การอนุพัทธ์ขั้นสูงและการอนุพัทธ์แบบจอร์แดนขั้นสูง
ของแกมมาริง

นางสาวจุฬาลักษณ์ แก้วหวังสกูล

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2553

HIGHER DERIVATIONS AND JORDAN HIGHER DERIVATIONS

OF Γ-RINGS

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A Thesis Submitted in Partial Fulfillment of the Requirements

for the Degree of Master of Science Program in Mathematics

Department of Mathematics

Faculty of Science

Chulalongkorn University

Academic Year 2010

Thesis Title	HIGHER DERIVATIONS AND JORDAN HIGHER DERIVATIONS OF Γ-RINGS
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Field of Study	Mathematics
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Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

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จุฬาลักษณ์ แก้วหวังสกูล : การอนุพัทธ์ขั้นสูงและการอนุพัทธ์แบบจอร์แดนขั้นสูงของแกมมาริง. (HIGHER DERIVATIONS AND JORDAN HIGHER DERIVATIONS OF Γ-RINGS) อ.ที่ปรึกษาวิทยานิพนธ์หลัก: ผศ.ดร.ศจี เพียรสกุล, 66 หน้า.

ให้ M และ Γ เป็นกรุปสลับที่ภายใต้การบวก ถ้ามีการส่ง $\cdot$ ซึ่งส่งจาก $M \times \Gamma \times M$ ไปยัง M โดยแทน $a \cdot \gamma \cdot b$ ด้วย $a\gamma b$ สำหรับทุก $a,b \in M$ และ $\gamma \in \Gamma$ ซึ่งสอดคล้องสมบัติต่อไปนี้คือ สำหรับแต่ละ $a,b,c \in M$ และแต่ละ $\gamma, \beta \in \Gamma$ (*i*) $(a\gamma b)\beta c = a\gamma(b\beta c)$ (ii) $(a+b)\gamma c = a\gamma c + b\gamma c$, $a(\gamma + \beta)c = a\gamma c + a\beta c$ และ $a\gamma(b+c) = a\gamma b + a\gamma c$ แล้วจะเรียก M ว่า Γ-ริง

กำหนดให้ M เป็น Γ-ริง และ U เป็นไอดีลของ M เราแนะนำแนวคิดของการอนุพัทธ์ขั้นสูง การอนุพัทธ์ขั้นสูงแบบจอร์แดนของ M การอนุพัทธ์ขั้นสูง การอนุพัทธ์ขั้นสูงแบบจอร์แดนของ U ไปยัง M และการอนุพัทธ์ขั้นสูงทั่วไป การอนุพัทธ์ขั้นสูงทั่วไปแบบจอร์แดนของ M สิ่งที่เราสนใจคือการหาเงื่อนไขที่เหมาะสมสำหรับ Γ-ริงที่ทำให้ได้ว่า

(1) การอนุพัทธ์ขั้นสูงแบบจอร์แดนและการอนุพัทธ์ขั้นสูงของ Γ-ริงเป็นสิ่งเดียวกัน

(2) การอนุพัทธ์ขั้นสูงแบบจอร์แดนและการอนุพัทธ์ขั้นสูงของไอดีลของ Γ-ริงเป็นสิ่งเดียวกัน

(3) การอนุพัทธ์ขั้นสูงทั่วไปแบบจอร์แดนและการอนุพัทธ์ขั้นสูงทั่วไปของ Γ-ริงเป็นสิ่งเดียวกัน

ภาควิชา............คณิตศาสตร์.......... ลายมือชื่อนิสิต จุฬาลักษณ์ แก้วหวังสกูล

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ปีการศึกษา............2553...............

5272263123 : MAJOR MATHEMATICS
KEYWORDS : DERIVATION / JORDAN DERIVATION / HIGHER DERIVATION / JORDAN HIGHER DERIVATION / GENERALIZED HIGHER DERIVATION / JORDAN GENERALIZED HIGHER DERIVATION / Γ-RINGS

JULALAK KAEWWANGSAKOON : HIGHER DERIVATIONS AND JORDAN HIGHER DERIVATIONS OF Γ-RINGS. ADVISOR : ASST. PROF. SAJEE PIANSKOOL, Ph.D., 66 pp.

Let M and Γ be additive abelain groups. If there exists a map $\cdot$ sending $M \times \Gamma \times M$ into M, denote the image of $a \cdot \gamma \cdot b$ by, simply, $a\gamma b$ for all $a, b \in M$ and $\gamma \in \Gamma$, satisfying the following properties: for each $a, b, c \in M$ and $\gamma, \beta \in \Gamma$, (i) $(a\gamma b)\beta c = a\gamma(b\beta c)$; (ii) $(a+b)\gamma c = a\gamma c + b\gamma c$, $a(\gamma+\beta)c = a\gamma c + a\beta c$ and $a\gamma(b+c) = a\gamma b + a\gamma c$, then M is called a Γ-*ring*.

Let M be a Γ-ring and U be an ideal of M. We introduce the concept of higher derivations, Jordan higher derivations of M, higher derivations, Jordan higher derivations of U into M and generalized higher derivations, Jordan generalized higher derivations of M. Our main interests are finding appropriate conditions for a Γ-ring in order to obtain that

1. Jordan higher derivations and higher derivations of a Γ-ring are the same,
2. Jordan higher derivations and higher derivations of an ideal of a Γ-ring are identical, and
3. Jordan generalized higher derivations and generalized higher derivations of a Γ-ring are coincide.

Department	: ....Mathematics....	Student's Signature	จุฬาลักษณ์ แก้ววังสกุล
Field of Study	: ....Mathematics....	Advisor's Signature	....
Academic Year : ....2010...........			

ACKNOWLEDGEMENTS

I am sincerely grateful to my advisor, Assistant Professor Dr. Sajee Pianskool for her consistent encouragement, invaluable comments and willingness to sacrifice her time to suggest and advice me throughout the completion of this thesis. Moreover, my thankfulness goes to Associate Professor Dr. Amorn Wasanawichit, the chairman, Assistant Professor Dr. Sureeporn Chaopraknoi and Assistant Professor Dr. Knograt Savettaseranee, the thesis commitee, for their suggestions and valuable comments.

In particular, I would like to express my sincere gratitude to the Development and Promotion of Science and Technology Talents Project (DPST) for the financial support.

Finally, I am also grateful to my family for love and encouragement during my study.

CONTENTS

CHAPTER I

INTRODUCTION

It is known that one of important algebraic structures is a ring and there are many areas of researches related to rings. Various properties of rings have been investigated as well as parallel structures of rings along with the corresponding properties. In 1957, I.N. Herstein [9] introduced the concept of derivations of rings. In fact, a derivation and a Jordan derivation of a ring were defined. No doubt that there must be some correlations between these. A *derivation* of a ring R is an additive map $d : R \to R$ such that $d(ab) = d(a)b + ad(b)$ for all $a, b \in R$. A *Jordan derivation* (JD) of a ring R is an additive map $d : R \to R$ such that $d(a^2) = d(a)a + ad(a)$ for all $a \in R$. Recall that an *additive* map of a ring R is a map $d : R \to R$ such that $d(a + b) = d(a) + d(b)$ for all $a, b \in R$, i.e., d preserves the addition.

One can see that a "Jordan"derivation d indicates that the value d of the product of any a and a is taken into account instead of the value d of the product of any a and b. Actually, a derivation is a Jordan derivation but not vice versa, see [5]. Naturally, the curiosity that "when Jordan derivation is a derivation"must take place. Herstein proved in [9] that every JD of a prime ring of characteristic different from 2 turns out to be a derivation. Recall that a ring R is *prime* if $aRb = 0$ implies $a = 0$ or $b = 0$ for all $a, b \in R$, see [9].

Let us give an obvious example of a derivation. Unsurprisingly, the zero map of any rings is, definitely, a derivation. The next example assures that a nonzero derivation of a given ring exists. Note that the familiar ring is the ring $\mathbb{R}$ of real numbers.

Example 1. Let $S = \{f : \mathbb{R} \to \mathbb{R} \mid f$ is differentiable$\}$. Define $d : S \to S$ by $d(f) = f'$, the first derivative of f, for all $f \in S$. Clearly, d is a derivation of S.

Example 2. Let R be a nonzero ring and $a \in R$. Define $d : R \to R$ by $d(x) = xa - ax$ for all $x \in R$. Then d is an additive map and $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Thus d is a derivation of R.

Later, in 1988, M. Bresar [3] gave another point of view that a JD of a certain ring is also a derivation. He presented that the condition "2-torsion free prime ring" can be replaced by "2-torsion free semiprime ring". A ring R is said to be 2-*torsion free* if $2x = 0$ implies $x = 0$ for all $x \in R$ (or the characteristic of R is different from 2); moreover, R is *semiprime* if $aRa = 0$ implies $a = 0$ for all $a \in R$. Notice that every prime ring is semiprime. In his paper [3], Bresar introduced a *Jordan triple derivation* (JTD) of a ring R, i.e., an additive map $d : R \to R$ satisfying $d(aba) = d(a)ba + ad(b)a + abd(a)$ for all $a, b \in R$. Moreover, he proved that every JTD of a 2-torsion free semiprime ring is a derivation and applied the result from Herstein that each JD of a 2-torsion free ring is a JTD (however, Herstein did not use the terminology "JTD").

Throughout this thesis, let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. M. Ferrero and C. Haetinger [6] studied higher derivations of rings. Let $D = (d_i)_{i \in \mathbb{N}_0}$ be a family of additive mappings of a ring R where d_0 is the identity map on R. Then D is a *higher derivation* (HD) of R if $d_n(ab) = \sum_{i+j=n} d_i(a)d_j(b)$ for all $a, b \in R$ and $n \in \mathbb{N}_0$, a *Jordan higher derivation* (JHD) of R if $d_n(a^2) = \sum_{i+j=n} d_i(a)d_j(a)$ for all $a \in R$ and $n \in \mathbb{N}_0$ and a *Jordan triple higher derivation* (JTHD) of R if $d_n(aba) = \sum_{i+j+k=n} d_i(a)d_j(b)d_k(a)$ for all $a, b \in R$ and $n \in \mathbb{N}_0$. An example of a JHD which is not a HD was given by them. Then They proved that every JHD of a 2-torsion free ring is a JTHD and every JTHD of a 2-torsion free semiprime ring is a HD. That is every JHD of a 2-torsion free semiprime ring is a HD. Next, let us see an example of a HD.

Example 3. ([7]) Let R be a ring and $a \in R$. For $n \in \mathbb{N}_0$, define a map $d_n : R \to R$

by

$$d_n(x) = \begin{cases} x, & \text{if } n = 0, \\ (-1)^n(a^n x - a^{n-1}xa), & \text{if } n \neq 0. \end{cases}$$

for all $x \in R$. Thus the family $D = (d_i)_{i\in\mathbb{N}_0}$ of additive maps is a higher derivation on R.

So far we have gathered some results regarding that JDs of certain rings are derivations as well as JHDs of specific rings are HDs. As mentioned above, studying analogous algebraic structures is another direction of doing research. It is the appropriate place to propose the notion of Γ-rings given by W.E. Barnes in [1]. For additive abelian groups $(M, +_M)$ and $(\Gamma, +_\Gamma)$, if there exists a map $\cdot : M \times \Gamma \times M \to M$, denote the image of $a \cdot \gamma \cdot b$ by, simply, $a\gamma b$ for all $a, b \in M$ and $\gamma \in \Gamma$, satisfying the following properties: for each $a, b, c \in M$ and $\gamma, \beta \in \Gamma$,
Associative Law $(a\gamma b)\beta c = a\gamma(b\beta c)$;
Distributive Law $(a +_M b)\gamma c = a\gamma c +_M b\gamma c$, $a(\gamma +_\Gamma \beta)c = a\gamma c +_M a\beta c$ and $a\gamma(b +_M c) = a\gamma b +_M a\gamma c$,
then M is called a Γ-*ring*. From now on, whenever M is a Γ-ring, both of the operations $+_M$ and $+_\Gamma$ will be written as $+$ since it should be clear from the context.

One can construct a Γ-ring from a given ring.

Example 4. Let $(R, +, \cdot)$ be a ring. Then R can be considered as a Γ-ring.

Solution. Let M be the abelian group $(R, +)$ and Γ be a subgroup of R under $+$. Then, obviously, the map $d : M \times \Gamma \times M \to M$ defined by $(a, \gamma, b) \mapsto a \cdot \gamma \cdot b$ satisfies the associative law and the distributive law. Thus R is always a Γ-ring when Γ is any additive subgroup of the group R. $\square$

On the other hand, a ring can be extracted from a given Γ-ring.

Example 5. Let M be a Γ-ring. Then M can be formed as a ring.

Solution. Note that $(M, +)$ is an abelian group. Fix $\gamma \in \Gamma$ and define a binary operation $\cdot$ on M by $a \cdot b = a\gamma b$ for any $a, b \in M$. Then $(M, \cdot)$ is a semigroup and $\cdot$ is distributive over $+$. Thus $(M, +, \cdot)$ is a ring. $\square$

The next example gives an idea that for a Γ-ring M, the abelian group Γ can be chosen from other abelian groups which is not a subgroup of the abelian group M.

Example 6. Let $(R, +, \cdot)$ be a ring with identity and $\mathbb{Z}$ be the set of all integers. Define a mapping from $R \times \mathbb{Z} \times R$ into R by $(x, k, y) \mapsto xky$ where $xky = x(ky)$ for all $x, y \in R$ and $k \in \mathbb{Z}$. Then R is a Γ-ring where $\Gamma = \mathbb{Z}$. Note here that for each $x, y \in R$ and $k \in \mathbb{Z}$, the element $xky \in R$ is, in fact, $x(ky) = (kx)y = k(xy)$.

In particular, any fields are Γ-rings where $\Gamma = \mathbb{Z}$.

For each $m, n \in \mathbb{N}_0$, let $M_{m\times n}(R)$ be the set of all $m \times n$ matrices over a ring R.

Example 7. Let $m, n \in \mathbb{N}$ and R be a ring. Consider $M = M_{m\times n}(R)$ and $\Gamma = M_{n\times m}(R)$. It is known that M and Γ are abelian groups under the usual addition of matrices. We define a map $(A, P, B) \mapsto APB$ for all $A, B \in M$ and $P \in \Gamma$, the ususal multiplication of matrices. Then M is a Γ-ring following from the properties of matrices.

The next two propositions are natural results in any Γ-rings which will be used in other chapters.

Proposition 8. Let M be a Γ-ring. Then

$$0_M\gamma a = a0_\Gamma b = a\gamma 0_M = 0_M$$

for all $a, b \in M, \gamma \in \Gamma$ where 0_M and 0_Γ are the identities of the abelian groups M and Γ, respectively.

Proof. Let $a, b \in M$ and $\gamma \in \Gamma$. Note that

$$0_M\gamma a = (0_M + 0_M)\gamma a = 0_M\gamma a + 0_M\gamma a,$$

$$a0_\Gamma b = a(0_\Gamma + 0_\Gamma)b = a0_\Gamma b + a0_\Gamma b \qquad \text{and}$$
$$a\gamma 0_M = a\gamma(0_M + 0_M) = a\gamma 0_M + a\gamma 0_M.$$

The cancellation of the group M implies that

$$0_M\gamma a = a0_\Gamma b = a\gamma 0_M = 0_M.$$

$\square$

Proposition 9. Let M be a Γ-ring. Then

$$(-a)\gamma b = a(-\gamma)b = a\gamma(-b) = -(a\gamma b)$$

for all $a, b \in M$ and $\gamma \in \Gamma$.

Proof. Let $a, b \in M$ and $\gamma \in \Gamma$. Then

$$a\gamma b + (-a)\gamma b = \big(a + (-a)\big)\gamma b = 0_M\gamma b = 0_M,$$
$$a\gamma b + a(-\gamma)b = a\big(\gamma + (-\gamma)\big)b = a0_\Gamma b = 0_M \qquad \text{and}$$
$$a\gamma b + a\gamma(-b) = a\gamma\big(b + (-b)\big) = a\gamma 0_M = 0_M.$$

This implies that $(-a)\gamma b = a(-\gamma)b = a\gamma(-b) = -(a\gamma b)$. $\square$

Moreover, we would like to give the definitions of the commutativity and the center of a Γ-ring which will be referred later. A Γ-ring M is said to be *commutative* if $a\gamma b = b\gamma a$ for all $a, b \in M$ and $\gamma \in \Gamma$. Moreover, the *center* of M, denoted by $Z(M)$, is defined by $Z(M) = \{c \in M \mid c\gamma m = m\gamma c$ for all $m \in M$ and $\gamma \in \Gamma\}$.

In 1997, M. Sapanci and A. Nakajima [11] introduced a derivation and a Jordan derivation of a Γ-ring. For a Γ-ring M, an *additive* map of M is a map $d : M \to M$ such that $d(a + b) = d(a) + d(b)$ for all $a, b \in M$. An additive map $d : M \to M$ of a Γ-ring M is called a *derivation* of M if $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$

for all $a, b \in M$ and $\alpha \in \Gamma$ and a *Jordan derivation* (JD) of M if $d(a\alpha a) = d(a)\alpha a + a\alpha d(a)$ for all $a \in M$ and $\alpha \in \Gamma$. Notice that derivations and JDs of a Γ-ring are defined analogously to those of a ring. Moreover, derivations of a Γ-ring are JDs. They proved that JDs of particular Γ-rings are derivations. M. Soyturk [12] and M.A. Ozturk and Y.B. Jun [10] gave definitions of prime Γ-ring and semiprime Γ-ring, respectively, as follow: M is a *prime* Γ-ring if for any $a, b \in M$, $a\Gamma M\Gamma b = 0$ implies that $a = 0$ or $b = 0$ and M is a *semiprime* Γ-ring if for any $a \in M$, $a\Gamma M\Gamma a = 0$ implies that $a = 0$. It is clear that a prime Γ-ring is a semiprime Γ-ring.

We give an example of a derivation of a Γ-ring arising from a derivation of a ring.

Example 10. Let R be a ring with identity, d be a derivation of R and $\Gamma = \mathbb{Z}$. Then R is a Γ-ring from Example 6. Since

$$d(xky) = d(k(xy)) = kd(xy) = k(d(x)y + xd(y)) = d(x)ky + xkd(y)$$

for all $x, y \in R$ and $k \in \mathbb{Z}$, it follows that d is a derivation of the Γ-ring R. Moreover, if d is a Jordan derivation of the ring R, then d is also a Jordan derivation of the Γ-ring R.

We observe that, for rings, derivations, JDs, HDs and JHDs have been studied. While, derivations and JDs of Γ-rings have been investigated. However, HDs and JHDs of Γ-rings have not been found. These motivated us to give notion of HDs and JHDs of Γ-rings and study whether they possess related results to those of rings.

Found in [12], M. Soyturk gave the definitions of one-sided ideals and an ideal of a Γ-ring. A *right (left) ideal* of a Γ-ring M is an additive subgroup U of M such that $U\Gamma M \subseteq U$ ($M\Gamma U \subseteq U$); moreover, if U is both a right and a left ideal of M, then U is called an *ideal* of M. Note that, for nonempty subsets A and B of a Γ-ring, the nonempty set $A\Gamma B$ is given by $A\Gamma B = \left\{ \sum_{i=1}^{n} a_i\gamma_i b_i \mid n \in \mathbb{N}, a_i \in A, \right.$

$\left.\gamma_i \in \Gamma, b_i \in B \text{ for all } i\right\}$. This brought us to another objective of doing this research. In the same manner of the first aim, we give definitions of a HD and a JHD of an ideal of a Γ-ring and are interested in the analogous results of HDS and JHDs of an ideal of a ring proposed by C. Haetinger [8].

In 2004, Y. Ceven and M.A. Ozturk [4] introduced a generalized derivation and a Jordan generalized derivation of a Γ-ring. For a Γ-ring M, an additive map $f : M \to M$ is called a *generalized derivation* (GD) of M if there exists a derivation $d : M \to M$ of M such that $f(x\gamma y) = f(x)\gamma y + x\gamma d(y)$ for all $x, y \in M$ and $\gamma \in \Gamma$, a *Jordan generalized derivation* (JGD) of M if there exists a derivation $d : M \to M$ of M such that $f(x\gamma x) = f(x)\gamma x + x\gamma d(x)$ for all $x \in M$ and $\gamma \in \Gamma$. Obsiously, derivations of a Γ-ring are GDs and GDs of a Γ-ring are JGDs but not another way round. They proved that JGDs of a Γ-ring satisfying some certain properties are GDs. As a result, we extend, correspondingly, this notion to generalized higher derivations and Jordan generalized higher derivations of Γ-rings.

This thesis is separated into 4 chapters. Chapter I contains general definitions and results mainly of Γ-rings which are basic ideas for the whole.

Chapter II pays attention to higher derivations and Jordan higher derivations of a Γ-ring. Some of their properties are explored. At the end, the main result stating that Jordan higher derivations and higher derivations of specific Γ-rings are identical will be provided.

Likewise Chapter II, higher derivations and Jordan higher derivations of an ideal of a Γ-ring are taken into account in Chapter III. We study their properties in order to obtain our aim which is the fact that Jordan higher derivations of an ideal of a specific Γ-ring are higher derivations.

Finally, we develop the notion of generalized derivations and Jordan generalized derivations to generalized higher derivations and Jordan generalized higher derivations, respectively, of a Γ-ring in Chaper IV. In the same fashion, for a certain Γ-ring, its Jordan generalized higher derivations are generalized higher derivations along with other properties are given.

CHAPTER II
HIGHER DERIVATIONS AND JORDAN HIGHER DERIVATIONS OF Γ-RINGS

Higher derivations and Jordan higher derivations of Γ-rings are main notions of this thesis. Consequently, we first give their definitions in the first section. Moreover, the major tools for this work are provided in the same section. Next section is focus on the results showing how Jordan higher derivations of Γ-rings are Jordan triple higher derivations. Finally, in the third section, the first main result is provided.

2.1 Essential Properties for Higher Derivations and Jordan Higher Derivations of Γ-Rings

We first provide definitions of a higher derivation, a Jordan higher derivation and a Jordan triple higher derivation of a Γ-ring.

Definition 2.1.1. Let $D = (d_i)_{i\in\mathbb{N}_0}$ be a family of additive mappings of a Γ-ring M (i.e., $d_i : M \to M$ preserves the addition for all i) where d_0 is the identity mapping. Then D is said to be a *higher derivation* (HD) of M if

$$d_n(a\gamma b) = \sum_{i+j=n} d_i(a)\gamma d_j(b) \qquad \text{for all } a, b \in M, \gamma \in \Gamma \text{ and } n \in \mathbb{N}_0;$$

a *Jordan higher derivation* (JHD) of M if

$$d_n(a\gamma a) = \sum_{i+j=n} d_i(a)\gamma d_j(a) \qquad \text{for all } a \in M, \gamma \in \Gamma \text{ and } n \in \mathbb{N}_0;$$

and a *Jordan triple higher derivation* (JTHD) of M if

$$d_n(a\gamma b\beta a) = \sum_{i+j+k=n} d_i(a)\gamma d_j(b)\beta d_k(a) \quad \text{for all } a, b \in M, \gamma, \beta \in \Gamma \text{ and } n \in \mathbb{N}_0.$$

It is clear that HDs are JHDs. Moreover, note that if D is a HD, then for any $a, b \in M, \gamma, \beta \in \Gamma$ and $n \in \mathbb{N}_0$,

$$\begin{aligned}
d_n(a\gamma b\beta a) &= d_n\big((a\gamma b)\beta a\big) \\
&= \sum_{i+j=n} d_i(a\gamma b)\beta d_j(a) \\
&= \sum_{i+j=n} \left(\sum_{k+l=i} d_k(a)\gamma d_l(b) \right) \beta d_j(a) \\
&= \sum_{k+l+j=n} d_k(a)\gamma d_l(b)\beta d_j(a).
\end{aligned}$$

This shows that a HD is also a JTHD. In summary, any HDs are JHDs and JTHDs. However, the converse does not hold.

Example 2.1.2. Let $\mathbb{F}$ be a field and $\Gamma = \mathbb{Z}$. Then $\mathbb{F}$ is a Γ-ring by Example 6 in Chapter I. Let $n \in \mathbb{N}_0$. Define a map $d_n : \mathbb{F} \to \mathbb{F}$ by

$$d_n(x) = \begin{cases} x & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

for all $x \in \mathbb{F}$. It is easy to see that the family of additive mappings $D = (d_i)_{i \in \mathbb{N}_0}$ is a HD on the Γ-ring.

Example 2.1.3. Let R be a ring with identity, $\Gamma = \mathbb{Z}$ and $D = (d_i)_{i \in \mathbb{N}_0}$ be a HD of R. Then R is a Γ-ring from Example 6. To show that D is a HD of the Γ-ring R, let $x, y \in R$ and $k \in \mathbb{Z}$. Then

$$d_n(xky) = kd_n(xy) = k \sum_{i+j=n} d_i(a)d_j(b) = \sum_{i+j=n} d_i(a)kd_j(b).$$

Thus D is a HD of the Γ-ring R. Similarly, if D is a JHD of the ring R, then D is also a JHD of the Γ-ring R.

As mentioned above, Proposition 2.1.4 and Proposition 2.1.5 play important roles for the rest of this thesis.

Proposition 2.1.4. *Assume that M is a 2-torsion free semiprime Γ-ring. Let $G_1, \ldots, G_n$ be additive groups, $S : G_1 \times \cdots \times G_n \to M$ and $T : G_1 \times \cdots \times G_n \to M$ be mappings which are additive in each argument. If $S(a_1, \ldots, a_n)\gamma x\beta T(a_1, \ldots, a_n) = 0$ for all $x \in M, \gamma, \beta \in \Gamma$ and $a_i \in G_i$ for all $i = 1, \ldots, n$, then*

$$S(a_1, \ldots, a_n)\gamma x\beta T(b_1, \ldots, b_n) = 0$$

for all $x \in M, \gamma, \beta \in \Gamma$ and $a_i, b_i \in G_i$ for all $i = 1, \ldots, n$.

Proof. We prove this result by using the induction on n. For basic step, assume that $S, T : G_1 \to M$ are additive mappings such that $S(a)\gamma x\beta T(a) = 0$ for all $x \in M$, $\gamma, \beta \in \Gamma$ and $a \in G_1$. We see that, for each $x \in M, \gamma, \beta \in \Gamma$ and $a \in G_1$,

$$\Big(T(a)\gamma x\beta S(a)\Big)\delta y\sigma\Big(T(a)\gamma x\beta S(a)\Big) = T(a)\gamma x\beta\Big(S(a)\delta y\sigma T(a)\Big)\gamma x\beta S(a) = 0$$

for all $y \in M$ and $\delta, \sigma \in \Gamma$ so that $T(a)\gamma x\beta S(a) = 0$ since M is semiprime. Observe that we now have

$$S(a)\gamma x\beta T(a) = 0 = T(a)\gamma x\beta S(a)$$

for all $x \in M, \gamma, \beta \in \Gamma$ and $a \in G_1$. Now, let $x \in M, \gamma, \beta \in \Gamma$ and $a, b \in G_1$. Then

$$\begin{aligned} 0 &= S(a+b)\gamma x\beta T(a+b) \\ &= S(a)\gamma x\beta T(a) + S(a)\gamma x\beta T(b) + S(b)\gamma x\beta T(a) + S(b)\gamma x\beta T(b) \\ &= S(a)\gamma x\beta T(b) + S(b)\gamma x\beta T(a). \end{aligned}$$

Consequently, $S(a)\gamma x\beta T(b) = -\Big(S(b)\gamma x\beta T(a)\Big)$. Furthermore, we can see that

$$\Big(S(a)\gamma x\beta T(b)\Big)\delta y\sigma\Big(S(a)\gamma x\beta T(b)\Big) = -\Big(S(b)\gamma x\beta T(a)\Big)\delta y\sigma\Big(S(a)\gamma x\beta T(b)\Big)$$
$$= -S(b)\gamma x\beta\Big(T(a)\delta y\sigma S(a)\Big)\gamma x\beta T(b)$$
$$= 0$$

for all $y \in M$ and $\delta, \sigma \in \Gamma$. Since M is semiprime, $S(a)\gamma x\beta T(b) = 0$ as desired.

Case $n = 2$. Assume that $S, T : G_1 \times G_2 \to M$ are mappings which additive in each argument such that $S(a_1, a_2)\gamma x\beta T(a_1, a_2) = 0$ for all $x \in M$, $\gamma, \beta \in \Gamma$ and $a_i \in G_i$ for all $i = 1, 2$. Let $b_2 \in G_2$. Define $\overline{S}, \overline{T} : G_1 \to M$ by $\overline{S}(a) = S(a, b_2)$ and $\overline{T}(a) = T(a, b_2)$ for all $a \in G_1$. Then, clearly, $\overline{S}$ and $\overline{T}$ are additive mappings. From the assumption, we see that

$$\overline{S}(a)\gamma x\beta\overline{T}(a) = S(a, b_2)\gamma x\beta T(a, b_2) = 0$$

for all $x \in M$, $\gamma, \beta \in \Gamma$ and $a \in G_1$. We obtain from the basic step that $\overline{S}(a)\gamma x\beta\overline{T}(b) = 0$ for all $x \in M$, $\gamma, \beta \in \Gamma$ and $a, b \in G_1$. This implies that $S(a, b_2)\gamma x\beta T(b, b_2) = 0$ for all $x \in M$, $\gamma, \beta \in \Gamma$ and $a, b \in G_1$. This shows that

$$S(a, b_2)\gamma x\beta T(b, b_2) = 0 \tag{1}$$

for all $x \in M, \gamma, \beta \in \Gamma, a, b \in G_1$ and $b_2 \in G_2$. Next, let $a_1, b_1 \in G_1$. Define $S', T' : G_2 \to M$ by $S'(a) = S(a_1, a)$ and $T'(a) = T(b_1, a)$ for all $a \in G_2$. Then S' and T' are additive mappings. It follows from (1) that

$$S'(a)\gamma x\beta T'(a) = S(a_1, a)\gamma x\beta T(b_1, a) = 0$$

for all $x \in M, \gamma, \beta \in \Gamma$ and $a \in G_2$. Applying the basic step yields $S'(a)\gamma x\beta T'(b) = 0$, i.e., $S(a_1, a)\gamma x\beta T(b_1, b) = 0$ for all $x \in M$, $\gamma, \beta \in \Gamma$ and $a, b \in G_2$. Hence $S(a_1, a)\gamma x\beta T(b_1, b) = 0$ for all $x \in M$, $\gamma, \beta \in \Gamma$, $a_1, b_1 \in G_1$ and $a, b \in G_2$.

For induction step, let $m \in \mathbb{N}$ and assume that for any $j \leq m$ if S, T :

$G_1 \times \cdots \times G_j \to M$ are mappings which are additive in each argument and $S(a_1, \ldots, a_j)\gamma x\beta T(a_1, \ldots, a_j) = 0$ for all $x \in M$, $\gamma, \beta \in \Gamma$ and $a_i \in G_i$ for all $i = 1, \ldots, j$, then

$$S(a_1, \ldots, a_j)\gamma x\beta T(b_1, \ldots, b_j) = 0$$

for all $x \in M, \gamma, \beta \in \Gamma$ and $a_i, b_i \in G_i$ for all $i = 1, \ldots, j$. Next, we assume further that $S, T : G_1 \times \cdots \times G_{m+1} \to M$ are mappings which are additive in each argument and $S(a_1, \ldots, a_{m+1})\gamma x\beta T(a_1, \ldots, a_{m+1}) = 0$ for all $x \in M, \gamma, \beta \in \Gamma$ and $a_i \in G_i$ for all $i = 1, \ldots, m+1$. Let $\overline{S}, \overline{T} : (G_1 \times \cdots \times G_m) \times G_{m+1} \to M$ be defined by

$$\overline{S}\big((a_1, \ldots, a_m), a_{m+1}\big) = S(a_1, \ldots, a_m, a_{m+1})$$

and

$$\overline{T}\big((a_1, \ldots, a_m), a_{m+1}\big) = T(a_1, \ldots, a_m, a_{m+1})$$

for all $(a_1, \ldots, a_m) \in G_1 \times \cdots \times G_m$ and $a_{m+1} \in G_{m+1}$. Then $\overline{S}$ and $\overline{T}$ are additive in $G_1 \times \cdots \times G_m$ and G_{m+1}. Besides, for each $x \in M, \gamma, \beta \in \Gamma, (a_1, \ldots, a_m) \in G_1 \times \cdots \times G_m$ and $a_{m+1} \in G_{m+1}$, we obtain that

$$\begin{aligned}\overline{S}\big((a_1, \ldots, a_m), a_{m+1}\big)&\gamma x\beta \overline{T}\big((a_1, \ldots, a_m), a_{m+1}\big)\\ &= S(a_1, \ldots, a_m, a_{m+1})\gamma x\beta T(a_1, \ldots, a_m, a_{m+1}) = 0.\end{aligned}$$

By induction hypothesis,

$$\begin{aligned}S\big(a_1, \ldots, a_m, a_{m+1}\big)&\gamma x\beta T\big(b_1, \ldots, b_m, b_{m+1}\big)\\ &= \overline{S}\big((a_1, \ldots, a_m), a_{m+1}\big)\gamma x\beta \overline{T}\big((b_1, \ldots, b_m), b_{m+1}\big) = 0\end{aligned}$$

for all $x \in M, \gamma, \beta \in \Gamma$ and $a_i, b_i \in G_i$ for all $i = 1, \ldots, m+1$. $\square$

Proposition 2.1.5. *Let M be a 2-torsion free semiprime Γ-ring. If $a, b \in M$ are such that $a\gamma x\beta b + b\gamma x\beta a = 0$ for all $x \in M$ and $\gamma, \beta \in \Gamma$, then $a\gamma x\beta b = b\gamma x\beta a = 0$ for all $x \in M$ and $\gamma, \beta \in \Gamma$.*

Proof. Let $a, b \in M$ be such that $a\gamma x\beta b + b\gamma x\beta a = 0$ for all $x \in M$ and $\gamma, \beta \in \Gamma$. Moreover, let $x, y \in M$ and $\gamma, \beta, \delta, \sigma \in \Gamma$. We consider $(a\gamma x\beta b)\delta y\sigma(a\gamma x\beta b)$ as follows:

$$
\begin{aligned}
(a\gamma x\beta b)\delta y\sigma(a\gamma x\beta b) &= a\gamma x\beta(b\delta y\sigma a)\gamma x\beta b \\
&= a\gamma x\beta(-a\delta y\sigma b)\gamma x\beta b && \text{(by assumption)} \\
&= -\big(a\gamma x\beta(a\delta y\sigma b)\gamma x\beta b\big) \\
&= -\big(a\gamma(x\beta a\delta y)\sigma b\big)\gamma x\beta b \\
&= \Big(-\big(a\gamma(x\beta a\delta y)\sigma b\big)\Big)\gamma x\beta b \\
&= \big(b\gamma(x\beta a\delta y)\sigma a\big)\gamma x\beta b && \text{(by assumption)} \\
&= (b\gamma x\beta a)\delta y\sigma(a\gamma x\beta b) \\
&= \big(-(a\gamma x\beta b)\big)\delta y\sigma(a\gamma x\beta b) && \text{(by assumption)} \\
&= -\Big((a\gamma x\beta b)\delta y\sigma(a\gamma x\beta b)\Big).
\end{aligned}
$$

Thus $2\Big((a\gamma x\beta b)\delta y\sigma(a\gamma x\beta b)\Big) = 0$. As a consequence, $(a\gamma x\beta b)\delta y\sigma(a\gamma x\beta b) = 0$ because M is 2-torsion free. Note that y, δ, σ are arbitrary and M is semiprime. We can conclude that $a\gamma x\beta b = 0$. $\square$

Now, we give examples of 2-torsion free semiprime Γ-rings.

Example 2.1.6. Let $\mathbb{F}$ be a field of characteristic different from 2 and $\Gamma = \mathbb{Z}$. Then $\mathbb{F}$ is a Γ-ring. Since the characteristic of $\mathbb{F}$ is not equal to 2, the Γ-ring $\mathbb{F}$ is 2-torsion free. Claim that $\mathbb{F}$ is semiprime. Let $a \in \mathbb{F}$ be such that $a\mathbb{Z}\mathbb{F}\mathbb{Z}a = 0$. This implies that $aa = 0$. Since $\mathbb{F}$ has no zero divisors, $a = 0$. Hence $\mathbb{F}$ is semiprime. Therefore, $\mathbb{F}$ is a 2-torsion free semiprime Γ-ring.

2.2 Relationships between JHDs and JTHDs

We know from the previous section that, for a Γ-ring, HDs are JHDs and JTHDs; however, the explicit connections between JHDs and JTHDs have not been seen. Our aim of this section is proving that JHDs of a certain Γ-ring are JTHDs. To achieve this, we first need Proposition 2.2.1 and Proposition 2.2.2.

Throughout this section, let $D = (d_i)_{i\in\mathbb{N}_0}$ be a JHD of a Γ-ring M.

Proposition 2.2.1. *For each $a, b \in M, \gamma \in \Gamma$ and $n \in \mathbb{N}_0$,*

$$d_n(a\gamma b + b\gamma a) = \sum_{i+j=n} \Big(d_i(a)\gamma d_j(b) + d_i(b)\gamma d_j(a)\Big).$$

Proof. Let $a, b \in M, \gamma \in \Gamma$ and $n \in \mathbb{N}_0$. Replacing a by $a + b$ in the definition of a JHD yields

$$\begin{aligned}
&d_n\big((a+b)\gamma(a+b)\big)\\
&\quad= \sum_{i+j=n} d_i(a+b)\gamma d_j(a+b)\\
&\quad= \sum_{i+j=n} \Big(d_i(a)\gamma d_j(a) + d_i(a)\gamma d_j(b) + d_i(b)\gamma d_j(a) + d_i(b)\gamma d_j(b)\Big)\\
&\quad= \sum_{i+j=n} d_i(a)\gamma d_j(a) + \sum_{i+j=n} \Big(d_i(a)\gamma d_j(b) + d_i(b)\gamma d_j(a)\Big) + \sum_{i+j=n} d_i(b)\gamma d_j(b).
\end{aligned}$$

On the other hand, we see that

$$\begin{aligned}
d_n\big((a+b)\gamma(a+b)\big) &= d_n(a\gamma a + a\gamma b + b\gamma a + b\gamma b)\\
&= d_n(a\gamma a) + d_n(a\gamma b + b\gamma a) + d_n(b\gamma b)\\
&= \sum_{i+j=n} d_i(a)\gamma d_j(a) + d_n(a\gamma b + b\gamma a) + \sum_{i+j=n} d_i(b)\gamma d_j(b).
\end{aligned}$$

Thus

$$\begin{aligned}
0 &= d_n\big((a+b)\gamma(a+b)\big) - d_n\big((a+b)\gamma(a+b)\big) \\
&= \left(\sum_{i+j=n} d_i(a)\gamma d_j(a) + \sum_{i+j=n}\Big(d_i(a)\gamma d_j(b) + d_i(b)\gamma d_j(a)\Big) + \sum_{i+j=n} d_i(b)\gamma d_j(b)\right) \\
&\quad - \left(\sum_{i+j=n} d_i(a)\gamma d_j(a) + d_n(a\gamma b + b\gamma a) + \sum_{i+j=n} d_i(b)\gamma d_j(b)\right) \\
&= \sum_{i+j=n}\Big(d_i(a)\gamma d_j(b) + d_i(b)\gamma d_j(a)\Big) - d_n(a\gamma b + b\gamma a).
\end{aligned}$$

This implies that

$$d_n(a\gamma b + b\gamma a) = \sum_{i+j=n}\Big(d_i(a)\gamma d_j(b) + d_i(b)\gamma d_j(a)\Big).$$

□

Proposition 2.2.2. *For each $a, b \in M, \gamma, \beta \in \Gamma$ amd $n \in \mathbb{N}_0$,*

$$d_n(a\beta b\gamma a + a\gamma b\beta a) = \sum_{i+j+k=n}\Big(d_i(a)\beta d_j(b)\gamma d_k(a) + d_i(a)\gamma d_j(b)\beta d_k(a)\Big).$$

Proof. Let $a, b \in M, \gamma, \beta \in \Gamma$ and $n \in \mathbb{N}_0$. For our convenience, put

$$w = a\beta(a\gamma b + b\gamma a) + (a\gamma b + b\gamma a)\beta a.$$

We apply Proposition 2.2.1 so that

$$\begin{aligned}
d_n(w) &= d_n(a\beta(a\gamma b + b\gamma a) + (a\gamma b + b\gamma a)\beta a) \\
&= \sum_{i+j=n}\Big(d_i(a)\beta d_j(a\gamma b + b\gamma a) + d_i(a\gamma b + b\gamma a)\beta d_j(a)\Big) \\
&= \sum_{i+j=n} d_i(a)\beta d_j\big(a\gamma b + b\gamma a\big) + \sum_{i+j=n} d_i\big(a\gamma b + b\gamma a\big)\beta d_j(a)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i+j=n} d_i(a)\beta\Big(\sum_{r+s=j}\big(d_r(a)\gamma d_s(b) + d_r(b)\gamma d_s(a)\big)\Big) \\
&\quad + \sum_{i+j=n}\Big(\sum_{t+u=i}\big(d_t(a)\gamma d_u(b) + d_t(b)\gamma d_u(a)\big)\Big)\beta d_j(a) \\
&= \sum_{i+j=n} d_i(a)\beta\Big(\sum_{r+s=j} d_r(a)\gamma d_s(b)\Big) + \sum_{i+j=n} d_i(a)\beta\Big(\sum_{r+s=j} d_r(b)\gamma d_s(a)\Big) \\
&\quad + \sum_{i+j=n}\Big(\sum_{t+u=i} d_t(a)\gamma d_u(b)\Big)\beta d_j(a) + \sum_{i+j=n}\Big(\sum_{t+u=i} d_t(b)\gamma d_u(a)\Big)\beta d_j(a) \\
&= \sum_{i+j=n}\sum_{r+s=j} d_i(a)\beta d_r(a)\gamma d_s(b) + \sum_{i+j=n}\sum_{r+s=j} d_i(a)\beta d_r(b)\gamma d_s(a) \\
&\quad + \sum_{i+j=n}\sum_{t+u=i} d_t(a)\gamma d_u(b)\beta d_j(a) + \sum_{i+j=n}\sum_{t+u=i} d_t(b)\gamma d_u(a)\beta d_j(a) \\
&= \sum_{i+r+s=n} d_i(a)\beta d_r(a)\gamma d_s(b) + \sum_{i+r+s=n} d_i(a)\beta d_r(b)\gamma d_s(a) \\
&\quad + \sum_{t+u+j=n} d_t(a)\gamma d_u(b)\beta d_j(a) + \sum_{t+u+j=n} d_t(b)\gamma d_u(a)\beta d_j(a) \\
&= \sum_{i+j+k=n}\Big(d_i(a)\beta d_j(a)\gamma d_k(b) + d_i(a)\beta d_j(b)\gamma d_k(a) \\
&\quad + d_i(a)\gamma d_j(b)\beta d_k(a) + d_i(b)\gamma d_j(a)\beta d_k(a)\Big) \\
&= \sum_{i+j+k=n}\Big(d_i(a)\beta d_j(a)\gamma d_k(b) + d_i(b)\gamma d_j(a)\beta d_k(a)\Big) \\
&\quad + \sum_{i+j+k=n}\Big(d_i(a)\beta d_j(b)\gamma d_k(a) + d_i(a)\gamma d_j(b)\beta d_k(a)\Big).
\end{aligned}
$$

We also obtain that

$$
\begin{aligned}
d_n(w) &= d_n\big(a\beta(a\gamma b + b\gamma a) + (a\gamma b + b\gamma a)\beta a\big) \\
&= d_n\big(a\beta a\gamma b + a\beta b\gamma a + a\gamma b\beta a + b\gamma a\beta a\big)
\end{aligned}
$$

$$
\begin{aligned}
&= d_n\big(a\beta a\gamma b + b\gamma a\beta a\big) + d_n\big(a\beta b\gamma a + a\gamma b\beta a\big)\\
&= d_n\big((a\beta a)\gamma b + b\gamma(a\beta a)\big) + d_n\big(a\beta b\gamma a + a\gamma b\beta a\big)\\
&= \sum_{i+j=n} \big(d_i(a\beta a)\gamma d_j(b) + d_i(b)\gamma d_j(a\beta a)\big) + d_n\big(a\beta b\gamma a + a\gamma b\beta a\big)\\
&= \sum_{i+j=n} d_i(a\beta a)\gamma d_j(b) + \sum_{i+j=n} d_i(b)\gamma d_j(a\beta a) + d_n\big(a\beta b\gamma a + a\gamma b\beta a\big)\\
&= \sum_{i+j=n}\Bigg(\sum_{p+q=i} d_p(a)\beta d_q(a)\Bigg)\gamma d_j(b) + \sum_{i+j=n} d_i(b)\gamma\Bigg(\sum_{u+v=j} d_u(a)\beta d_v(a)\Bigg)\\
&\quad + d_n\big(a\beta b\gamma a + a\gamma b\beta a\big)\\
&= \sum_{i+j=n}\sum_{p+q=i} d_p(a)\beta d_q(a)\gamma d_j(b) + \sum_{i+j=n}\sum_{u+v=j} d_i(b)\gamma d_u(a)\beta d_v(a)\\
&\quad + d_n\big(a\beta b\gamma a + a\gamma b\beta a\big)\\
&= \sum_{p+q+j=n} d_p(a)\beta d_q(a)\gamma d_j(b) + \sum_{i+u+v=n} d_i(b)\gamma d_u(a)\beta d_v(a)\\
&\quad + d_n\big(a\beta b\gamma a + a\gamma b\beta a\big)\\
&= \sum_{i+j+k=n} \Big(d_i(a)\beta d_j(a)\gamma d_k(b) + d_i(b)\gamma d_j(a)\beta d_k(a)\Big) + d_n\big(a\beta b\gamma a + a\gamma b\beta a\big).
\end{aligned}
$$

As a result,

$$
\begin{aligned}
0 &= d_n(w) - d_n(w)\\
&= \Bigg[\sum_{i+j+k=n} \Big(d_i(a)\beta d_j(b)\gamma d_k(a) + d_i(a)\gamma d_j(b)\beta d_k(a)\Big)\\
&\quad + \sum_{i+j+k=n}\Big(d_i(a)\beta d_j(a)\gamma d_k(b) + d_i(b)\gamma d_j(a)\beta d_k(a)\Big)\Bigg]\\
&\quad - \Bigg[\sum_{i+j+k=n} \Big(d_i(a)\beta d_j(b)\gamma d_k(b) + d_i(b)\gamma d_k(a)\beta d_k(a)\Big) + d_n\big(a\beta b\gamma a + a\gamma b\beta a\big)\Bigg]\\
&= \sum_{i+j+k=n} \Big(d_i(a)\beta d_j(b)\gamma d_k(a) + d_i(a)\gamma d_j(b)\beta d_k(a)\Big) - d_n(a\beta b\gamma a + a\gamma b\beta a).
\end{aligned}
$$

Hence

$$d_n(a\beta b\gamma a + a\gamma b\beta a) = \sum_{i+j+k=n} \Big(d_i(a)\beta d_j(b)\gamma d_k(a) + d_i(a)\gamma d_j(b)\beta d_k(a) \Big).$$

$\square$

Next, the conclusion of this section is given. The requirement of a Γ-ring M to possess the property that any JHDs are JTHDs is not only that M is 2-torsion free but also the ability of interchanging between any elements γ and β in Γ.

Theorem 2.2.3. *Let M be a 2-torsion free Γ-ring such that $a\gamma b\beta c = a\beta b\gamma c$ for all $a, b, c \in M$ and $\gamma, \beta \in \Gamma$. Then every Jordan higher derivation of M is a Jordan triple higher derivation of M.*

Proof. Let $D = (d_i)_{i\in\mathbb{N}_0}$ be a JHD of M. To show that D is a JTHD, let $a, b \in M$ and $\gamma, \beta \in \Gamma$. It follows from Proposition 2.2.2 that

$$d_n(a\beta b\gamma a + a\gamma b\beta a) = \sum_{i+j+k=n} \Big(d_i(a)\beta d_j(b)\gamma d_k(a) + d_i(a)\gamma d_j(b)\beta d_k(a) \Big).$$

Making use of the assumption to the above equation implies

$$d_n(2a\beta b\gamma a) = \sum_{i+j+k=n} 2d_i(a)\beta d_j(b)\gamma d_k(a)$$

$$2d_n(a\beta b\gamma a) = 2\sum_{i+j+k=n} d_i(a)\beta d_j(b)\gamma d_k(a).$$

Since M is 2-torsion free,

$$d_n(a\beta b\gamma a) = \sum_{i+j+k=n} d_i(a)\beta d_j(b)\gamma d_k(a).$$

Therefore, D is a JTHD. $\square$

2.3 The Main Result of JHDs and HDs of Γ-Rings

We present the first major result of doing this research in this section. In a ring, its JHDs are HDs provided that the ring is 2-torsion free semiprime and this can be proved via the fact that JHDs are JTHDs and JTHDs are HDs. Analogous to rings, we can show that JHDs and HDs of a Γ-ring M are identical provided that M is 2-torsion free semiprime; however, another condition for M must be required.

Throughout this section, let $D = (d_i)_{i\in\mathbb{N}_0}$ be a JTHD of a Γ-ring M. First, we extend Proposition 2.2.1 to Proposition 2.3.1, namely, the choice of γ in $a\gamma c + c\gamma a$ is extended to be $\gamma b\beta$.

Proposition 2.3.1. *For each $a, b, c \in M, \gamma, \beta \in \Gamma$ and $n \in \mathbb{N}_0$,*

$$d_n(a\gamma b\beta c + c\gamma b\beta a) = \sum_{i+j+k=n} \Big(d_i(a)\gamma d_j(b)\beta d_k(c) + d_i(c)\gamma d_j(b)\beta d_k(a) \Big).$$

Proof. Let $a, b, c \in M, \gamma, \beta \in \Gamma$ and $n \in \mathbb{N}_0$. Replacing a by $a + c$ in the definition of JTHD yields

$$\begin{aligned}
&d_n\big((a+c)\gamma b\beta(a+c)\big)\\
&\quad= \sum_{i+j+k=n} d_i(a+c)\gamma d_j(b)\beta d_k(a+c)\\
&\quad= \sum_{i+j+k=n} \Big(d_i(a)\gamma d_j(b)\beta d_k(a) + d_i(a)\gamma d_j(b)\beta d_k(c)\\
&\qquad\quad + d_i(c)\gamma d_j(b)\beta d_k(a) + d_i(c)\gamma d_j(b)\beta d_k(c) \Big)\\
&\quad= \sum_{i+j+k=n} d_i(a)\gamma d_j(b)\beta d_k(a) + \sum_{i+j+k=n} d_i(c)\gamma d_j(b)\beta d_k(c)\\
&\qquad\quad + \sum_{i+j+k=n} \Big(d_i(a)\gamma d_j(b)\beta d_k(c) + d_i(c)\gamma d_j(b)\beta d_k(a) \Big).
\end{aligned}$$

We also have

$$
\begin{aligned}
&d_n\big((a+c)\gamma b\beta(a+c)\big)\\
&\quad= d_n\big(a\gamma b\beta a + a\gamma b\beta c + c\gamma b\beta a + c\gamma b\beta c\big)\\
&\quad= d_n(a\gamma b\beta a) + d_n(a\gamma b\beta c + c\gamma b\beta a) + d_n(c\gamma b\beta c)\\
&\quad= \sum_{i+j+k=n} d_i(a)\gamma d_j(b)\beta d_k(a) + \sum_{i+j+k=n} d_i(c)\gamma d_j(b)\beta d_k(c)\\
&\qquad\quad + d_n(a\gamma b\beta c + c\gamma b\beta a).
\end{aligned}
$$

As a result,

$$
\begin{aligned}
0 &= d_n\big((a+c)\gamma b\beta(a+c)\big) - d_n\Big((a+c)\gamma b\beta(a+c)\Big)\\
&= \sum_{i+j+k=n}\Big(d_i(a)\gamma d_j(b)\beta d_k(c) + d_i(c)\gamma d_j(b)\beta d_k(a)\Big) - d_n(a\gamma b\beta c + c\gamma b\beta a).
\end{aligned}
$$

Thus

$$
d_n(a\gamma b\beta c + c\gamma b\beta a) = \sum_{i+j+k=n}\Big(d_i(a)\gamma d_j(b)\beta d_k(c) + d_i(c)\gamma d_j(b)\beta d_k(a)\Big).
$$

□

Before going further, we would like to set up two types of elements of M and provide some of their basic properties separately. For each $a, b, c \in M$, $\gamma, \beta \in \Gamma$ and $n \in \mathbb{N}_0$, let $\varphi_n(a,b,c)_{\gamma,\beta}$ and $[a,b,c]_{\gamma,\beta}$ be the elements of M defined by

$$
\begin{aligned}
\varphi_n(a,b,c)_{\gamma,\beta} &= d_n(a\gamma b\beta c) - \sum_{i+j+k=n} d_i(a)\gamma d_j(b)\beta d_k(c)\\
[a,b,c]_{\gamma,\beta} &= a\gamma b\beta c - c\gamma b\beta a.
\end{aligned}
$$

Obviously, $[a,b,a]_{\gamma,\beta} = 0$ for all $a, b \in M$ and $\gamma, \beta \in \Gamma$. Besides, it follows that $\varphi_n(a,b,a)_{\gamma,\beta} = 0$ for all $a, b \in M$, $\gamma, \beta \in \Gamma$ and $n \in \mathbb{N}_0$ since D is a JTHD of M.

Next, we aim to show that $\varphi_n(a,b,c)_{\gamma,\beta}=0$ for all $a,b,c\in M,\gamma,\beta\in\Gamma$ and $n\in\mathbb{N}_0$. However, the fact that $\varphi_n(a,b,c)_{\gamma,\beta}$ and $[a,b,c]_{\gamma,\beta}$ are additive in each argument is shown first, respectively.

Proposition 2.3.2. *For each $a,b,c,x\in M$ and $\gamma,\beta,\zeta\in\Gamma$,*

(i) $\varphi_n(a+x,b,c)_{\gamma,\beta}=\varphi_n(a,b,c)_{\gamma,\beta}+\varphi_n(x,b,c)_{\gamma,\beta}$,

(ii) $\varphi_n(a,b+x,c)_{\gamma,\beta}=\varphi_n(a,b,c)_{\gamma,\beta}+\varphi_n(a,x,c)_{\gamma,\beta}$,

(iii) $\varphi_n(a,b,c+x)_{\gamma,\beta}=\varphi_n(a,b,c)_{\gamma,\beta}+\varphi_n(a,b,x)_{\gamma,\beta}$,

(iv) $\varphi_n(a,b,c)_{\gamma+\zeta,\beta}=\varphi_n(a,b,c)_{\gamma,\beta}+\varphi_n(a,b,c)_{\zeta,\beta}$,

(v) $\varphi_n(a,b,c)_{\gamma,\beta+\zeta}=\varphi_n(a,b,c)_{\gamma,\beta}+\varphi_n(a,b,c)_{\gamma,\zeta}$.

Proof. Let $a,b,c,x\in M$ and $\gamma,\beta,\zeta\in\Gamma$.

(i) $\varphi_n(a+x,b,c)_{\gamma,\beta}$

$$\begin{aligned}
&= d_n\big((a+x)\gamma b\beta c\big)-\sum_{i+j+k=n} d_i(a+x)\gamma d_j(b)\beta d_k(c)\\
&= d_n\big(a\gamma b\beta c+x\gamma b\beta c\big)-\sum_{i+j+k=n}\Big(d_i(a)\gamma d_j(b)\beta d_k(c)+d_i(x)\gamma d_j(b)\beta d_k(c)\Big)\\
&= \Bigg(d_n(a\gamma b\beta c)-\sum_{i+j+k=n} d_i(a)\gamma d_j(b)\beta d_k(c)\Bigg)\\
&\qquad+\Bigg(d_n(x\gamma b\beta c)-\sum_{i+j+k=n} d_i(x)\gamma d_j(b)\beta d_k(c)\Bigg)\\
&= \varphi_n(a,b,c)_{\gamma,\beta}+\varphi_n(x,b,c)_{\gamma,\beta}.
\end{aligned}$$

(ii) and *(iii)* are obtained similary to *(i)*.

(iv) $\varphi_n(a,b,c)_{\gamma+\zeta,\beta}$

$$\begin{aligned}
&= d_n\big(a(\gamma+\zeta)b\beta c\big)-\sum_{i+j+k=n} d_i(a)(\gamma+\zeta)d_j(b)\beta d_k(c)\\
&= d_n\big(a\gamma b\beta c+a\zeta b\beta c\big)-\sum_{i+j+k=n}\Big(d_i(a)\gamma d_j(b)\beta d_k(c)+d_i(a)\zeta d_j(b)\beta d_k(c)\Big)
\end{aligned}$$

$$= \left(d_n(a\gamma b\beta c) - \sum_{i+j+k=n} d_i(a)\gamma d_j(b)\beta d_k(c)\right)$$

$$+ \left(d_n(a\zeta b\beta c) - \sum_{i+j+k=n} d_i(a)\zeta d_j(b)\beta d_k(c)\right)$$

$$= \varphi_n(a,b,c)_{\gamma,\beta} + \varphi_n(a,b,c)_{\zeta,\beta}.$$

(v) is obtained similary to *(iv)*. □

Proposition 2.3.3. *For each $a, b, c, x \in M$ and $\gamma, \beta, \zeta \in \Gamma$,*

(i) $[a+x, b, c]_{\gamma,\beta} = [a,b,c]_{\gamma,\beta} + [x,b,c]_{\gamma,\beta}$,

(ii) $[a, b+x, c]_{\gamma,\beta} = [a,b,c]_{\gamma,\beta} + [a,x,c]_{\gamma,\beta}$,

(iii) $[a, b, c+x]_{\gamma,\beta} = [a,b,c]_{\gamma,\beta} + [a,b,x]_{\gamma,\beta}$,

(iv) $[a,b,c]_{\gamma+\zeta,\beta} = [a,b,c]_{\gamma,\beta} + [a,b,c]_{\zeta,\beta}$,

(v) $[a,b,c]_{\gamma,\beta+\zeta} = [a,b,c]_{\gamma,\beta} + [a,b,c]_{\gamma,\zeta}$.

Proof. Let $a, b, c, x \in M$ and $\gamma, \beta, \zeta \in \Gamma$.

(i) $[a+x, b, c]_{\gamma,\beta}$

$$= (a+x)\gamma b\beta c - c\gamma b\beta(a+x)$$

$$= \big(a\gamma b\beta c + x\gamma b\beta c\big) - \big(c\gamma b\beta a + c\gamma b\beta x\big)$$

$$= \big(a\gamma b\beta c - c\gamma b\beta a\big) + \big(x\gamma b\beta c - c\gamma b\beta x\big)$$

$$= [a,b,c]_{\gamma,\beta} + [x,b,c]_{\gamma,\beta}$$

(ii) and *(iii)* are obtained similary to *(i)*.

(iv) $[a,b,c]_{\gamma+\zeta,\beta}$

$$= a\big(\gamma+\zeta\big)b\beta c - c\big(\gamma+\zeta\big)b\beta a$$

$$= \big(a\gamma b\beta c + a\zeta b\beta c\big) - \big(c\gamma b\beta a + c\zeta b\beta a\big)$$

$$= \big(a\gamma b\beta c - \gamma b\beta a\big) + \big(a\zeta b\beta c + c\zeta b\beta a\big)$$

$$= [a,b,c]_{\gamma,\beta} + [a,b,c]_{\zeta,\beta}.$$

(v) is obtained similary to *(iv)*. □

Next proposition provides results needed to prove our main theorem.

Proposition 2.3.4. *For each* $a, b, c \in M, \gamma, \beta \in \Gamma$ *and* $n \in \mathbb{N}_0$,

(i) $\varphi_n(a,b,c)_{\gamma,\beta} = -\varphi_n(c,b,a)_{\gamma,\beta}$,

(ii) $[a,b,c]_{\gamma,\beta} = -[c,b,a]_{\gamma,\beta}$,

(iii) $2\varphi_n(a,b,c)_{\gamma,\beta} = d_n\big([a,b,c]_{\gamma,\beta}\big) + \sum_{i+j+k=n}[d_i(c), d_j(b), d_k(a)]_{\gamma,\beta}$.

Proof. Let $a, b, c \in M, \gamma, \beta \in \Gamma$ and $n \in \mathbb{N}_0$.

(i) Applying Proposition 2.3.2 *(i)* and *(iii)* and the fact that $\varphi_n(x,y,x)_{\gamma,\beta} = 0$ for all $x, y \in M$ gives

$$\begin{aligned}
0 &= \varphi_n\Big((a+c), b, (a+c)\Big)_{\gamma,\beta} \\
&= \varphi_n(a,b,a)_{\gamma,\beta} + \varphi_n(a,b,c)_{\gamma,\beta} + \varphi_n(c,b,a)_{\gamma,\beta} + \varphi_n(c,b,c)_{\gamma,\beta} \\
&= 0 + \varphi_n(a,b,c)_{\gamma,\beta} + \varphi_n(c,b,a)_{\gamma,\beta} + 0 \\
&= \varphi_n(a,b,c)_{\gamma,\beta} + \varphi_n(c,b,a)_{\gamma,\beta}.
\end{aligned}$$

Thus $\varphi_n(a,b,c)_{\gamma,\beta} = -\varphi_n(c,b,a)_{\gamma,\beta}$.

(ii) is straightforward.

(iii) We obtain from *(i)* that

$$\begin{aligned}
2\varphi_n(a,b,c)_{\gamma,\beta} &= \varphi_n(a,b,c)_{\gamma,\beta} + \varphi_n(a,b,c)_{\gamma,\beta} \\
&= \varphi_n(a,b,c)_{\gamma,\beta} - \varphi_n(c,b,a)_{\gamma,\beta} \\
&= \Big(d_n(a\gamma b\beta c) - \sum_{i+j+k=n} d_i(a)\gamma d_j(b)\beta d_k(c)\Big) \\
&\quad - \Big(d_n(c\gamma b\beta a) - \sum_{i+j+k=n} d_i(c)\gamma d_j(b)\beta d_k(a)\Big) \\
&= d_n(a\gamma b\beta c - c\gamma b\beta a) + \sum_{i+j+k=n}\Big(d_i(c)\gamma d_j(b)\beta d_k(a) - d_i(a)\gamma d_j(b)\beta d_k(c)\Big) \\
&= d_n\big([a,b,c]_{\gamma,\beta}\big) + \sum_{i+j+k=n}[d_i(c), d_j(b), d_k(a)]_{\gamma,\beta}.
\end{aligned}$$

$\square$

The following series of propositions are main ideas for our main theorem in this chapter. We first show that

$$\varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma[a,b,c]_{\gamma,\beta}+[a,b,c]_{\gamma,\beta}\delta r\sigma\varphi_n(a,b,c)_{\gamma,\beta}=0.$$

Then this result is extended to the case that the elements a,b,c appeared in the terms $\varphi_n(a,b,c)_{\gamma,\beta}$ and $[a,b,c]_{\gamma,\beta}$, in fact, need not be the same.

Proposition 2.3.5. *Let $n\in\mathbb{N}_0$. If $\varphi_m(a,b,c)_{\gamma,\beta}=0$ for all $a,b,c\in M$, $\gamma,\beta\in\Gamma$ and $m<n$, then*

$$\varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma[a,b,c]_{\gamma,\beta}+[a,b,c]_{\gamma,\beta}\delta r\sigma\varphi_n(a,b,c)_{\gamma,\beta}=0$$

for all $a,b,c,r\in M$ and $\gamma,\beta,\delta,\sigma\in\Gamma$.

Proof. Assume that $\varphi_m(a,b,c)_{\gamma,\beta}=0$ for all $a,b,c\in M$, $\gamma,\beta\in\Gamma$ and $m<n$. Let $a,b,c,r\in M$ and $\gamma,\beta,\delta,\sigma\in\Gamma$. Since $D=(d_i)_{i\in\mathbb{N}_0}$ is a JTHD,

$$\begin{aligned}
&d_n\Big(a\gamma\big(b\beta c\delta r\sigma c\gamma b\big)\beta a+c\gamma\big(b\beta a\delta r\sigma a\gamma b\big)\beta c\Big)\\
&\quad=d_n\Big(a\gamma\big(b\beta c\delta r\sigma c\gamma b\big)\beta a\Big)+d_n\Big(c\gamma\big(b\beta a\delta r\sigma a\gamma b\big)\beta c\Big)\\
&\quad=\sum_{i+j+k=n}d_i(a)\gamma d_j\big(b\beta c\delta r\sigma c\gamma b\big)\beta d_k(a)+\sum_{i+j+k=n}d_i(c)\gamma d_j\big(b\beta a\delta r\sigma a\gamma b\big)\beta d_k(c)\\
&\quad=\sum_{i+j+k=n}d_i(a)\gamma\Bigg(\sum_{p+q+l=j}d_p(b)\beta d_q\big(c\delta r\sigma c\big)\gamma d_l(b)\Bigg)\beta d_k(a)\\
&\qquad+\sum_{i+j+k=n}d_i(c)\gamma\Bigg(\sum_{p+q+l=j}d_p(b)\beta d_q\big(a\delta r\sigma a\big)\gamma d_l(b)\Bigg)\beta d_k(a)\\
&\quad=\sum_{i+p+q+l+k=n}d_i(a)\gamma d_p(b)\beta\Bigg(\sum_{s+t+z=q}d_s(c)\delta d_t(r)\sigma d_z(c)\Bigg)\gamma d_l(b)\beta d_k(a)\\
&\qquad+\sum_{i+p+q+l+k=n}d_i(c)\gamma d_p(b)\beta\Bigg(\sum_{s+t+z=q}d_s(a)\delta d_t(r)\sigma d_z(a)\Bigg)\gamma d_l(b)\beta d_k(a)
\end{aligned}$$

$$= \sum_{i+p+s+t+z+l+k=n} d_i(a)\gamma d_p(b)\beta d_s(c)\delta d_t(r)\sigma d_z(c)\gamma d_l(b)\beta d_k(a)$$
$$+ \sum_{i+p+s+t+z+l+k=n} d_i(c)\gamma d_p(b)\beta d_s(a)\delta d_t(r)\sigma d_z(a)\gamma d_l(b)\beta d_k(a). \quad (1)$$

On the other hand, it follows from Proposition 2.3.1 that

$$d_n\Big((a\gamma b\beta c)\delta r\sigma(c\gamma b\beta a) + (c\gamma b\beta a)\delta r\sigma(a\gamma b\beta c)\Big)$$
$$= \sum_{u+v+w=n} \Big(d_u(a\gamma b\beta c)\delta d_v(r)\sigma d_w(c\gamma b\beta a) + d_u(c\gamma b\beta a)\delta d_v(r)\sigma d_w(a\gamma b\beta c)\Big).$$

Let

$$x = \sum_{u+v+w=n} d_u(a\gamma b\beta c)\delta d_v(r)\sigma d_w(c\gamma b\beta a),$$
$$y = \sum_{u+v+w=n} d_u(c\gamma b\beta a)\delta d_v(r)\sigma d_w(a\gamma b\beta c) \qquad \text{and}$$
$$g = (a\gamma b\beta c)\delta r\sigma(c\gamma b\beta a) + (c\gamma b\beta a)\delta r\sigma(a\gamma b\beta c).$$

Then $d_n(g) = x + y$ and we obtain from (1) that

$$\sum_{i+p+s+t+z+l+k=n} d_i(a)\gamma d_p(b)\beta d_s(c)\delta d_t(r)\sigma d_z(c)\gamma d_l(b)\beta d_k(a) - x$$
$$= -\Big(\sum_{i+p+s+t+z+l+k=n} d_i(c)\gamma d_p(b)\beta d_s(a)\delta d_t(r)\sigma d_z(a)\gamma d_l(b)\beta d_k(a) - y\Big). \quad (2)$$

Note that, for each $m < n$, $\varphi_m(a,b,c)_{\gamma,\beta} = 0$ and $\varphi_m(c,b,a)_{\gamma,\beta} = 0$, i.e.,

$$d_m(a\gamma b\beta c) = \sum_{i+j+k=m} d_i(a)\gamma d_j(b)\beta d_k(c) \qquad \text{and}$$
$$d_m(c\gamma b\beta a) = \sum_{i+j+k=m} d_i(c)\gamma d_j(b)\beta d_k(a). \quad (3)$$

Thus

$$
\begin{aligned}
x &= d_n(a\gamma b\beta c)\delta d_0(r)\sigma d_0(c\gamma b\beta a) + \sum_{\substack{u+v+w=n\\ u\neq n, w\neq n}} d_u(a\gamma b\beta c)\delta d_v(r)\sigma d_w(c\gamma b\beta a)\\
&\qquad + d_0(a\gamma b\beta c)\delta d_0(r)\sigma d_n(c\gamma b\beta a)\\
&= d_n(a\gamma b\beta c)\delta r\sigma c\gamma b\beta a + \sum_{\substack{u+v+w=n\\ u\neq n, w\neq n}} d_u(a\gamma b\beta c)\delta d_v(r)\sigma d_w(c\gamma b\beta a)\\
&\qquad + a\gamma b\beta c\delta r\sigma d_n(c\gamma b\beta a).
\end{aligned}
$$

We consider the term $\sum_{\substack{u+v+w=n\\ u\neq n, w\neq n}} d_u(a\gamma b\beta c)\delta d_v(r)\sigma d_w(c\gamma b\beta a)$ of x only. For $u<n$ and $w<n$, from (3)

$$
\begin{aligned}
&\sum_{\substack{u+v+w=n\\ u\neq n, w\neq n}} d_u\big(a\gamma b\beta c\big)\delta d_v(r)\sigma d_w\big(c\gamma b\beta a\big)\\
&\qquad = \sum_{\substack{(i+p+s)+v+(z+l+k)=n\\ i+p+s\neq n, z+l+k\neq n}} \big(d_i(a)\gamma d_p(b)\beta d_s(c)\big)\delta d_v(r)\sigma\big(d_z(c)\gamma d_l(b)\beta d_k(a)\big). \quad (4)
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\sum_{i+p+s+t+z+l+k=n} d_i(a)\gamma d_p(b)\beta d_s(c)\delta d_t(r)\sigma d_z(c)\gamma d_l(b)\beta d_k(a)\\
&\qquad = \sum_{i+p+s=n} d_i(a)\gamma d_p(b)\beta d_s(c)\delta r\sigma c\gamma b\beta a\\
&\qquad\quad + \sum_{\substack{i+p+s+t+z+l+k=n\\ i+p+s\neq n, z+l+k\neq n}} \Big(d_i(a)\gamma d_p(b)\beta d_s(c)\Big)\delta d_t(r)\sigma\Big(d_z(c)\gamma d_l(b)\beta d_k(a)\Big)\\
&\qquad\quad + \sum_{z+l+k=n} a\gamma b\beta c\delta r\sigma d_z(c)\gamma d_l(b)\beta d_k(a).
\end{aligned}
$$

We can see that

$$\sum_{\substack{i+p+s+v+z+l+k=n \\ i+p+s\neq n, z+l+k\neq n}} \Big(d_i(a)\gamma d_p(b)\beta d_s(c)\Big)\delta d_v(r)\sigma\Big(d_z(c)\gamma d_l(b)\beta d_k(a)\Big)$$
$$- \sum_{\substack{u+v+w=n \\ u\neq n, w\neq n}} d_u\big(a\gamma b\beta c\big)\delta d_v(r)\sigma d_w\big(c\gamma b\beta a\big) = 0.$$

Hence

$$\begin{aligned}
&\sum_{i+p+s+v+z+l+k=n} d_i(a)\gamma d_p(b)\beta d_s(c)\delta d_t(r)\sigma d_z(c)\gamma d_l(b)\beta d_k(a) - x \\
&= \left(\sum_{i+p+s=n} d_i(a)\gamma d_p(b)\beta d_s(c)\delta r\sigma c\gamma b\beta a - d_n(a\gamma b\beta c)\delta r\sigma c\gamma b\beta a\right) \\
&\quad + \left(\sum_{z+l+k=n} a\gamma b\beta c\delta r\sigma d_z(c)\gamma d_l(b)\beta d_k(a) - a\gamma b\beta c\delta r\sigma d_n(c\gamma b\beta a)\right) \\
&= \left(\sum_{i+p+s=n} d_i(a)\gamma d_p(b)\beta d_s(c) - d_n(a\gamma b\beta c)\right)\delta r\sigma c\gamma b\beta a \\
&\quad + a\gamma b\beta c\delta r\sigma\left(\sum_{z+l+k=n} d_z(c)\gamma d_l(b)\beta d_k(a) - d_n(c\gamma b\beta a)\right) \\
&= -\varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma c\gamma b\beta a - a\gamma b\beta c\delta r\sigma\varphi_n(c,b,a)_{\gamma,\beta}. \qquad (5)
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\sum_{i+p+s+t+z+l+k=n} d_i(c)\gamma d_p(b)\beta d_s(a)\delta d_t(r)\sigma d_z(a)\gamma d_l(b)\beta d_k(a) - y \\
&= -\varphi_n(c,b,a)_{\gamma,\beta}\delta r\sigma a\gamma b\beta c - c\gamma b\beta a\delta r\sigma\varphi_n(a,b,c)_{\gamma,\beta}. \qquad (6)
\end{aligned}$$

Recall that $d_n(g) = x + y$. From (2),(5) and (6), we have

$$\begin{aligned}
0 &= \Big(-\varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma c\gamma b\beta a - a\gamma b\beta c\delta r\sigma\varphi_n(c,b,a)_{\gamma,\beta}\Big) \\
&\quad + \Big(-\varphi_n(c,b,a)_{\gamma,\beta}\delta r\sigma a\gamma b\beta c - c\gamma b\beta a\delta r\sigma\varphi_n(a,b,c)_{\gamma,\beta}\Big)
\end{aligned}$$

$$= -\Big(\varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma c\gamma b\beta a + \varphi_n(c,b,a)_{\gamma,\beta}\delta r\sigma a\gamma b\beta c\Big)$$
$$- \Big(a\gamma b\beta c\delta r\sigma\varphi_n(c,b,a)_{\gamma,\beta} + c\gamma b\beta a\delta r\sigma\varphi_n(a,b,c)_{\gamma,\beta}\Big)$$
$$= -\Big(\varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma c\gamma b\beta a - \varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma a\gamma b\beta c\Big)$$
$$- \Big(-a\gamma b\beta c\delta r\sigma\varphi_n(a,b,c)_{\gamma,\beta} + c\gamma b\beta a\delta r\sigma\varphi_n(a,b,c)_{\gamma,\beta}\Big) \quad \text{(by Proposition 2.3.4)}$$
$$= \varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma\Big(a\gamma b\beta c - c\gamma b\beta a\Big) + \Big(a\gamma b\beta c - c\gamma b\beta a\Big)\delta r\sigma\varphi_n(a,b,c)_{\gamma,\beta}$$
$$= \varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma[a,b,c]_{\gamma,\beta} + [a,b,c]_{\gamma,\beta}\delta r\sigma\varphi_n(a,b,c)_{\gamma,\beta}.$$

□

Proposition 2.3.6. *Let M be 2-torsion free semiprime Γ-ring and $n \in \mathbb{N}_0$. If the element $\varphi_m(a,b,c)_{\gamma,\beta} = 0$ for all $a,b,c \in M$, $\gamma,\beta \in \Gamma$ and $m < n$, then*

$$\varphi_n(a_1,b_1,c_1)_{\gamma_1,\beta_1}\delta r\sigma[a_2,b_2,c_2]_{\gamma_2,\beta_2} = [a_2,b_2,c_2]_{\gamma_2,\beta_2}\delta r\sigma\varphi_n(a_1,b_1,c_1)_{\gamma_1,\beta_1} = 0$$

for all $a_i,b_i,c_i,r \in M$ and $\gamma_i,\beta_i,\delta,\sigma \in \Gamma$ for all $i = 1,2$.

Proof. Assume that $\varphi_m(a,b,c)_{\gamma,\beta} = 0$ for all $a,b,c \in M, \gamma,\beta \in \Gamma$ and $m < n$. Proposition 2.3.5 yields

$$\varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma[a,b,c]_{\gamma,\beta} + [a,b,c]_{\gamma,\beta}\delta r\sigma\varphi_n(a,b,c)_{\gamma,\beta} = 0$$

for all $a,b,c,r \in M$ and $\gamma,\beta,\delta,\sigma \in \Gamma$. Since M is 2-torsion free semiprime, by Proposition 2.1.5,

$$\varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma[a,b,c]_{\gamma,\beta} = [a,b,c]_{\gamma,\beta}\delta r\sigma\varphi_n(a,b,c)_{\gamma,\beta} = 0 \qquad (1)$$

for all $a,b,c,r \in M$ and $\gamma,\beta,\delta,\sigma \in \Gamma$. Let $S,T : M\times\Gamma\times M\times\Gamma\times M \to M$ be defined by

$$S(a,\gamma,b,\beta,c) = \varphi_n(a,b,c)_{\gamma,\beta} \quad \text{and} \quad T(a,\gamma,b,\beta,c) = [a,b,c]_{\gamma,\beta}$$

for all $a, b, c \in M$ and$\gamma, \beta \in \Gamma$. Proposition 2.3.2 and Proposition 2.3.3 show that both S and T are additive in each argument. Moreover, (1) provides that $S(a, \gamma, b, \beta, c)\delta r\sigma T(a, \gamma, b, \beta, c) = 0$ for all $a, b, c, r \in M$ and $\gamma, \beta, \delta, \sigma \in \Gamma$. It, then, follows from Proposition 2.1.4 that

$$S(a_1, \gamma_1, b_1, \beta_1, c_1)\delta r\sigma T(a_2, \gamma_2, b_2, \beta_2, c_2) = 0,$$

i.e.,

$$\varphi_n(a_1, b_1, c_1)_{\gamma_1,\beta_1}\delta r\sigma[a_2, b_2, c_2]_{\gamma_2,\beta_2} = 0$$

for all $a_i, b_i, c_i, r \in M$ and $\gamma_i, \beta_i, \delta, \sigma \in \Gamma$ for all $i = 1, 2$. Similarly, we obtain

$$[a_2, b_2, c_2]_{\gamma_2,\beta_2}\delta r\sigma\varphi_n(a_1, b_1, c_1)_{\gamma_1,\beta_1} = 0$$

for all $a_i, b_i, c_i, r \in M$ and $\gamma_i, \beta_i, \delta, \sigma \in \Gamma$ for all $i = 1, 2$. □

Next proposition shows that the element of the form $[a, b, c]_{\gamma,\beta}$ in the result of Proposition 2.3.6 is able to be replaced by the element of the form $d_m([a, b, c]_{\gamma,\beta})$. However, the new result is slightly different.

Proposition 2.3.7. *Assume that M is a 2-torsion free semiprime Γ-ring and $n \in \mathbb{N}_0$. If $\varphi_m(a, b, c)_{\gamma,\beta} = 0$ for all $a, b, c \in M$, $\gamma, \beta \in \Gamma$ and $m < n$, then*

$$\varphi_n(a, b, c)_{\gamma,\beta}\delta r\sigma d_m([a, b, c]_{\gamma,\beta}) = d_m([a, b, c]_{\gamma,\beta})\delta r\sigma\varphi_n(a, b, c)_{\gamma,\beta} = 0$$

for all $a, b, c, r \in M$, $\gamma, \beta, \delta, \sigma \in \Gamma$ and $m < n$.

Proof. Assume that $\varphi_m(a, b, c)_{\gamma,\beta} = 0$, i.e.,

$$d_m(a\gamma b\beta c) = \sum_{i+j+k=n} d_i(a)\gamma d_j(b)\beta d_k(c)$$

for all $a, b, c \in M$, $\gamma, \beta \in \Gamma$ and $m < n$. Let $a, b, c, r \in M$, $\gamma, \beta, \delta, \sigma \in \Gamma$ and $m < n$. Then

$$\begin{aligned}
&\varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma d_m([a,b,c]_{\gamma,\beta}) \\
&\quad = \varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma d_m(a\gamma b\beta c - c\gamma b\beta a) \\
&\quad = \varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma \big(d_m(a\gamma b\beta c) - d_m(c\gamma b\beta a)\big) \\
&\quad = \varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma \left(\sum_{i+j+k=m} \Big(d_i(a)\gamma d_j(b)\beta d_k(c) - d_i(c)\gamma d_j(b)\beta d_k(a)\Big)\right) \\
&\quad = \varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma \sum_{i+j+k=m} \big[d_i(a), d_j(b), d_k(c)\big]_{\gamma,\beta} \\
&\quad = \sum_{i+j+k=m} \varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma \big[d_i(a), d_j(b), d_k(c)\big]_{\gamma,\beta} \\
&\quad = 0
\end{aligned}$$

where the last equation holds because of Proposition 2.3.6. □

For any Γ-ring M, we know that

$$\varphi_n(a,b,a)_{\gamma,\beta} = 0 \qquad \text{for all } a, b \in M, \gamma, \beta \in \Gamma \text{ and } n \in \mathbb{N}_0.$$

Recall that our aim here is to generalize the above result to

$$\varphi_n(a,b,c)_{\gamma,\beta} = 0 \qquad \text{for all } a, b, c \in M, \gamma, \beta \in \Gamma \text{ and } n \in \mathbb{N}_0.$$

To acheive that, Lemma 2.3.8 and Lemma 2.3.9 are required.

Lemma 2.3.8. *Let M be a 2-torsion free semiprime Γ-ring. For each $n \in \mathbb{N}_0$, if $\varphi_m(a,b,c)_{\gamma,\beta} = 0$ for all $a, b, c \in M, \gamma, \beta \in \Gamma$ and $m < n$, then*

$$\varphi_n(a,b,c)_{\gamma,\beta}\zeta s\alpha\Big(\varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma d_n([a,b,c]_{\gamma,\beta})\Big)\zeta s\alpha\varphi_n(a,b,c)_{\gamma,\beta}$$
$$+\varphi_n(a,b,c)_{\gamma,\beta}\zeta s\alpha\Big(d_n([a,b,c]_{\gamma,\beta})\delta r\sigma\varphi_n(a,b,c)_{\gamma,\beta}\Big)\zeta s\alpha\varphi_n(a,b,c)_{\gamma,\beta}=0$$

for all $a,b,c,r,s\in M$ *and* $\gamma,\beta,\zeta,\alpha,\delta,\sigma\in\Gamma$.

Proof. We prove by induction on $n\in\mathbb{N}_0$. Note that the statement holds if $n=0$. Let $n=1$ and assume that

$$\varphi_m(a,b,c)_{\gamma,\beta}=0 \qquad \text{for all } a,b,c\in M,\gamma,\beta\in\Gamma \text{ and } m<1.$$

It means that $\varphi_0(a,b,c)_{\gamma,\beta}=0$, i.e., $d_0(a\gamma b\beta c)=a\gamma b\beta c$ for all $a,b,c\in M$ and $\gamma,\beta\in\Gamma$. As a result, the statement holds.

For the induction step, assume that

$$\varphi_m(a,b,c)_{\gamma,\beta}=0 \qquad \text{for all } a,b,c\in M,\gamma,\beta\in\Gamma \text{ and } m<n.$$

Let $a,b,c,r,s\in M$ and $\gamma,\beta,\zeta,\alpha,\delta,\sigma\in\Gamma$. Then Proposition 2.3.5 provides

$$\varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma[a,b,c]_{\gamma,\beta}+[a,b,c]_{\gamma,\beta}\delta r\sigma\varphi_n(a,b,c)_{\gamma,\beta}=0. \tag{1}$$

Note that $d_n(0)=0$. We apply Proposition 2.3.1 to (1) and obtain

$$\begin{aligned}0&=d_n\Big(\varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma[a,b,c]_{\gamma,\beta}+[a,b,c]_{\gamma,\beta}\delta r\sigma\varphi_n(a,b,c)_{\gamma,\beta}\Big)\\&=\sum_{i+j+k=n}\Bigg(d_i\big(\varphi_n(a,b,c)_{\gamma,\beta}\big)\delta d_j(r)\sigma d_k\big([a,b,c]_{\gamma,\beta}\big)\\&\qquad+d_i\big([a,b,c]_{\gamma,\beta}\big)\delta d_j(r)\sigma d_k\big(\varphi_n(a,b,c)_{\gamma,\beta}\big)\Bigg).\end{aligned}$$

Moreover,

$$0 = \varphi_n(a,b,c)_{\gamma,\beta}\zeta s\alpha\Bigg[\sum_{i+j+k=n}\Bigg(d_i\big(\varphi_n(a,b,c)_{\gamma,\beta}\big)\delta d_j(r)\sigma d_k\big([a,b,c]_{\gamma,\beta}\big)$$

$$+ d_i\big([a,b,c]_{\gamma,\beta}\big)\delta d_j(r)\sigma d_k\big(\varphi_n(a,b,c)_{\gamma,\beta}\big)\Bigg)\Bigg]\zeta s\alpha\varphi_n(a,b,c)_{\gamma,\beta}$$

$$= \sum_{i+j+k=n}\varphi_n(a,b,c)_{\gamma,\beta}\zeta s\alpha\Bigg(d_i\big(\varphi_n(a,b,c)_{\gamma,\beta}\big)\delta d_j(r)\sigma d_k\big([a,b,c]_{\gamma,\beta}\big)\Bigg)\zeta s\alpha\varphi_n(a,b,c)_{\gamma,\beta}$$

$$+ \sum_{i+j+k=n}\varphi_n(a,b,c)_{\gamma,\beta}\zeta s\alpha\Bigg(d_i\big([a,b,c]_{\gamma,\beta}\big)\delta d_j(r)\sigma d_k\big(\varphi_n(a,b,c)_{\gamma,\beta}\big)\Bigg)\zeta s\alpha\varphi_n(a,b,c)_{\gamma,\beta}$$

$$= \sum_{i+j+k=n}\varphi_n(a,b,c)_{\gamma,\beta}\zeta s\alpha d_i\big(\varphi_n(a,b,c)_{\gamma,\beta}\big)\delta d_j(r)\sigma\Bigg(d_k\big([a,b,c]_{\gamma,\beta}\big)\zeta s\alpha\varphi_n(a,b,c)_{\gamma,\beta}\Bigg)$$

$$+ \sum_{i+j+k=n}\Bigg(\varphi_n(a,b,c)_{\gamma,\beta}\zeta s\alpha d_i\big([a,b,c]_{\gamma,\beta}\big)\Bigg)\delta d_j(r)\sigma d_k\big(\varphi_n(a,b,c)_{\gamma,\beta}\big)\zeta s\alpha\varphi_n(a,b,c)_{\gamma,\beta}$$

$$= \Bigg[\sum_{\substack{(i+j)+k=n \\ k\neq n}}\varphi_n(a,b,c)_{\gamma,\beta}\zeta s\alpha d_i\big(\varphi_n(a,b,c)_{\gamma,\beta}\big)\delta d_j(r)\sigma\Bigg(d_k\big([a,b,c]_{\gamma,\beta}\big)\zeta s\alpha\varphi_n(a,b,c)_{\gamma,\beta}\Bigg)$$

$$+ \varphi_n(a,b,c)_{\gamma,\beta}\zeta s\alpha d_0\big(\varphi_n(a,b,c)_{\gamma,\beta}\big)\delta d_0(r)\sigma\Bigg(d_n\big([a,b,c]_{\gamma,\beta}\big)\zeta s\alpha\varphi_n(a,b,c)_{\gamma,\beta}\Bigg)\Bigg]$$

$$+ \Bigg[\Bigg(\varphi_n(a,b,c)_{\gamma,\beta}\zeta s\alpha d_n\big([a,b,c]_{\gamma,\beta}\big)\Bigg)\delta d_0(r)\sigma d_0\big(\varphi_n(a,b,c)_{\gamma,\beta}\big)\zeta s\alpha\varphi_n(a,b,c)_{\gamma,\beta}$$

$$+ \sum_{\substack{i+(j+k)=n \\ i\neq n}}\Bigg(\varphi_n(a,b,c)_{\gamma,\beta}\zeta s\alpha d_i\big([a,b,c]_{\gamma,\beta}\big)\Bigg)\delta d_j(r)\sigma d_k\big(\varphi_n(a,b,c)_{\gamma,\beta}\big)\zeta s\alpha\varphi_n(a,b,c)_{\gamma,\beta}\Bigg].$$

Note that d_0 is the identity map and from Proposition 2.3.7, we obtain

$$d_k([a,b,c]_{\gamma,\beta})\zeta s\delta\varphi_n(a,b,c)_{\gamma,\beta} = 0 = \varphi_n(a,b,c)_{\gamma,\beta}\zeta s\delta d_i([a,b,c]_{\gamma,\beta})$$

for all $k < n$ and $i < n$. Consequently,

$$0 = \varphi_n(a,b,c)_{\gamma,\beta}\zeta s\alpha\varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma\Big(d_n\big([a,b,c]_{\gamma,\beta}\big)\zeta s\alpha\varphi_n(a,b,c)_{\gamma,\beta}\Big)$$
$$+\Big(\varphi_n(a,b,c)_{\gamma,\beta}\zeta s\alpha d_n\big([a,b,c]_{\gamma,\beta}\big)\Big)\delta r\sigma\varphi_n(a,b,c)_{\gamma,\beta}\zeta s\alpha\varphi_n(a,b,c)_{\gamma,\beta}.$$

Hence the statement holds as desired. □

Lemma 2.3.9. *Let M be a 2-torsion free semiprime Γ-ring and $n \in \mathbb{N}_0$. If the element $\varphi_m(a,b,c)_{\gamma,\beta} = 0$ for all $a,b,c \in M, \gamma,\beta \in \gamma$ and $m < n$, then*

$$\begin{aligned} 2\varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma\varphi_n(a,b,c)_{\gamma,\beta} &= \varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma d_n([a,b,c]_{\gamma,\beta}) \\ &= d_n([a,b,c]_{\gamma,\beta})\delta r\sigma\varphi_n(a,b,c)_{\gamma,\beta} \end{aligned}$$

for all $a,b,c,r \in M$ and $\gamma,\beta,\delta,\sigma \in \Gamma$.

Proof. Let $a,b,c,r \in M$ and $\gamma,\beta,\delta,\sigma \in \Gamma$. Proposition 2.3.4 (ii) gives

$$\begin{aligned} &2\varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma\varphi_n(a,b,c)_{\gamma,\beta} \\ &= \varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma\Big(2\varphi_n(a,b,c)_{\gamma,\beta}\Big) \\ &= \varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma\Big(d_n\big([a,b,c]_{\gamma,\beta}\big) + \sum_{i+j+k=n}[d_i(c),d_j(b),d_k(a)]_{\gamma,\beta}\Big) \\ &= \varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma d_n\big([a,b,c]_{\gamma,\beta}\big) + \sum_{i+j+k=n}\varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma[d_i(c),d_j(b),d_k(a)]_{\gamma,\beta}. \end{aligned}$$

Moreover, Proposition 2.3.6 yields

$$\sum_{i+j+k=n}\varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma[d_i(c),d_j(b),d_k(a)]_{\gamma,\beta} = 0.$$

Hence

$$2\varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma\varphi_n(a,b,c)_{\gamma,\beta} = \varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma d_n([a,b,c]_{\gamma,\beta})$$

as desired. Similarly, the other result follows. □

Lemma 2.3.10. *Let M be a 2-torsion free semiprime Γ-ring. Then $\varphi_n(a,b,c)_{\gamma,\beta} = 0$ for all $a,b,c \in M, \gamma,\beta \in \Gamma$ and $n \in \mathbb{N}_0$.*

Proof. We show this by induction on n. This is clear for the case $n = 0$. For the induction step, assume that $\varphi_m(a,b,c)_{\gamma,\beta} = 0$ for all $a,b,c \in M, \gamma,\beta \in \Gamma$ and $m < n$. Lemma 2.3.8 and Lemma 2.3.9 show that, for all $a,b,c,r,s \in M$ and γ, $\beta,\zeta,\alpha,\delta,\sigma \in \Gamma$,

$$\begin{aligned} 0 &= \varphi_n(a,b,c)_{\gamma,\beta}\zeta s\alpha\Big(2\varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma\varphi_n(a,b,c)_{\gamma,\beta}\Big)\zeta s\alpha\varphi_n(a,b,c)_{\gamma,\beta} \\ &\quad + \varphi_n(a,b,c)_{\gamma,\beta}\zeta s\alpha\Big(2\varphi_n(a,b,c)_{\gamma,\beta}\delta r\sigma\varphi_n(a,b,c)_{\gamma,\beta}\Big)\zeta s\alpha\varphi_n(a,b,c)_{\gamma,\beta} \\ &= 4\Bigg(\Big(\varphi_n(a,b,c)_{\gamma,\beta}\zeta s\alpha\varphi_n(a,b,c)_{\gamma,\beta}\Big)\delta r\sigma\Big(\varphi_n(a,b,c)_{\gamma,\beta}\zeta s\alpha\varphi_n(a,b,c)_{\gamma,\beta}\Big)\Bigg). \end{aligned}$$

Since M is 2-torsion free,

$$\Big(\varphi_n(a,b,c)_{\gamma,\beta}\zeta s\alpha\varphi_n(a,b,c)_{\gamma,\beta}\Big)\delta r\sigma\Big(\varphi_n(a,b,c)_{\gamma,\beta}\zeta s\alpha\varphi_n(a,b,c)_{\gamma,\beta}\Big) = 0$$

for all $a,b,c,r,s \in M$ and $\gamma,\beta,\zeta,\alpha,\delta,\sigma \in \Gamma$. Note that r,δ,σ are arbitary and M is semiprime,

$$\varphi_n(a,b,c)_{\gamma,\beta}\zeta s\alpha\varphi_n(a,b,c)_{\gamma,\beta} = 0$$

for all $a,b,c,s \in M$ and $\gamma,\beta,\zeta,\alpha \in \Gamma$. Again, because of the semiprimeness of M and the arbitary choices of s,ζ,α,

$$\varphi_n(a,b,c)_{\gamma,\beta} = 0$$

for all $a,b,c \in M$ and $\gamma,\beta \in \Gamma$. □

Corollary 2.3.11. *Assume that M is a 2-torsion free semiprime Γ-ring. Then*

$$d_n(a\gamma b\beta c) = \sum_{i+j+k=n} d_i(a)\gamma d_j(b)\beta d_k(c)$$

for all $a, b, c \in M, \gamma, \beta \in \Gamma$ *and* $n \in \mathbb{N}_0$.

Now we are ready to prove the main result. Nevertheless, the following theorem is required. This theorem provides that for a certain Γ-ring, its JTHDs are HDs.

Theorem 2.3.12. *Let* M *be a 2-torsion free semiprime* Γ*-ring. Then every Jordan triple higher derivation of* M *is a higher derivation of* M.

Proof. Let $D = (d_i)_{i \in \mathbb{N}_0}$ be a JTHD of M. We apply the induction on n in order to prove that $d_n(a\gamma b) = \sum_{i+j=n} d_i(a)\gamma d_j(b)$ for all $a, b \in M, \gamma \in \Gamma$ and $n \in \mathbb{N}_0$. It is clear for $n = 0$.

Assume for the induction step that $d_m(a\gamma b) = \sum_{i+j=m} d_i(a)\gamma d_j(b)$ for all $a, b \in M$, $\gamma \in \Gamma$ and $m < n$. Let $a, b, x \in M$ and $\gamma, \delta, \sigma \in \Gamma$. Replacing a by $a\gamma b$ in the definition of a JTHD yields

$$d_n\big((a\gamma b)\delta x\sigma(a\gamma b)\big) = \sum_{i+j+k=n} d_i(a\gamma b)\delta d_j(x)\sigma d_k(a\gamma b)$$

and applying Corollary 2.3.11 obtains

$$\begin{aligned} d_n\big((a\gamma b)\delta x\sigma(a\gamma b)\big) &= d_n\big(a\gamma(b\delta x\sigma a)\gamma b\big) \\ &= \sum_{p+r+v=n} d_p(a)\gamma d_r(b\delta x\sigma a)\gamma d_v(b) \\ &= \sum_{p+r+v=n} d_p(a)\gamma\Big(\sum_{q+s+t=r} d_q(b)\delta d_s(x)\sigma d_t(a)\Big)\gamma d_v(b) \\ &= \sum_{p+q+s+t+v=n} d_p(a)\gamma d_q(b)\delta d_s(x)\sigma d_t(a)\gamma d_v(b). \end{aligned}$$

Then

$$0 = \sum_{i+j+k=n} d_i(a\gamma b)\delta d_j(x)\sigma d_k(a\gamma b) \; - \sum_{p+q+s+t+v=n} d_p(a)\gamma d_q(b)\delta d_s(x)\sigma d_t(a)\gamma d_v(b)$$

$$= \Big(d_n(a\gamma b) \; - \sum_{p+q=n} d_p(a)\gamma d_q(b)\Big)\delta x\sigma a\gamma b$$

$$+ \left(\sum_{\substack{i+j+k=n \\ i\neq n, k\neq n}} d_i(a\gamma b)\delta d_j(x)\sigma d_k(a\gamma b) \;\; - \sum_{\substack{p+q+s+t+v=n \\ p+q\neq n, t+v\neq n}} d_p(a)\gamma d_q(b)\delta d_s(x)\sigma d_t(a)\gamma d_v(b) \right)$$

$$+ \, a\gamma b\delta x\sigma\Big(d_n(a\gamma b) - \sum_{t+v=n} d_t(a)\gamma d_v(b)\Big).$$

The induction hypothesis provides

$$\sum_{\substack{i+j+k=n \\ i\neq n, k\neq n}} d_i(a\gamma b)\delta d_j(x)\sigma d_k(a\gamma b) \;\; - \sum_{\substack{p+q+s+t+v=n \\ p+q\neq n, t+v\neq n}} d_p(a)\gamma d_q(b)\delta d_s(x)\sigma d_t(a)\gamma d_v(b) = 0.$$

Hence

$$0 = \Big(d_n(a\gamma b) - \sum_{p+q=n} d_p(a)\gamma d_q(b)\Big)\delta x\sigma a\gamma b + a\gamma b\delta x\sigma\Big(d_n(a\gamma b) - \sum_{t+v=n} d_t(a)\gamma d_v(b)\Big).$$

Since x, δ, σ are arbitrary, we obtain from Proposition 2.1.5 that

$$\Big(d_n(a\gamma b) - \sum_{p+q=n} d_p(a)\gamma d_q(b)\Big)\delta x\sigma(a\gamma b) = 0$$

for all $x \in M$ and $\delta, \sigma \in \Gamma$. In order to apply Proposition 2.1.4, define mappings $S, T : M \times \Gamma \times M \to \; M$ by

$$S(a_1, \gamma_1, b_1) = d_n(a_1\gamma_1 b_1) - \sum_{p+q=n} d_p(a_1)\gamma_1 d_q(b_1) \quad \text{and} \quad T(a_1, \gamma_1, b_1) = a_1\gamma_1 b_1$$

for all $a_1, b_1 \in M$ and $\gamma_1 \in \Gamma$. It is obvious that S and T satisfy the hypothesis of Proposition 2.1.4 so that

$$\Big(d_n(a_1\gamma_1 b_1) - \sum_{p+q=n} d_p(a_1)\gamma_1 d_q(b_1)\Big)\delta x\sigma(a_2\gamma_2 b_2) = 0$$

for all $a_1, b_1, x, a_2, b_2 \in M$ and $\gamma_1, \gamma_2, \delta, \sigma \in \Gamma$. In particular, for any $x, y \in M$ and

$\beta, \delta, \sigma \in \Gamma$.

$$\begin{aligned} 0 &= \Big(d_n(a\gamma b) - \sum_{p+q=n} d_p(a)\gamma d_q(b)\Big)\delta x\sigma\Big(y\beta\Big(\Big(d_n(a\gamma b) - \sum_{p+q=n} d_p(a)\gamma d_q(b)\Big)\delta x\Big)\Big) \\ &= \Big(\Big(d_n(a\gamma b) - \sum_{p+q=n} d_p(a)\gamma d_q(b)\Big)\delta x\Big)\sigma y\beta\Big(\Big(d_n(a\gamma b) - \sum_{p+q=n} d_p(a)\gamma d_q(b)\Big)\delta x\Big). \end{aligned}$$

Note that y, σ, β are arbitrary and M is semiprime. Hence for each $x \in M$ and $\delta \in \Gamma$, we see that

$$\Big(d_n(a\gamma b) - \sum_{p+q=n} d_p(a)\gamma d_q(b)\Big)\delta x = 0.$$

Especially, for each $c \in M$ and $\delta, \sigma \in \Gamma$

$$\Big(d_n(a\gamma b) - \sum_{p+q=n} d_p(a)\gamma d_q(b)\Big)\delta\Big(c\sigma\Big(d_n(a\gamma b) - \sum_{p+q=n} d_p(a)\gamma d_q(b)\Big)\Big) = 0,$$

i.e.,

$$\Big(d_n(a\gamma b) - \sum_{p+q=n} d_p(a)\gamma d_q(b)\Big)\delta c\sigma\Big(d_n(a\gamma b) - \sum_{p+q=n} d_p(a)\gamma d_q(b)\Big) = 0.$$

Observe, again, that c, δ, σ are abitrary, we can conclude similarly that

$$d_n(a\gamma b) - \sum_{p+q=n} d_p(a)\gamma d_q(b) = 0,$$

i.e.,

$$d_n(a\gamma b) = \sum_{p+q=n} d_p(a)\gamma d_q(b).$$

Then D is a HD. $\square$

Our main theorem is now provided.

Theorem 2.3.13. *Let M be a 2-torsion free semiprime Γ-ring such that $a\gamma b\beta c = a\beta b\gamma c$ for all $a, b, c \in M$ and $\gamma, \beta \in \Gamma$. Then every Jordan higher derivation of M is a higher derivation of M.*

Proof. Theorem 2.2.3 yields that every JHD of M is a JTHD. Theorem 2.3.12 shows that every JTHD of M is a HD. Consequently, the conclusion that every JHD of M is a HD is obtained. $\square$

CHAPTER III

HIGHER DERIVATIONS AND JORDAN HIGHER DERIVATIONS OF IDEALS OF Γ-RINGS

There are two sections in this chapter. The first section provides the definitions of a higher derivation and a Jordan higher derivation of an ideal of a Γ-ring. Moreover, some results regarding ideals of a Γ-ring proved by M. Soyturk in [12] which are related to our work are included in this section. In the second section, we show that a Jordan higher derivation of an ideal of a Γ-ring is a higher derivation. This, in fact, is another main result of our thesis.

3.1 Essential Properties for Higher Derivations and Jordan Higher Derivations of Ideals of Γ-Rings

We begin with giving the definition of a higher derivation and a Jordan higher derivation of an ideal of a Γ-ring.

Definition 3.1.1. Let U be an ideal of a Γ-ring M and $D = (d_i)_{i\in\mathbb{N}_0}$ be a family of additive mappings of M. Then D is said to be a *higher derivation* (HD) of U (into M) if

$$d_n(u\gamma v) = \sum_{i+j=n} d_i(u)\gamma d_j(v) \qquad \text{for all } u, v \in U, \gamma \in \Gamma \text{ and } n \in \mathbb{N}_0;$$

a *Jordan higher derivation* (JHD) of U (into M) if

$$d_n(u\gamma u) = \sum_{i+j=n} d_i(u)\gamma d_j(u) \qquad \text{for all } u \in U, \gamma \in \Gamma \text{ and } n \in \mathbb{N}_0.$$

For a Γ-ring M and an ideal U of M, it is clear that a HD of U is a JHD of U. Let us have a close look at the difference of HDs (JHDs) of M and those of U. For a HD $D = (d_i)_{i\in\mathbb{N}_0}$ of an ideal U of a Γ-ring M, the property that

$$d_n(u\gamma v) = \sum_{i+j=n} d_i(u)\gamma d_j(v) \qquad \text{for all } u, v \in U, \gamma \in \Gamma \text{ and } n \in \mathbb{N}_0$$

holds for arbitary elements of the ideal U but not necessary for the whole M as a HD of M.

Note that an ideal U of a ring R with identity is also an ideal of the Γ-ring R defined in Example 6 where $\Gamma = \mathbb{Z}$. Let R be a ring with identity, U an ideal of R and $\Gamma = \mathbb{Z}$. Then R is a Γ-ring and U is an ideal of the Γ-ring. It can be shown that a HD and a JHD of U into the ring R is a HD and a JHD of U into the Γ-ring R, respectively.

The following propositions found in [12] are major tools for this chapter.

Proposition 3.1.2. *[12] Let U be a nonzero right ideal of M. If $U \subseteq Z(M)$, then M is commutative.*

Proposition 3.1.3. *[12] Let M be a prime Γ-ring, U be a nonzero right (left) ideal of M and $a \in M$. If $U\Gamma a = 0$ $\big(a\Gamma U = 0\big)$, then $a = 0$.*

Proposition 3.1.4. *[12] Let M be a prime Γ-ring, U be a nonzero ideal of M and $a, b \in M$. If $a\Gamma U\Gamma b = 0$, then $a = 0$ or $b = 0$.*

We extend Proposition 2.1.5 to the next proposition.

Proposition 3.1.5. *Let M be a 2-torsion free prime Γ-ring and U be a nonzero ideal of M. If $a, b \in M$ such that $a\gamma w\beta b + b\gamma w\beta a = 0$ for all $w \in U$ and $\gamma, \beta \in \Gamma$, then $a\gamma w\beta b = b\gamma w\beta a = 0$ for all $w \in U$ and $\gamma, \beta \in \Gamma$.*

Proof. Let $a, b \in M$ be such that $a\gamma w\beta b + b\gamma w\beta a = 0$, i.e., $a\gamma w\beta b = -b\gamma w\beta a$ for all $w \in U$ and $\gamma, \beta \in \Gamma$. Moreover, let $w, u \in U$ and $\gamma, \beta, \zeta, \lambda \in \Gamma$. We can see that

$$\begin{aligned}
\big(a\gamma w\beta b\big)\zeta u\lambda\big(a\gamma w\beta b\big) &= \big(-b\gamma w\beta a\big)\zeta u\lambda a\gamma w\beta b \\
&= \Big(-b\gamma\big(w\beta a\zeta u\big)\lambda a\Big)\gamma w\beta b \\
&= \Big(a\gamma\big(w\beta a\zeta u\big)\lambda b\Big)\gamma w\beta b \\
&= a\gamma w\beta\big(a\zeta u\lambda b\big)\gamma w\beta b \\
&= a\gamma w\beta\big(-b\zeta u\lambda a\big)\gamma w\beta b \\
&= -\Big(\big(a\gamma w\beta b\big)\zeta u\lambda\big(a\gamma w\beta b\big)\Big).
\end{aligned}$$

Thus $2\big(a\gamma w\beta b\big)\zeta u\lambda\big(a\gamma w\beta b\big) = 0$ so that $\big(a\gamma w\beta b\big)\zeta u\lambda\big(a\gamma w\beta b\big) = 0$ as M is 2-torsion free. Hence $\big(a\gamma w\beta b\big)\zeta u\lambda\big(a\gamma w\beta b\big) = 0$ for all $u \in U$ and $\zeta, \lambda \in \Gamma$. This implies that $\Big(a\gamma w\beta b\Big)\Gamma U\Gamma\Big(a\gamma w\beta b\Big) = 0$. Proposition 3.1.4 yields $a\gamma w\beta b = 0$ so that $a\gamma w\beta b = b\gamma w\beta a = 0$. $\square$

3.2 The Main Result of JHDs and HDs of Ideals of Γ-Rings

The steps of proofs for the main conclusion follow similarly to those for the main result in Chapter II. Throughout this section, let $D = (d_i)_{i\in\mathbb{N}_0}$ be a JHD of an ideal U into a Γ-ring M.

Let us summarize what we are going to prove where the arbitrary elements whether they belong to U or Γ should be understood from the context. First, we study what $d_n(u\gamma v + v\gamma u)$ and $d_n(u\gamma v\beta u)$ are. Then extend this to $d_n(u\gamma v\beta w + w\gamma v\beta u)$. Next, we set up particular types of elements of M, namely, $\varphi_n(u, v)_\gamma$ and $[a, b]_\gamma$ and prove the crucial results which are

$$\varphi_n(u, v)_\beta\gamma w\zeta[u, v]_\beta + [u, v]_\beta\gamma w\zeta\varphi_n(u, v)_\beta = 0$$

$$\varphi_n(u, v)_\beta\gamma w\zeta[a, b]_\lambda = [a, b]_\lambda\gamma w\zeta\varphi_n(u, v)_\beta = 0.$$

Finally, the main result is obtained.

Proposition 3.2.1. *For each $u, v, w \in U, \gamma \in \Gamma$ and $n \in \mathbb{N}_0$,*

$$d_n(u\gamma v + v\gamma u) = \sum_{i+j=n} \Big(d_i(u)\gamma d_j(v) + d_i(v)\gamma d_j(u) \Big).$$

Proof. The result follows from replacing u by $u+v$ in the definition of a JHD of U into M and applying the similar process in the proof of Proposition 2.2.1. $\square$

Proposition 3.2.2. *Let M be a 2-torsion free Γ-ring and $a\gamma b\beta c = a\beta b\gamma c$ for all $a, b, c \in M$ and $\gamma, \beta \in \Gamma$. Then*

$$d_n(u\gamma v\beta u) = \sum_{i+j+k=n} d_i(u)\gamma d_j(v)\beta d_k(u)$$

for all $u, v \in U, \gamma, \beta \in \Gamma$ and $n \in \mathbb{N}_0$.

Proof. Let $u, v \in U, \gamma, \beta \in \Gamma$ and $n \in \mathbb{N}_0$. We put

$$x = u\gamma(u\beta v + v\beta u) + (u\beta v + v\beta u)\gamma u.$$

By Proposition 3.2.1 and assumption, we obtain that

$$\begin{aligned} d_n(x) &= d_n(u\gamma(u\beta v + v\beta u) + (u\beta v + v\beta u)\gamma u) \\ &= \sum_{i+j=n} \Big(d_i(u)\gamma d_j(u\beta v + v\beta u) + d_i(u\beta v + v\beta u)\gamma d_j(u) \Big) \\ &= \sum_{i+j=n} d_i(u)\gamma \left(\sum_{p+q=j} \Big(d_p(u)\beta d_q(v) + d_p(v)\beta d_q(u) \Big) \right) \\ &\quad + \sum_{i+j=n} \left(\sum_{s+t=i} \Big(d_s(u)\beta d_t(v) + d_s(v)\beta d_t(u) \Big) \right) \gamma d_j(u) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i+j+k=n} d_i(u)\gamma d_j(u)\beta d_k(v) + \sum_{i+j+k=n} d_i(u)\gamma d_j(v)\beta d_k(u) \\
&\quad + \sum_{i+j+k=n} d_i(u)\beta d_j(v)\gamma d_k(u) + \sum_{i+j+k=n} d_i(v)\beta d_j(u)\gamma d_k(u) \\
&= \sum_{i+j+k=n} \Big(d_i(u)\gamma d_j(u)\beta d_k(v) + d_i(v)\beta d_j(u)\gamma d_k(u)\Big) \\
&\quad + \sum_{i+j+k=n} \Big(d_i(u)\gamma d_j(v)\beta d_k(u) + d_i(u)\beta d_j(v)\gamma d_k(u)\Big) \\
&= \sum_{i+j+k=n} \Big(d_i(u)\gamma d_j(u)\beta d_k(v) + d_i(v)\beta d_j(u)\gamma d_k(u)\Big) \\
&\quad + 2\sum_{i+j+k=n} d_i(u)\gamma d_j(v)\beta d_k(u)
\end{aligned}
$$

and

$$
\begin{aligned}
d_n(x) &= d_n(u\gamma(u\beta v + v\beta u) + (u\beta v + v\beta u)\gamma u) \\
&= d_n(u\gamma u\beta v + v\beta u\gamma u) + d_n(u\gamma v\beta u + u\beta v\gamma u) \\
&= d_n\big((u\gamma u)\beta v + v\beta(u\gamma u)\big) + 2d_n(u\gamma v\beta u) \\
&= \sum_{i+j=n} \big(d_i(u\gamma u)\beta d_j(v) + d_i(v)\beta d_j(u\gamma u)\big) + 2d_n(u\gamma v\beta u) \\
&= \sum_{i+j+k=n} \Big(d_i(u)\gamma d_j(u)\beta d_k(v) + d_i(v)\beta d_j(u)\gamma d_k(u)\Big) + 2d_n(u\gamma v\beta u).
\end{aligned}
$$

This implies that

$$0 = d_n(x) - d_n(x) = 2\sum_{i+j+k=n} d_i(u)\gamma d_j(v)\beta d_k(u) - 2d_n(u\gamma v\beta u).$$

Since M is 2-torsion free, $\sum_{i+j+k=n} d_i(u)\gamma d_j(v)\beta d_k(u) - d_n(u\gamma v\beta u) = 0$. Hence $d_n(u\gamma v\beta u) = \sum_{i+j+k=n} d_i(u)\gamma d_j(v)\beta d_k(u)$. □

Proposition 3.2.3. *Let M be a 2-torsion free Γ-ring and $a\gamma b\beta c = a\beta b\gamma c$ for all $a, b, c \in$*

M and $\gamma, \beta \in \Gamma$. Then, for all $u, v, w \in U, \gamma, \beta \in \Gamma$ and $n \in \mathbb{N}_0$,

$$d_n(u\gamma v\beta w + w\gamma v\beta u) = \sum_{i+j+k=n} \Big(d_i(u)\gamma d_j(v)\beta d_k(w) + d_i(w)\gamma d_j(v)\beta d_k(u) \Big).$$

Proof. Let $u, v, w \in U, \gamma, \beta \in \Gamma$ and $n \in \mathbb{N}_0$. By Proposition 3.2.2,

$$\begin{aligned} & d_n\big((u+w)\gamma v\beta(u+w)\big) \\ &= \sum_{i+j+k=n} d_i(u+w)\gamma d_j(v)\beta d_k(u+w) \\ &= \sum_{i+j+k=n} d_i(u)\gamma d_j(v)\beta d_k(u) + \sum_{i+j+k=n} d_i(w)\gamma d_j(v)\beta d_k(w) \\ &\quad + \sum_{i+j+k=n} \Big(d_i(u)\gamma d_j(v)\beta d_k(w) + d_i(w)\gamma d_j(v)\beta d_k(u) \Big). \end{aligned}$$

and

$$\begin{aligned} & d_n\big((u+w)\gamma v\beta(u+w)\big) \\ &= d_n(u\gamma v\beta u) + d_n(u\gamma v\beta w + w\gamma v\beta u) + d_n(w\gamma v\beta w) \\ &= \sum_{i+j+k=n} d_i(u)\gamma d_j(v)\beta d_k(u) + \sum_{i+j+k=n} d_i(w)\gamma d_j(v)\beta d_k(w) \\ &\quad + d_n(u\gamma v\beta w + w\gamma v\beta u). \end{aligned}$$

That is ,

$$\sum_{i+j+k=n} \Big(d_i(u)\gamma d_j(v)\beta d_k(w) + d_i(w)\gamma d_j(v)\beta \Big) d_k(u) - d_n(u\gamma v\beta w + w\gamma v\beta u) = 0.$$

Hence

$$d_n(u\gamma v\beta w + w\gamma v\beta u) = \sum_{i+j+k=n} \Big(d_i(u)\gamma d_j(v)\beta d_k(w) + d_i(w)\gamma d_j(v)\beta d_k(u) \Big).$$

□

For all $a, b \in M, u, v \in U, \gamma \in \Gamma$ and $n \in \mathbb{N}_0$, let

$$\varphi_n(u, v)_\gamma = d_n(u\gamma v) - \sum_{i+j=n} d_i(u)\gamma d_j(v) \qquad \text{and}$$

$$[a, b]_\gamma = a\gamma b - b\gamma a.$$

Then we obtain similar results of the propoties of $\varphi_n(u, v)_\gamma$ and $[a, b]_\gamma$ to those of $\varphi_n(a, b, c)_{\gamma,\beta}$ and $[a, b, c]_{\gamma,\beta}$ in Chapter II.

Proposition 3.2.4. *For all $a, b, c \in M, u, v, w \in U, \gamma, \beta \in \Gamma$ and $n \in \mathbb{N}_0$,*

(i) $\varphi_n(u + w, v)_\gamma = \varphi_n(u, v)_\gamma + \varphi_n(w, v)_\gamma$,

(ii) $\varphi_n(u, v + w)_\gamma = \varphi_n(u, v)_\gamma + \varphi_n(u, w)_\gamma$,

(iii) $\varphi_n(u, v)_{\gamma+\beta} = \varphi_n(u, v)_\gamma + \varphi_n(u, v)_\beta$,

(iv) $\varphi_n(u, v)_\gamma = -\varphi_n(v, u)_\gamma$,

(v) $[a + c, b]_\gamma = [a, b]_\gamma + [c, b]_\gamma$,

(vi) $[a, b + c]_\gamma = [a, b]_\gamma + [a, c]_\gamma$,

(vii) $[a, b]_{\gamma+\beta} = [a, b]_\gamma + [a, b]_\beta$,

(viii) $[a, b]_\gamma = -[b, a]_\gamma$.

Proof. These can be shown in the same way of the proofs of Proposition 2.3.2, Proposition 2.3.3 and Proposition 2.3.4 *(i)*. □

Proposition 3.2.5. *Let M be a 2-torsion free Γ-ring, U be a nonzero ideal of M, $n \in \mathbb{N}_0$ and $a\gamma b\beta c = a\beta b\gamma c$ for all $a, b, c \in M$ and $\gamma, \beta \in \Gamma$. If $\varphi_m(u, v)_\beta = 0$ for all $u, v \in U, \beta \in \Gamma$ and $m < n$, then, for all $u, v, w \in U$ and $\gamma, \beta, \zeta \in \Gamma$,*

$$\varphi_n(u, v)_\beta \gamma w \zeta [u, v]_\beta + [u, v]_\beta \gamma w \zeta \varphi_n(u, v)_\beta = 0.$$

Proof. Assume that $\varphi_m(u, v)_\beta = 0$ for all $u, v \in U, \beta \in \Gamma$ and $m < n$. Let

$u, v, w \in U$ and $\gamma, \beta, \zeta \in \Gamma$. By Proposition 3.2.2, we have

$$\begin{aligned}
&d_n\big(u\beta(v\gamma w\zeta v)\beta u + v\beta(u\gamma w\zeta u)\beta v\big) \\
&\quad = d_n\big(u\beta(v\gamma w\zeta v)\beta u\big) + d_n\big(v\beta(u\gamma w\zeta u)\beta v\big) \\
&\quad = \sum_{h+i+j+k+l=n} d_h(u)\beta d_i(v)\gamma d_j(w)\zeta d_k(v)\beta d_l(u) \\
&\qquad + \sum_{h+i+j+k+l=n} d_h(v)\beta d_i(u)\gamma d_j(w)\zeta d_k(u)\beta d_l(v). \qquad (1)
\end{aligned}$$

On the other hand, by Proposition 3.2.3, we obtain

$$\begin{aligned}
&d_n\big(u\beta(v\gamma w\zeta v)\beta u + v\beta(u\gamma w\zeta u)\beta v\big) \\
&\quad = d_n\big((u\beta v)\gamma w\zeta(v\beta u) + (v\beta u)\gamma w\zeta(u\beta v)\big) \\
&\quad = \sum_{y+r+z=n} \Big(d_y(u\beta v)\gamma d_r(w)\zeta d_z(v\beta u) + d_y(v\beta u)\gamma d_r(w)\zeta d_z(u\beta v\Big) \\
&\quad = \sum_{y+r+z=n} d_y(u\beta v)\gamma d_r(w)\zeta d_z(v\beta u) + \sum_{y+r+z=n} d_y(v\beta u)\gamma d_r(w)\zeta d_z(u\beta v).
\end{aligned}$$

Let

$$\begin{aligned}
a &= \sum_{y+r+z=n} d_y(u\beta v)\gamma d_r(w)\zeta d_z(v\beta u) \\
b &= \sum_{y+r+z=n} d_y(v\beta u)\gamma d_r(w)\zeta d_z(u\beta v) \qquad \text{and} \\
x &= (u\beta v)\gamma w\zeta(v\beta u) + (v\beta u)\gamma w\zeta(u\beta v).
\end{aligned}$$

Then $d_n(x) = a + b$ and we obtain from (1) that

$$\begin{aligned}
&\sum_{h+i+j+k+l=n} d_h(u)\beta d_i(v)\gamma d_j(w)\zeta d_k(v)\beta d_l(u) - a \\
&\qquad = -\left(\sum_{h+i+j+k+l=n} d_h(v)\beta d_i(u)\gamma d_j(w)\zeta d_k(u)\beta d_l(v) - b\right). \qquad (2)
\end{aligned}$$

Note that, for $m < n$, we have $\varphi_m(u,v)_\beta = 0$ and $\varphi_m(v,u)_\beta = 0$, i.e.,

$$d_m(u\beta v) = \sum_{i+j=m} d_i(u)\beta d_j(v) \quad \text{and} \quad d_m(v\beta u) = \sum_{i+j=m} d_i(v)\beta d_j(u). \tag{3}$$

Now,

$$a = d_n(u\beta v)\gamma w\zeta(v\beta u) + \sum_{\substack{y+r+z=n \\ y\neq n, z\neq n}} d_y(u\beta v)\gamma d_r(w)\zeta d_z(v\beta u) + (u\beta v)\gamma w\zeta d_n(v\beta u).$$

We consider the term $\sum_{\substack{y+r+z=n \\ y\neq n, z\neq n}} d_y(u\beta v)\gamma d_r(w)\zeta d_z(v\beta u)$ of a. For $y < n$ and $z < n$, we obtain from (3) that

$$\sum_{\substack{y+r+z=n \\ y\neq n, z\neq n}} d_y(u\beta v)\gamma d_r(w)\zeta d_z(v\beta u) = \sum_{\substack{p+q+r+s+t=n \\ p+q\neq n, s+t\neq n}} \Big(d_p(u)\beta d_q(v)\Big)\gamma d_r(w)\zeta\Big(d_s(v)\beta d_t(u)\Big).$$

Thus

$$\sum_{\substack{p+q+r+s+t=n \\ p+q\neq n, s+t\neq n}} d_p(u)\beta d_q(v)\gamma d_r(w)\zeta d_s(v)\beta d_t(u) - \sum_{\substack{y+r+z=n \\ y\neq n, z\neq n}} d_y(u\beta v)\gamma d_r(w)\zeta d_z(v\beta u) = 0.$$

Then we consider

$$\begin{aligned}
&\sum_{h+i+j+k+l=n} d_h(u)\beta d_i(v)\gamma d_j(w)\zeta d_k(v)\beta d_l(u) \\
&\qquad = \sum_{h+i=n} d_h(u)\beta d_i(v)\gamma w\zeta v\beta u \ + \sum_{\substack{h+i+j+k+l=n \\ h+i\neq n, k+l\neq n}} d_h(u)\beta d_i(v)\gamma d_j(w)\zeta d_k(v)\beta d_l(u) \\
&\qquad\quad + u\beta v\gamma w\zeta \sum_{k+l=n} d_k(v)\beta d_l(u).
\end{aligned}$$

Hence

$$
\begin{aligned}
\sum_{h+i+j+k+l=n} & d_h(u)\beta d_i(v)\gamma d_j(w)\zeta d_k(v)\beta d_l(u) - a \\
&= \left(\sum_{h+i=n} d_h(u)\beta d_i(v) - d_n(u\beta v)\right)\gamma w\zeta v\beta u \\
&\quad + u\beta v\gamma w\zeta\left(\sum_{k+l=n} d_k(v)\beta d_l(u) - d_n(v\beta u)\right) \\
&= -\varphi_n(u,v)_\beta\gamma w\zeta v\beta u - u\beta v\gamma w\zeta\varphi_n(v,u)_\beta. \qquad (4)
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
\sum_{h+i+j+k+l=n} & d_h(v)\beta d_i(u)\gamma d_j(w)\zeta d_k(u)\beta d_l(v) - b \\
&= -\varphi_n(v,u)_\beta\gamma w\zeta u\beta v - v\beta u\gamma w\zeta\varphi_n(u,v)_\beta. \qquad (5)
\end{aligned}
$$

Recall that $d_n(x) = a + b$. From (2),(4) and (5), we have

$$
\begin{aligned}
0 &= \Big(-\varphi_n(u,v)_\beta\gamma w\zeta v\beta u - u\beta v\gamma w\zeta\varphi_n(v,u)_\beta\Big) \\
&\quad + \Big(-\varphi_n(v,u)_\beta\gamma w\zeta u\beta v - v\beta u\gamma w\zeta\varphi_n(u,v)_\beta\Big) \\
&= -\Big(\varphi_n(u,v)_\beta\gamma w\zeta v\beta u + \varphi_n(v,u)_\beta\gamma w\zeta u\beta v\Big) \\
&\quad - \Big(u\beta v\gamma w\zeta\varphi_n(v,u)_\beta + v\beta u\gamma w\zeta\varphi_n(u,v)_\beta\Big) \\
&= -\Big(\varphi_n(u,v)_\beta\gamma w\zeta v\beta u - \varphi_n(u,v)_\beta\gamma w\zeta u\beta v\Big) \\
&\quad - \Big(u\beta v\gamma w\zeta\varphi_n(v,u)_\beta - v\beta u\gamma w\zeta\varphi_n(v,u)_\beta\Big) \\
&= -\varphi_n(u,v)_\beta\gamma w\zeta\big(u\beta v - v\beta u\big) - \big(u\beta v - v\beta u\big)\gamma w\zeta\varphi_n(u,v)_\beta \\
&= -\varphi_n(u,v)_\beta\gamma w\zeta[u,v]_\beta - [u,v]_\beta\gamma w\zeta\varphi_n(u,v)_\beta.
\end{aligned}
$$

Thus $\varphi_n(u,v)_\beta\gamma w\zeta[u,v]_\beta + [u,v]\gamma w\zeta\varphi_n(u,v)_\beta = 0.$ $\square$

In order to make use of Proposition 2.1.4 to prove the next result, the semiprimeness of a Γ-ring is required.

Proposition 3.2.6. *Let M be a 2-torsion free semiprime Γ-ring, U be a nonzero ideal of M, $n \in \mathbb{N}_0$ and $a\gamma b\beta c = a\beta b\gamma c$ for all $a, b, c \in M$ and $\gamma, \beta \in \Gamma$. If $\varphi_m(u, v)_\beta = 0$ for all $u, v \in U, \beta \in \Gamma$ and $m < n$, then*

$$\varphi_n(u, v)_\beta \gamma w \zeta [a, b]_\lambda = [a, b]_\lambda \gamma w \zeta \varphi_n(u, v)_\beta = 0$$

for all $u, v, w, a, b \in U$ and $\gamma, \beta, \zeta, \lambda \in \Gamma$.

Proof. Assume that $\varphi_m(u, v)_\beta = 0$ for all $u, v \in U, \beta \in \Gamma$ and $m < n$. We obtain from Proposition 3.2.5 that

$$\varphi_n(u, v)_\beta \gamma w \zeta [u, v]_\beta + [u, v]_\beta \gamma w \zeta \varphi_n(u, v)_\beta = 0$$

for all $u, v, w \in U$ and $\gamma, \beta, \zeta \in \Gamma$. By Proposition 2.1.5,

$$\varphi_n(u, v)_\beta \gamma w \zeta [u, v]_\beta = [u, v]_\beta \gamma w \zeta \varphi_n(u, v)_\beta = 0$$

for all $u, v, w \in U$ and $\gamma, \beta, \zeta \in \Gamma$. Let $S, T : U \times \Gamma \times U \to U$ be defined by

$$S(u, \beta, v) = \varphi_n(u, v)_\beta \qquad \text{and} \qquad T(u, \beta, v) = [u, v]_\beta$$

for all $u, v \in U$ and $\beta \in \Gamma$. Then S and T satisfy the hypothesis of Proposition 2.1.4. This implies that

$$\varphi_n(u, v)_\beta \gamma w \zeta [a, b]_\lambda = 0$$

for all $u, v, w, a, b \in U$ and $\gamma, \beta, \zeta, \lambda, \in \Gamma$. Similarly, another result is obtained. $\square$

Proposition 3.2.7. *Let M be a 2-torsion free prime Γ-ring, U be a nonzero ideal of M,*

$n \in \mathbb{N}_0$ and $a\gamma b\beta c = a\beta b\gamma c$ for all $a, b, c \in M$ and $\gamma, \beta \in \Gamma$. If $\varphi_m(u, v)_\beta = 0$ for all $u, v \in U, \gamma, \beta \in \Gamma$ and $m < n$, then, for all $u, v, w, a, b \in U$ and $\gamma, \beta, \zeta, \lambda \in \Gamma$,

$$\varphi_n(u, v)_\beta \gamma w \zeta [a, b]_\lambda = [a, b]_\lambda \gamma w \zeta \varphi_n(u, v)_\beta = 0.$$

Proof. This follows from Proposition 3.2.6 and the fact that a prime Γ-ring is also a semiprime Γ-ring. $\square$

Now, we are ready to present the main theorem. Nevertheless, the following lemma is given.

Lemma 3.2.8. *Let M be a commutative 2-torsion free prime Γ-ring, U be a nonzero ideal of M and $a\gamma b\beta c = a\beta b\gamma c$ for all $a, b, c \in M$ and $\gamma, \beta \in \Gamma$. Then a Jordan higher derivation of U into M is a higher derivation of U into M.*

Proof. Let $D = (d_i)_{i \in \mathbb{N}_0}$ be a JHD of U into M, $u, v, w \in U$, $\gamma \in \Gamma$ and $n \in \mathbb{N}_0$, Proposition 3.2.1 provides that

$$d_n(u\gamma v + v\gamma u) = \sum_{i+j=n} \Big(d_i(u)\gamma d_j(v) + d_i(v)\gamma d_j(u) \Big).$$

Since M is commutative,

$$\begin{aligned} 2d_n(u\gamma v) = d_n(u\gamma v + u\gamma v) = d_n(u\gamma v + v\gamma u) &= \sum_{i+j=n} \Big(d_i(u)\gamma d_j(v) + d_i(v)\gamma d_j(u) \Big) \\ &= \sum_{i+j=n} \Big(d_i(u)\gamma d_j(v) + d_j(u)\gamma d_i(v) \Big) \\ &= 2 \sum_{i+j=n} d_i(u)\gamma d_j(v). \end{aligned}$$

Since M is 2-torsion free, $d_n(u\gamma v) = \sum_{i+j=n} d_i(u)\gamma d_j(v)$. Hence D is a HD of U into M. $\square$

Theorem 3.2.9. *Let M be a 2-torsion free prime Γ-ring, U be a nonzero ideal of M and*

$a\gamma b\beta c = a\beta b\gamma c$ for all $a, b, c \in M$ and $\gamma, \beta \in \Gamma$. Then a Jordan higher derivation of U into M is a higher derivation of U into M.

Proof. Let $D = (d_i)_{i\in\mathbb{N}_0}$ be a JHD of U into M. Lemma 3.2.8 shows that the statement holds if M is commutative. As a result, we assume that M is noncommutative.

To prove that D is a HD, we show that $d_n(u\gamma v) = \sum_{i+j=n} d_i(u)\gamma d_j(v)$, i.e., $\varphi_n(u, v)_\gamma = 0$ for all $u, v \in U, \gamma \in \Gamma$ and $n \in \mathbb{N}_0$. This obviously holds when $n = 0$. Now, let $n \in \mathbb{N}_0$ and assume that $d_m(u\gamma v) = \sum_{i+j=m} d_i(u)\gamma d_j(v)$ for all $u, v \in U$, $\gamma \in \Gamma$ and $m < n$. By Proposition 3.2.7,

$$\varphi_n(u, v)_\gamma \beta w \zeta [a, b]_\lambda = 0$$

for all $u, v, w, a, b \in U$ and $\gamma, \beta, \zeta, \lambda \in \Gamma$. Since M is prime and w, β, ζ are arbitrary, $\varphi_n(u, v)_\gamma = 0$ for all $u, v \in U$ and $\gamma \in \Gamma$ or $[a, b]_\lambda = 0$ for all $a, b \in U$ and $\lambda \in \Gamma$.

Suppose that $[a, b]_\lambda = 0$, i.e., $a\lambda b = b\lambda a$ for all $a, b \in U$ and $\lambda \in \Gamma$. Let $a, b \in U, m \in M$ and $\gamma, \beta \in \Gamma$. Then $a\gamma m \in U$ because U is an ideal of M so that $[a\gamma m, b]_\beta = 0$. Note that

$$\begin{aligned}
0 &= [a\gamma m, b]_\beta \\
&= a\gamma m\beta b - (b\beta a)\gamma m \\
&= a\gamma m\beta b - (a\beta b)\gamma m \\
&= \big(a\gamma m\beta b - a\beta b\gamma m\big) + \big(a\gamma b\beta m - a\gamma b\beta m\big) \\
&= \big(a\gamma m\beta b - a\gamma b\beta m\big) + \big(a\gamma b\beta m - a\beta b\gamma m\big) \\
&= a\gamma\big(m\beta b - b\beta m\big) + 0 && \text{(by assumption)} \\
&= a\gamma[m, b]_\beta.
\end{aligned}$$

Since a and γ are arbitrary, it means that $U\Gamma[m, b]_\beta = 0$. By Proposition 3.1.3, we have $[m,b]_\beta = 0$. This shows that $[m,b]_\beta = 0$ for all $b \in U, m \in M$ and $\beta \in \Gamma$. As a result, $U \subseteq Z(M)$. As a consequence of Proposition 3.1.2, M is commutative leading to a contradiction. Thus $\varphi_n(u,v)_\gamma = 0$ for all $u, v \in U$ and $\gamma \in \Gamma$. Therefore D is a HD of U into M. $\square$

CHAPTER IV
GENERALIZED HIGHER DERIVATIONS AND JORDAN GENERALIZED HIGHER DERIVATIONS OF Γ-RINGS

In this chapter, we define a generalized higher derivation and a Jordan generalized higher derivation of a Γ-ring. These terminologies are extended from a generalized derivation and a Jordan generalized derivation of a Γ-ring. For the sake of the consistency of the format of writing this thesis, we still divide this chapter into two sections. The first section provides the definition and simple example. The second section devotes to our last main result for this thesis.

4.1 Definition

Only definitions of a generalized higher derivation and a Jordan generalized higher derivation of a Γ-ring are given because the main tools for this chapter are results from Chapter II.

Definition 4.1.1. Let $G = (f_i)_{i\in\mathbb{N}_0}$ be a family of additive mappings of a Γ-ring M (i.e., $f_i : M \to M$ preserves the addition for all i) where f_0 is the identity mapping. Then G is said to be a *generalized higher derivation* (GHD) of M if there is a higher derivation $D = (d_i)_{i\in\mathbb{N}_0}$ of M such that

$$f_n(a\gamma b) = \sum_{i+j=n} f_i(a)\gamma d_j(b) \qquad \text{for all } a, b \in M,\ \gamma \in \Gamma \text{ and } n \in \mathbb{N}_0,$$

a *Jordan generalized higher derivation* (JGHD) of M if there is a higher derivation $D = (d_i)_{i\in\mathbb{N}_0}$ of M such that

$$f_n(a\gamma a) = \sum_{i+j=n} f_i(a)\gamma d_j(a) \qquad \text{for all } a \in M, \gamma \in \Gamma \text{ and } n \in \mathbb{N}_0.$$

It is clear that a GHD is a JGHD in general but the converse is not true. Observe that a GHD and a GJHD of a ring R with identity is also a GHD and a GJHD of a Γ-ring R where $\Gamma = \mathbb{Z}$, respectively.

4.2 The Main Result of JGHDs and GHDs of Γ-Rings

Throughout this section, let $G = (f_i)_{i\in\mathbb{N}_0}$ be a JGHD of M and $D = (d_i)_{i\in\mathbb{N}_0}$ be a HD of M such that $f_n(a\gamma a) = f_i(a)\gamma d_j(a)$ for all $a \in M, \gamma \in \Gamma$ and $n \in \mathbb{N}_0$. The following results are needed.

Proposition 4.2.1. *For each $a, b \in M, \gamma \in \Gamma$ and $n \in \mathbb{N}_0$,*

$$f_n(a\gamma b + b\gamma a) = \sum_{i+j=n} \Big(f_i(a)\gamma d_j(b) + f_i(b)\gamma d_j(a)\Big).$$

Proof. The result follows from using the similar process in the proof of Proposition 2.2.1 by replacing a by $a + b$ in the definition of a JGHD of M. $\square$

Proposition 4.2.2. *For each $a, b \in M, \gamma, \beta \in \Gamma$ and $n \in \mathbb{N}_0$,*

$$f_n(a\gamma b\beta a + a\beta b\gamma a) = \sum_{i+j+k=n} \Big(f_i(a)\gamma d_j(b)\beta d_k(a) + f_i(a)\beta d_j(b)\gamma d_k(a)\Big).$$

Proof. The same process in the proof of Proposition 2.2.2 is applied. Let $a, b \in M$, $\gamma, \beta \in \Gamma$ and $n \in \mathbb{N}_0$. We put $x = a\gamma(a\beta b + b\beta a) + (a\beta b + b\beta a)\gamma a$. By Proposition 4.2.1,

$$\begin{aligned}
f_n(x) &= \sum_{i+j=n} \Big(f_i(a)\gamma d_j(a\beta b + b\beta a) + f_i(a\beta b + b\beta a)\gamma d_j(a) \Big) \\
&= \sum_{i+j=n} \Big(f_i(a)\gamma \sum_{k+l=j} \Big(d_k(a)\beta d_l(b) + d_k(b)\beta d_l(a) \Big) && \text{(since } D \text{ is a HD)} \\
&\quad + \sum_{p+q=i} \Big(f_p(a)\beta d_q(b) + f_p(b)\beta d_q(a) \Big)\gamma d_j(a) \Big) && \text{(by Proprsition 4.2.1)} \\
&= \sum_{i+j+k=n} \Big(f_i(a)\gamma d_j(a)\beta d_k(b) + f_i(b)\beta d_j(a)\gamma d_k(a) \Big) \\
&\quad + \sum_{i+j+k=n} \Big(f_i(a)\gamma d_j(b)\beta d_k(a) + f_i(a)\beta d_j(b)\gamma d_k(a) \Big)
\end{aligned}$$

and

$$\begin{aligned}
f_n(x) &= f_n\big(a\gamma(a\beta b + b\beta a) + (a\beta b + b\beta a)\gamma a\big) \\
&= f_n\big(a\gamma a\beta b + b\beta a\gamma a\big) + f_n\big(a\gamma b\beta a + a\beta b\gamma a\big) \\
&= \sum_{i+j=n} \Big(f_i(a\gamma a)\beta d_j(b) + f_i(b)\beta d_j(a\gamma a) \Big) + f_n\big(a\gamma b\beta a + a\beta b\gamma a\big) \\
&= \sum_{i+j+k=n} \Big(f_i(a)\gamma d_j(a)\beta d_k(b) + f_i(b)\beta d_j(a)\gamma d_k(a) \Big) + f_n\big(a\gamma b\beta a + a\beta b\gamma a\big).
\end{aligned}$$

This implies that

$$\begin{aligned}
0 &= f_n(x) - f_n(x) \\
&= \sum_{i+j+k=n} \Big(f_i(a)\gamma d_j(b)\beta d_k(a) + f_i(a)\beta d_j(b)\gamma d_k(a) \Big) - f_n\big(a\gamma b\beta a + a\beta b\gamma a\big).
\end{aligned}$$

Thus

$$f_n\big(a\gamma b\beta a + a\beta b\gamma a\big) = \sum_{i+j+k=n} \Big(f_i(a)\gamma d_j(b)\beta d_k(a) + f_i(a)\beta d_j(b)\gamma d_k(a) \Big).$$

$\square$

Proposition 4.2.3. *Let M be a 2-torsion free Γ-ring. Then*

$$f_n(a\gamma b\gamma a) = \sum_{i+j+k=n} \Big(f_i(a)\gamma d_j(b)\gamma d_k(a)\Big)$$

for each $a, b \in M, \gamma \in \Gamma$ and $n \in \mathbb{N}_0$.

Proof. Let $a, b \in M, \gamma \in \Gamma$ and $n \in \mathbb{N}_0$. By Proposition 4.2.2, we obtain that

$$f_n(a\gamma b\gamma a + a\gamma b\gamma a) = \sum_{i+j+k=n} \Big(f_i(a)\gamma d_j(b)\gamma d_k(a) + f_i(a)\gamma d_j(b)\gamma d_k(a)\Big).$$

That is

$$2f_n(a\gamma b\gamma a) = 2\left(\sum_{i+j+k=n} f_i(a)\gamma d_j(b)\gamma d_k(a)\right).$$

Since M is 2-torsion free,

$$f_n(a\gamma b\gamma a) = \sum_{i+j+k=n} \Big(f_i(a)\gamma d_j(b)\gamma d_k(a)\Big).$$

□

Proposition 4.2.4. *Let M be a 2-torsion free Γ-ring. Then*

$$f_n(a\gamma b\gamma c + c\gamma b\gamma a) = \sum_{i+j+k=n} \Big(f_i(a)\gamma d_j(b)\gamma d_k(c) + f_i(c)\gamma d_j(b)\gamma d_k(a)\Big)$$

for each $a, b, c \in M, \gamma \in \Gamma$ and $n \in \mathbb{N}_0$.

Proof. Let $a, b, c \in M, \gamma \in \Gamma$ and $n \in \mathbb{N}_0$. Apply Proposition 4.2.3 and obtain that

$$\begin{aligned} & f_n\Big((a+c)\gamma b\gamma(a+c)\Big) \\ &\quad = \sum_{i+j+k=n} \Big(f_i(a+c)\gamma d_j(b)\gamma d_k(a+c)\Big) \end{aligned}$$

$$= \sum_{i+j+k=n} \Big(f_i(a)\gamma d_j(b)\gamma d_k(a) \Big) + \sum_{i+j+k=n} \Big(f_i(c)\gamma d_j(b)\gamma d_k(c) \Big)$$
$$+ \sum_{i+j+k=n} \Big(f_i(a)\gamma d_j(b)\gamma d_k(c) + f_i(c)\gamma d_j(b)\gamma d_k(a) \Big)$$

and

$$\begin{aligned} & f_n\Big((a+c)\gamma b\gamma(a+c)\Big) \\ &= f_n(a\gamma b\gamma a) + f_n(c\gamma b\gamma c) + f_n(a\gamma b\gamma c + c\gamma b\gamma a) \\ &= \sum_{i+j+k=n} \Big(f_i(a)\gamma d_j(b)\gamma d_k(a) \Big) + \sum_{i+j+k=n} \Big(f_i(c)\gamma d_j(b)\gamma d_k(c) \Big) \\ &\quad + f_n(a\gamma b\gamma c + c\gamma b\gamma a). \end{aligned}$$

This implies that

$$\begin{aligned} 0 &= f_n\Big((a+c)\gamma b\gamma(a+c)\Big) - f_n\Big((a+c)\gamma b\gamma(a+c)\Big) \\ &= f_n(a\gamma b\gamma c + c\gamma b\gamma a) - \sum_{i+j+k=n} \Big(f_i(a)\gamma d_j(b)\gamma d_k(c) + f_i(c)\gamma d_j(b)\gamma d_k(a) \Big) \end{aligned}$$

Hence

$$f_n(a\gamma b\gamma c + c\gamma b\gamma a) = \sum_{i+j+k=n} \Big(f_i(a)\gamma d_j(b)\gamma d_k(c) + f_i(c)\gamma d_j(b)\gamma d_k(a) \Big).$$

□

For each $n \in \mathbb{N}_0$, let $F_n(a,b)_\gamma = f_n(a\gamma b) - \sum_{i+j=n} f_i(a)\gamma d_j(b)$ for all $a, b \in M$ and $\gamma \in \Gamma$. It is easy to see that $F_n(a,a)_\gamma = 0$ for all $a \in M$, $\gamma \in \Gamma$ and $n \in \mathbb{N}_0$.

Proposition 4.2.5. *For every $a, b, c \in M, \gamma, \beta \in \Gamma$ and $n \in \mathbb{N}_0$,*

(i) $F_n(a+c,b)_\gamma = F_n(a,b)_\gamma + F_n(c,b)_\gamma$,

(ii) $F_n(a,b+c)_\gamma = F_n(a,b)_\gamma + F_n(a,c)_\gamma$,

(iii) $F_n(a,b)_{\gamma,\beta} = F_n(a,b)_\gamma + F_n(a,b)_\beta$,

(iv) $F_n(a,b)_\gamma = -F_n(b,a)_\gamma$.

Proof. These can be shown in the same way of the proof of Proposition 2.3.2 and Proposition 2.3.4. $\square$

Lemma 4.2.6. *Let M be a 2-torsion free Γ-ring and $n \in \mathbb{N}_0$. If $F_m(a,b)_\gamma = 0$ for all $a, b \in M$, $\gamma \in \Gamma$ and $m < n$, then, for all $a, b, x \in M$ and $\gamma, \beta \in \Gamma$,*

$$F_n(a,b)_\gamma \beta x \beta [a,b]_\gamma = 0.$$

Proof. Let $a, b, x \in M$ and $\gamma, \beta \in \Gamma$. By Proposition 4.2.3 and the definition of a HD D,

$$\begin{aligned}
& f_n\Big(a\gamma\big(b\beta x\beta b\big)\gamma a + b\gamma\big(a\beta x\beta a\big)\gamma b\Big) \\
&\quad = \sum_{i+j+k=n}\Big(f_i(a)\gamma d_j(b\beta x\beta b)\gamma d_k(a)\Big) + \sum_{i+j+k=n}\Big(f_i(b)\gamma d_j(a\beta x\beta a)\gamma d_k(b)\Big) \\
&\quad = \sum_{i+j+k=n} f_i(a)\gamma\Big(\sum_{u+v+w=j} d_u(b)\beta d_v(x)\beta d_w(b)\Big)\gamma d_k(a) \\
&\qquad + \sum_{i+j+k=n} f_i(b)\gamma\Big(\sum_{u+v+w=j} d_u(a)\beta d_v(x)\beta d_w(a)\Big)\gamma d_k(b) \\
&\quad = \sum_{i+u+v+w+k=n} f_i(a)\gamma d_u(b)\beta d_v(x)\beta d_w(b)\gamma d_k(a) \\
&\qquad + \sum_{i+u+v+w+k=n} f_i(b)\gamma d_u(a)\beta d_v(x)\beta d_w(a)\gamma d_k(b).
\end{aligned}$$

On the other hand, applying Proposition 4.2.4 and the definition of a HD D yields

$$\begin{aligned}
& f_n\Big(\big(a\gamma b\big)\beta x\beta\big(b\gamma a\big) + \big(b\gamma a\big)\beta x\beta\big(a\gamma b\big)\Big) \\
&\quad = \sum_{r+s+t=n}\Big(f_r\big(a\gamma b\big)\beta d_s(x)\beta d_t\big(b\gamma a\big) + f_r\big(b\gamma a\big)\beta d_s(x)\beta d_t\big(a\gamma b\big)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{r+s+t=n} f_r\big(a\gamma b\big)\beta d_s(x)\beta\left(\sum_{p+q=t} d_p(b)\gamma d_q(a)\right) \\
&\qquad + \sum_{r+s+t=n} f_r\big(b\gamma a\big)\beta d_s(x)\beta\left(\sum_{p+q=t} d_p(a)\gamma d_q(b)\right) \\
&= \sum_{r+s+p+q=n} f_r\big(a\gamma b\big)\beta d_s(x)\beta d_p(b)\gamma d_q(a) + \sum_{r+s+p+q=n} f_r\big(b\gamma a\big)\beta d_s(x)\beta d_p(a)\gamma d_q(b).
\end{aligned}
$$

Let

$$
\begin{aligned}
y &= \sum_{r+s+p+q=n} f_r\big(a\gamma b\big)\beta d_s(x)\beta d_p(b)\gamma d_q(a) \qquad \text{and} \\
z &= \sum_{r+s+p+q=n} f_r\big(b\gamma a\big)\beta d_s(x)\beta d_p(a)\gamma d_q(b).
\end{aligned}
$$

Then $f_n\Big(a\gamma\big(b\beta x\beta b\big)\gamma a + b\gamma\big(a\beta x\beta a\big)\gamma b\Big) = y + z$ and

$$
\begin{aligned}
&\sum_{i+u+v+w+k=n} f_i(a)\gamma d_u(b)\beta d_v(x)\beta d_w(b)\gamma d_k(a) - y \\
&\qquad\qquad = -\left(\sum_{i+u+v+w+k=n} f_i(b)\gamma d_u(a)\beta d_v(x)\beta d_w(a)\gamma d_k(b) - z\right).
\end{aligned}
$$

Note that, for $m < n$, we have $F_m(a,b)_\gamma = 0$, i.e.,

$$
f_m(a\gamma b) = \sum_{i+j=m} f_i(a)\gamma d_j(b).
$$

Now,

$$
\begin{aligned}
y &= f_n(a\gamma b)\beta x\beta b\gamma a + \sum_{\substack{r+s+p+q=n \\ r\neq n}} f_r\big(a\gamma b\big)\beta d_s(x)\beta d_p(b)\gamma d_q(a) \\
&= f_n(a\gamma b)\beta x\beta b\gamma a + \sum_{\substack{l+g+s+p+q=n \\ l+g\neq n}} f_l(a)\gamma d_g(b)\beta d_s(x)\beta d_p(b)\gamma d_q(a).
\end{aligned}
$$

We can see that

$$
\begin{aligned}
&\sum_{i+u+v+w+k=n} f_i(a)\gamma d_u(b)\beta d_v(x)\beta d_w(b)\gamma d_k(a) - y\\
&= \left(\sum_{i+u=n} f_i(a)\gamma d_u(b)\beta x\beta b\gamma a + \sum_{\substack{i+u+v+w+k=n\\ i+u\neq n}} f_i(a)\gamma d_u(b)\beta d_v(x)\beta d_w(b)\gamma d_k(a)\right) - y\\
&= \sum_{i+u=n} f_i(a)\gamma d_u(b)\beta x\beta b\gamma a - f_n(a\gamma b)\beta x\beta b\gamma a\\
&= \left(\sum_{i+u=n} f_i(a)\gamma d_u(b) - f_n(a\gamma b)\right)\beta x\beta b\gamma a\\
&= -F_n(a,b)_\gamma\beta x\beta\big(b\gamma a\big)
\end{aligned}
$$

and, similarly,

$$
\sum_{i+u+v+w+k=n} f_i(b)\gamma d_u(a)\beta d_v(x)\beta d_w(a)\gamma d_k(b) - z = -F_n(b,a)_\gamma\beta x\beta\big(a\gamma b\big).
$$

Hence, applying Proposition 4.2.5 (i), we have

$$
\begin{aligned}
0 &= f_n\Big(a\gamma\big(b\beta x\beta b\big)\gamma a + b\gamma\big(a\beta x\beta a\big)\gamma b\Big) - \Big(y+z\Big)\\
&= \left(\sum_{i+u+v+w+k=n} f_i(a)\gamma d_u(b)\beta d_v(x)\beta d_w(b)\gamma d_k(a) - y\right)\\
&\quad + \left(\sum_{i+u+v+w+k=n} f_i(b)\gamma d_u(a)\beta d_v(x)\beta d_w(a)\gamma d_k(b) - z\right)\\
&= -F_n(a,b)_\gamma\beta x\beta b\gamma a - F_n(b,a)_\gamma\beta x\beta a\gamma b\\
&= -F_n(a,b)_\gamma\beta x\beta b\gamma a + F_n(a,b)_\gamma\beta x\beta a\gamma b\\
&= F_n(a,b)_\gamma\beta x\beta\big(a\gamma b - b\gamma a\big)\\
&= F_n(a,b)_\gamma\beta x\beta[a,b]_\gamma.
\end{aligned}
$$

□

Lemma 4.2.7. *Let M be a 2-torsion free semiprime Γ-ring and $n \in \mathbb{N}_0$. If the element $F_m(a,b)_\gamma = 0$ for all $a, b \in M, \gamma \in \Gamma$ and $m < n$, then*

$$F_n(a,b)_\gamma \beta x \zeta [a,b]_\gamma = 0$$

for all $a, b, x \in M, \gamma, \beta, \zeta \in \Gamma$.

Proof. Let $a, b, x, y \in M$ and $\gamma, \beta, \zeta, \alpha, \delta \in \Gamma$. Lemma 4.2.6 provides

$$\begin{aligned}
0 &= F_n(a,b)_\gamma(\beta+\zeta)x(\beta+\zeta)[a,b]_\gamma \\
&= F_n(a,b)_\gamma \beta x \beta [a,b]_\gamma + F_n(a,b)_\gamma \beta x \zeta [a,b]_\gamma + F_n(a,b)_\gamma \zeta\, x \beta [a,b]_\gamma + F_n(a,b)_\gamma \zeta x \zeta [a,b]_\gamma \\
&= F_n(a,b)_\gamma \beta x \zeta [a,b]_\gamma + F_n(a,b)_\gamma \zeta\, x \beta [a,b]_\gamma.
\end{aligned}$$

This implies that

$$F_n(a,b)_\gamma \beta x \zeta [a,b]_\gamma = -F_n(a,b)_\gamma \zeta\, x \beta [a,b]_\gamma.$$

Consequently,

$$\begin{aligned}
&\Big(F_n(a,b)_\gamma \beta x \zeta [a,b]_\gamma\Big) \alpha y \delta \Big(F_n(a,b)_\gamma \beta x \zeta [a,b]_\gamma\Big) \\
&\quad = \Big(-F_n(a,b)_\gamma \zeta x \beta [a,b]_\gamma\Big) \alpha y \delta \Big(F_n(a,b)_\gamma \beta x \zeta [a,b]_\gamma\Big) \\
&\quad = -\Big(F_n(a,b)_\gamma \zeta \Big(x \beta [a,b]_\gamma \alpha y \delta F_n(a,b)_\gamma \beta x\Big) \zeta [a,b]_\gamma\Big) \\
&\quad = 0
\end{aligned}$$

where the last equality is obtained from Lemma 4.2.6. Since the choices of y, α, δ are arbitrary and M is semiprime,

$$F_n(a,b)_\gamma \beta x \zeta [a,b]_\gamma = 0.$$

$\square$

Lemma 4.2.8. *Let M be a 2-torsion free semiprime Γ-ring and $n \in \mathbb{N}_0$. If the element $F_m(a,b)_\gamma = 0$ for all $a, b \in M, \gamma \in \Gamma$ and $m < n$, then*

$$F_n(a,b)_\gamma \beta x \zeta [u,v]_\lambda = 0$$

for all $a, b, x, u, v \in M$ and $\gamma, \beta, \zeta, \lambda \in \Gamma$.

Proof. Lemma 4.2.7 provides that $F_n(a,b)_\gamma \beta x \zeta [a,b]_\gamma = 0$ for all $a, b, x \in M$ and $\gamma, \beta, \zeta \in \Gamma$. Define $S, T : M \times \Gamma \times M \to M$ by $S(a,\gamma,b) = F_n(a,b)_\gamma$ and $T(a,\gamma,b) = [a,b]_\gamma$. Then S and T satisfy the hypothesis of Proposition 2.1.4. As a result,

$$F_n(a,b)_\gamma \beta x \zeta [u,v]_\lambda = 0$$

for all $a, b, x, u, v \in M$ and $\gamma, \beta, \zeta, \lambda \in \Gamma$. □

Corollary 4.2.9. *Let M be a 2-torsion free prime Γ-ring and $n \in \mathbb{N}_0$. If the element $F_m(a,b)_\gamma = 0$ for all $a, b \in M, \gamma \in \Gamma$ and $m < n$, then*

$$F_n(a,b)_\gamma \beta x \zeta [u,v]_\lambda = 0$$

for all $a, b, x, v, u \in M$ and $\gamma, \beta, \zeta, \lambda \in \Gamma$.

Proof. This is obtained from Lemma 4.2.8 and the fact that primeness implies semiprimeness. □

Finally, the main theorem of this chapter is provided.

Theorem 4.2.10. *Let M be a noncommutative 2-torsion free prime Γ-ring. Then Jordan generalized higher derivation of M is a generalized higher derivation of M.*

Proof. Let $F = (f_i)_{i \in \mathbb{N}_0}$ be a JGHD of M. Then there is a HD $D = (d_i)_{i \in \mathbb{N}_0}$ such that $f_n(a\gamma a) = \sum_{i+j=n} f_i(a)\gamma d_j(a)$. To prove that F is a GHD of M, it suffices to show that $f_n(a\gamma b) = \sum_{i+j=n} f_i(a)\gamma d_j(b)$ for all $a, b \in M, \gamma \in \Gamma$ and $n \in \mathbb{N}_0$.

We prove this by induction on n. First, notice that this holds when $n = 0$ because f_0 is the identity map. Now, let $n \in \mathbb{N}_0$ and assume that $f_m(a\gamma b) = \sum_{i+j=m} f_i(a)\gamma d_j(b)$ for all $a, b \in M$, $\gamma \in \Gamma$ and $m < n$. Applying Corollary 4.2.9, we obtain that

$$F_n(a,b)_\gamma \beta x \zeta [u,v]_\lambda = 0$$

for all $a, b, x, u, v \in M$ and $\gamma, \beta, \zeta, \lambda \in \Gamma$. Since M is prime and x, β, ζ are arbitrary, $F_n(a,b)_\gamma = 0$ for all $a, b \in M$ and $\gamma \in \Gamma$ or $[u,v]_\lambda = 0$ for all $u, v \in M$ and $\lambda \in \Gamma$. Suppose that $[u,v]_\lambda = 0$, i.e., $u\gamma v = v\gamma u$ for all $u, v \in M$ and $\lambda \in \Gamma$. Then M is commutative which leads to a contradiction. Thus $F_n(a,b)_\gamma = 0$ for all $a, b \in M$ and $\gamma \in \Gamma$. Therefore, F is a GHD of M. $\square$

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