties:*

(i) $|x\alpha^{-1}| = |x(\alpha^2)^{-1}|$ *for all $x \in X$;*

(ii) $|y\alpha^{-1}|$ *is infinite for some $y \in X$.*

Proof. Let $\alpha \in \mathrm{LReg}(E(X) \smallsetminus G(X))$. Then $\alpha \in \mathrm{LReg}(E(X))$. By Theorem 3.7, we have $|x\alpha^{-1}| = |x(\alpha^2)^{-1}|$ for all $x \in X$. Suppose that for all $y \in X, |y\alpha^{-1}|$ is finite. Let $y \in X$. Since $y(\alpha^2)^{-1} = (y\alpha^{-1})\alpha^{-1} = \dot{\bigcup\limits_{z \in y\alpha^{-1}}} z\alpha^{-1}$, it follows that

$$|y\alpha^{-1}| = |y(\alpha^2)^{-1}| = \left| \dot{\bigcup_{z \in y\alpha^{-1}}} z\alpha^{-1} \right| = \sum_{z \in y\alpha^{-1}} |z\alpha^{-1}|.$$

Since α is onto, $z\alpha^{-1} \neq \varnothing$ for all $z \in y\alpha^{-1}$. This shows that $|z\alpha^{-1}| = 1$ for all $z \in y\alpha^{-1}$ and for all $y \in X$. But $X = \dot{\bigcup\limits_{y \in X}} y\alpha^{-1}$, so $|z\alpha^{-1}| = 1$ for all $z \in X$. Hence α is 1-1. Thus $\alpha \in G(X)$, a contradiction.

For the converse, we assume that $\alpha \in E(X)$ such that $|x\alpha^{-1}| = |x(\alpha^2)^{-1}|$ for all $x \in X$ and $|y\alpha^{-1}|$ is infinite for some $y \in X$. Let $a \in y\alpha^{-1}$ be given. Then $|y\alpha^{-1} \smallsetminus \{a\}| = |y\alpha^{-1}| = |y(\alpha^2)^{-1}|$. Thus there exists a bijection φ from $y\alpha^{-1} \smallsetminus \{a\}$ onto $y(\alpha^2)^{-1}$. Fix $b \in y(\alpha^2)^{-1}$ and let $\gamma_y : y\alpha^{-1} \to y(\alpha^2)^{-1}$ be defined by

$$\gamma_y = \begin{pmatrix} a & c \\ b & c\varphi \end{pmatrix}_{c \in y\alpha^{-1} \smallsetminus \{a\}}.$$

Since $a\gamma_y = b = c\varphi = c\gamma_y$ for some $c \in y\alpha^{-1} \smallsetminus \{a\}$, we have that γ_y is not 1-1. For each $x \in X \smallsetminus \{y\}$, let $\gamma_x : x\alpha^{-1} \to x(\alpha^2)^{-1}$ be a bijection. Define $\gamma : X \to X$ by

$$\gamma = \begin{pmatrix} z \\ z\gamma_x \end{pmatrix}_{\substack{x\in X \\ z\in x\alpha^{-1}}}.$$

Since $X = \dot{\bigcup\limits_{x\in X}} x\alpha^{-1} = \dot{\bigcup\limits_{x\in X}} x(\alpha^2)^{-1}$, we have that γ is onto. If $x \in X$ and $z \in x\alpha^{-1}$, then $z\gamma = z\gamma_x \in x(\alpha^2)^{-1}$, so $z(\gamma\alpha^2) = (z\gamma)\alpha^2 = x = z\alpha$. Since $X = \dot{\bigcup\limits_{x\in X}} x\alpha^{-1}$, it follows that $\gamma\alpha^2 = \alpha$. Since $X = \dot{\bigcup\limits_{x\in X}} x\alpha^{-1} = \dot{\bigcup\limits_{x\in X}} x(\alpha^2)^{-1}$ and γ_y is not 1-1, it follows that γ is not 1-1. Thus $\gamma \in E(X) \smallsetminus G(X)$. This proves that $\alpha \in \mathrm{LReg}(E(X) \smallsetminus G(X))$, as desired.

Therefore the proof is completed. □

Corollary 3.10. $\mathrm{RReg}(E(X) \smallsetminus G(X)) = \varnothing$.

Proof. If $\alpha \in \mathrm{RReg}(E(X) \smallsetminus G(X))$, then $\alpha \in \mathrm{RReg}(E(X))$, so $\alpha \in G(X)$ by Theorem 3.8. This is impossible. Hence $\mathrm{RReg}(E(X) \smallsetminus G(X)) = \varnothing$. □

We recall the Baer-Levi semigroup of type $(|X|, q)$ on the set X and its dual as follows:

$$BL(X,q) = \{\alpha \in T(X) \mid \alpha \text{ is 1-1 and } |X \smallsetminus \operatorname{ran}\alpha| = q\},$$
$$DBL(X,q) = \{\alpha \in T(X) \mid \alpha \text{ is onto and } |x\alpha^{-1}| = q \text{ for all } x \in X\}$$

where $|X| \geq q \geq \aleph_0$.

Theorem 3.11.

(i) $\mathrm{LReg}(BL(X,q)) = \varnothing$.

(ii) $\mathrm{RReg}(BL(X,q)) = BL(X,q)$.

Proof. (i) Suppose $\mathrm{LReg}(BL(X,q)) \neq \varnothing$. Let $\alpha \in \mathrm{LReg}(BL(X,q))$ be given. Then $\alpha = \beta\alpha^2$ for some $\beta \in BL(X,q)$. Since α is 1-1, $1_X = \beta\alpha$. This implies that α is onto, contradicting the definition of $BL(X,q)$.

(ii) We have that $BL(X,q)$ is right simple from Theorem 2.2. By Theorem 2.1(ii), $BL(X,q) = \alpha^2 BL(X,q)$ for all $\alpha \in BL(X,q)$. Let $\alpha \in BL(X,q)$. Then $\alpha = \alpha^2\beta$ for some $\beta \in BL(X,q)$. Thus $\alpha \in \mathrm{RReg}(BL(X,q))$. □

The following dual version of Theorem 3.11 can be shown in a similar manner.

Theorem 3.12.

(i) $\mathrm{LReg}(DBL(X,q)) = DBL(X,q)$.

(ii) $\mathrm{RReg}(DBL(X,q)) = \varnothing$.

Remark 3.13. Since $BL(X,q)$ and $DBL(X,q)$ do not contain idempotents by Theorem 2.2 and Theorem 2.3, respectively, we have that all elements of $BL(X,q)$ and $DBL(X,q)$ are not regular.

Theorem 3.11 shows that every element of $BL(X,q)$ is right regular but not left regular. Therefore every element of $BL(X,q)$ is right regular but neither regular nor left regular.

From Theorem 3.12, we have that every element of $DBL(X,q)$ is left regular but not right regular. Then every element of $DBL(X,q)$ is left regular but neither regular nor right regular.

The another semigroup which has the same results as $BL(X,q)$ is $KN(X,q)$. Recall that

$$KN(X,q) = \{\alpha \in T(X) \mid \alpha \text{ is 1-1 and } |X \setminus \operatorname{ran}\alpha| \geq q\}$$

where $|X| \geq q \geq \aleph_0$.

Theorem 3.14.

(i) $\mathrm{LReg}(KN(X,q)) = \varnothing$.

(ii) $\mathrm{RReg}(KN(X,q)) = KN(X,q)$.

Proof. (i) Suppose $\mathrm{LReg}(KN(X,q)) \neq \varnothing$. Let $\alpha \in \mathrm{LReg}(KN(X,q))$ be given. Then $\alpha = \beta\alpha^2$ for some $\beta \in KN(X,q)$. Since α is 1-1, $1_X = \beta\alpha$. Thus α is onto, which is contrary to $|X \smallsetminus \operatorname{ran}\alpha| \geq q$.

(ii) Let $\alpha \in KN(X,q)$. Then $|X \smallsetminus \operatorname{ran}\alpha| \geq q$, so $X \smallsetminus \operatorname{ran}\alpha$ is an infinite set. By Theorem 3.4, $\alpha \in \mathrm{RReg}(M(X))$. That is, $\alpha\mathcal{R}\alpha^2$ in $M(X)$. By Lemma 3.3, $|X \smallsetminus \operatorname{ran}\alpha| = |X \smallsetminus \operatorname{ran}\alpha^2|$. Since $X \smallsetminus \operatorname{ran}\alpha$ is infinite, there are $A, B \subseteq X \smallsetminus \operatorname{ran}\alpha$ such that $X \smallsetminus \operatorname{ran}\alpha = A \,\dot{\cup}\, B$ and $|A| = |B| = |X \smallsetminus \operatorname{ran}\alpha|$. Then we have $|X \smallsetminus \operatorname{ran}\alpha^2| = |A|$. Let $\varphi : X \smallsetminus \operatorname{ran}\alpha^2 \to A$ be a bijection. Define $\gamma \in T(X)$ by

$$\gamma = \begin{pmatrix} x\alpha^2 & y \\ x\alpha & y\varphi \end{pmatrix}_{\substack{x \in X \\ y \in X \smallsetminus \operatorname{ran}\alpha^2}}.$$

For $x_1, x_2 \in X$, $x_1\alpha^2 = x_2\alpha^2$ if and only if $x_1\alpha = x_2\alpha$ since α is 1-1. This shows that γ is well-defined and the mapping $x\alpha^2 \mapsto x\alpha$ $(x \in X)$ is 1-1. But since φ is 1-1 and $X\alpha \cap A = \operatorname{ran}\alpha \cap A = \varnothing$, it follows that $\gamma \in M(X)$. We have that $\alpha = \alpha^2\gamma$ and

$$\begin{aligned}
\operatorname{ran}\gamma &= X\gamma \\
&= (\operatorname{ran}\alpha^2 \,\dot{\cup}\, (X \smallsetminus \operatorname{ran}\alpha^2))\gamma \\
&= (\operatorname{ran}\alpha^2)\gamma \,\dot{\cup}\, (X \smallsetminus \operatorname{ran}\alpha^2)\gamma \\
&= \operatorname{ran}\alpha \,\dot{\cup}\, (X \smallsetminus \operatorname{ran}\alpha^2)\varphi \\
&= \operatorname{ran}\alpha \,\dot{\cup}\, A.
\end{aligned}$$

Then $X \smallsetminus \operatorname{ran}\gamma = B$, so $|X \smallsetminus \operatorname{ran}\gamma| = |B| = |X \smallsetminus \operatorname{ran}\alpha| \geq q$. This implies that $\gamma \in KN(X,q)$. Hence $\alpha \in \mathrm{RReg}(KN(X,q))$, and the desired result follows. □

For the remainder of this chapter, we will consider the left regular and right regular elements of $Trf(X)$, $Prf(X)$ and $Irf(X)$. We recall that

$$\begin{aligned}
Trf(X) &= \{\alpha \in T(X) \mid \operatorname{ran}\alpha \text{ is finite}\}, \\
Prf(X) &= \{\alpha \in P(X) \mid \operatorname{ran}\alpha \text{ is finite}\}, \\
Irf(X) &= \{\alpha \in I(X) \mid \operatorname{ran}\alpha \text{ is finite}\}.
\end{aligned}$$

We use the following lemma to prove our desired results for the left regular elements of $Trf(X)$, $Prf(X)$ and $Irf(X)$.

Lemma 3.15. *Let $S(X)$ be $Trf(X), Prf(X)$ or $Irf(X)$. Then for $\alpha, \beta \in S(X)$,*

$$\alpha \mathcal{L} \beta \text{ in } S(X) \Leftrightarrow \operatorname{ran}\alpha = \operatorname{ran}\beta.$$

Proof. Let $\alpha, \beta \in S(X)$. Assume that $\alpha\mathcal{L}\beta$ in $S(X)$. Then $\alpha = \gamma\beta$ and $\beta = \lambda\alpha$ for some $\gamma, \lambda \in S(X)^1$. It follows that $\operatorname{ran}\alpha = \operatorname{ran}(\gamma\beta) \subseteq \operatorname{ran}\beta = \operatorname{ran}(\lambda\alpha) \subseteq \operatorname{ran}\alpha$, so $\operatorname{ran}\alpha = \operatorname{ran}\beta$.

To show the converse, we assume that $\operatorname{ran}\alpha = \operatorname{ran}\beta$. For each $x \in \operatorname{ran}\alpha$, choose $d_x \in x\beta^{-1}$. Then $d_x\beta = x$ for all $x \in \operatorname{ran}\alpha$. Define $\gamma : \operatorname{dom}\alpha \to X$ by

$$\gamma = \begin{pmatrix} x\alpha^{-1} \\ d_x \end{pmatrix}_{x \in \operatorname{ran}\alpha}.$$

Thus $\gamma \in P(X)$, $\operatorname{dom}\gamma = \operatorname{dom}\alpha$, $\operatorname{ran}\gamma \subseteq \operatorname{dom}\beta$ and $|\operatorname{ran}\gamma| = |\{d_x \mid x \in \operatorname{ran}\alpha\}| = |\operatorname{ran}\alpha|$. If $\alpha \in Trf(X)$, then $\gamma \in Trf(X)$. If $\alpha \in Prf(X)$, then $\gamma \in Prf(X)$. If $\alpha \in Irf(X)$, then $\gamma \in Irf(X)$ since $|x\alpha^{-1}| = 1$ for all $x \in \operatorname{ran}\alpha$. Hence $\gamma \in S(X)$. We also have that $\operatorname{dom}(\gamma\beta) = (\operatorname{ran}\gamma \cap \operatorname{dom}\beta)\gamma^{-1} = (\operatorname{ran}\gamma)\gamma^{-1} = \operatorname{dom}\gamma = \operatorname{dom}\alpha$. For $x \in \operatorname{dom}\alpha$, $x \in (x\alpha)\alpha^{-1}$, so $x\gamma\beta = d_{x\alpha}\beta = x\alpha$. Hence $\alpha = \gamma\beta$. We can show similarly that $\beta = \lambda\alpha$ for some $\lambda \in S(X)$. This proves that $\alpha\mathcal{L}\beta$ in $S(X)$, as desired. □

The proof of the next lemma is slightly different from that of Theorem 2.4(ii) given in [5], p. 52. It is needed to determine the right regular elements of $Trf(X)$.

Lemma 3.16. *For any $\alpha, \beta \in Trf(X)$,*

$$\alpha \mathcal{R} \beta \text{ in } Trf(X) \Leftrightarrow \pi_\alpha = \pi_\beta.$$

Proof. Let $\alpha, \beta \in Trf(X)$ be such that $\alpha\mathcal{R}\beta$ in $Trf(X)$. Then $\alpha\mathcal{R}\beta$ in $T(X)$, so by Theorem 2.4(ii), $\pi_\alpha = \pi_\beta$.

Conversely, let $\alpha, \beta \in Trf(X)$ be such that $\pi_\alpha = \pi_\beta$. Let $a \in X$ be fixed. Define $\gamma : X \to X$ by

$$\gamma = \begin{pmatrix} x\beta & y \\ x\alpha & a \end{pmatrix}_{\substack{x \in X \\ y \in X \smallsetminus \operatorname{ran}\beta}}.$$

Since $\pi_\beta \subseteq \pi_\alpha$, γ is well-defined. We also have that $\alpha = \beta\gamma$ and $\operatorname{ran}\gamma = \operatorname{ran}\alpha \cup \{a\}$ which is finite. By using $\pi_\alpha \subseteq \pi_\beta$, we obtain similarly that $\beta = \alpha\lambda$ for some $\lambda \in Trf(X)$. Therefore $\alpha\mathcal{R}\beta$ in $Trf(X)$. □

The following lemma enables us to give the result that LReg($Trf(X)$) = RReg($Trf(X)$). Moreover, we make use of this lemma to show the result of $Prf(X)$.

Lemma 3.17. *For any $\alpha \in Prf(X)$ and $\beta \in P(X)$,*

$$\operatorname{ran}\alpha = \operatorname{ran}\alpha\beta\alpha \Leftrightarrow \pi_\alpha = \pi_{\alpha\beta\alpha}.$$

In particular, for any $\alpha \in Prf(X)$,

$$\operatorname{ran}\alpha = \operatorname{ran}\alpha^2 \Leftrightarrow \pi_\alpha = \pi_{\alpha^2}.$$

Proof. Let $\alpha \in Prf(X)$ and $\beta \in P(X)$. Assume that $\operatorname{ran}\alpha = \operatorname{ran}\alpha\beta\alpha$. Then $\operatorname{ran}\alpha = (\operatorname{ran}\alpha \cap \operatorname{dom}\beta\alpha)\beta\alpha$, so $|\operatorname{ran}\alpha| \le |\operatorname{ran}\alpha \cap \operatorname{dom}\beta\alpha|$. But $\operatorname{ran}\alpha \cap \operatorname{dom}\beta\alpha \subseteq \operatorname{ran}\alpha$, $|\operatorname{ran}\alpha| \ge |\operatorname{ran}\alpha \cap \operatorname{dom}\beta\alpha|$. It follows that $|\operatorname{ran}\alpha| = |\operatorname{ran}\alpha \cap \operatorname{dom}\beta\alpha|$. Since $\operatorname{ran}\alpha$ is finite, we have that $\operatorname{ran}\alpha = \operatorname{ran}\alpha \cap \operatorname{dom}\beta\alpha$. Thus $(\operatorname{ran}\alpha)\beta\alpha = (\operatorname{ran}\alpha \cap \operatorname{dom}\beta\alpha)\beta\alpha = \operatorname{ran}\alpha\beta\alpha = \operatorname{ran}\alpha$, so $(\beta\alpha)_{|\operatorname{ran}\alpha} : \operatorname{ran}\alpha \to \operatorname{ran}\alpha$ is onto. Hence $(\beta\alpha)_{|\operatorname{ran}\alpha}$ is 1-1 since $\operatorname{ran}\alpha$ is finite.

Next, we will prove that $\pi_\alpha = \pi_{\alpha\beta\alpha}$. Since $\operatorname{ran}\alpha \cap \operatorname{dom}\beta\alpha = \operatorname{ran}\alpha$, it follows that $\operatorname{dom}\alpha\beta\alpha = (\operatorname{ran}\alpha \cap \operatorname{dom}\beta\alpha)\alpha^{-1} = (\operatorname{ran}\alpha)\alpha^{-1} = \operatorname{dom}\alpha$. If $(x, y) \in \pi_\alpha$, then $x\alpha = y\alpha$, so $x\alpha\beta\alpha = y\alpha\beta\alpha$. Let $(x, y) \in \pi_{\alpha\beta\alpha}$. Then $x\alpha\beta\alpha = y\alpha\beta\alpha$. Since $(\beta\alpha)_{|\operatorname{ran}\alpha}$ is 1-1, we have that $x\alpha = y\alpha$, i.e., $(x, y) \in \pi_\alpha$. Hence $\pi_\alpha = \pi_{\alpha\beta\alpha}$.

To prove necessity, we assume that $\pi_\alpha = \pi_{\alpha\beta\alpha}$. This implies that

$$\begin{aligned} |\operatorname{ran}\alpha| &= \text{the number of the equivalence classes of } \pi_\alpha \\ &= \text{the number of the equivalence classes of } \pi_{\alpha\beta\alpha} \\ &= |\operatorname{ran}\alpha\beta\alpha|. \end{aligned}$$

Since $\operatorname{ran}\alpha\beta\alpha \subseteq \operatorname{ran}\alpha$ and $\operatorname{ran}\alpha$ is finite, it follows that $\operatorname{ran}\alpha = \operatorname{ran}\alpha\beta\alpha$. □

From the previous series of lemmas, we have the following theorem for $Trf(X)$.

Theorem 3.18. $\operatorname{LReg}(Trf(X)) = \{\alpha \in Trf(X) \mid \alpha_{|\operatorname{ran}\alpha} \in G(\operatorname{ran}\alpha)\}$
$= \operatorname{RReg}(Trf(X))$.

Proof. By Lemma 3.15, $\operatorname{LReg}(Trf(X)) = \{\alpha \in Trf(X) \mid \operatorname{ran}\alpha = \operatorname{ran}\alpha^2\}$. By Lemma 3.16, $\operatorname{RReg}(Trf(X)) = \{\alpha \in Trf(X) \mid \pi_\alpha = \pi_{\alpha^2}\}$. By Lemma 3.17, $\operatorname{LReg}(Trf(X)) = \operatorname{RReg}(Trf(X))$.

Next, to prove that $\operatorname{LReg}(Trf(X)) = \{\alpha \in Trf(X) \mid \alpha_{|\operatorname{ran}\alpha} \in G(\operatorname{ran}\alpha)\}$, let $\alpha \in Trf(X)$. Assume that $\alpha \in \operatorname{LReg}(Trf(X))$. Then $\operatorname{ran}\alpha = \operatorname{ran}\alpha^2 = (\operatorname{ran}\alpha)\alpha$. But since $\operatorname{ran}\alpha$ is finite, it follows that $\alpha_{|\operatorname{ran}\alpha} \in G(\operatorname{ran}\alpha)$. Conversely, if $\alpha_{|\operatorname{ran}\alpha} \in G(\operatorname{ran}\alpha)$, then $\operatorname{ran}\alpha^2 = (\operatorname{ran}\alpha)\alpha = \operatorname{ran}\alpha$, so $\alpha \in \operatorname{LReg}(Trf(X))$.

Hence the result follows. □

We already have the lemma for determining the left regular elements of $Prf(X)$. To obtain the theorem for $Prf(X)$ which is similar to that of $Trf(X)$, we first give the Green's relation $\mathcal{R}$ on $Prf(X)$ as a lemma.

Lemma 3.19. *For any $\alpha, \beta \in Prf(X)$,*

$$\alpha\mathcal{R}\beta \text{ in } Prf(X) \Leftrightarrow \pi_\alpha = \pi_\beta.$$

Proof. Let $\alpha, \beta \in Prf(X)$ be such that $\alpha\mathcal{R}\beta$ in $Prf(X)$. Then $\alpha\mathcal{R}\beta$ in $P(X)$. By Theorem 2.5(ii), $\pi_\alpha = \pi_\beta$.

For the converse, let $\alpha, \beta \in Prf(X)$ be such that $\pi_\alpha = \pi_\beta$. Then $\operatorname{dom}\alpha = \operatorname{dom}\beta$. We define $\gamma : \operatorname{ran}\beta \to X$ by

$$\gamma = \begin{pmatrix} x\beta \\ x\alpha \end{pmatrix}_{x \in \operatorname{dom} \beta}.$$

If $x, y \in \operatorname{dom} \beta$ $(= \operatorname{dom} \alpha)$ are such that $x\beta = y\beta$, then $(x, y) \in \pi_\beta$, so $(x, y) \in \pi_\alpha$ and hence $x\alpha = y\alpha$. Thus γ is well-defined. Since $\operatorname{ran} \gamma = (\operatorname{dom} \beta)\alpha = (\operatorname{dom} \alpha)\alpha = \operatorname{ran} \alpha$ which is finite, $\gamma \in Prf(X)$. We also have that $\operatorname{dom}(\beta\gamma) = (\operatorname{ran} \beta \cap \operatorname{dom} \gamma)\beta^{-1} = (\operatorname{ran} \beta)\beta^{-1} = \operatorname{dom} \beta = \operatorname{dom} \alpha$. If $x \in \operatorname{dom} \alpha$ $(= \operatorname{dom} \beta)$, then $x\alpha = x\beta\gamma$. It follows that $\alpha = \beta\gamma$. It can be shown analogously that $\beta = \alpha\lambda$ where $\lambda : \operatorname{ran} \alpha \to X$ is defined by

$$\lambda = \begin{pmatrix} x\alpha \\ x\beta \end{pmatrix}_{x \in \operatorname{dom} \alpha}.$$

Therefore the lemma is obtained. □

Theorem 3.20. $\operatorname{LReg}(Prf(X)) = \{0\} \cup \{\alpha \in Prf(X) \mid \varnothing \neq \operatorname{ran} \alpha \subseteq \operatorname{dom} \alpha$ *and* $\alpha_{|\operatorname{ran} \alpha} \in G(\operatorname{ran} \alpha)\}$
$= \operatorname{RReg}(Prf(X))$.

Proof. By Lemma 3.15, Lemma 3.19 and Lemma 3.17, we have respectively that

$$\begin{aligned} \operatorname{LReg}(Prf(X)) &= \{\alpha \in Prf(X) \mid \operatorname{ran} \alpha = \operatorname{ran} \alpha^2\}, \\ \operatorname{RReg}(Prf(X)) &= \{\alpha \in Prf(X) \mid \pi_\alpha = \pi_{\alpha^2}\}, \\ \operatorname{LReg}(Prf(X)) &= \operatorname{RReg}(Prf(X)). \end{aligned}$$

Next, we will show that $\operatorname{LReg}(Prf(X)) = \{0\} \cup \{\alpha \in Prf(X) \mid \varnothing \neq \operatorname{ran} \alpha \subseteq \operatorname{dom} \alpha$ and $\alpha_{|\operatorname{ran} \alpha} \in G(\operatorname{ran} \alpha)\}$. Let $\alpha \in \operatorname{LReg}(Prf(X)) \smallsetminus \{0\}$. Since $\operatorname{ran} \alpha^2 = \operatorname{ran} \alpha$, it follows that

$$\begin{aligned} |\operatorname{ran} \alpha \cap \operatorname{dom} \alpha| &\geq |(\operatorname{ran} \alpha \cap \operatorname{dom} \alpha)\alpha| \\ &= |\operatorname{ran} \alpha^2| \\ &= |\operatorname{ran} \alpha| \\ &\geq |\operatorname{ran} \alpha \cap \operatorname{dom} \alpha|, \end{aligned}$$

so $|\operatorname{ran}\alpha \cap \operatorname{dom}\alpha| = |\operatorname{ran}\alpha|$. Since $\operatorname{ran}\alpha$ is finite, $\operatorname{ran}\alpha \cap \operatorname{dom}\alpha = \operatorname{ran}\alpha$. It follows that $\varnothing \neq \operatorname{ran}\alpha \subseteq \operatorname{dom}\alpha$ and $(\operatorname{ran}\alpha)\alpha = (\operatorname{ran}\alpha \cap \operatorname{dom}\alpha)\alpha = \operatorname{ran}\alpha^2 = \operatorname{ran}\alpha$. This means that $\alpha_{|\operatorname{ran}\alpha} : \operatorname{ran}\alpha \to \operatorname{ran}\alpha$ is onto. Since $\operatorname{ran}\alpha$ is finite, $\alpha_{|\operatorname{ran}\alpha} \in G(\operatorname{ran}\alpha)$.

The element 0 clearly belongs to $\operatorname{LReg}(Prf(X))$. We assume that $\alpha \in Prf(X)$ such that $\varnothing \neq \operatorname{ran}\alpha \subseteq \operatorname{dom}\alpha$ and $\alpha_{|\operatorname{ran}\alpha} \in G(\operatorname{ran}\alpha)$. Then $\operatorname{ran}\alpha \subseteq \operatorname{dom}\alpha$ and $(\operatorname{ran}\alpha)\alpha = \operatorname{ran}\alpha$. Thus $\operatorname{ran}\alpha = (\operatorname{ran}\alpha)\alpha =(\operatorname{ran}\alpha \cap \operatorname{dom}\alpha)\alpha = \operatorname{ran}\alpha^2$, so $\operatorname{ran}\alpha = \operatorname{ran}\alpha^2$. Hence $\alpha \in \operatorname{LReg}(Prf(X))$, and the theorem holds. $\square$

The next two theorems show that the set of all left regular elements and the set of all right regular elements of $Irf(X)$ coincide. However, to determine $\operatorname{RReg}(Irf(X))$, the Green's relation $\mathcal{R}$ on $Irf(X)$ is first provided.

Theorem 3.21. $\operatorname{LReg}(Irf(X)) = \{\alpha \in Irf(X) \mid \operatorname{dom}\alpha = \operatorname{ran}\alpha\}$.

Proof. Let $\alpha \in \operatorname{LReg}(Irf(X))$. Then $\alpha\mathcal{L}\alpha^2$ in $Irf(X)$. By Lemma 3.15, $\operatorname{ran}\alpha = \operatorname{ran}\alpha^2$. Thus $(\operatorname{dom}\alpha)\alpha = \operatorname{ran}\alpha = \operatorname{ran}\alpha^2 = (\operatorname{ran}\alpha \cap \operatorname{dom}\alpha)\alpha$. Since α is 1-1, $\operatorname{dom}\alpha = \operatorname{ran}\alpha \cap \operatorname{dom}\alpha$. This means that $\operatorname{dom}\alpha \subseteq \operatorname{ran}\alpha$. Since $|\operatorname{dom}\alpha| = |\operatorname{ran}\alpha|$ and $\operatorname{ran}\alpha$ is finite, we have that $\operatorname{dom}\alpha = \operatorname{ran}\alpha$.

For the reverse inclusion, let $\alpha \in Irf(X)$ be such that $\operatorname{dom}\alpha = \operatorname{ran}\alpha$. Then $\operatorname{ran}\alpha = (\operatorname{dom}\alpha)\alpha = (\operatorname{ran}\alpha \cap \operatorname{dom}\alpha)\alpha = \operatorname{ran}\alpha^2$. Using Lemma 3.15, we obtain $\alpha\mathcal{L}\alpha^2$ in $Irf(X)$. Therefore $\alpha \in \operatorname{LReg}(Irf(X))$, as required. $\square$

Lemma 3.22. *For any* $\alpha, \beta \in Irf(X)$,

$$\alpha\mathcal{R}\beta \text{ in } Irf(X) \Leftrightarrow \operatorname{dom}\alpha = \operatorname{dom}\beta.$$

Proof. If $\alpha\mathcal{R}\beta$ in $Irf(X)$, then $\alpha\mathcal{R}\beta$ in $I(X)$, so by Theorem 2.7(ii), $\operatorname{dom}\alpha = \operatorname{dom}\beta$. Assume that $\alpha, \beta \in Irf(X)$ and $\operatorname{dom}\alpha = \operatorname{dom}\beta$. Let $\gamma = \beta^{-1}\alpha$. Then $\gamma \in I(X)$ and $\operatorname{ran}\gamma \subseteq \operatorname{ran}\alpha$ which is finite. Therefore we have that $\gamma \in Irf(X)$ and $\alpha = 1_{\operatorname{dom}\alpha}\alpha = 1_{\operatorname{dom}\beta}\alpha = \beta\beta^{-1}\alpha = \beta\gamma$. If $\lambda = \alpha^{-1}\beta$, then we also have that $\lambda \in Irf(X)$ and $\beta = \alpha\lambda$. Hence $\alpha\mathcal{R}\beta$ in $Irf(X)$. $\square$

Theorem 3.23. $\text{RReg}(Irf(X)) = \{\alpha \in Irf(X) \mid \operatorname{dom}\alpha = \operatorname{ran}\alpha\}$.

Proof. Let $\alpha \in Irf(X)$ be such that $\alpha\mathcal{R}\alpha^2$ in $Irf(X)$. By Lemma 3.22, $\operatorname{dom}\alpha = \operatorname{dom}\alpha^2$, i.e., $(\operatorname{ran}\alpha)\alpha^{-1} = (\operatorname{ran}\alpha \cap \operatorname{dom}\alpha)\alpha^{-1}$. Since α is 1-1, $\operatorname{ran}\alpha = \operatorname{ran}\alpha \cap \operatorname{dom}\alpha$, so $\operatorname{ran}\alpha \subseteq \operatorname{dom}\alpha$. Since α is 1-1, $|\operatorname{dom}\alpha| = |\operatorname{ran}\alpha|$. Thus $\operatorname{dom}\alpha = \operatorname{ran}\alpha$ since $\operatorname{ran}\alpha$ is finite.

For the reverse inclusion, assume that $\operatorname{dom}\alpha = \operatorname{ran}\alpha$. Then $\operatorname{dom}\alpha^2 = (\operatorname{ran}\alpha \cap \operatorname{dom}\alpha)\alpha^{-1} = (\operatorname{ran}\alpha)\alpha^{-1} = \operatorname{dom}\alpha$. By Lemma 3.22, $\alpha\mathcal{R}\alpha^2$ in $Irf(X)$, i.e., $\alpha \in \text{RReg}(Irf(X))$. □

Remark 3.24. We have that for any $\alpha, \beta \in I(X)$,

$$\operatorname{ran}(\alpha\beta) \subseteq \operatorname{ran}\beta \quad \text{and} \quad \operatorname{ran}(\alpha\beta) = (\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\beta.$$

It follows that for all $\alpha, \beta \in I(X)$,

$$|\operatorname{ran}(\alpha\beta)| \leq |\operatorname{ran}\beta|$$

and

$$|\operatorname{ran}(\alpha\beta)| = |(\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\beta| = |\operatorname{ran}\alpha \cap \operatorname{dom}\beta| \leq |\operatorname{ran}\alpha|.$$

Consequently, $Irf(X)$ is an ideal of $I(X)$. Since $I(X)$ is a regular semigroup, $Irf(X)$ is a regular semigroup.

It is evident from Theorem 3.21 and Theorem 3.23 that an element of $Irf(X)$ need be neither left regular nor right regular.

CHAPTER IV
SEMIGROUPS OF LINEAR TRANSFORMATIONS

In this chapter, V is assumed to be an infinite-dimensional vector space over a field F. We consider the left regular and right regular elements of the following semigroups:

$$M_F(V), M_F(V) \smallsetminus G_F(V), E_F(V), E_F(V) \smallsetminus G_F(V),$$
$$BL_F(V,q), DBL_F(V,q), KN_F(V,q) \text{ and } Lrf_F(V)$$

where $\dim_F V \geq q \geq \aleph_0$.

Comparing with the results in Chapter III, the sets of left regular elements and the sets of right regular elements of the semigroups $M_F(V)$, $M_F(V) \smallsetminus G_F(V)$, $E_F(V)$, $E_F(V) \smallsetminus G_F(V)$, $BL_F(V,q)$, $DBL_F(V,q)$, $KN_F(V,q)$ and $Lrf_F(V)$ are obtained accordingly in this chapter. However, each of the theorems for $\mathrm{LReg}(E_F(V))$ and $\mathrm{LReg}(E_F(V) \smallsetminus G_F(V))$ is obtained in a better form. In addition, some more lemmas are required.

Lemma 4.1. *For any $\alpha, \beta \in M_F(V)$,*

$$\alpha \mathcal{L} \beta \textit{ in } M_F(V) \Leftrightarrow \operatorname{ran}\alpha = \operatorname{ran}\beta.$$

Proof. Note that if $\alpha \in M_F(V)$, then $\alpha^{-1} : \operatorname{ran}\alpha \to V$ is linear. It can be seen from the proof of Lemma 3.1 that the lemma holds. □

Theorem 4.2. $\mathrm{LReg}(M_F(V)) = G_F(V)$.

Proof. From Lemma 4.1 and the proof of Theorem 3.2, we can see that the theorem holds. □

Lemma 4.3. *For any $\alpha, \beta \in M_F(V)$,*

$$\alpha \mathcal{R} \beta \text{ in } M_F(V) \Leftrightarrow \dim_F(V/\operatorname{ran}\alpha) = \dim_F(V/\operatorname{ran}\beta).$$

Proof. Let $\alpha, \beta \in M_F(V)$ be arbitrary. First, assume that $\alpha \mathcal{R} \beta$ in $M_F(V)$. Then $\alpha = \beta\gamma$ and $\beta = \alpha\lambda$ for some $\gamma, \lambda \in M_F(V)$. Thus $(\operatorname{ran}\beta)\gamma = \operatorname{ran}\alpha$ and $(\operatorname{ran}\alpha)\lambda = \operatorname{ran}\beta$. It follows that $\dim_F(V/\operatorname{ran}\beta) = \dim_F(V/V\beta) = \dim_F(V\gamma/(V\beta)\gamma) = \dim_F(\operatorname{ran}\gamma/(\operatorname{ran}\beta)\gamma)$ since γ is a 1-1 linear transformation. Consequently,

$$\begin{aligned} \dim_F(V/\operatorname{ran}\beta) &= \dim_F(\operatorname{ran}\gamma/(\operatorname{ran}\beta)\gamma) \\ &= \dim_F(\operatorname{ran}\gamma/\operatorname{ran}\alpha) \\ &\leq \dim_F(V/\operatorname{ran}\alpha). \end{aligned}$$

We obtain similarly from $\beta = \alpha\lambda$ that $\dim_F(V/\operatorname{ran}\alpha) \leq \dim_F(V/\operatorname{ran}\beta)$. Hence $\dim_F(V/\operatorname{ran}\alpha) = \dim_F(V/\operatorname{ran}\beta)$.

Conversely, assume that $\dim_F(V/\operatorname{ran}\alpha) = \dim_F(V/\operatorname{ran}\beta)$. Let B be a basis of V. Since α and β are 1-1 linear transformations, we have that $B\alpha$ and $B\beta$ are bases of $\operatorname{ran}\alpha$ and $\operatorname{ran}\beta$, respectively. Let B' be a basis of V containing $B\beta$ and B'' a basis of V containing $B\alpha$. Since $\dim_F(V/\operatorname{ran}\beta) = \dim_F(V/\operatorname{ran}\alpha)$, $\dim_F(V/\operatorname{ran}\beta) = |B' \smallsetminus B\beta|$ and $\dim_F(V/\operatorname{ran}\alpha) = |B'' \smallsetminus B\alpha|$, it follows that $|B' \smallsetminus B\beta| = |B'' \smallsetminus B\alpha|$. Let $\varphi : B' \smallsetminus B\beta \to B'' \smallsetminus B\alpha$ be a bijection. Define $\gamma, \lambda \in L_F(V)$ on B' and B'', respectively by

$$\gamma = \begin{pmatrix} v\beta & u \\ v\alpha & u\varphi \end{pmatrix}_{\substack{v \in B \\ u \in B' \smallsetminus B\beta}} \quad \text{and} \quad \lambda = \begin{pmatrix} v\alpha & u \\ v\beta & u\varphi^{-1} \end{pmatrix}_{\substack{v \in B \\ u \in B'' \smallsetminus B\alpha}}.$$

We have that γ and λ are well-defined and 1-1 since α and β are 1-1. Since $\gamma_{|_{B'}} : B' \to B''$ and $\lambda_{|_{B''}} : B'' \to B'$ are bijections, we have that $\gamma, \lambda \in G_F(V)$. Hence the equalities $\beta\gamma = \alpha$ and $\alpha\lambda = \beta$ hold since $v\beta\gamma = v\alpha$ and $v\alpha\lambda = v\beta$ for all $v \in B$. Therefore $\alpha \mathcal{R} \beta$ in $M_F(V)$, as required. $\square$

Theorem 4.4.

$$\mathrm{RReg}(M_F(V)) = \{\alpha \in M_F(V) \mid \operatorname{ran}\alpha = V \text{ or } \dim_F(V/\operatorname{ran}\alpha) \text{ is infinite}\}.$$

Proof. By Lemma 4.3, we have that

$$\mathrm{RReg}(M_F(V)) = \{\alpha \in M_F(V) \mid \dim_F(V/\operatorname{ran}\alpha) = \dim_F(V/\operatorname{ran}\alpha^2)\}.$$

It suffices to show that for $\alpha \in M_F(V)$, $\dim_F(V/\operatorname{ran}\alpha) = \dim_F(V/\operatorname{ran}\alpha^2)$ if and only if $\operatorname{ran}\alpha = V$ or $\dim_F(V/\operatorname{ran}\alpha)$ is infinite.

First, let $\alpha \in M_F(V)$ be such that $\dim_F(V/\operatorname{ran}\alpha) = \dim_F(V/\operatorname{ran}\alpha^2)$ and assume that $\dim_F(V/\operatorname{ran}\alpha)$ is finite. Note that $\operatorname{ran}\alpha^2 \subseteq \operatorname{ran}\alpha \subseteq V$. Let B_1 be a basis of $\operatorname{ran}\alpha^2$, B_2 a basis of $\operatorname{ran}\alpha$ containing B_1 and B a basis of V containing B_2. Since $\dim_F(V/\operatorname{ran}\alpha) = \dim_F(V/\operatorname{ran}\alpha^2)$, $\dim_F(V/\operatorname{ran}\alpha) = |B \smallsetminus B_2|$ and $\dim_F(V/\operatorname{ran}\alpha^2) = |B \smallsetminus B_1|$, we have that $|B \smallsetminus B_2| = |B \smallsetminus B_1|$. We also have that $B \smallsetminus B_2$ is finite since $\dim_F(V/\operatorname{ran}\alpha)$ is finite. But $B \smallsetminus B_2 \subseteq B \smallsetminus B_1$, so we have $B \smallsetminus B_2 = B \smallsetminus B_1$ and hence $B_1 = B_2$. It follows that $\operatorname{ran}\alpha^2 = \operatorname{ran}\alpha$, i.e., $(V\alpha)\alpha = V\alpha$. This implies that $V\alpha = V$ since α is 1-1. Thus $\operatorname{ran}\alpha = V$.

For the converse, let $\alpha \in M_F(V)$ be such that $\operatorname{ran}\alpha = V$ or $\dim_F(V/\operatorname{ran}\alpha)$ is infinite. If $\operatorname{ran}\alpha = V$, then $\operatorname{ran}\alpha^2 = V$, so $\dim_F(V/\operatorname{ran}\alpha) = 0 = \dim_F(V/\operatorname{ran}\alpha^2)$. Next, we assume that $\dim_F(V/\operatorname{ran}\alpha)$ is infinite. Since $\operatorname{ran}\alpha^2 \subseteq \operatorname{ran}\alpha \subseteq V$, we have that $\operatorname{ran}\alpha/\operatorname{ran}\alpha^2$ is a subspace of $V/\operatorname{ran}\alpha^2$, so

$$\begin{aligned}
\dim_F(V/\operatorname{ran}\alpha^2) &= \dim_F\big((V/\operatorname{ran}\alpha^2)/(\operatorname{ran}\alpha/\operatorname{ran}\alpha^2)\big) + \dim_F(\operatorname{ran}\alpha/\operatorname{ran}\alpha^2)\\
&= \dim_F(V/\operatorname{ran}\alpha) + \dim_F(\operatorname{ran}\alpha/\operatorname{ran}\alpha^2)\\
&= \dim_F(V/\operatorname{ran}\alpha) + \dim_F(V/\operatorname{ran}\alpha) \qquad (\text{since } \alpha \in M_F(V))\\
&= 2\dim_F(V/\operatorname{ran}\alpha)\\
&= \dim_F(V/\operatorname{ran}\alpha).
\end{aligned}$$

Therefore the theorem is proved. □

Corollary 4.5.

(i) $\mathrm{LReg}(M_F(V) \smallsetminus G_F(V)) = \varnothing$.

(ii) $\mathrm{RReg}(M_F(V) \smallsetminus G_F(V)) = \{\alpha \in M_F(V) \mid \dim_F(V/\operatorname{ran}\alpha) \text{ is infinite}\}$.

Proof. (i) The proof can be obtained in the same way as that of Corollary 3.5(i) by using Theorem 4.2 instead of Theorem 3.2.

(ii) Let $\alpha \in \mathrm{RReg}(M_F(V) \smallsetminus G_F(V))$. Then $\alpha \in \mathrm{RReg}(M_F(V))$. By Theorem 4.4, $\operatorname{ran}\alpha = V$ or $\dim_F(V/\operatorname{ran}\alpha)$ is infinite. Since $\alpha \notin G_F(V)$, $\dim_F(V/\operatorname{ran}\alpha)$ is infinite.

For the reverse inclusion, let $\alpha \in M_F(V)$ be such that $\dim_F(V/\operatorname{ran}\alpha)$ is infinite. Again by Theorem 4.4, $\alpha \in \mathrm{RReg}(M_F(V))$. That is, $\alpha = \alpha^2\beta$ for some $\beta \in M_F(V)$. Let B_1 be a basis of $\operatorname{ran}\alpha^2$, B_2 a basis of $\operatorname{ran}\alpha$ containing B_1 and B a basis of V containing B_2. Then $\dim_F(V/\operatorname{ran}\alpha^2) = |B \smallsetminus B_1|$ and $\dim_F(V/\operatorname{ran}\alpha)$ $= |B \smallsetminus B_2|$. Since $\alpha\mathcal{R}\alpha^2$ in $M_F(V)$, by Lemma 4.3, $|B \smallsetminus B_1| = |B \smallsetminus B_2|$. Note that $|B \smallsetminus B_2|$ is infinite by assumption. Fix $z \in B \smallsetminus B_2$. Then $|B \smallsetminus (B_2 \cup \{z\})| =$ $|B \smallsetminus B_2| = |B \smallsetminus B_1|$. Thus there is a bijection $\lambda : B \smallsetminus B_1 \to B \smallsetminus (B_2 \cup \{z\})$. Define $\gamma \in L_F(V)$ on B by

$$\gamma = \begin{pmatrix} u & v \\ u\beta & v\lambda \end{pmatrix}_{\substack{u \in B_1 \\ v \in B \smallsetminus B_1}}.$$

We claim that $\gamma \in M_F(V)$. Since $\beta \in M_F(V)$, we have that $B_1\beta$ is linearly independent. Since $\alpha = \alpha^2\beta$ and B_1 is a basis of $\operatorname{ran}\alpha^2$, it follows that $B_1\beta \subseteq$ $\operatorname{ran}\alpha^2\beta = \operatorname{ran}\alpha$, so $\langle B_1\beta\rangle \subseteq \langle B_2\rangle$. We also have that $(B \smallsetminus B_1)\lambda = B \smallsetminus (B_2 \cup \{z\})$ and $\langle B_2\rangle \cap \langle B \smallsetminus (B_2 \cup \{z\})\rangle = \{0\}$. Consequently, $\langle B_1\beta\rangle \cap \langle (B \smallsetminus B_1)\lambda\rangle = \{0\}$. This implies that $B_1\beta \cup (B \smallsetminus B_1)\lambda$ is linearly independent (Remark 2.9(2)). It follows that $\gamma_{|B}$ is 1-1, and hence $\gamma \in M_F(V)$ (Remark 2.9(8)). Next, we claim that $v\alpha^2\gamma = v\alpha^2\beta$ for all $v \in V$. Let $v \in V$. Then $v\alpha^2 \in \operatorname{ran}\alpha^2$. Thus $v\alpha^2$ can be written as a finite sum of the form $\sum_{u \in B_1} a_u u$ where $a_u \in F$ and $u \in B_1$. Hence

$$\begin{aligned} v\alpha^2\gamma &= \left(\sum_{u \in B_1} a_u u\right)\gamma \\ &= \sum_{u \in B_1} a_u(u\gamma) \\ &= \sum_{u \in B_1} a_u(u\beta) \end{aligned}$$

$$= \left(\sum_{u \in B_1} a_u u \right) \beta$$
$$= v\alpha^2\beta,$$

so $v\alpha^2\gamma = v\alpha^2\beta = v\alpha$ for all $v \in V$. Since

$$\begin{aligned} V\gamma &= \langle B \rangle \gamma \\ &= \langle B_1 \cup (B \smallsetminus B_1) \rangle \gamma \\ &= \langle B_1\gamma \rangle + \langle (B \smallsetminus B_1)\gamma \rangle \\ &= \langle B_1\beta \rangle + \langle (B \smallsetminus B_1)\lambda \rangle \\ &\subseteq \langle B_2 \rangle + \langle B \smallsetminus (B_2 \cup \{z\}) \rangle \\ &= \langle B_2 \cup (B \smallsetminus (B_2 \cup \{z\})) \rangle \\ &= \langle B \smallsetminus \{z\} \rangle \\ &\subsetneq \langle B \rangle = V, \end{aligned}$$

we have that γ is not onto. Therefore $\gamma \in M_F(V) \smallsetminus G_F(V)$. This shows that $\alpha \in \mathrm{RReg}(M_F(V) \smallsetminus G_F(V))$.

The proof is thereby completed. □

Lemma 4.6. *For any* $\alpha \in E_F(V)$, $\ker \alpha^2 / \ker \alpha \cong \ker \alpha$.

Proof. First, we note that $\ker \alpha$ is a subspace of $\ker \alpha^2$. We will show that $(\ker \alpha^2)\alpha = \ker \alpha$. If $v \in \ker \alpha^2$, then $(v\alpha)\alpha = v\alpha^2 = 0$, so $v\alpha \in \ker \alpha$. Let $v \in \ker \alpha$. Since α is onto, $w\alpha = v$ for some $w \in V$. Thus $w\alpha^2 = (w\alpha)\alpha = v\alpha = 0$, so $w \in \ker \alpha^2$. Hence $v = w\alpha \in (\ker \alpha^2)\alpha$. Therefore $(\ker \alpha^2)\alpha = \ker \alpha$, so we have $\alpha_{|_{\ker \alpha^2}} : \ker \alpha^2 \to \ker \alpha$ is an onto linear transformation. Consequently, $\ker \alpha^2 / \ker(\alpha_{|_{\ker \alpha^2}}) \cong \ker \alpha$. It is easily seen that $\ker(\alpha_{|_{\ker \alpha^2}}) = \ker \alpha$. Hence $\ker \alpha^2 / \ker \alpha \cong \ker \alpha$. □

Lemma 4.7. *For any* $\alpha, \beta \in E_F(V)$,

$$\alpha \mathcal{L} \beta \textit{ in } E_F(V) \Leftrightarrow \dim_F \ker \alpha = \dim_F \ker \beta.$$

Proof. Let $\alpha, \beta \in E_F(V)$ be arbitrary. Assume that $\alpha\mathcal{L}\beta$ in $E_F(V)$. Then $\alpha = \gamma\beta$ and $\beta = \lambda\alpha$ for some $\gamma, \lambda \in E_F(V)$. If $v \in \ker\alpha$, then $v\gamma\beta = v\alpha = 0$, which implies that $v\gamma \in \ker\beta$. It follows that $(\ker\alpha)\gamma \subseteq \ker\beta$. If $v \in V \setminus \ker\alpha$, then $v\gamma\beta = v\alpha \neq 0$, so $v\gamma \notin \ker\beta$. This shows that $(V \setminus \ker\alpha)\gamma \subseteq V \setminus \ker\beta$. Since γ is onto, $(\ker\alpha)\gamma = \ker\beta$. This means that $\gamma_{|\ker\alpha} : \ker\alpha \to \ker\beta$ is an onto linear transformation, so $\dim_F \ker\alpha \geq \dim_F \ker\beta$. Similarly, $\dim_F \ker\beta \geq \dim_F \ker\alpha$ by the fact that $\beta = \lambda\alpha$.

Conversely, we assume that $\dim_F \ker\alpha = \dim_F \ker\beta$. Let B_1 and B_2 be bases of $\ker\alpha$ and $\ker\beta$, respectively. By assumption, there exists a bijection $\varphi : B_1 \to B_2$. Let B be a basis of V. Since α and β are onto, for each $v \in B$, we can choose $v' \in v\alpha^{-1}$ and $v'' \in v\beta^{-1}$. Then $v'\alpha = v = v''\beta$ for all $v \in B$. Note that $|B| = |\{v' \mid v \in B\}| = |\{v'' \mid v \in B\}|$. We have $B_1 \dot\cup \{v' \mid v \in B\}$ and $B_2 \dot\cup \{v'' \mid v \in B\}$ are bases of V. Define $\gamma \in L_F(V)$ on $B_1 \dot\cup \{v' \mid v \in B\}$ by

$$\gamma = \begin{pmatrix} u & v' \\ u\varphi & v'' \end{pmatrix}_{\substack{u \in B_1 \\ v \in B}}.$$

Since $B_1\varphi = B_2$ which is disjoint to $\{v'' \mid v \in B\}$, we have that the restriction of γ to $B_1 \dot\cup \{v' \mid v \in B\}$ is 1-1. Moreover, $(B_1 \dot\cup \{v' \mid v \in B\})\gamma = (B_1\gamma) \dot\cup (\{v' \mid v \in B\}\gamma) = B_2 \dot\cup \{v'' \mid v \in B\}$. These imply that $\gamma \in G_F(V)$. If $v \in B_1$, then $v\gamma\beta = v\varphi\beta = 0 = v\alpha$ since $v\varphi \in B_2 \subseteq \ker\beta$. If $v \in B$, then $v'\gamma\beta = v''\beta = v = v'\alpha$. These show that $\gamma\beta = \alpha$. Then $\gamma^{-1}\alpha = \beta$. Hence $\alpha\mathcal{L}\beta$ in $E_F(V)$. $\square$

Theorem 4.8.

$$\mathrm{LReg}(E_F(V)) = \{\alpha \in E_F(V) \mid \ker\alpha = \{0\} \textit{ or } \dim_F \ker\alpha \textit{ is infinite}\}.$$

Proof. Let $\alpha \in \mathrm{LReg}(E_F(V))$. Then $\alpha\mathcal{L}\alpha^2$ in $E_F(V)$. By Lemma 4.7, $\dim_F \ker\alpha = \dim_F \ker\alpha^2$. Suppose $\dim_F \ker\alpha$ is finite. Since $\ker\alpha \subseteq \ker\alpha^2$, $\ker\alpha = \ker\alpha^2$. Since $\ker\alpha = 0\alpha^{-1}$ and $\ker\alpha^2 = 0(\alpha^2)^{-1} = (0\alpha^{-1})\alpha^{-1} = (\ker\alpha)\alpha^{-1} = \dot{\bigcup\limits_{x \in \ker\alpha}} x\alpha^{-1}$ $= \left(\dot{\bigcup\limits_{x \in \ker\alpha \setminus \{0\}}} x\alpha^{-1} \right) \dot\cup\, 0\alpha^{-1}$, it follows that

$$\ker\alpha = \ker\alpha^2 = \left(\dot{\bigcup_{x\in\ker\alpha\smallsetminus\{0\}}} x\alpha^{-1}\right) \dot\cup\, 0\alpha^{-1} = \left(\dot{\bigcup_{x\in\ker\alpha\smallsetminus\{0\}}} x\alpha^{-1}\right) \dot\cup\, \ker\alpha.$$

This implies that $\ker\alpha = \{0\}$.

For the converse, let $\alpha \in E_F(V)$ be such that $\ker\alpha = \{0\}$ or $\dim_F \ker\alpha$ is infinite. If $\ker\alpha = \{0\}$, then $\alpha \in G_F(V) \subseteq \mathrm{LReg}(E_F(V))$. Assume that $\dim_F \ker\alpha$ is infinite. We have $\dim_F(\ker\alpha^2/\ker\alpha) = \dim_F \ker\alpha$ by Lemma 4.6. Thus

$$\begin{aligned}\dim_F \ker\alpha^2 &= \dim_F(\ker\alpha^2/\ker\alpha) + \dim_F \ker\alpha\\ &= \dim_F \ker\alpha + \dim_F \ker\alpha\\ &= \dim_F \ker\alpha.\end{aligned}$$

By Lemma 4.7, $\alpha\mathcal{L}\alpha^2$ in $E_F(V)$. Hence $\alpha \in \mathrm{LReg}(E_F(V))$. □

Theorem 4.9. $\mathrm{RReg}(E_F(V)) = G_F(V)$.

Proof. Using the same argument as the proof of Theorem 3.8, we obtain the desired result. □

Corollary 4.10. $\mathrm{LReg}(E_F(V)\smallsetminus G_F(V)) = \{\alpha \in E_F(V) \mid \dim_F \ker\alpha$ *is infinite*$\}$.

Proof. Let $\alpha \in \mathrm{LReg}(E_F(V) \smallsetminus G_F(V))$. Then $\alpha \in \mathrm{LReg}(E_F(V))$ and α is not 1-1. By Theorem 4.8, $\ker\alpha = \{0\}$ or $\dim_F \ker\alpha$ is infinite. But α is not 1-1, so $\dim_F \ker\alpha$ is infinite.

Conversely, let $\alpha \in E_F(V)$ be such that $\dim_F \ker\alpha$ is infinite. By Theorem 4.8, $\alpha \in \mathrm{LReg}(E_F(V))$. Then $\dim_F \ker\alpha = \dim_F \ker\alpha^2$ by Lemma 4.7. Let B_1 be a basis of $\ker\alpha$ and B_2 a basis of $\ker\alpha^2$ containing B_1. Then B_1 and B_2 are infinite and $|B_1| = |B_2|$. Fix $w \in B_1$. We have $|B_1 \smallsetminus \{w\}| = |B_1| = |B_2|$. This implies that there exists a bijection φ from $B_1 \smallsetminus \{w\}$ onto B_2. Let B be a basis of V. For each $v \in B$, we choose $v' \in v\alpha^{-1}$ and $v'' \in v(\alpha^2)^{-1}$. Then $B_1 \dot\cup \{v' \mid v \in B\}$ and $B_2 \,\dot\cup\, \{v'' \mid v \in B\}$ are bases of V. Define $\beta \in L_F(V)$ on $B_1 \,\dot\cup\, \{v' \mid v \in B\}$ by

$$\beta = \begin{pmatrix} w & u & v' \\ 0 & u\varphi & v'' \end{pmatrix}_{\substack{u \in B_1 \setminus \{w\} \\ v \in B}}.$$

Next, we will show that $\alpha = \beta\alpha^2$ on $B_1 \dot{\cup} \{v' \mid v \in B\}$. If $u \in B_1 \setminus \{w\}$, then $u\varphi \in B_2 \subseteq \ker\alpha^2$, so $u\beta\alpha^2 = (u\varphi)\alpha^2 = 0 = u\alpha$. We also have that $(w\beta)\alpha^2 = 0\alpha^2 = 0 = w\alpha$ and for any $v \in B$, $v'\alpha = v = v''\alpha^2 = (v'\beta)\alpha^2$. It follows that $\alpha = \beta\alpha^2$. Since $(B_1 \dot{\cup} \{v' \mid v \in B\})\beta = \{w\beta\} \cup (B_1 \setminus \{w\})\beta \cup (\{v' \mid v \in B\})\beta = \{0\} \cup B_2 \cup \{v'' \mid v \in B\} \supseteq B_2 \cup \{v'' \mid v \in B\}$, we have that

$$\begin{aligned} V\beta &= \langle (B_1 \dot{\cup} \{v' \mid v \in B\})\beta \rangle \\ &\supseteq \langle B_2 \dot{\cup} \{v'' \mid v \in B\} \rangle \\ &= V, \end{aligned}$$

so β is onto. Since $0 \neq w \in \ker\beta$, β is not 1-1. Consequently, $\beta \in E_F(V) \setminus G_F(V)$ and $\alpha = \beta\alpha^2$. Hence $\alpha \in \mathrm{LReg}(E_F(V) \setminus G_F(V))$.

This completes the proof of the corollary. □

Corollary 4.11. $\mathrm{RReg}(E_F(V) \setminus G_F(V)) = \varnothing$.

Proof. This can be proved in the same way as the proof of Corollary 3.10 by using Theorem 4.9 instead of Theorem 3.8. □

Next, recall that

$$\begin{aligned} BL_F(V,q) &= \{\alpha \in L_F(V) \mid \alpha \text{ is 1-1 and } \dim_F(V/\operatorname{ran}\alpha) = q\}, \\ DBL_F(V,q) &= \{\alpha \in L_F(V) \mid \alpha \text{ is onto and } \dim_F \ker\alpha = q\} \end{aligned}$$

where $\dim_F V \geq q \geq \aleph_0$.

Theorem 4.12.

(i) $\mathrm{LReg}(BL_F(V,q)) = \varnothing$.

(ii) $\mathrm{RReg}(BL_F(V,q)) = BL_F(V,q)$.

Proof. (i) The proof can be given in the same way as that of Theorem 3.11(i).

(ii) From Theorem 2.8, the proof can be given in the same way as that of Theorem 3.11(ii). □

Lemma 4.13. *$DBL_F(V,q)$ is a left simple semigroup.*

Proof. Let $\alpha \in DBL_F(V,q)$. We will show that $DBL_F(V,q) \subseteq DBL_F(V,q)\alpha$. Let $\beta \in DBL_F(V,q)$. Then $\dim_F \ker\beta = q = \dim_F \ker\alpha$. Let B_1 be a basis of $\ker\beta$ and B_2 a basis of $\ker\alpha$. Thus B_1 and B_2 are infinite and $|B_1| = |B_2|$. Let C, D be disjoint subsets of B_1 such that $B_1 = C \,\dot\cup\, D$ and $|C| = |D| = |B_1| = q$. Thus $|D| = |B_2|$, so there exists a bijection $\varphi : D \to B_2$. Let B be a basis of V. For each $v \in B$, we choose $v' \in v\beta^{-1}$ and $v'' \in v\alpha^{-1}$. Then $B_1 \,\dot\cup\, \{v' \mid v \in B\}$ and $B_2 \,\dot\cup\, \{v'' \mid v \in B\}$ are bases of V. Define $\gamma \in L_F(V)$ on $B_1 \,\dot\cup\, \{v' \mid v \in B\}$ by

$$\gamma = \begin{pmatrix} C & u & v' \\ 0 & u\varphi & v'' \end{pmatrix}_{\substack{u\in D\\ v\in B}}.$$

Then we have that

$$\begin{aligned} V\gamma &= \langle B_1 \,\dot\cup\, \{v' \mid v \in B\}\rangle\gamma \\ &= \langle (C\gamma) \cup (D\gamma) \cup (\{v' \mid v \in B\}\gamma)\rangle \\ &= \langle \{0\} \cup B_2 \cup \{v'' \mid v \in B\}\rangle \\ &= V \end{aligned}$$

and hence γ is onto. By the definition of γ, $\gamma|_{D \,\dot\cup\, \{v' \mid v \in B\}}$ is a 1-1 linear transformation and $(D \,\dot\cup\, \{v' \mid v \in B\})\gamma = B_2 \,\dot\cup\, \{v'' \mid v \in B\}$, so $\ker\gamma = \langle C\rangle$ (Remark 2.9(7)). Since $C \subseteq B_1$, C is a basis of $\ker\gamma$. Hence $\dim_F \ker\gamma = |C| = q$, so $\gamma \in DBL_F(V,q)$. Next, we claim that $\beta = \gamma\alpha$ on $B_1 \,\dot\cup\, \{v' \mid v \in B\}$. If $u \in C$, then $u \in B_1$, so $u\beta = 0 = 0\alpha = (u\gamma)\alpha = u\gamma\alpha$. If $u \in D$, then $u \in B_1$, so $u\beta = 0 = (u\varphi)\alpha = (u\gamma)\alpha = u\gamma\alpha$. If $v \in B$, then $v'\beta = v = v''\alpha = (v'\gamma)\alpha = v'\gamma\alpha$. These show that $\beta = \gamma\alpha$ on $B_1 \,\dot\cup\, \{v' \mid v \in B\}$, so $\beta = \gamma\alpha$. This implies that $DBL_F(V,q) \subseteq DBL_F(V,q)\alpha$. Thus $DBL_F(V,q)\alpha = DBL_F(V,q)$ for all $\alpha \in DBL_F(V,q)$. By Theorem 2.1(i), $DBL_F(V,q)$ is left simple, as desired. □

Theorem 4.14.

(i) $\mathrm{LReg}(DBL_F(V,q)) = DBL_F(V,q)$.

(ii) $\mathrm{RReg}(DBL_F(V,q)) = \varnothing$.

Proof. (i) Let $\alpha \in DBL_F(V,q)$. By Lemma 4.13, $DBL_F(V,q)$ is left simple. By Theorem 2.1(i), $DBL_F(V,q) = DBL_F(V,q)\alpha^2$. Then $\alpha = \beta\alpha^2$ for some $\beta \in DBL_F(V,q)$. Thus $\alpha \in \mathrm{LReg}(DBL_F(V,q))$.

(ii) Suppose that $\mathrm{RReg}(DBL_F(V,q)) \neq \varnothing$. Let $\alpha \in \mathrm{RReg}(DBL_F(V,q))$. Then $\alpha = \alpha^2\beta$ for some $\beta \in DBL_F(V,q)$. Since α is onto, we have $1_V = \alpha\beta$. This implies that α is 1-1, which is contrary to that $\dim_F \ker\alpha = q$. □

The definition of $KN_F(V,q)$ is recalled as follows:

$$KN_F(V,q) = \{\alpha \in L_F(V) \mid \alpha \text{ is 1-1 and } \dim_F(V/\operatorname{ran}\alpha) \geq q\}$$

where $\dim_F V \geq q \geq \aleph_0$.

Theorem 4.15.

(i) $\mathrm{LReg}(KN_F(V,q)) = \varnothing$.

(ii) $\mathrm{RReg}(KN_F(V,q)) = KN_F(V,q)$.

Proof. (i) The proof of Theorem 3.14(i) shows that (i) holds.

(ii) Let $\alpha \in KN_F(V,q)$. Then $\dim_F(V/\operatorname{ran}\alpha) \geq q$, so $\dim_F(V/\operatorname{ran}\alpha)$ is infinite. Since $\alpha \in M_F(V)$, we have that $\dim_F(V/\operatorname{ran}\alpha^2) = \dim_F(V/\operatorname{ran}\alpha) + \dim_F(V/\operatorname{ran}\alpha)$ (see p. 9), so $\dim_F(V/\operatorname{ran}\alpha^2) = \dim_F(V/\operatorname{ran}\alpha)$. Let B be a basis of V. Since α is a 1-1 linear transformation, we have that $B\alpha$ and $B\alpha^2$ are bases of $\operatorname{ran}\alpha$ and $\operatorname{ran}\alpha^2$, respectively. Let B' and B'' be bases of V containing $B\alpha$ and $B\alpha^2$, respectively. Then $|B' \setminus B\alpha| = \dim_F(V/\operatorname{ran}\alpha) = \dim_F(V/\operatorname{ran}\alpha^2) =$

$|B'' \smallsetminus B\alpha^2|$. Since $B' \smallsetminus B\alpha$ is infinite, $B' \smallsetminus B\alpha = C \dot{\cup} D$ for some $C, D \subseteq B' \smallsetminus B\alpha$ such that $|C| = |D| = |B' \smallsetminus B\alpha|$. But $|B'' \smallsetminus B\alpha^2| = |B' \smallsetminus B\alpha|$, we have a bijection φ from $B'' \smallsetminus B\alpha^2$ onto C. Define $\beta \in L_F(V)$ on B'' by

$$\beta = \begin{pmatrix} u\alpha^2 & v \\ u\alpha & v\varphi \end{pmatrix}_{\substack{u \in B \\ v \in B'' \smallsetminus B\alpha^2}}.$$

Since α is 1-1, we have that β is well-defined. Note that $B\alpha \dot{\cup} C$ is linearly independent and $B''\beta = B\alpha \dot{\cup} C$. It follows that $\beta \in M_F(V)$ (Remark 2.9(8)). By the definition of β, $\alpha = \alpha^2\beta$ on B, so $\alpha = \alpha^2\beta$ on V. Since β is a 1-1 linear transformation, we have $B''\beta$ is a basis of $\operatorname{ran}\beta$. Since $B''\beta = B\alpha \dot{\cup} C$, we have $B\alpha \dot{\cup} C$ is a basis of $\operatorname{ran}\beta$. It follows that $\dim_F(V/\operatorname{ran}\beta) = |B' \smallsetminus (B\alpha \dot{\cup} C)| = |D| = |B' \smallsetminus B\alpha| = \dim_F(V/\operatorname{ran}\alpha) \geq q$. This means that $\beta \in KN_F(V, q)$ and $\alpha = \alpha^2\beta$. Therefore $\alpha \in \operatorname{RReg}(KN_F(V, q))$, as desired. □

Finally, recall that

$$Lrf_F(V) = \{\alpha \in L_F(V) \mid \dim_F \operatorname{ran}\alpha \text{ is finite}\}.$$

Lemma 4.16. *For any $\alpha, \beta \in Lrf_F(V)$,*

$$\alpha\mathcal{L}\beta \text{ in } Lrf_F(V) \Leftrightarrow \operatorname{ran}\alpha = \operatorname{ran}\beta.$$

Proof. For any $\alpha, \beta \in Lrf_F(V)$, if $\alpha\mathcal{L}\beta$ in $Lrf_F(V)$, then we also have $\alpha\mathcal{L}\beta$ in $L_F(V)$. By Theorem 2.6(i), $\operatorname{ran}\alpha = \operatorname{ran}\beta$.

Next, we will prove the converse by using the proof of Lemma 2 in [17]. Let $\alpha, \beta \in Lrf_F(V)$, B_1 a basis of $\ker\alpha$ and B a basis of V containing B_1. Then $\{v\alpha \mid v \in B \smallsetminus B_1\}$ is a basis of $\operatorname{ran}\alpha$ $(= \operatorname{ran}\beta)$. For each $v \in B \smallsetminus B_1$, we choose $v' \in (v\alpha)\beta^{-1}$. Define $\gamma \in L_F(V)$ on B by

$$\gamma = \begin{pmatrix} u & v \\ 0 & v' \end{pmatrix}_{\substack{u \in B_1 \\ v \in B \smallsetminus B_1}}.$$

If $u \in B_1$, then $u\alpha = 0 = (u\gamma)\beta$. If $v \in B \setminus B_1$, then $v\gamma\beta = v'\beta = v\alpha$. This shows that $\alpha = \gamma\beta$ on B. Moreover, we will prove $\{v' \mid v \in B \setminus B_1\}$ is a basis of $\operatorname{ran}\gamma$. To verify that $\{v' \mid v \in B \setminus B_1\}$ is linearly independent, let $\sum_{v \in B \setminus B_1} a_v v' = 0$ where $a_v \in F$ for all $v \in B \setminus B_1$. Then $\sum_{v \in B \setminus B_1} a_v(v\alpha) = \sum_{v \in B \setminus B_1} a_v(v'\beta) = \left(\sum_{v \in B \setminus B_1} a_v v'\right)\beta = 0$, so $a_v = 0$ for all $v \in B \setminus B_1$. By the definition of γ, we have that $\{v' \mid v \in B \setminus B_1\}$ is a basis of $\operatorname{ran}\gamma$. Note that $|\{v' \mid v \in B \setminus B_1\}| = |B \setminus B_1|$. Since $(B \setminus B_1)\alpha$ is a basis of $\operatorname{ran}\alpha$ and $|B \setminus B_1| = |(B \setminus B_1)\alpha|$ (Remark 2.9(9)), it follows that $\{v' \mid v \in B \setminus B_1\}$ is finite. Therefore $\gamma \in Lrf_F(V)$ and $\alpha = \gamma\beta$, as required. A similar argument implies that $\beta = \lambda\alpha$ for some $\lambda \in Lrf_F(V)$. Hence $\alpha\mathcal{L}\beta$ in $Lrf_F(V)$. □

Lemma 4.17. *For any $\alpha, \beta \in Lrf_F(V)$,*

$$\alpha\mathcal{R}\beta \text{ in } Lrf_F(V) \Leftrightarrow \ker\alpha = \ker\beta.$$

Proof. Let $\alpha, \beta \in Lrf_F(V)$ be such that $\alpha\mathcal{R}\beta$ in $Lrf_F(V)$. Then $\alpha\mathcal{R}\beta$ in $L_F(V)$. By Theorem 2.6(ii), $\ker\alpha = \ker\beta$.

We will prove the converse by using the proof of Lemma 3 in [17]. Let B_1 be a basis of $\ker\alpha\,(= \ker\beta)$, B a basis of V containing B_1. We know that $(B \setminus B_1)\alpha$ and $(B \setminus B_1)\beta$ are bases of $\operatorname{ran}\alpha$ and $\operatorname{ran}\beta$, respectively and $\dim_F \operatorname{ran}\alpha = |(B \setminus B_1)\alpha| = |B \setminus B_1| = |(B \setminus B_1)\beta| = \dim_F \operatorname{ran}\beta$. Let B' and B'' be bases of V containing $(B \setminus B_1)\alpha$ and $(B \setminus B_1)\beta$, respectively. Define $\gamma \in L_F(V)$ on B'' and $\lambda \in L_F(V)$ on B' by

$$\gamma = \begin{pmatrix} v\beta & u \\ v\alpha & 0 \end{pmatrix}_{\substack{v \in B \setminus B_1 \\ u \in B'' \setminus ((B \setminus B_1)\beta)}} \quad \text{and} \quad \lambda = \begin{pmatrix} v\alpha & u \\ v\beta & 0 \end{pmatrix}_{\substack{v \in B \setminus B_1 \\ u \in B' \setminus ((B \setminus B_1)\alpha)}}.$$

Since $\ker\alpha = \ker\beta$, γ and λ are well-defined. We also have that $\alpha = \beta\gamma$ and $\beta = \alpha\lambda$ on B. Then $\alpha = \beta\gamma$ and $\beta = \alpha\lambda$ on V. Since $\alpha, \beta \in Lrf_F(V)$, $(B \setminus B_1)\alpha$

and $(B \setminus B_1)\beta$ are finite. But $\operatorname{ran}\gamma = \langle (B \setminus B_1)\alpha \rangle$ and $\operatorname{ran}\lambda = \langle (B \setminus B_1)\beta \rangle$, so we have that $\dim_F \operatorname{ran}\gamma = |(B \setminus B_1)\alpha|$ and $\dim_F \operatorname{ran}\lambda = |(B \setminus B_1)\beta|$. Hence $\gamma, \lambda \in Lrf_F(V)$. This proves that $\alpha \mathcal{R} \beta$ in $Lrf_F(V)$. $\square$

Lemma 4.18. *For any $\alpha \in Lrf_F(V)$ and $\beta \in L_F(V)$,*

$$\operatorname{ran}\alpha = \operatorname{ran}\alpha\beta\alpha \Leftrightarrow \ker\alpha = \ker\alpha\beta\alpha.$$

In particular, for any $\alpha \in Lrf_F(V)$,

$$\operatorname{ran}\alpha = \operatorname{ran}\alpha^2 \Leftrightarrow \ker\alpha = \ker\alpha^2.$$

Proof. Let $\alpha \in Lrf_F(V)$ and $\beta \in L_F(V)$. We assume that $\operatorname{ran}\alpha = \operatorname{ran}\alpha\beta\alpha$. Let B_1 be a basis of $\ker\alpha$, B_2 a basis of $\ker\alpha\beta\alpha$ containing B_1 and B a basis of V containing B_2. Then $(B \setminus B_1)\alpha$ is a basis of $\operatorname{ran}\alpha$, $|(B \setminus B_1)\alpha| = |B \setminus B_1|$, $(B \setminus B_2)\alpha\beta\alpha$ is a basis of $\operatorname{ran}\alpha\beta\alpha$ and $|(B \setminus B_2)\alpha\beta\alpha| = |B \setminus B_2|$. Since $\operatorname{ran}\alpha = \operatorname{ran}\alpha\beta\alpha$, it follows that

$$\begin{aligned} |B \setminus B_2| &= |(B \setminus B_2)\alpha\beta\alpha| \\ &= |(B \setminus B_1)\alpha| \\ &= |B \setminus B_1| \\ &= |B \setminus B_2| + |B_2 \setminus B_1|. \end{aligned}$$

But $\dim_F \operatorname{ran}\alpha$ is finite, so $B \setminus B_2$ is a finite set. This implies that $|B_2 \setminus B_1| = 0$. Thus $B_1 = B_2$. Consequently, $\ker\alpha = \langle B_1 \rangle = \langle B_2 \rangle = \ker\alpha\beta\alpha$.

To show the converse, assume that $\ker\alpha = \ker\alpha\beta\alpha$. Let B_1 be a basis of $\ker\alpha$ $(= \ker\alpha\beta\alpha)$. Then $(B \setminus B_1)\alpha$ is a basis of $\operatorname{ran}\alpha$, $(B \setminus B_1)\alpha\beta\alpha$ is a basis of $\operatorname{ran}\alpha\beta\alpha$ and $|(B \setminus B_1)\alpha| = |B \setminus B_1| = |(B \setminus B_1)\alpha\beta\alpha|$. Thus $\dim_F \operatorname{ran}\alpha = \dim_F \operatorname{ran}\alpha\beta\alpha$. Since $\dim_F \operatorname{ran}\alpha$ is finite and $\operatorname{ran}\alpha\beta\alpha$ is a subspace of $\operatorname{ran}\alpha$, it follows that $\operatorname{ran}\alpha = \operatorname{ran}\alpha\beta\alpha$.

Therefore the lemma is proved. $\square$

Theorem 4.19. $\mathrm{LReg}(Lrf_F(V)) = \{\alpha \in Lrf_F(V) \mid \alpha_{|\mathrm{ran}\,\alpha} \in G_F(\mathrm{ran}\,\alpha)\}$
$= \mathrm{RReg}(Lrf_F(V))$.

Proof. By Lemma 4.16, $\mathrm{LReg}(Lrf_F(V)) = \{\alpha \in Lrf_F(V) \mid \mathrm{ran}\,\alpha = \mathrm{ran}\,\alpha^2\}$. By Lemma 4.17, $\mathrm{RReg}(Lrf_F(V)) = \{\alpha \in Lrf_F(V) \mid \ker\alpha = \ker\alpha^2\}$. By Lemma 4.18, $\mathrm{LReg}(Lrf_F(V)) = \mathrm{RReg}(Lrf_F(V))$.

Next, we will show that $\mathrm{LReg}(Lrf_F(V)) = \{\alpha \in Lrf_F(V) \mid \alpha_{|\mathrm{ran}\,\alpha} \in G_F(\mathrm{ran}\,\alpha)\}$. If $\alpha_{|\mathrm{ran}\,\alpha} \in G_F(\mathrm{ran}\,\alpha)$, then $\mathrm{ran}\,\alpha = (\mathrm{ran}\,\alpha)\alpha = \mathrm{ran}\,\alpha^2$, so $\alpha \in \mathrm{LReg}(Lrf_F(V))$. Let $\alpha \in \mathrm{LReg}(Lrf_F(V))$. Then $\mathrm{ran}\,\alpha = \mathrm{ran}\,\alpha^2$. Thus $(\mathrm{ran}\,\alpha)\alpha = \mathrm{ran}\,\alpha^2 = \mathrm{ran}\,\alpha$, i.e., $\alpha_{|\mathrm{ran}\,\alpha} : \mathrm{ran}\,\alpha \to \mathrm{ran}\,\alpha$ is onto. Let B be a basis of $\mathrm{ran}\,\alpha$. Then $\langle B\rangle = \mathrm{ran}\,\alpha = \mathrm{ran}\,\alpha^2 = (\mathrm{ran}\,\alpha)\alpha = \langle B\rangle\alpha = \langle B\alpha\rangle$. Since $\langle B\alpha\rangle = \mathrm{ran}\,\alpha^2$, we have that there exists a basis C of $\mathrm{ran}\,\alpha^2$ contained in $B\alpha$. Then $|B| = |C| \leq |B\alpha| \leq |B|$, so $|B| = |C| = |B\alpha|$. Since B is finite and $C \subseteq B\alpha$, it follows that $B\alpha = C$ which is a finite basis of $\mathrm{ran}\,\alpha^2$. Then $B\alpha$ is linearly independent and $v\alpha \neq w\alpha$ for all distinct $v, w \in B$. Thus $\alpha_{|B} : B \to B\alpha$ is a bijection. This implies that $\alpha_{|\mathrm{ran}\,\alpha}$ is a 1-1 linear transformation from $\mathrm{ran}\,\alpha$ onto $\langle B\alpha\rangle$. But $\mathrm{ran}\,\alpha = \mathrm{ran}\,\alpha^2 = \langle B\alpha\rangle$, so $\alpha_{|\mathrm{ran}\,\alpha} : \mathrm{ran}\,\alpha \to \mathrm{ran}\,\alpha$ is an isomorphism. Hence $\alpha_{|\mathrm{ran}\,\alpha} \in G_F(\mathrm{ran}\,\alpha)$.

The proof is thereby completed. □

CHAPTER V

VARIANTS OF SEMIGROUPS OF TRANSFORMATIONS OF SETS

In this chapter, the left regular and right regular elements of the variants of the well-known transformation semigroups $T(X)$, $P(X)$ and $I(X)$ on a nonempty set X and those semigroups in Chapter III are determined.

Assume that X is a nonempty set. We first determine $\mathrm{LReg}(S(X),\theta)$ and $\mathrm{RReg}(S(X),\theta)$ where $S(X)$ is $T(X), P(X)$ or $I(X)$ and $\theta \in S(X)$.

Theorem 5.1. *For any* $\theta \in T(X)$,

(i) $\mathrm{LReg}(T(X),\theta) = \{\alpha \in T(X) \mid \operatorname{ran}\alpha = \operatorname{ran}\alpha\theta\alpha\}$;

(ii) $\mathrm{RReg}(T(X),\theta) = \{\alpha \in T(X) \mid \pi_\alpha = \pi_{\alpha\theta\alpha}\}$.

Proof. Let $\theta \in T(X)$.

(i) Let $\alpha \in \mathrm{LReg}(T(X),\theta)$. Then $\alpha = \beta\theta(\alpha\theta\alpha)$ for some $\beta \in T(X)$, so $\alpha\mathcal{L}\alpha\theta\alpha$ in $T(X)$. By Theorem 2.4(i), $\operatorname{ran}\alpha = \operatorname{ran}\alpha\theta\alpha$.

For the converse, assume $\alpha \in T(X)$ such that

$$\operatorname{ran}\alpha = \operatorname{ran}\alpha\theta\alpha.$$

Since $\operatorname{ran}\alpha = \operatorname{ran}\alpha\theta\alpha \subseteq \operatorname{ran}\theta\alpha \subseteq \operatorname{ran}\alpha$, we have that $\operatorname{ran}\alpha = \operatorname{ran}\theta\alpha$. Thus

$$\operatorname{ran}\alpha\theta\alpha = (\operatorname{ran}\alpha)\theta\alpha = (\operatorname{ran}\theta\alpha)\theta\alpha = \operatorname{ran}\theta\alpha\theta\alpha.$$

It follows that $\operatorname{ran}\alpha = \operatorname{ran}\theta\alpha\theta\alpha$, so $\alpha\mathcal{L}\theta\alpha\theta\alpha$ in $T(X)$ by Theorem 2.4(i). Then $\alpha = \beta\theta\alpha\theta\alpha$ for some $\beta \in T(X)$. This means that $\alpha \in \mathrm{LReg}(T(X),\theta)$.

(ii) If $\alpha \in \mathrm{RReg}(T(X),\theta)$, then $\alpha = (\alpha\theta\alpha)\theta\beta$ for some $\beta \in T(X)$. By Theorem 2.4(ii), $\pi_\alpha = \pi_{\alpha\theta\alpha}$.

Conversely, let $\alpha \in T(X)$ be such that $\pi_\alpha = \pi_{\alpha\theta\alpha}$. By Theorem 2.4(ii), $\alpha\mathcal{R}\alpha\theta\alpha$ in $T(X)$. But $\mathcal{R}$ is left compatible,$(\alpha\theta)\alpha\mathcal{R}(\alpha\theta)\alpha\theta\alpha$ in $T(X)$, so $\alpha\mathcal{R}\alpha\theta\alpha\theta\alpha$ in $T(X)$. Thus $\alpha = \alpha\theta\alpha\theta\alpha\beta$ for some $\beta \in T(X)$. This implies that $\alpha \in \mathrm{RReg}(T(X), \theta)$. □

Theorem 5.2. *For any* $\theta \in P(X)$,

(i) $\mathrm{LReg}(P(X), \theta) = \{\alpha \in P(X) \mid \operatorname{ran}\alpha = \operatorname{ran}\alpha\theta\alpha\}$;

(ii) $\mathrm{RReg}(P(X), \theta) = \{\alpha \in P(X) \mid \pi_\alpha = \pi_{\alpha\theta\alpha}\}$.

Proof. Let $\theta \in P(X)$.

(i) Let $\alpha \in \mathrm{LReg}(P(X), \theta)$. Then there is $\beta \in P(X)$ such that $\alpha = \beta\theta(\alpha\theta\alpha)$. Thus $\alpha\mathcal{L}\alpha\theta\alpha$ in $P(X)$. By Theorem 2.5(i), $\operatorname{ran}\alpha = \operatorname{ran}\alpha\theta\alpha$.

For the reverse inclusion, assume $\alpha \in P(X)$ such that

$$\operatorname{ran}\alpha = \operatorname{ran}\alpha\theta\alpha.$$

Then $\operatorname{ran}\alpha = \operatorname{ran}\alpha\theta\alpha \subseteq \operatorname{ran}\theta\alpha \subseteq \operatorname{ran}\alpha$, so $\operatorname{ran}\alpha = \operatorname{ran}\theta\alpha$. Thus

$$\operatorname{ran}\alpha\theta\alpha = (\operatorname{ran}\alpha \cap \operatorname{dom}\theta\alpha)\theta\alpha = (\operatorname{ran}\theta\alpha \cap \operatorname{dom}\theta\alpha)\theta\alpha = \operatorname{ran}\theta\alpha\theta\alpha.$$

It follows that $\operatorname{ran}\alpha = \operatorname{ran}\theta\alpha\theta\alpha$. Again by Theorem 2.5(i), $\alpha\mathcal{L}\theta\alpha\theta\alpha$ in $P(X)$, so there is $\beta \in P(X)$ such that $\alpha = \beta\theta\alpha\theta\alpha$. This implies that $\alpha \in \mathrm{LReg}(P(X), \theta)$, so the result follows.

(ii) It can be proved in the same way as the proof of Theorem 5.1(ii) by using Theorem 2.5(ii) instead of Theorem 2.4(ii). □

Theorem 5.3. *For any* $\theta \in I(X)$,

(i) $\mathrm{LReg}(I(X), \theta) = \{\alpha \in I(X) \mid \operatorname{ran}\alpha = \operatorname{ran}\alpha\theta\alpha\}$;

(ii) $\mathrm{RReg}(I(X), \theta) = \{\alpha \in I(X) \mid \operatorname{dom}\alpha = \operatorname{dom}\alpha\theta\alpha\}$.

Proof. Let $\theta \in I(X)$.

(i) By using Theorem 2.7(i) instead of Theorem 2.5(i), the proof is given in the same way as that of Theorem 5.2(i).

(ii) If $\alpha \in \mathrm{RReg}(I(X), \theta)$, then $\alpha \in \mathrm{RReg}(P(X), \theta)$, so by Theorem 5.2(ii), $\pi_\alpha = \pi_{\alpha\theta\alpha}$, and hence $\mathrm{dom}\,\alpha = \mathrm{dom}\,\alpha\theta\alpha$.

For the converse, assume that $\mathrm{dom}\,\alpha = \mathrm{dom}\,\alpha\theta\alpha$. By Theorem 2.7(ii), $\alpha\mathcal{R}\alpha\theta\alpha$ in $I(X)$. Then $(\alpha\theta)\alpha\mathcal{R}(\alpha\theta)\alpha\theta\alpha$ in $I(X)$. These imply that $\alpha\mathcal{R}\alpha\theta\alpha\theta\alpha$ in $I(X)$. Thus $\alpha = \alpha\theta\alpha\theta\alpha\beta$ for some $\beta \in I(X)$. This means that $\alpha \in \mathrm{RReg}(I(X), \theta)$. □

In the remainder, assume that X is infinite. We shall determine $\mathrm{LReg}(S(X), \theta)$ and $\mathrm{RReg}(S(X), \theta)$ where $S(X) = M(X), M(X) \setminus G(X), E(X), E(X) \setminus G(X), BL(X, q), DBL(X, q), KN(X, q), Trf(X), Prf(X)$ and $Irf(X)$ where $|X| \geq q \geq \aleph_0$ and $\theta \in S(X)$.

Theorem 5.4. *The following statements hold for* $\theta \in M(X)$.

(i) *If* $\theta \in G(X)$, *then* $\mathrm{LReg}(M(X), \theta) = \mathrm{LReg}(M(X))$.

(ii) *If* $\theta \notin G(X)$, *then* $\mathrm{LReg}(M(X), \theta) = \varnothing$.

(iii) *If* $\theta \in G(X)$, *then* $\mathrm{RReg}(M(X), \theta) = \mathrm{RReg}(M(X))$.

(iv) *If* $\theta \notin G(X)$, *then* $\mathrm{RReg}(M(X), \theta) = \{\alpha \in M(X) \mid |X \setminus \mathrm{ran}\,\alpha| = |X \setminus \mathrm{ran}\,\alpha\theta\alpha|\}$.

Proof. Let $\theta \in M(X)$.

(i) Assume that $\theta \in G(X)$. Let $\alpha \in \mathrm{LReg}(M(X), \theta)$. Then $\alpha = \beta\theta(\alpha\theta\alpha)$ for some $\beta \in M(X)$. Thus $1_X = \beta\theta\alpha\theta$ since α is 1-1, so $\beta\theta\alpha = \theta^{-1} \in G(X)$. This implies that α is onto. Hence $\alpha \in G(X)$, so $\alpha \in \mathrm{LReg}(M(X))$.

Conversely, let $\alpha \in \mathrm{LReg}(M(X))$. By Theorem 3.2, $\alpha \in G(X)$, so $(\theta\alpha\theta)^{-1} \in G(X) \subseteq M(X)$. Since $\alpha = (\theta\alpha\theta)^{-1}\theta(\alpha\theta\alpha)$, we have that $\alpha \in \mathrm{LReg}(M(X), \theta)$.

(ii) Assume $\theta \notin G(X)$. Then θ is not onto. Suppose that $\mathrm{LReg}(M(X), \theta) \neq \varnothing$. Let $\alpha \in \mathrm{LReg}(M(X), \theta)$. Then $\alpha = \beta\theta(\alpha\theta\alpha)$ for some $\beta \in M(X)$, so $1_X = \beta\theta\alpha\theta$ since α is 1-1. Thus θ is onto, a contradiction.

(iii) By Theorem 3.4, we have that $\mathrm{RReg}(M(X)) = \{\alpha \in M(X) \mid \mathrm{ran}\,\alpha = X$ or $X \setminus \mathrm{ran}\,\alpha$ is infinite$\}$. Assume $\theta \in G(X)$. Let $\alpha \in \mathrm{RReg}(M(X), \theta)$. Then $\alpha = (\alpha\theta\alpha)\theta\beta$ for some $\beta \in M(X)$. Since $\theta\beta \in M(X)$, $\alpha\mathcal{R}\alpha\theta\alpha$ in $M(X)$. Then $\theta\alpha\mathcal{R}\theta\alpha\theta\alpha$ in $M(X)$ and thus $\theta\alpha \in \mathrm{RReg}(M(X))$. This means that $\mathrm{ran}\,\theta\alpha = X$ or $X \setminus \mathrm{ran}\,\theta\alpha$ is infinite. Since θ is onto, $\mathrm{ran}\,\theta\alpha = \mathrm{ran}\,\alpha$. Therefore $\mathrm{ran}\,\alpha = X$ or

$X \setminus \operatorname{ran} \alpha$ is infinite. That is, $\alpha \in \operatorname{RReg}(M(X))$.

For the reverse inclusion, let $\alpha \in \operatorname{RReg}(M(X))$. Since θ is onto, $\operatorname{ran} \theta\alpha = \operatorname{ran} \alpha$. Thus $|X \setminus \operatorname{ran} \theta\alpha| = |X \setminus \operatorname{ran} \alpha|$, so $\alpha \mathcal{R} \theta\alpha$ in $M(X)$ by Lemma 3.3. Then $\alpha^2 \mathcal{R} \alpha\theta\alpha$ in $M(X)$. Since $\alpha \mathcal{R} \alpha^2$ in $M(X)$, we have $\alpha \mathcal{R} \alpha\theta\alpha$ in $M(X)$. Hence there exists $\beta \in M(X)$ such that $\alpha = \alpha\theta\alpha\beta$. Then $\alpha = \alpha\theta\alpha\theta(\theta^{-1}\beta)$. Since $\theta^{-1}\beta \in M(X)$, $\alpha \in \operatorname{RReg}(M(X), \theta)$.

(iv) Assume $\theta \notin G(X)$. Then θ is not onto. Let $\alpha \in \operatorname{RReg}(M(X), \theta)$. Then there exists $\beta \in M(X)$ such that $\alpha = (\alpha\theta\alpha)\theta\beta$. Thus $\alpha \mathcal{R} \alpha\theta\alpha$ in $M(X)$. That is, $|X \setminus \operatorname{ran} \alpha| = |X \setminus \operatorname{ran} \alpha\theta\alpha|$ by Lemma 3.3.

Conversely, let $\alpha \in M(X)$ be such that $|X \setminus \operatorname{ran} \alpha| = |X \setminus \operatorname{ran} \alpha\theta\alpha|$. Then $\alpha \mathcal{R} \alpha\theta\alpha$ in $M(X)$. Thus $(\alpha\theta)\alpha \mathcal{R} (\alpha\theta)\alpha\theta\alpha$ in $M(X)$. It follows that $\alpha \mathcal{R} \alpha\theta\alpha\theta\alpha$ in $M(X)$. Hence $\alpha = \alpha\theta\alpha\theta\alpha\beta$ for some $\beta \in M(X)$. Since $\alpha\beta \in M(X)$, $\alpha \in \operatorname{RReg}(M(X), \theta)$. $\square$

Lemma 5.5. *For $\theta \in M(X)$, if $\theta \notin G(X)$, then* $\operatorname{RReg}(M(X), \theta) \subseteq \operatorname{RReg}(M(X))$.

Proof. Let $\theta \in M(X) \setminus G(X)$ and $\alpha \in \operatorname{RReg}(M(X), \theta)$. By Theorem 5.4(iv), $|X \setminus \operatorname{ran} \alpha| = |X \setminus \operatorname{ran} \alpha\theta\alpha|$. We have that $\operatorname{ran} \alpha\theta\alpha = X\alpha\theta\alpha \subseteq X\theta\alpha \subsetneq X\alpha = \operatorname{ran} \alpha$ since θ is not onto and α is 1-1. Then $X \setminus \operatorname{ran} \alpha \subsetneq X \setminus \operatorname{ran} \alpha\theta\alpha$. But $|X \setminus \operatorname{ran} \alpha| = |X \setminus \operatorname{ran} \alpha\theta\alpha|$, so we have $X \setminus \operatorname{ran} \alpha$ is infinite. By Theorem 3.4, $\alpha \in \operatorname{RReg}(M(X))$. This proves that $\operatorname{RReg}(M(X), \theta) \subseteq \operatorname{RReg}(M(X))$. $\square$

Corollary 5.6. *For any $\theta \in M(X) \setminus G(X)$,*

(i) $\operatorname{LReg}(M(X) \setminus G(X), \theta) = \varnothing$;

(ii) $\operatorname{RReg}(M(X) \setminus G(X), \theta) = \{\alpha \in M(X) \mid X \setminus \operatorname{ran} \alpha$ *is infinite and* $|X \setminus \operatorname{ran} \alpha| \geq |X \setminus \operatorname{ran} \theta|\}$.

Proof. Let $\theta \in M(X) \setminus G(X)$.

(i) Since $\operatorname{LReg}(M(X) \setminus G(X), \theta) \subseteq \operatorname{LReg}(M(X), \theta)$, by Theorem 5.4(ii), $\operatorname{LReg}(M(X) \setminus G(X), \theta) = \varnothing$.

(ii) Let $\alpha \in \operatorname{RReg}(M(X) \setminus G(X), \theta)$. Since $\operatorname{RReg}(M(X) \setminus G(X), \theta) \subseteq \operatorname{RReg}(M(X), \theta)$, $\alpha \in \operatorname{RReg}(M(X), \theta)$. By Theorem 5.4(iv), $|X \setminus \operatorname{ran} \alpha| = |X \setminus$

$\operatorname{ran}\alpha\theta\alpha|$. We also have that $\alpha \in \operatorname{RReg}(M(X))$ by Lemma 5.5. But $\operatorname{ran}\alpha \neq X$, by Theorem 3.4, $X \setminus \operatorname{ran}\alpha$ is infinite. Since $\operatorname{ran}\alpha\theta\alpha \subseteq \operatorname{ran}\theta\alpha \subseteq \operatorname{ran}\alpha$, $X \setminus \operatorname{ran}\alpha \subseteq X \setminus \operatorname{ran}\theta\alpha \subseteq X \setminus \operatorname{ran}\alpha\theta\alpha$. It follows that $|X \setminus \operatorname{ran}\alpha| = |X \setminus \operatorname{ran}\theta\alpha|$. Consequently,

$$\begin{aligned}
|X \setminus \operatorname{ran}\alpha| &= |X \setminus \operatorname{ran}\theta\alpha| \\
&= |X \setminus \operatorname{ran}\alpha| + |\operatorname{ran}\alpha \setminus \operatorname{ran}\theta\alpha| \\
&= |X \setminus \operatorname{ran}\alpha| + |X\alpha \setminus X\theta\alpha| \\
&= |X \setminus \operatorname{ran}\alpha| + |(X \setminus X\theta)\alpha| \qquad \text{(since } \alpha \text{ is 1-1)} \\
&= |X \setminus \operatorname{ran}\alpha| + |X \setminus X\theta| \qquad \text{(since } \alpha \text{ is 1-1)} \\
&= |X \setminus \operatorname{ran}\alpha| + |X \setminus \operatorname{ran}\theta|,
\end{aligned}$$

which implies that $|X \setminus \operatorname{ran}\alpha| \geq |X \setminus \operatorname{ran}\theta|$.

For the reverse inclusion, let $\alpha \in M(X)$ be such that $X \setminus \operatorname{ran}\alpha$ is infinite and $|X \setminus \operatorname{ran}\alpha| \geq |X \setminus \operatorname{ran}\theta|$. Since $X \setminus \operatorname{ran}\alpha \subseteq X \setminus \operatorname{ran}\theta\alpha$, we have that $X \setminus \operatorname{ran}\theta\alpha$ is also infinite. By Corollary 3.5, $\theta\alpha \in \operatorname{RReg}(M(X) \setminus G(X))$, i.e., $\theta\alpha\mathcal{R}(\theta\alpha)^2$ in $M(X)\setminus G(X)$, so $\theta\alpha\mathcal{R}(\theta\alpha)^2$ in $M(X)$. By Lemma 3.3, $|X\setminus\operatorname{ran}\theta\alpha| = |X \setminus \operatorname{ran}(\theta\alpha)^2| = |X \setminus \operatorname{ran}\theta\alpha\theta\alpha|$. Since $\operatorname{ran}\theta\alpha\theta\alpha \subseteq \operatorname{ran}\alpha\theta\alpha \subseteq \operatorname{ran}\theta\alpha$, we have $|X\setminus\operatorname{ran}\theta\alpha| \leq |X\setminus\operatorname{ran}\alpha\theta\alpha| \leq |X\setminus\operatorname{ran}\theta\alpha\theta\alpha| = |X\setminus\operatorname{ran}\theta\alpha|$. This implies that $|X\setminus\operatorname{ran}\theta\alpha| = |X\setminus\operatorname{ran}\alpha\theta\alpha|$. Since $X\setminus\operatorname{ran}\alpha$ is infinite, $|X\setminus\operatorname{ran}\theta| \leq |X\setminus\operatorname{ran}\alpha|$ and α is 1-1, it follows that

$$\begin{aligned}
|X \setminus \operatorname{ran}\theta\alpha| &= |X \setminus \operatorname{ran}\alpha| + |\operatorname{ran}\alpha \setminus \operatorname{ran}\theta\alpha| \\
&= |X \setminus \operatorname{ran}\alpha| + |X\alpha \setminus X\theta\alpha| \\
&= |X \setminus \operatorname{ran}\alpha| + |(X \setminus X\theta)\alpha| \\
&= |X \setminus \operatorname{ran}\alpha| + |X \setminus X\theta| \\
&= |X \setminus \operatorname{ran}\alpha| + |X \setminus \operatorname{ran}\theta| \\
&= |X \setminus \operatorname{ran}\alpha|.
\end{aligned}$$

Hence $|X \setminus \operatorname{ran}\alpha| = |X \setminus \operatorname{ran}\theta\alpha| = |X \setminus \operatorname{ran}\alpha\theta\alpha|$. By Theorem 5.4(iv), $\alpha \in \operatorname{RReg}(M(X), \theta)$. Thus $\alpha = (\alpha\theta\alpha)\theta\beta$ for some $\beta \in M(X)$. It follows that $\alpha =$

$\alpha\theta\alpha\theta\beta = \alpha\theta(\alpha\theta\alpha\theta\beta)\theta\beta = (\alpha\theta\alpha)\theta(\alpha\theta\beta\theta\beta)$. Since $\alpha \in M(X) \smallsetminus G(X)$ and $M(X) \smallsetminus G(X)$ is an ideal of $M(X)$, we have that $\alpha\theta\beta\theta\beta \in M(X) \smallsetminus G(X)$. Therefore $\alpha \in \mathrm{RReg}(M(X) \smallsetminus G(X), \theta)$, as required. □

Theorem 5.7. *For any* $\theta \in E(X)$,

$$\mathrm{LReg}(E(X), \theta) = \{\alpha \in E(X) \mid |x\alpha^{-1}| = |x(\alpha\theta\alpha)^{-1}| \textit{ for all } x \in X\}.$$

Proof. Let $\theta \in E(X)$ and $\alpha = \beta\theta(\alpha\theta\alpha)$ for some $\beta \in E(X)$. Since $\beta\theta \in E(X)$, $\alpha\mathcal{L}\alpha\theta\alpha$ in $E(X)$. By Lemma 3.6, $|x\alpha^{-1}| = |x(\alpha\theta\alpha)^{-1}|$ for all $x \in X$.

For the converse, we assume that $\alpha \in E(X)$ and $|x\alpha^{-1}| = |x(\alpha\theta\alpha)^{-1}|$ for all $x \in X$. By Lemma 3.6, we have $\alpha\mathcal{L}\alpha\theta\alpha$ in $E(X)$. Since $\mathcal{L}$ is right compatible, $\alpha(\theta\alpha)\mathcal{L}\alpha\theta\alpha(\theta\alpha)$ in $E(X)$. Then $\alpha\mathcal{L}\alpha\theta\alpha\theta\alpha$ in $E(X)$, so $\alpha = \beta\alpha\theta\alpha\theta\alpha$ for some $\beta \in E(X)$. This means that $\alpha \in \mathrm{LReg}(E(X), \theta)$. □

Theorem 5.8. *The following statements hold for* $\theta \in E(X)$.

(i) *If* $\theta \in G(X)$, *then* $\mathrm{RReg}(E(X), \theta) = \mathrm{RReg}(E(X))$.

(ii) *If* $\theta \notin G(X)$, *then* $\mathrm{RReg}(E(X), \theta) = \varnothing$.

Proof. Let $\theta \in E(X)$.

(i) Assume that $\theta \in G(X)$. Let $\alpha \in \mathrm{RReg}(E(X), \theta)$. Then $\alpha = (\alpha\theta\alpha)\theta\beta$ for some $\beta \in E(X)$. Thus $1_X = \theta\alpha\theta\beta$ since α is onto. This implies that $\alpha\theta\beta = \theta^{-1} \in G(X)$. It follows that α is 1-1, which implies that $\alpha \in G(X)$. Consequently, $\alpha \in \mathrm{RReg}(E(X))$.

Conversely, if $\alpha \in \mathrm{RReg}(E(X))$, then by Theorem 3.8, $\alpha \in G(X)$, so $\theta\alpha\theta \in G(X)$. Hence $(\theta\alpha\theta)^{-1} \in G(X) \subseteq E(X)$ and $\alpha = \alpha\theta\alpha\theta(\theta\alpha\theta)^{-1}$. This means that $\alpha \in \mathrm{RReg}(E(X), \theta)$.

(ii) Assume that $\alpha \in \mathrm{RReg}(E(X), \theta)$. Then $\alpha = (\alpha\theta\alpha)\theta\beta$ for some $\beta \in E(X)$. Since α is onto, $1_X = \theta\alpha\theta\beta$. This implies that θ is 1-1, so $\theta \in G(X)$. This proves that if $\theta \notin G(X)$, then $\mathrm{RReg}(E(X), \theta) = \varnothing$. □

Corollary 5.9. *For any* $\theta \in E(X) \setminus G(X)$,

(i) $\mathrm{LReg}(E(X) \setminus G(X), \theta) = \{\alpha \in E(X) \setminus G(X) \mid |x\alpha^{-1}| = |x(\alpha\theta\alpha)^{-1}|$ *for all* $x \in X\}$;

(ii) $\mathrm{RReg}(E(X) \setminus G(X), \theta) = \varnothing$.

Proof. Let $\theta \in E(X) \setminus G(X)$.

(i) Let $\alpha \in \mathrm{LReg}(E(X) \setminus G(X), \theta)$. Then $\alpha \in \mathrm{LReg}(E(X), \theta)$. By Theorem 5.7, $|x\alpha^{-1}| = |x(\alpha\theta\alpha)^{-1}|$ for all $x \in X$.

For the reverse inclusion, let $\alpha \in E(X) \setminus G(X)$ be such that $|x\alpha^{-1}| = |x(\alpha\theta\alpha)^{-1}|$ for all $x \in X$. By Theorem 5.7, $\alpha \in \mathrm{LReg}(E(X), \theta)$. Then $\alpha = \beta\theta(\alpha\theta\alpha)$ for some $\beta \in E(X)$, so $\alpha = \beta\theta\alpha\theta\alpha = \beta\theta(\beta\theta\alpha\theta\alpha)\theta\alpha = (\beta\theta\beta\theta\alpha)\theta\alpha\theta\alpha$. Since $\alpha \in E(X) \setminus G(X)$ and $E(X) \setminus G(X)$ is an ideal of $E(X)$, we have that $\beta\theta\beta\theta\alpha \in E(X) \setminus G(X)$. This implies that $\alpha \in \mathrm{LReg}(E(X) \setminus G(X), \theta)$.

(ii) Since $\mathrm{RReg}(E(X) \setminus G(X), \theta) \subseteq \mathrm{RReg}(E(X), \theta)$, by Theorem 5.8(ii), the result follows. □

Theorem 5.10. *For any* $\theta \in BL(X, q)$,

(i) $\mathrm{LReg}(BL(X, q), \theta) = \varnothing$;

(ii) $\mathrm{RReg}(BL(X, q), \theta) = BL(X, q)$.

Proof. Let $\theta \in BL(X, q)$. Then $|X \setminus \operatorname{ran}\theta| = q \geq \aleph_0$.

(i) Suppose that there exists $\alpha \in \mathrm{LReg}(BL(X, q), \theta)$. Then $\alpha = \beta\theta(\alpha\theta\alpha)$ for some $\beta \in BL(X, q)$. Since α is 1-1, $1_X = \beta\theta\alpha\theta$. Hence θ is onto, which is contrary to $|X \setminus \operatorname{ran}\theta| = q \geq \aleph_0$. Consequently, $\mathrm{LReg}(BL(X, q), \theta) = \varnothing$.

(ii) Let $\alpha \in BL(X, q)$. We know that $BL(X, q)$ is right simple from Theorem 2.2. By Theorem 2.1(ii), $BL(X, q) = (\alpha\theta\alpha\theta)BL(X, q)$. Then $\alpha = \alpha\theta\alpha\theta\beta$ for some $\beta \in BL(X, q)$. This means that $\alpha \in \mathrm{RReg}(BL(X, q), \theta)$. Therefore $\mathrm{RReg}(BL(X, q), \theta) = BL(X, q)$. □

A dual version of the previous theorem can be shown in a similar manner.

Theorem 5.11. *For any* $\theta \in DBL(X,q)$,

(i) $\operatorname{LReg}(DBL(X,q),\theta) = DBL(X,q)$;

(ii) $\operatorname{RReg}(DBL(X,q),\theta) = \varnothing$.

Theorem 5.12. *For any* $\theta \in KN(X,q)$,

(i) $\operatorname{LReg}(KN(X,q),\theta) = \varnothing$;

(ii) $\operatorname{RReg}(KN(X,q),\theta) = \{\alpha \in KN(X,q) \mid |X \smallsetminus \operatorname{ran}\alpha| \geq |X \smallsetminus \operatorname{ran}\theta|\}$.

Proof. Let $\theta \in KN(X,q)$. Then $|X \smallsetminus \operatorname{ran}\theta| \geq q \geq \aleph_0$.

(i) If $\alpha \in \operatorname{LReg}(KN(X,q),\theta)$, then $\alpha = \beta\theta(\alpha\theta\alpha)$ for some $\beta \in KN(X,q)$, thus $1_X = \beta\theta\alpha\theta$ since α is 1-1 and hence θ is onto, a contradiction. Therefore $\operatorname{LReg}(KN(X,q),\theta) = \varnothing$.

(ii) Let $\alpha \in \operatorname{RReg}(KN(X,q),\theta)$. Since $KN(X,q) \subseteq M(X) \smallsetminus G(X)$, $\alpha \in \operatorname{RReg}(M(X) \smallsetminus G(X),\theta)$. By Corollary 5.6(ii), $|X \smallsetminus \operatorname{ran}\alpha| \geq |X \smallsetminus \operatorname{ran}\theta|$.

For the converse, let $\alpha \in KN(X,q)$ such that $|X \smallsetminus \operatorname{ran}\alpha| \geq |X \smallsetminus \operatorname{ran}\theta|$. By Corollary 5.6(ii), $\alpha \in \operatorname{RReg}(M(X) \smallsetminus G(X),\theta)$. Then $\alpha = (\alpha\theta\alpha)\theta\beta$ for some $\beta \in M(X) \smallsetminus G(X)$, so $\alpha = \alpha\theta\alpha\theta\beta = \alpha\theta(\alpha\theta\alpha\theta\beta)\theta\beta = \alpha\theta\alpha\theta(\alpha\theta\beta\theta\beta)$. We will consider $|X \smallsetminus \operatorname{ran}\alpha\theta\beta\theta\beta|$. Since $\operatorname{ran}\alpha\theta\beta\theta\beta \subseteq \operatorname{ran}\theta\beta\theta\beta$, we have that

$$
\begin{aligned}
|X \smallsetminus \operatorname{ran}\alpha\theta\beta\theta\beta| &= |X \smallsetminus \operatorname{ran}\theta\beta\theta\beta| + |\operatorname{ran}\theta\beta\theta\beta \smallsetminus \operatorname{ran}\alpha\theta\beta\theta\beta| \\
&= |X \smallsetminus \operatorname{ran}\theta\beta\theta\beta| + |X\theta\beta\theta\beta \smallsetminus X\alpha\theta\beta\theta\beta| \\
&= |X \smallsetminus \operatorname{ran}\theta\beta\theta\beta| + |(X \smallsetminus X\alpha)\theta\beta\theta\beta| \quad \text{(since } \theta\beta\theta\beta \text{ is 1-1)} \\
&= |X \smallsetminus \operatorname{ran}\theta\beta\theta\beta| + |X \smallsetminus X\alpha| \qquad\quad \text{(since } \theta\beta\theta\beta \text{ is 1-1)} \\
&\geq |X \smallsetminus X\alpha| \\
&= |X \smallsetminus \operatorname{ran}\alpha| \geq q.
\end{aligned}
$$

From this, we obtain $\alpha\theta\beta\theta\beta \in KN(X,q)$ such that $\alpha = \alpha\theta\alpha\theta(\alpha\theta\beta\theta\beta)$. This means that $\alpha \in \operatorname{RReg}(KN(X,q),\theta)$, as required. □

Theorem 5.13. *For any* $\theta \in Trf(X)$,

$$\begin{aligned}\mathrm{LReg}(Trf(X),\theta) &= \{\alpha \in Trf(X) \mid (\theta\alpha)_{|\mathrm{ran}\,\theta\alpha} \in G(\mathrm{ran}\,\theta\alpha) \ \textit{and} \\ &\qquad \mathrm{ran}\,\theta\alpha = \mathrm{ran}\,\alpha\} \\ &= \mathrm{RReg}(Trf(X),\theta).\end{aligned}$$

Proof. Let $\theta \in Trf(X)$ and $\alpha \in \mathrm{LReg}(Trf(X),\theta)$. Then $\alpha = \beta\theta(\alpha\theta\alpha)$ for some $\beta \in Trf(X)$. This means that $\alpha\mathcal{L}\theta\alpha\theta\alpha$ in $Trf(X)$. By Lemma 3.15, $\mathrm{ran}\,\alpha = \mathrm{ran}\,\theta\alpha\theta\alpha$. Since $\alpha = \beta\theta\alpha\theta\alpha$, we have $\theta\alpha = \theta\beta\theta\alpha\theta\alpha = \theta\beta(\theta\alpha)^2$, so $\theta\alpha \in \mathrm{LReg}(Trf(X))$. By Theorem 3.18, $(\theta\alpha)_{|\mathrm{ran}\,\theta\alpha} \in G(\mathrm{ran}\,\theta\alpha)$, which implies that $\mathrm{ran}\,\theta\alpha\theta\alpha = \mathrm{ran}\,\theta\alpha$. Hence $\mathrm{ran}\,\theta\alpha = \mathrm{ran}\,\alpha$.

Conversely, let $\alpha \in Trf(X)$ be such that $(\theta\alpha)_{|\mathrm{ran}\,\theta\alpha} \in G(\mathrm{ran}\,\theta\alpha)$ and $\mathrm{ran}\,\theta\alpha = \mathrm{ran}\,\alpha$. Then $\mathrm{ran}\,\theta\alpha\theta\alpha = (\mathrm{ran}\,\theta\alpha)\theta\alpha = \mathrm{ran}\,\theta\alpha = \mathrm{ran}\,\alpha$. By Lemma 3.15, we have $\alpha\mathcal{L}\theta\alpha\theta\alpha$ in $Trf(X)$, so $\alpha = \beta\theta\alpha\theta\alpha$ for some $\beta \in Trf(X)$. This means that $\alpha \in \mathrm{LReg}(Trf(X),\theta)$.

Next, we will show that $\mathrm{LReg}(Trf(X),\theta) = \mathrm{RReg}(Trf(X),\theta)$.

Let $\alpha \in \mathrm{LReg}(Trf(X),\theta)$. Then there exists $\beta \in Trf(X)$ such that $\alpha = \beta\theta(\alpha\theta\alpha)$. Thus $\alpha\mathcal{L}\alpha\theta\alpha$ in $Trf(X)$. By Lemma 3.15, $\mathrm{ran}\,\alpha = \mathrm{ran}\,\alpha\theta\alpha$. Hence $\pi_\alpha = \pi_{\alpha\theta\alpha}$ by Lemma 3.17. By Lemma 3.16, $\alpha\mathcal{R}\alpha\theta\alpha$ in $Trf(X)$, so $\alpha = (\alpha\theta\alpha)\gamma$ for some $\gamma \in Trf(X)$. Therefore $\alpha = \alpha\theta\alpha\gamma = \alpha\theta(\alpha\theta\alpha\gamma)\gamma = \alpha\theta\alpha\theta(\alpha\gamma\gamma)$. This implies that $\alpha \in \mathrm{RReg}(Trf(X),\theta)$.

For the reverse inclusion, let $\alpha \in \mathrm{RReg}(Trf(X),\theta)$. Then $\alpha = (\alpha\theta\alpha)\theta\beta$ for some $\beta \in Trf(X)$, so $\alpha\mathcal{R}\alpha\theta\alpha$ in $Trf(X)$. By Lemma 3.16, $\pi_\alpha = \pi_{\alpha\theta\alpha}$. By Lemma 3.17, $\mathrm{ran}\,\alpha = \mathrm{ran}\,\alpha\theta\alpha$. Thus we have that $\alpha\mathcal{L}\alpha\theta\alpha$ in $Trf(X)$ by Lemma 3.15, so $\alpha = \gamma\alpha\theta\alpha$ for some $\gamma \in Trf(X)$. Hence $\alpha = \gamma\alpha\theta\alpha = \gamma(\gamma\alpha\theta\alpha)\theta\alpha = (\gamma\gamma\alpha)\theta\alpha\theta\alpha$. This means that $\alpha \in \mathrm{LReg}(Trf(X),\theta)$.

This completes the proof of the theorem. □

Theorem 5.14. *For any* $\theta \in Prf(X)$,

$$\begin{aligned}\mathrm{LReg}(Prf(X),\theta) &= \{0\} \cup \{\alpha \in Prf(X) \mid \varnothing \neq \mathrm{ran}\,\alpha = \mathrm{ran}\,\theta\alpha \subseteq \mathrm{dom}\,\theta\alpha \\ &\qquad \textit{and} \ (\theta\alpha)_{|\mathrm{ran}\,\theta\alpha} \in G(\mathrm{ran}\,\theta\alpha)\} \\ &= \mathrm{RReg}(Prf(X),\theta).\end{aligned}$$

Proof. Let $\theta \in Prf(X)$. We assume that $\alpha \in \mathrm{LReg}(Prf(X), \theta)$. Then there is $\beta \in Prf(X)$ such that $\alpha = \beta\theta(\alpha\theta\alpha)$, so $\alpha\mathcal{L}\theta\alpha\theta\alpha$ in $Prf(X)$. Thus $\operatorname{ran}\alpha = \operatorname{ran}\theta\alpha\theta\alpha$ by Lemma 3.15. Since $\theta\alpha = \theta\beta\theta\alpha\theta\alpha$, $\theta\alpha \in \mathrm{LReg}(Prf(X))$, i.e., $\theta\alpha\mathcal{L}(\theta\alpha)^2$ in $Prf(X)$. By Lemma 3.15, $\operatorname{ran}\theta\alpha = \operatorname{ran}\theta\alpha\theta\alpha$ and hence $\operatorname{ran}\theta\alpha = \operatorname{ran}\alpha$. By Theorem 3.20, $\theta\alpha = 0$ or $\varnothing \neq \operatorname{ran}\alpha = \operatorname{ran}\theta\alpha \subseteq \operatorname{dom}\theta\alpha$ and $(\theta\alpha)_{|\operatorname{ran}\theta\alpha} \in G(\operatorname{ran}\theta\alpha)$. If $\theta\alpha = 0$, then $\alpha = \beta\theta\alpha\theta\alpha = 0$.

For the converse, if $\alpha = 0$, then we are done. Assume that $\alpha \in Prf(X)$ and $\varnothing \neq \operatorname{ran}\alpha = \operatorname{ran}\theta\alpha \subseteq \operatorname{dom}\theta\alpha$ and $(\theta\alpha)_{|\operatorname{ran}\theta\alpha} \in G(\operatorname{ran}\theta\alpha)$. By Theorem 3.20, $\theta\alpha \in \mathrm{LReg}(Prf(X))$. By Lemma 3.15, $\operatorname{ran}\theta\alpha\theta\alpha = \operatorname{ran}\theta\alpha$. Since $\operatorname{ran}\theta\alpha = \operatorname{ran}\alpha$, we have that $\operatorname{ran}\theta\alpha\theta\alpha = \operatorname{ran}\alpha$. By Lemma 3.15, $\alpha\mathcal{L}\theta\alpha\theta\alpha$ in $Prf(X)$, so $\alpha = \beta\theta\alpha\theta\alpha$ for some $\beta \in Prf(X)$. This means that $\alpha \in \mathrm{LReg}(Prf(X), \theta)$.

The proof of that $\mathrm{LReg}(Prf(X), \theta) = \mathrm{RReg}(Prf(X), \theta)$ is given in the same way as the proof of that $\mathrm{LReg}(Trf(X), \theta) = \mathrm{RReg}(Trf(X), \theta)$ by using Lemma 3.19 instead of Lemma 3.16.

Therefore the theorem is obtained. □

Theorem 5.15. *For any* $\theta \in Irf(X)$,

(i) $\mathrm{LReg}(Irf(X), \theta) = \{\alpha \in Irf(X) \mid \operatorname{dom}\theta\alpha = \operatorname{ran}\theta\alpha = \operatorname{ran}\alpha\}$;

(ii) $\mathrm{RReg}(Irf(X), \theta) = \{\alpha \in Irf(X) \mid \operatorname{dom}\alpha = \operatorname{dom}\alpha\theta = \operatorname{ran}\alpha\theta\}$.

Proof. Let $\theta \in Irf(X)$.

(i) Let $\alpha \in \mathrm{LReg}(Irf(X), \theta)$. Then $\alpha = \beta\theta(\alpha\theta\alpha)$ where $\beta \in Irf(X)$, so $\alpha\mathcal{L}\theta\alpha\theta\alpha$ in $Irf(X)$. By Lemma 3.15, $\operatorname{ran}\alpha = \operatorname{ran}\theta\alpha\theta\alpha$. Since $\theta\alpha = \theta\beta\theta\alpha\theta\alpha$, $\theta\alpha\mathcal{L}\theta\alpha\theta\alpha$ in $Irf(X)$, so $\operatorname{ran}\theta\alpha = \operatorname{ran}\theta\alpha\theta\alpha$. Moreover, $\theta\alpha \in \mathrm{LReg}(Irf(X))$. By Theorem 3.21, $\operatorname{dom}\theta\alpha = \operatorname{ran}\theta\alpha$. It follows that $\operatorname{dom}\theta\alpha = \operatorname{ran}\theta\alpha = \operatorname{ran}\theta\alpha\theta\alpha = \operatorname{ran}\alpha$.

For the reverse inclusion, let $\alpha \in Irf(X)$ be such that $\operatorname{dom}\theta\alpha = \operatorname{ran}\theta\alpha = \operatorname{ran}\alpha$. By Theorem 3.21, $\theta\alpha \in \mathrm{LReg}(Irf(X))$, i.e., $\theta\alpha\mathcal{L}\theta\alpha\theta\alpha$ in $Irf(X)$. We also have that $\alpha\mathcal{L}\theta\alpha$ in $Irf(X)$ by Lemma 3.15. Then $\alpha\mathcal{L}\theta\alpha\theta\alpha$ in $Irf(X)$. Therefore $\alpha = \beta\theta\alpha\theta\alpha$ for some $\beta \in Irf(X)$. This implies that $\alpha \in \mathrm{LReg}(Irf(X), \theta)$.

(ii) Let $\alpha = \alpha\theta\alpha\theta\beta$ where $\beta \in Irf(X)$. Then $\alpha\mathcal{R}\alpha\theta\alpha\theta$ in $Irf(X)$. By Lemma

3.22, $\operatorname{dom}\alpha = \operatorname{dom}\alpha\theta\alpha\theta$. We also have that $\alpha\theta = \alpha\theta\alpha\theta\beta\theta$. This implies that $\alpha\theta \in \operatorname{RReg}(Irf(X))$. By Lemma 3.22 and Theorem 3.23, we have respectively that

$$\operatorname{dom}\alpha\theta = \operatorname{dom}\alpha\theta\alpha\theta \text{ and } \operatorname{dom}\alpha\theta = \operatorname{ran}\alpha\theta.$$

It follows that $\operatorname{dom}\alpha = \operatorname{dom}\alpha\theta\alpha\theta = \operatorname{dom}\alpha\theta = \operatorname{ran}\alpha\theta$.

For the converse, let $\alpha \in Irf(X)$ be such that $\operatorname{dom}\alpha = \operatorname{dom}\alpha\theta = \operatorname{ran}\alpha\theta$. By Lemma 3.22 and Theorem 3.23, $\alpha\mathcal{R}\alpha\theta$ and $\alpha\theta\mathcal{R}\alpha\theta\alpha\theta$ in $Irf(X)$, respectively. Then $\alpha\mathcal{R}\alpha\theta\alpha\theta$ in $Irf(X)$. Thus $\alpha = \alpha\theta\alpha\theta\beta$ for some $\beta \in Irf(X)$. This means that $\alpha \in \operatorname{RReg}(Irf(X), \theta)$. $\square$

CHAPTER VI

VARIANTS OF SEMIGROUPS OF LINEAR TRANSFORMATIONS

In the last chapter, the left regular and right regular elements of the variants of the semigroup $L_F(V)$ and those semigroups in Chapter IV are characterized. Comparing with the results in Chapter V, we obtain the results in this chapter accordingly.

Throughout this chapter, let V be a vector space over a field F.

Theorem 6.1. *For any* $\theta \in L_F(V)$,

(i) $\mathrm{LReg}(L_F(V), \theta) = \{\alpha \in L_F(V) \mid \operatorname{ran} \alpha = \operatorname{ran} \alpha\theta\alpha\}$;

(ii) $\mathrm{RReg}(L_F(V), \theta) = \{\alpha \in L_F(V) \mid \ker \alpha = \ker \alpha\theta\alpha\}$.

Proof. Let $\theta \in L_F(V)$.

(i) Let $\alpha \in \mathrm{LReg}(L_F(V), \theta)$. Then $\alpha = \beta\theta(\alpha\theta\alpha)$ for some $\beta \in L_F(V)$. Thus $\alpha\mathcal{L}\alpha\theta\alpha$ in $L_F(V)$. By Theorem 2.6(i), $\operatorname{ran} \alpha = \operatorname{ran} \alpha\theta\alpha$.

For the converse, let $\alpha \in L_F(V)$ be such that $\operatorname{ran} \alpha = \operatorname{ran} \alpha\theta\alpha$. By Theorem 2.6(i), $\alpha\mathcal{L}\alpha\theta\alpha$ in $L_F(V)$. Then $\alpha(\theta\alpha)\mathcal{L}\alpha\theta\alpha(\theta\alpha)$ in $L_F(V)$, so $\alpha\mathcal{L}\alpha\theta\alpha\theta\alpha$ in $L_F(V)$. Therefore $\alpha = \beta\alpha\theta\alpha\theta\alpha$ for some $\beta \in L_F(V)$. This means that $\alpha \in \mathrm{LReg}(L_F(V), \theta)$.

(ii) Let $\alpha \in \mathrm{RReg}(L_F(V), \theta)$. Then $\alpha = (\alpha\theta\alpha)\theta\beta$ for some $\beta \in L_F(V)$. Thus $\alpha\mathcal{R}\alpha\theta\alpha$ in $L_F(V)$. By Theorem 2.6(ii), $\ker \alpha = \ker \alpha\theta\alpha$.

Conversely, let $\alpha \in L_F(V)$ be such that $\ker \alpha = \ker \alpha\theta\alpha$. By Theorem 2.6(ii), $\alpha\mathcal{R}\alpha\theta\alpha$ in $L_F(V)$. Thus $(\alpha\theta)\alpha\mathcal{R}(\alpha\theta)\alpha\theta\alpha$ in $L_F(V)$. Then $\alpha\mathcal{R}\alpha\theta\alpha\theta\alpha$ in $L_F(V)$, so $\alpha = \alpha\theta\alpha\theta\alpha\beta$ for some $\beta \in L_F(V)$. This implies that $\alpha \in \mathrm{RReg}(L_F(V), \theta)$. $\square$

From now on, we assume that V is infinite-dimensional. We will characterize $\mathrm{LReg}(S_F(V),\theta)$ and $\mathrm{RReg}(S_F(V),\theta)$ where $S_F(V) = M_F(V)$, $M_F(V) \smallsetminus G_F(V)$, $E_F(V)$, $E_F(V)\smallsetminus G_F(V)$, $BL_F(V,q)$, $DBL_F(V,q)$, $KN_F(V,q)$ and $Lrf_F(V)$ where $\dim_F V \geq q \geq \aleph_0$ and $\theta \in S_F(V)$.

Theorem 6.2. *The following statements hold for $\theta \in M_F(V)$.*

(i) *If $\theta \in G_F(V)$, then* $\mathrm{LReg}(M_F(V),\theta) = \mathrm{LReg}(M_F(V))$.

(ii) *If $\theta \notin G_F(V)$, then* $\mathrm{LReg}(M_F(V),\theta) = \varnothing$.

(iii) *If $\theta \in G_F(V)$, then* $\mathrm{RReg}(M_F(V),\theta) = \mathrm{RReg}(M_F(V))$.

(iv) *If $\theta \notin G_F(V)$, then* $\mathrm{RReg}(M_F(V),\theta) = \{\alpha \in M_F(V) \mid \dim_F(V/\operatorname{ran}\alpha) = \dim_F(V/\operatorname{ran}\alpha\theta\alpha)\}$.

Proof. The proof is given in the same way as that of Theorem 5.4 by using Theorem 4.2, Theorem 4.4 and Lemma 4.3 instead of Theorem 3.2, Theorem 3.4 and Lemma 3.3, respectively. □

Lemma 6.3. *If $\theta \in M_F(V) \smallsetminus G_F(V)$, then* $\mathrm{RReg}(M_F(V),\theta) \subseteq \mathrm{RReg}(M_F(V))$.

Proof. Let $\theta \in M_F(V) \smallsetminus G_F(V)$ and $\alpha \in \mathrm{RReg}(M_F(V),\theta)$. By Theorem 6.2(iv), $\dim_F(V/\operatorname{ran}\alpha) = \dim_F(V/\operatorname{ran}\alpha\theta\alpha)$. Since θ is not onto and α is 1-1, we have $V\alpha\theta\alpha \subseteq V\theta\alpha \subsetneq V\alpha$, so $\operatorname{ran}\alpha\theta\alpha \subsetneq \operatorname{ran}\alpha$. Suppose that $\dim_F(V/\operatorname{ran}\alpha)$ is finite. Let B_1 be a basis of $\operatorname{ran}\alpha\theta\alpha$, B_2 a basis of $\operatorname{ran}\alpha$ containing B_1 and B a basis of V containing B_2. Then

$$\begin{aligned} |B \smallsetminus B_2| &= \dim_F(V/\operatorname{ran}\alpha) \\ &= \dim_F(V/\operatorname{ran}\alpha\theta\alpha) \\ &= |B \smallsetminus B_1| \\ &= |B \smallsetminus B_2| + |B_2 \smallsetminus B_1|. \end{aligned}$$

Since $B \smallsetminus B_2$ is finite, we have $|B_2 \smallsetminus B_1| = 0$, so $B_1 = B_2$. This contradicts the fact that $\operatorname{ran}\alpha\theta\alpha \subsetneq \operatorname{ran}\alpha$. Hence $\dim_F(V/\operatorname{ran}\alpha)$ is infinite. By Theorem 4.4, $\alpha \in \mathrm{RReg}(M_F(V))$. □

Corollary 6.4. *For any $\theta \in M_F(V) \smallsetminus G_F(V)$,*

(i) $\operatorname{LReg}(M_F(V) \smallsetminus G_F(V), \theta) = \varnothing$;

(ii) $\operatorname{RReg}(M_F(V) \smallsetminus G_F(V), \theta) = \{\alpha \in M_F(V) \mid \dim_F(V/\operatorname{ran}\alpha)$ *is infinite and* $\dim_F(V/\operatorname{ran}\alpha) \geq \dim_F(V/\operatorname{ran}\theta)\}$.

Proof. Let $\theta \in M_F(V) \smallsetminus G_F(V)$.

(i) Since $\operatorname{LReg}(M_F(V) \smallsetminus G_F(V), \theta) \subseteq \operatorname{LReg}(M_F(V), \theta)$, by Theorem 6.2(ii), we have that $\operatorname{LReg}(M_F(V) \smallsetminus G_F(V), \theta) = \varnothing$.

(ii) Let $\alpha \in \operatorname{RReg}(M_F(V) \smallsetminus G_F(V), \theta)$. Then $\alpha \in \operatorname{RReg}(M_F(V), \theta)$. By Lemma 6.3, $\alpha \in \operatorname{RReg}(M_F(V))$. Since α is not onto, by Theorem 4.4, $\dim_F(V/\operatorname{ran}\alpha)$ is infinite. Since $\alpha \in \operatorname{RReg}(M_F(V), \theta)$, by Theorem 6.2(iv), $\dim_F(V/\operatorname{ran}\alpha) = \dim_F(V/\operatorname{ran}\alpha\theta\alpha)$. Since $\alpha, \theta \in M_F(V)$, it follows that

$$\begin{aligned}
\dim_F(V/\operatorname{ran}\alpha) &= \dim_F(V/\operatorname{ran}\alpha\theta\alpha) \\
&= \dim_F(V/\operatorname{ran}\alpha) + \dim_F(V/\operatorname{ran}\theta\alpha) \qquad \text{(see p. 9)} \\
&= \dim_F(V/\operatorname{ran}\alpha) + \dim_F(V/\operatorname{ran}\theta) + \dim_F(V/\operatorname{ran}\alpha) \\
&= \dim_F(V/\operatorname{ran}\alpha) + \dim_F(V/\operatorname{ran}\theta).
\end{aligned}$$

This implies that $\dim_F(V/\operatorname{ran}\theta) \leq \dim_F(V/\operatorname{ran}\alpha)$.

For the reverse inclusion, let $\alpha \in M_F(V)$ be such that $\dim_F(V/\operatorname{ran}\alpha)$ is infinite and $\dim_F(V/\operatorname{ran}\alpha) \geq \dim_F(V/\operatorname{ran}\theta)$. Since $\alpha, \theta \in M_F(V)$, we have

$$\begin{aligned}
\dim_F(V/\operatorname{ran}\alpha\theta\alpha) &= \dim_F(V/\operatorname{ran}\alpha) + \dim_F(V/\operatorname{ran}\theta) + \dim_F(V/\operatorname{ran}\alpha) \\
&= \dim_F(V/\operatorname{ran}\alpha) + \dim_F(V/\operatorname{ran}\alpha) \\
&= \dim_F(V/\operatorname{ran}\alpha)
\end{aligned}$$

By Theorem 6.2(iv), $\alpha \in \operatorname{RReg}(M_F(V), \theta)$. Then $\alpha = (\alpha\theta\alpha)\theta\beta$ for some $\beta \in M_F(V)$. Thus $\alpha = \alpha\theta\alpha\theta\beta = \alpha\theta(\alpha\theta\alpha\theta\beta)\theta\beta = (\alpha\theta\alpha)\theta(\alpha\theta\beta\theta\beta)$. Since $\alpha \in M_F(V) \smallsetminus G_F(V)$ and $M_F(V) \smallsetminus G_F(V)$ is an ideal of $M_F(V)$, we have $\alpha\theta\beta\theta\beta \in M_F(V) \smallsetminus G_F(V)$. Hence $\alpha \in \operatorname{RReg}(M_F(V) \smallsetminus G_F(V), \theta)$, as desired. □

Theorem 6.5. *For any $\theta \in E_F(V)$,*

$$\mathrm{LReg}(E_F(V),\theta) = \{\alpha \in E_F(V) \mid \dim_F \ker\alpha = \dim_F \ker\alpha\theta\alpha\}.$$

In particular, if $\theta \in G_F(V)$, then

$$\mathrm{LReg}(E_F(V),\theta) = \{\alpha \in E_F(V) \mid \ker\alpha\theta = \{0\} \text{ or } \dim_F \ker\alpha\theta \text{ is infinite}\}.$$

Proof. Let $\theta \in E_F(V)$ and $\alpha \in \mathrm{LReg}(E_F(V),\theta)$. Then $\alpha = \beta\theta(\alpha\theta\alpha)$ for some $\beta \in E_F(V)$, so $\alpha\mathcal{L}\alpha\theta\alpha$ in $E_F(V)$. By Lemma 4.7, $\dim_F \ker\alpha = \dim_F \ker\alpha\theta\alpha$.

Conversely, we assume that $\alpha \in E_F(V)$ and $\dim_F \ker\alpha = \dim_F \ker\alpha\theta\alpha$. Then $\alpha\mathcal{L}\alpha\theta\alpha$ in $E_F(V)$ by Lemma 4.7, so there exists $\beta \in E_F(V)$ such that $\alpha = \beta\alpha\theta\alpha = \beta(\beta\alpha\theta\alpha)\theta\alpha = (\beta\beta\alpha)\theta\alpha\theta\alpha$. This implies that $\alpha \in \mathrm{LReg}(E_F(V),\theta)$.

Next, assume that $\theta \in G_F(V)$.

Let $\alpha \in \mathrm{LReg}(E_F(V),\theta)$. By Lemma 4.7, $\alpha\mathcal{L}\alpha\theta\alpha$ in $E_F(V)$. Thus $\alpha\theta\mathcal{L}\alpha\theta\alpha\theta$ in $E_F(V)$, i.e., $\alpha\theta \in \mathrm{LReg}(E_F(V))$. By Theorem 4.8, $\ker\alpha\theta = \{0\}$ or $\dim_F \ker\alpha\theta$ is infinite.

For the converse, let $\alpha \in E_F(V)$ be such that $\ker\alpha\theta = \{0\}$ or $\dim_F \ker\alpha\theta$ is infinite. By Theorem 4.8, $\alpha\theta \in \mathrm{LReg}(E_F(V))$. Thus $\alpha\theta = \beta\alpha\theta\alpha\theta$ for some $\beta \in E_F(V)$. Since $\theta \in G_F(V)$, $\alpha = (\alpha\theta)\theta^{-1} = (\beta\alpha\theta\alpha\theta)\theta^{-1} = \beta\alpha\theta\alpha$, so $\alpha = \beta\alpha\theta\alpha = \beta(\beta\alpha\theta\alpha)\theta\alpha = (\beta\beta\alpha)\theta\alpha\theta\alpha$. This implies that $\alpha \in \mathrm{LReg}(E_F(V),\theta)$, as desired.

This completes the proof of the theorem. □

Theorem 6.6. *The following statements hold for $\theta \in E_F(V)$.*

(i) *If $\theta \in G_F(V)$, then* $\mathrm{RReg}(E_F(V),\theta) = \mathrm{RReg}(E_F(V))$.

(ii) *If $\theta \notin G_F(V)$, then* $\mathrm{RReg}(E_F(V),\theta) = \varnothing$.

Proof. By using Theorem 4.9 instead of Theorem 3.8, we can prove the theorem in the same way as the proof of Theorem 5.8. □

Corollary 6.7. *For any $\theta \in E_F(V) \setminus G_F(V)$,*

(i) $\mathrm{LReg}(E_F(V) \smallsetminus G_F(V), \theta) = \{\alpha \in E_F(V) \mid \dim_F \ker \alpha = \dim_F \ker \alpha\theta\alpha\}$;

(ii) $\mathrm{RReg}(E_F(V) \smallsetminus G_F(V), \theta) = \varnothing$.

Proof. Let $\theta \in E_F(V) \smallsetminus G_F(V)$.

(i) If $\alpha \in \mathrm{LReg}(E_F(V) \smallsetminus G_F(V), \theta)$, then $\alpha \in \mathrm{LReg}(E_F(V), \theta)$, so $\dim_F \ker \alpha = \dim_F \ker \alpha\theta\alpha$ by Theorem 6.5.

Conversely, let $\alpha \in E_F(V) \smallsetminus G_F(V)$ be such that $\dim_F \ker \alpha = \dim_F \ker \alpha\theta\alpha$. By Theorem 6.5, $\alpha \in \mathrm{LReg}(E_F(V), \theta)$. Thus there is $\beta \in E_F(V)$ such that $\alpha = \beta\theta(\alpha\theta\alpha)$, so $\alpha = \beta\theta(\beta\theta\alpha\theta\alpha)\theta\alpha = (\beta\theta\beta\theta\alpha)\theta\alpha\theta\alpha$. Since $\alpha \in E_F(V) \smallsetminus G_F(V)$ and $E_F(V) \smallsetminus G_F(V)$ is an ideal of $E_F(V)$, we have $\beta\theta\beta\theta\alpha \in E_F(V) \smallsetminus G_F(V)$. Therefore the desired result follows.

(ii) Since $\mathrm{RReg}(E_F(V) \smallsetminus G_F(V), \theta) \subseteq \mathrm{RReg}(E_F(V), \theta)$ and $\theta \notin G_F(V)$, by Theorem 6.6(ii), we have $\mathrm{RReg}(E_F(V) \smallsetminus G_F(V), \theta) = \varnothing$. □

Theorem 6.8. *For any* $\theta \in (BL_F(V, q), \theta)$,

(i) $\mathrm{LReg}(BL_F(V, q), \theta) = \varnothing$;

(ii) $\mathrm{RReg}(BL_F(V, q), \theta) = BL_F(V, q)$.

Proof. We can provide the proof in the same way as that of Theorem 5.10 by using Theorem 2.8 instead of Theorem 2.2. □

A dual version of the previous theorem can be shown in a similar manner.

Theorem 6.9. *For any* $\theta \in DBL_F(V, q)$,

(i) $\mathrm{LReg}(DBL_F(V, q), \theta) = DBL_F(V, q)$;

(ii) $\mathrm{RReg}(DBL_F(V, q), \theta) = \varnothing$.

Theorem 6.10. *For any* $\theta \in KN_F(V, q)$,

(i) $\mathrm{LReg}(KN_F(V, q), \theta) = \varnothing$;

(ii) $\mathrm{RReg}(KN_F(V, q), \theta) = \{\alpha \in KN_F(V, q) \mid \dim_F(V/\operatorname{ran}\alpha) \geq \dim_F(V/\operatorname{ran}\theta)\}$.

Proof. Let $\theta \in KN_F(V,q)$.

(i) Since $\mathrm{LReg}(KN_F(V,q),\theta) \subseteq \mathrm{LReg}(KN(V,q),\theta)$, by Theorem 5.12(i), the result follows.

(ii) Let $\alpha \in \mathrm{RReg}(KN_F(V,q),\theta)$. Since $KN_F(V,q) \subseteq M_F(V) \smallsetminus G_F(V)$, by Corollary 6.4(ii), $\dim_F(V/\operatorname{ran}\alpha) \geq \dim_F(V/\operatorname{ran}\theta)$.

Conversely, let $\alpha \in KN_F(V,q)$ be such that $\dim_F(V/\operatorname{ran}\alpha) \geq \dim_F(V/\operatorname{ran}\theta)$. By Corollary 6.4(ii), $\alpha \in \mathrm{RReg}(M_F(V) \smallsetminus G_F(V))$ since $\dim_F(V/\operatorname{ran}\alpha) \geq q$. Then $\alpha = (\alpha\theta\alpha)\theta\beta$ for some $\beta \in M_F(V) \smallsetminus G_F(V)$. Thus $\alpha = \alpha\theta\alpha\theta\beta = \alpha\theta(\alpha\theta\alpha\theta\beta)\theta\beta = \alpha\theta\alpha\theta(\alpha\theta\beta\theta\beta)$. Since $\alpha,\theta,\beta \in M_F(V)$, we have that

$$\begin{aligned}\dim_F(V/\operatorname{ran}\alpha\theta\beta\theta\beta) &= \dim_F(V/\operatorname{ran}\alpha) + \dim_F(V/\operatorname{ran}\theta\beta\theta\beta)\\ &\geq \dim_F(V/\operatorname{ran}\alpha)\\ &\geq q,\end{aligned}$$

so $\alpha\theta\beta\theta\beta \in KN_F(V,q)$. Hence $\alpha \in \mathrm{RReg}(KN_F(V,q),\theta)$, as desired.

Therefore the result follows. □

Theorem 6.11. *For any $\theta \in Lrf_F(V)$,*

$$\begin{aligned}\mathrm{LReg}(Lrf_F(V),\theta) &= \{\alpha \in Lrf_F(V) \mid (\theta\alpha)_{|\operatorname{ran}\theta\alpha} \in G_F(\operatorname{ran}\theta\alpha) \text{ and}\\ &\qquad\qquad \operatorname{ran}\theta\alpha = \operatorname{ran}\alpha\}\\ &= \mathrm{RReg}(Lrf_F(V),\theta).\end{aligned}$$

Proof. Let $\theta \in Lrf_F(V)$ and $\alpha \in \mathrm{LReg}(Lrf_F(V),\theta)$. Then there is $\beta \in Lrf_F(V)$ such that $\alpha = \beta\theta(\alpha\theta\alpha)$ for some $\beta \in Lrf_F(V)$. Thus $\alpha\mathcal{L}\alpha\theta\alpha$ in $Lrf_F(V)$. By Lemma 4.16, $\operatorname{ran}\alpha = \operatorname{ran}\alpha\theta\alpha$. Thus $\operatorname{ran}\alpha = \operatorname{ran}\alpha\theta\alpha \subseteq \operatorname{ran}\theta\alpha \subseteq \operatorname{ran}\alpha$, so $\operatorname{ran}\theta\alpha = \operatorname{ran}\alpha$. Since $\alpha = \beta\theta\alpha\theta\alpha$, we have $\theta\alpha = \theta\beta\theta\alpha\theta\alpha = (\theta\beta)(\theta\alpha)^2$, so $\theta\alpha \in \mathrm{LReg}(Lrf_F(V))$. By Theorem 4.19, $(\theta\alpha)_{|\operatorname{ran}\theta\alpha} \in G_F(\operatorname{ran}\theta\alpha)$.

For the converse, let $\alpha \in Lrf_F(V)$ be such that $(\theta\alpha)_{|\operatorname{ran}\theta\alpha} \in G_F(\operatorname{ran}\theta\alpha)$ and $\operatorname{ran}\theta\alpha = \operatorname{ran}\alpha$. Then $\operatorname{ran}\theta\alpha\theta\alpha = (\operatorname{ran}\theta\alpha)\theta\alpha = \operatorname{ran}\theta\alpha = \operatorname{ran}\alpha$. By Lemma 4.16, $\alpha\mathcal{L}\theta\alpha\theta\alpha$ in $Lrf_F(V)$. This implies that $\alpha \in \mathrm{LReg}(Lrf_F(V),\theta)$, as required.

Finally, we will show that $\mathrm{LReg}(Lrf_F(V),\theta) = \mathrm{RReg}(Lrf_F(V),\theta)$.

Let $\alpha \in \mathrm{LReg}(Lrf_F(V), \theta)$. Then $\alpha = \beta\theta(\alpha\theta\alpha)$ for some $\beta \in Lrf_F(V)$. Thus $\alpha\mathcal{L}\alpha\theta\alpha$ in $Lrf_F(V)$. By Lemma 4.16, $\operatorname{ran}\alpha = \operatorname{ran}\alpha\theta\alpha$. By Lemma 4.18, $\ker\alpha = \ker\alpha\theta\alpha$. By Lemma 4.17, $\alpha\mathcal{R}\alpha\theta\alpha$ in $Lrf_F(V)$, so $\alpha = \alpha\theta\alpha\gamma$ for some $\gamma \in Lrf_F(V)$. It follows that $\alpha = \alpha\theta\alpha\gamma = \alpha\theta(\alpha\theta\alpha\gamma)\gamma = \alpha\theta\alpha\theta(\alpha\gamma\gamma)$. This implies that $\alpha \in \mathrm{RReg}(Lrf_F(V), \theta)$.

Conversely, let $\alpha \in \mathrm{RReg}(Lrf_F(V), \theta)$. Then there exists $\beta \in Lrf_F(V)$ such that $\alpha = (\alpha\theta\alpha)\theta\beta$, so $\alpha\mathcal{R}\alpha\theta\alpha$ in $Lrf_F(V)$. By Lemma 4.17, $\ker\alpha = \ker\alpha\theta\alpha$. By Lemma 4.18, $\operatorname{ran}\alpha = \operatorname{ran}\alpha\theta\alpha$. By Lemma 4.16, $\alpha\mathcal{L}\alpha\theta\alpha$ in $Lrf_F(V)$. Hence there exists $\gamma \in Lrf_F(V)$ such that $\alpha = \gamma\alpha\theta\alpha$. Therefore $\alpha = \gamma\alpha\theta\alpha = \gamma(\gamma\alpha\theta\alpha)\theta\alpha = (\gamma\gamma\alpha)\theta\alpha\theta\alpha$. This shows that $\alpha \in \mathrm{LReg}(Lrf_F(V), \theta)$. Thus $\mathrm{LReg}(Lrf_F(V), \theta) = \mathrm{RReg}(Lrf_F(V), \theta)$.

Therefore the theorem is proved. $\square$

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