CHAPTER IV

VECTOR-VALUED SUBHARMONIC FUNCTIONS

1. Introduction

Let $(E, \| \ \|)$ be a Banach lattice. (See Appendix at the end of this chapter for its definition.) The real field R endowed with its absolute value and its usual ordering is an example of a Banach lattice. The subset $E_+ := \{x \in E \mid x \geqslant 0\}$ is called the positive cone of E; elements $x \in E_+$ are called positive. We denote by E' the set of all continuous linear functionals on E.

By an extended Banach lattice, $\bar{E}$, we shall mean a structure obtained by adjoining to the Banach lattice E the ideal elements $+\infty$ and $-\infty$ and making the operational conventions:

$x + (+\infty) = +\infty$, $x + (-\infty) = -\infty$ for all $x \in E$;

$\lambda(+\infty) = +\infty$ if $\lambda > 0$; $= -\infty$ if $\lambda < 0$; $= 0$ if $\lambda = 0$ where $\lambda \in R$;

$(+\infty) + (+\infty) = +\infty$, $(-\infty) + (-\infty) = -\infty$.

Also $-\infty < x < +\infty$ for all $x \in E$.

Let m be a positive integer and Ω be a subset of the euclidean space R^m. In this chapter, we wish to find a good definition of a vector-valued subharmonic function $f: \Omega \longrightarrow \bar{E}$ in the sense that it should be a natural generalization of the classical one. We then use this definition to prove some related theorems on subharmonic functions.

To this end, we begin with an attempt to understand how a sequence (a_n) in E converges to $-\infty$. In §2, we present several possible definitions that can explain the nature of convergence to $-\infty$. With the motivation from §2, we present a definition of vector-valued upper semi-continuous function in §3. We study integration of functions with values in extended Banach lattices in §4. Then we shall define and study vector-valued subharmonic functions in §5. Finally, we prove some theorems on hyperplane means of vector-valued subharmonic functions in §6.

2. Convergence to $-\infty$ in Banach lattices

Let E be a Banach lattice and (a_n) a sequence in E. Some possible definitions of "$a_n \to -\infty$ as $n \to +\infty$" are the following:

Definition A. The sequence (a_n) is not minorized, i.e. there is no $a_0 \in E$ such that $a_0 \leqslant a_n$ for all n.

Definition B. $(\forall n \in N\ [a_n \leqslant 0])$ & $(\|a_n\| \to +\infty$ as $n \to +\infty)$.

Definition C. $(\exists M \in R_+, \forall n \in N, [\|a_n^+\| \leqslant M])$ & $(\|a_n^-\| \to +\infty$ as $n \to +\infty)$.

Definition D. $\forall p \in E_+, \exists n_0 \in N, \forall n \geqslant n_0\ [a_n \leqslant -p]$.

Definition E. $\exists M \in R_+, \forall p \in E_+, \exists n_0 \in N, \forall n \geqslant n_0, \exists q \in E$

$[a_n \leqslant -p + q$, where $\|q\| < M]$.

Let (a_n) and (b_n) be two sequences in E. In fact, we need a definition of convergence to $-\infty$ such that we can prove the following properties.

Property 1. $(a_n \to -\infty \text{ as } n \to +\infty)$ & $(b_n \to -\infty \text{ as } n \to +\infty) \Longrightarrow$

$(a_n \vee b_n \to -\infty \text{ as } n \to +\infty)$ & $(a_n \wedge b_n \to -\infty \text{ as } n \to +\infty)$.

Property 2. $(a_n \to -\infty \text{ as } n \to +\infty)$ & $(\exists M \in R_+, \forall n \in N\ [\ \|b_n^+\| \leqslant M])$

$\Longrightarrow (a_n + b_n \to -\infty \text{ as } n \to +\infty)$.

Property 3. $(a_n \to -\infty \text{ as } n \to +\infty) \Rightarrow \forall e' \in E'_+ \setminus \{0\}\ [e'(a_n) \to -\infty$ as $n \to +\infty\ ]$.

By studying the above definitions and the given three properties together, we get the following results.

Theorem 4.1. For Definition A, we have

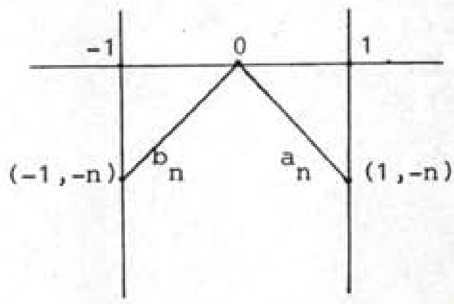
(4.1.1) Property 1 is true only in the case of minimum.

(4.1.2) Property 2 is not true.

(4.1.3) property 3 is not true.

Proof. To prove (4.1.1), let (a_n) and (b_n) be two sequences in E, since $a_n \wedge b_n \leqslant a_n$, then $(a_n \wedge b_n)$ is not minorized if (a_n) is not minorized. The case of maximum is not true by considering the following example.

The space C[-1,1] of continuous real functions on [-1,1], endowed with its canonical order defined by "$f \leqslant g$ iff $f(t) \leqslant g(t)$ for all $t \in [-1,1]$" and the supremum norm, is a Banach lattice. Define two sequences (a_n) and (b_n) in C[-1,1] as follows:

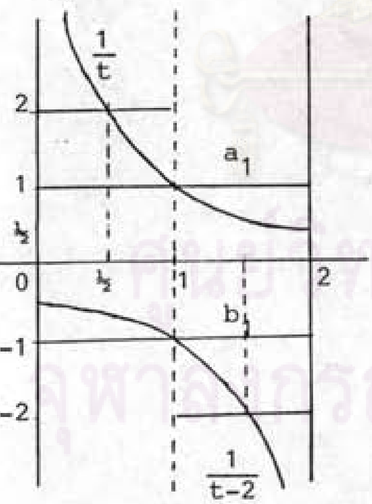


$$a_n(t) = \begin{cases} 0 & (-1 \leqslant t \leqslant 0), \\ -nt & (0 < t \leqslant 1), \end{cases}$$

$$b_n(t) = \begin{cases} nt & (-1 \leqslant t < 0), \\ 0 & (0 \leqslant t \leqslant 1), \end{cases}$$

where $n = 1,2,\ldots$ It is obvious that the sequence (a_n) and (b_n) are not minorized. But $a_n \vee b_n = 0$ for all n, then $(a_n \vee b_n)$ is minorized.

For (4.1.2), consider the following example: We note that the space $L^1(0,2)$ of Lebesgue integrable real functions on $(0,2)$, endowed with its canonical order defined by "$f \leqslant g$ iff $f(t) \leqslant g(t)$ a.e. on $(0,2)$" and L^1 norm, is a Banach lattice. Define two sequences (a_n) and (b_n) in $L^1(0,2)$ as follows:



$a_1(t) \equiv 1, \quad b_1(t) \equiv -1,$

$$a_n(t) = \begin{cases} n & (0 < t < 1/(n-1)), \\ -n & (2 - (1/(n-1)) < t < 2), \\ 0 & (\text{otherwise}), \end{cases}$$

$$b_n(t) = \begin{cases} -n & (0 < t < 1/(n-1)), \\ n & (2 - (1/(n-1)) < t < 2), \\ 0 & (\text{otherwise}), \end{cases}$$

where $n = 2,3,\ldots$ We note that the sequence (a_n) is not minorized. For, suppose not, we can find $g \in L^1(0,2)$ such that $g \leqslant a_n$ for all n. Thus, by definition of a_n, $g(t) < 1/(t-2)$ for all $t \in (1,2)$ and then

$|g(t)| \geqslant |1/(t-2)|$ for $t \in (1,2)$. Hence

$$\int_0^2 |g(t)|\,dt \geqslant \int_1^2 |1/(t-2)|\,dt = +\infty .$$

This contradicts to the fact that $g \in L^1(0,2)$. So, we have the required result. Moreover, for each $n \in N$, we have

$$\| b_n^+ \|_{L^1} \leqslant \int_0^2 |b_n(t)|\,dt = \begin{cases} 2 & (n = 1), \\ 2n/(n-1) & (n \geqslant 2). \end{cases}$$

Hence $\| b_n^+ \|_{L^1} \leqslant 4$ for all n. But $a_n + b_n = 0$ for all n, so $(a_n + b_n)$ is minorized.

For (4.1.3), consider the following example. Let $E = C[-1,1]$ as in (4.1.1). Define a sequence (a_n) in E as follow:

$$a_n(t) = nt \quad (t \in [-1,1],\ n = 1,2,\ldots).$$

It is obvious that (a_n) is not minorized. By choosing $e' \in E'_+ \setminus \{0\}$ to be the evaluation map at zero, i.e. $e'(f) = f(0)$. We get $e'(a_n) = 0$ for all n. Hence $e'(a_n) \not\to -\infty$ as $n \to +\infty$. This proves Theorem 4.1.

<u>Theorem</u> 4.2. For Definition B, we have

(4.2.1) Property 1 is true only in the case of minimum.

(4.2.2) Property 2 is not true.

(4.2.3) Property 3 is not true.

<u>Proof</u>. To prove (4.2.1), let (a_n) and (b_n) be two sequences in E which converges to $-\infty$ according to Definition B. For each $n \in N$,

we have

$$a_n \wedge b_n \leqslant a_n.$$

Then, by assumption, we have

$$-(a_n \wedge b_n) \geqslant -a_n \geqslant 0.$$

Hence, for each n,

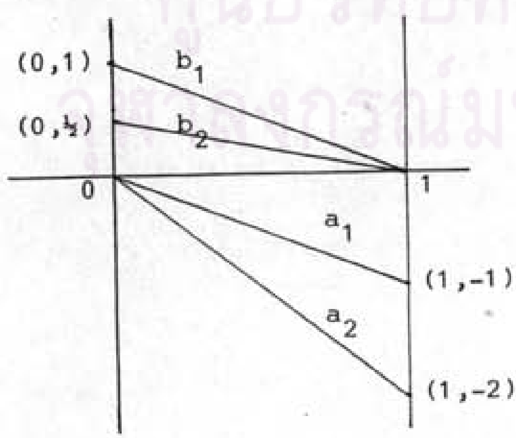
$$a_n \wedge b_n \leqslant 0 \quad \text{and} \quad \|a_n \wedge b_n\| \geqslant \|a_n\|.$$

This implies $a_n \wedge b_n \to -\infty$ as $n \to +\infty$ provided that $a_n \to -\infty$ as $n \to +\infty$. The case of maximum is not true. Consider the Banach lattice $E = C[-1,1]$ and the sequences (a_n) and (b_n) as defined in (4.1.1). It is easily seen that $a_n \to -\infty$ and $b_n \to -\infty$ as $n \to +\infty$. But $a_n \vee b_n = 0$ for all n, so $a_n \vee b_n \not\to -\infty$ as $n \to +\infty$.

For (4.2.2), consider the Banach lattice $E = C[0,1]$. Define two sequences (a_n) and (b_n) in E as follows:

$$a_n(t) = -nt \qquad (t \in [0,1],\ n = 1,2,\ldots),$$

$$b_n(t) = -\frac{1}{n}(t-1) \qquad (t \in [0,1],\ n = 1,2,\ldots).$$



We find that $\|b_n\|_{\sup} \leqslant 1$ for all n and $a_n \to -\infty$ as $n \to +\infty$. But $a_n + b_n$ is not comparable with 0 for each n, so $a_n + b_n \not\to -\infty$.

For (4.2.3), consider the Banach lattice $E = C[0,1]$ and define (a_n) as in (4.2.2). Choose the evaluation map $e' \in E'_+ \setminus \{0\}$ such that $e'(f) = f(0)$ where $f \in E$. We get that $e'(a_n) = 0$ for all n. So $e'(a_n) \not\to -\infty$ as $n \to +\infty$.

Theorem 4.3. For Definition C, we have

(4.3.1) Property 1 is true only in the case of minimum.

(4.3.2) Property 2 is true.

(4.3.3) Property 3 is not true.

Proof. To prove (4.3.1), Let (a_n) and (b_n) be two sequences in E which converge to $-\infty$ according to Definition C. Then there exists $M \in R_+$ such that, for each $n \in N$,

$$\|a_n^+\| \leqslant M \text{ and } \|a_n^-\| \to +\infty \text{ as } n \to +\infty$$

Since $a_n \wedge b_n \leqslant a_n$, then $(a_n \wedge b_n)^+ \leqslant a_n^+$ and $(a_n \wedge b_n)^- \geqslant a_n^- \geqslant 0$. Hence $\|(a_n \wedge b_n)^+\| \leqslant M$ and $\|(a_n \wedge b_n)^-\| \geqslant \|a_n^-\|$ for all $n \in N$. This proves that $a_n \wedge b_n \to -\infty$ as $n \to +\infty$. The case of maximum is not true. Consider the Banach lattice $E = C[-1,1]$ and the sequences (a_n) and (b_n) as defined in (4.1.1). It is easily seen that $a_n \to -\infty$ and $b_n \to -\infty$ as $n \to +\infty$. But $a_n \vee b_n = 0$ for all n, so $a_n \vee b_n \not\to -\infty$ as $n \to +\infty$.

For (4.3.2), let (a_n) and (b_n) be two sequences in E. Suppose that $a_n \to -\infty$ as $n \to +\infty$ and there exists $M_2 \in R_+$ such that, for each n, $\|b_n^+\| \leqslant M_2$. Then, by Definition C, there exists $M_1 \in R_+$ such that $\|a_n^+\| \leqslant M_1$ for all n. We observe that, for each n,

$$\||a_n + b_n|\| = \|(a_n + b_n)^+ + (a_n + b_n)^-\|$$

$$\leqslant \|a_n^+\| + \|b_n^+\| + \|(a_n + b_n)^-\|$$

$$\leqslant M_1 + M_2 + \|(a_n + b_n)^-\|,$$

and

$$\||a_n + b_n|\| = \|a_n + b_n\|$$

$$= \|(a_n^+ + b_n^+) - (a_n^- + b_n^-)\|$$

$$\geqslant |\|a_n^+ + b_n^+\| - \|a_n^- + b_n^-\||.$$

Hence

$$|\|a_n^+ + b_n^+\| - \|a_n^- + b_n^-\|| \leqslant M_1 + M_2 + \|(a_n + b_n)^-\|$$

Since $\|a_n^+ + b_n^+\| \leqslant M_1 + M_2$ for all n and $\|a_n^- + b_n^-\| \geqslant \|a_n^-\| \to +\infty$ as $n \to +\infty$, then $\|(a_n + b_n)^-\| \to +\infty$ as $n \to +\infty$. This proves that $a_n + b_n \to -\infty$ as $n \to +\infty$.

For (4.3.3), consider $E = C[-1,1]$ and define (a_n) as in (4.1.1), we see that $a_n \to -\infty$ as $n \to +\infty$ and $e'(a_n) = 0$ for all n where e' is the evaluation map at $x_0 = 0$.

Theorem 4.4. For Definition D, we have

(4.4.1) Property 1 is true.

(4.4.2) Property 2 is true in a Banach lattice which has an additional structure, i.e. in a Banach lattice where every norm bounded set is order bounded.

(4.4.3) Property 3 is true.

Proof. (4.4.1) is obvious from Definition D.

For (4.4.2), let (a_n) and (b_n) be two sequences in E with $a_n \to -\infty$ as $n \to +\infty$ and there exists $M \in R_+$ such that, for each n, $\|b_n^+\| \leqslant M$. By assumption, we can find $q \in E_+$ such that

(1) $$b_n \leqslant b_n^+ \leqslant q \quad \text{(for all n).}$$

Let $p \in E_+$, we can find $n_0 \in N$ such that

(2) $$a_n \leqslant -p - q \quad (n \geqslant n_0).$$

By adding (1) to (2), we get the required result.

For (4.4.3), let (a_n) be a sequence in E with $a_n \to -\infty$ as $n \to +\infty$ and let $e' \in E'_+ \setminus \{0\}$. Let c be a positive real number. Since $e' \neq 0$, there exists $x \in E$ such that $e'(x) \neq 0$. So $e'(x^+) - e'(x^-) \neq 0$. This implies that $e'(x^+)$ or $e'(x^-) \neq 0$. Thus there exists $y \in E_+$ such that $e'(y) > 0$. Let $n_0 \in N$ be so large that $e'(n_0 y) > c$. Hence, by Definition D, there exists $n_1 \in N$ such that

$$a_n \leqslant -n_0 y \quad (n \geqslant n_1).$$

We note that $a_n < -n_0 y$ is equivalent to $-a_n \geqslant n_0 y > 0$. Thus

$$e'(-a_n) \geqslant e'(n_0 y) > c.$$

This proves that $e'(a_n) < -c$ for all $n \geqslant n_1$. Hence $e'(a_n) \to -\infty$ as $n \to +\infty$. This completes the proof of Theorem 4.4.

We observe that it is difficult to find a counter example for (4.4.2) since there hardly examples for $a_n \to -\infty$ according to Definition D. However, there are such examples for Definition E In fact, $-ne \to -\infty$ for almost units e of E (see Definition 4.3 in §3 of this chapter for the definition of unit).

Theorem 4.5. For Definition E, we have

(4.5.1) Property 1 is true.

(4.5.2) Property 2 is true.

(4.5.3) Property 3 is true.

Proof. To prove (4.5.1), let (a_n) and (b_n) be two sequences in E such that $a_n \to -\infty$ and $b_n \to -\infty$ as $n \to +\infty$. Then there exists $M \in R_+$ such that for each $p \in E_+$ we can find $n_1, n_2 \in N$ such that

(3) $\forall n \geq n_1, \exists q_1 \in E\ [a_n \leq -p + q_1$, where $\|q_1\| \leq M]$, and

(4) $\forall n \geq n_2, \exists q_2 \in E\ [a_n \leq -p + q_2$, where $\|q_2\| \leq M]$.

since, for each n, $a_n \wedge b_n \leq a_n$, it follows from (3) that $a_n \wedge b_n \to -\infty$ as $n \to +\infty$. For the case of maximum: We note that, for each $n \geq \max(n_1, n_2)$, we have

$$a_n \leq -p + q_1 \leq -p + |q_1|,$$

and

$$b_n \leq -p + q_2 \leq -p + |q_2|.$$

Hence

(5) $$a_n \vee b_n \leq -p + |q_1| + |q_2|,$$

where $|q_1| + |q_2| \in E$ and $\| \, |q_1| + |q_2| \, \| \leq M + M = 2M$. Therefore (5) implies $a_n \vee b_n \to -\infty$ as $n \to +\infty$.

For (4.5.2), let (a_n) and (b_n) be two sequences in E such that

$a_n \to -\infty$ as $n \to +\infty$ and there exists $M_1 \in R_+$ such that $\|b_n^+\| \leqslant M_1$ for all n. Then there exists $M_2 \in R_+$ such that for each $p \in E_+$ we can find $n_0 \in N$ such that

(6) $\quad \forall n \geqslant n_0, \exists q \in E\ [\ a_n \leqslant -p + q$, where $\|q\| \leqslant M_2]$.

Thus, for each $n \geqslant n_0$, we have

$$a_n + b_n \leqslant -p + b_n^+ + q,$$

where $b_n^+ + q \in E$ and $\|b_n^+ + q\| \leqslant M_1 + M_2$. This proves that $a_n + b_n \to -\infty$ as $n \to +\infty$.

For (4.5.3.), let $e' \in E'_+ \setminus \{0\}$ and assume that $a_n \to -\infty$ as $n \to +\infty$. Then there exists $M \in R_+$ such that for each $p \in E_+$ we can find $n_0 \in N$ such that

(7) $\quad \forall n \geqslant n_0, \exists q \in E\ [\ a_n \leqslant -p + q$, where $\|q\| \leqslant M]$.

Since e' is continuous, the set $\{e'(x) |\ x \in E, \|x\| \leqslant M\}$ is bounded by M_1, say. Let $y \in E_+$ be such that $e'(y) > 0$. (Such a y exists since $e' \neq 0$.) Let $c \in R_+$, choose $n_1 \in N$ be so large that

$$e'(n_1 y) - M_1 > c.$$

Replace p in (7) by $n_1 y$, we can find $n_2 \in N$ such that

$$\forall n \geqslant n_2, \exists r \in E\ [\ a_n \leqslant -n_1 y + r, \text{ where } \|r\| \leqslant M].$$

Hence, for all $n \geqslant n_2$, we get

$$-a_n - (n_1 y - r) \geqslant 0.$$

Thus

$$-e'(a_n) \geqslant e'(n_1 y - r),$$

and then

$$e'(a_n) \leqslant -e'(n_1 y) + e'(r) \leqslant -e'(n_1 y) + M_1 \leqslant -c.$$

This proves that $e'(a_n) \to -\infty$ as $n \to +\infty$.

Remark 4.1.

(i) It follows from Theorem 4.5 that Definition E has all necessary properties that we should have for a sequence which converges to $-\infty$. So we shall adopt Definition E for the following discussion.

(ii) Consider the Banach lattice $E = L^1(0,2)$ as in the proof of (4.1.2). Define a sequence (a_n) in E as follow:

$$a_n(t) \equiv -ne \qquad (t \in (0,2),\ n = 1,2,\ldots),$$

where $e \in R_+ \setminus \{0\}$. We find that $a_n \not\to -\infty$ as $n \to +\infty$ according to Definition D as, for each n, a_n is not comparable with the negative element $-p$ where $p(t) = 1/\sqrt{t}$. But we still have $a_n \to -\infty$ as $n \to +\infty$ according to Definition E. To prove this, let $p \in L^1(0,2)_+$ and choose $M = 1$. For each $n \in N$, we define

$$A_n = \{t \in (0,2) \mid p(t) \geqslant ne\}.$$

Since $p \in L^1(0,2)$, then $m(A_n) \to 0$ as $n \to +\infty$ (where m is the Lebesgue measure on the real interval $(0,2)$). So we can find $n_0 \in N$ such that

$$\int_{A_{n_0}} p(t)dm(t) \leqslant 1.$$

Define q on (0,2) by

$$q(t) = \begin{cases} 0 & (t \notin A_{n_0}), \\ p(t) & (t \in A_{n_0}). \end{cases}$$

Thus $q \in L^1(0,2)$ and we have

$$-ne = a_n \leqslant -n_0 e \leqslant -p(t) + q(t) \qquad (n \geqslant n_0),$$

where $\|q\|_{L^1} \leqslant 1$. This implies that $a_n \to -\infty$ as $n \to +\infty$.

3. Vector-valued semi-continuous functions

Definition 5.1. Let Ω be a non-empty subset of R^m and E a Banach lattice. Let $f: \Omega \to \bar{E}$ be a function. Then f is said to be (vector-valued) upper semi-continuous (u.s.c.) on Ω if

(i) $-\infty \leqslant f(x) < +\infty$ (for all $x \in \Omega$).

(ii) There exists a sequence (f_n) of continuous functions from Ω into E such that

$f_1(x) \geqslant f_2(x) \geqslant \ldots \geqslant f(x)$ and $f_n(x) \to f(x)$ as $n \to +\infty$

for each $x \in \Omega$ where the convergence to $-\infty$ means that it converges to $-\infty$ according to Definition E.

A function f(x) is said to be (vector-valued) lower semi-continuous (l.s.c.) on Ω if $-f(x)$ is u.s.c. on Ω.

We note some properties of u.s.c. functions. The first one is very simple; it follows directly from Definition 4.1 and the properties of Definition E.

<u>Theorem</u> 4.6. If $f_1, \ldots, f_k$ are u.s.c. on Ω and $\lambda_1, \ldots, \lambda_k$ are non-negative real numbers, then

$$f = \sum_{n=1}^{k} \lambda_n f_n$$

is u.s.c. on Ω .

By using the relations (6) and (7) in the Appendix, Definition 4.1 and the properties of Definition E, one can prove

<u>Theorem</u> 4.7. If $f_1, \ldots, f_k$ are u.s.c. on Ω , then so are

$$f(x) = \sup_{n=1 \text{ to } k} f_n(x) \quad \text{and} \quad g(x) = \inf_{n=1 \text{ to } k} f_n(x).$$

Now, we shall give a necessary condition for vector-valued upper semi-continuity.

<u>Theorem</u> 4.8. Assume f is u.s.c. on Ω .

(i) If $f(x_0) > -\infty$, then, for each $\varepsilon > 0$, there exists a neighborhood U of x_0 in Ω such that

$$\forall x \in U \; [\; \| (f(x) - f(x_0))^+ \| < \varepsilon \;].$$

(ii) If $f(x_0) = -\infty$, then there exists $M \in R_+$ such that for each $p \in E_+$ there exists a neighborhood U of x_0 in Ω

such that

$$\forall x \in U,\ \exists q(x) \in E\ [\ f(x) \leqslant -p + q(x),\ \text{where } \|q(x)\| \leqslant M].$$

Proof. Since f is u.s.c. on Ω , then there exists a decreasing sequence (f_n) of continuous functions on Ω such that

(1) $$f_n(x) \to f(x) \text{ as } n \to +\infty \quad (\text{for all } x \in \Omega).$$

To prove (i), let $x_0 \in \Omega$ and suppose that $f(x_0) > -\infty$. Let $\varepsilon > 0$ be given, then there exists $n_0 \in N$ such that

(2) $$\|f_{n_0}(x_0) - f(x_0)\| < \varepsilon/2.$$

Since f_{n_0} is continuous at x_0, there exists a neighborhood U of x_0 in Ω such that

(3) $$\|f_{n_0}(x) - f_{n_0}(x_0)\| < \varepsilon/2 \quad (x \in U \cap \Omega).$$

Hence (2) and (3) imply

(4) $$\|f_{n_0}(x) - f(x_0)\| < \varepsilon \quad (x \in U \cap \Omega).$$

Since the sequence (f_n) is decreasing, for each $x \in \Omega$,

$$f(x) - f(x_0) \leqslant f_{n_0}(x) - f(x_0)$$

and hence

(5) $$(f(x) - f(x_0))^+ \leqslant (f_{n_0}(x) - f(x_0))^+.$$

Thus (4) and (5) yield

(6) $$\|(f(x) - f(x_0))^+\| \leqslant \|f_{n_0}(x) - f(x_0)\| < \varepsilon \quad (x \in U \cap \Omega).$$

This proves (i). Now, we go on to prove (ii). Let $x_0 \in \Omega$ with $f(x_0) = -\infty$. Then, by (1), we have

$$f_n(x_0) \to f(x_0) = -\infty \quad \text{as } n \to +\infty .$$

Thus, by Definition E, there exists $M \in R_+$ such that for each $p \in E_+$ we can find $n_0 \in N$ such that

(7) $\quad \forall\, n \geqslant n_0,\ \exists q \in E\ [f_n(x_0) \leqslant -p + q, \text{ where } \|q\| \leqslant M].$

Let $\varepsilon > 0$ be given, by continuity of f_{n_0} at x_0, there exists a neighborhood U of x_0 in Ω such that

(8) $\quad \|f_{n_0}(x) - f_{n_0}(x_0)\| < \varepsilon \quad (x \in U \cap \Omega).$

For each $x \in \Omega$, we have

$$f_{n_0}(x) - f_{n_0}(x_0) \leqslant (f_{n_0}(x) - f_{n_0}(x_0))^+.$$

Hence

(9) $\quad f(x) \leqslant f_{n_0}(x) \leqslant f_{n_0}(x_0) + (f_{n_0}(x) - f_{n_0}(x_0))^+.$

Thus, for each $x \in U \cap \Omega$, (7), (8), and (9) imply

$$f(x) \leqslant -p + q + (f_{n_0}(x) - f_{n_0}(x_0))^+,$$

where $\|q + (f_{n_0}(x) - f_{n_0}(x_0))^+\| \leqslant M + \varepsilon$. This proves (ii). The proof of Theorem 4.8 is now complete.

<u>Corollary to Theorem 4.8</u>. Let E be a Banach lattice and $f: \Omega \subset R^m \to E$. Then f is continuous on Ω iff f is u.s.c. and l.s.c. on Ω.

Proof. If f is continuous on Ω then, by Definition 4.1, f is both u.s.c. and l.s.c. on Ω. Conversely, let $x_0 \in \Omega$ and assume that f is u.s.c. and l.s.c. at x_0. Then, by Theorem 4.8(i), we can find a neighborhood U of x_0 in Ω such that

(10) $\forall x \in U\ [\ \|(f(x) - f(x_0))^+\| < \varepsilon/2$ and $\|(f(x) - f(x_0))^-\| < \varepsilon/2]$.

Hence (10) implies that

$$\|f(x) - f(x_0)\| < \varepsilon$$

for all $x \in U \cap \Omega$. Since x_0 is arbitrary, then f is continuous on Ω.

The converse of Theorem 4.8 is true in Banach lattices which have some addition structures, i.e. in a Banach lattice which is order complete and has a unit.

Definition 4.2. A Banach lattice E is said to be (order) complete if for each non-empty majorized set $B \subset E$, sup B exists in E.

Definition 4.3. A unit in a Banach lattice E is an element $e \in E_+ \setminus \{0\}$ such that e is the supremum of the closed unit ball $\{x \in E \mid \|x\| \leqslant 1\}$.

It follows easily from Definition 4.3 that, for $x \in E$ and $\varepsilon > 0$, $\|x\| \leqslant \varepsilon$ iff $|x| \leqslant \varepsilon e$. Let $(X, \mathcal{B}, \mu)$ be a measure space. The vector space $L^\infty(X)$ of all essentially bounded functions on X, endowed with its caconical order and essential supremum norm, is an example of a complete Banach lattice with unit e (where $e(x) = 1$, all $x \in X$). Now, we are ready to state and prove the partial converse of Theorem 4.8.

Theorem 4.9. Let E be a complete Banach lattice with unit e. Let Ω be an open subset of R^m. Let $f : \Omega \to \bar{E}$ be a function that satisfies (i) and (ii) of Theorem 4.8. Then there is a decreasing sequence (f_n) of continuous functions on Ω such that $f_n(x) \to f(x)$ as $n \to +\infty$ for all $x \in \Omega$.

Proof. We divide the proof of this theorem into two parts.

Part 1. Assume that $f(\Omega) := \{f(x) \mid x \in \Omega\}$ is majorized. For each n, we define f_n on Ω as follows:

$$(11) \qquad f_n(x) = \sup_{y \in \Omega} \{f(y) - n|x - y|e\} \qquad (x \in \Omega).$$

Thé funcfion f_n exists since $f(\Omega)$ is majorized and E is complete. Moreover, the set $f_n(\Omega)$ is also majorized as $f_n(x) \leqslant \sup_{y \in \Omega} \{f(y)\}$ for all $x \in \Omega$. We shall prove that the sequence (f_n) has the required properties. In the first instance, we have

$$(12) \qquad f_n(x) \geqslant f_{n+1}(x)$$

and

$$(13) \qquad f_n(x) \geqslant f(x).$$

for all $x \in \Omega$ and for all $n \in N$. To prove this, let $x \in \Omega$ and $n \in N$ be fixed. Then, for each $y \in \Omega$, we have

$$(14) \qquad f(y) - n|x-y|e \geqslant f(y) - (n+1)|x-y|e.$$

Taking supremum to both side of (14) through $y \in \Omega$, we get (12).

For (13), we note that

$$f_n(x) \geqslant f(x) - n|x-x| = f(x).$$

Hence (12) and (13) yield

$$f_1(x) \geqslant f_2(x) \geqslant \dots \geqslant f(x) \qquad (x \in \Omega).$$

Now, we shall show that $f_n(x)$ is continuous for each n. Let $\varepsilon > 0$ be given, choose $x_1, x_2 \in \Omega$ with $|x_1 - x_2| < \varepsilon/n$. Then, for each $y \in \Omega$, we have

(15) $$|x_2 - y| - |x_1 - y| \leqslant |x_2 - y - x_1 + y| = |x_2 - x_1| < \varepsilon/n.$$

It follows from (15) that

$$-|x_1 - y| < -|x_2 - y| + \varepsilon/n,$$

and then

$$-n|x_1 - y|e < (-n|x_2 - y| + \varepsilon)e.$$

Therefore

(16) $$f(y) - n|x_1 - y|e < f(y) - n|x_2 - y|e + \varepsilon e.$$

By taking supremum to both side of (16) through all $y \in \Omega$, we get

(17) $$f_n(x_1) \leqslant f_n(x_2) + \varepsilon e.$$

By interchanging the role of x_1, x_2, we also get

(18) $$f_n(x_2) \leqslant f_n(x_1) + \varepsilon e.$$

Hence (16) and (17) give

$$0 \leqslant |f_n(x_1) - f_n(x_2)| < \varepsilon e.$$

Thus $\|f_n(x_1) - f_n(x_2)\| \leqslant \|\varepsilon e\| = \varepsilon \|e\|$. This proves that f_n is continuous on Ω. It remains to prove that

$$(19) \qquad f_n(x) \to f(x) \text{ as } n \to +\infty \qquad (x \in \Omega).$$

Let $x_0 \in \Omega$ be fixed and suppose that $\sup\limits_{y \in \Omega} \{f(y)\} = \omega$. We shall split the proof of (19) into two cases.

<u>Case</u> 1. Suppose that $f(x_0) > -\infty$. Then we can assume that $f(x_0) = 0$. (For, if not, we can replace $f(x)$ by $g(x) = f(x) - f(x_0)$). Let $\varepsilon > 0$ be given, by (i) of Theorem 4.8, we can find $\delta > 0$ such that, for all $x \in B(x_0, \delta) \subset \Omega$,

$$(20) \qquad \|(f(x) - f(x_0))^+\| = \|f(x)^+\| < \varepsilon$$

where $B(x_0, \delta)$ is the ball in R^m centered at x_0 and of radius δ. Since e is a unit, (20) implies

$$f(x)^+ \leqslant \varepsilon e \qquad (x \in B(x_0, \delta)).$$

Thus

$$0 \leqslant \sup_{|x - x_0| < \delta} \{f(x)^+\} \leqslant \varepsilon e.$$

Hence

$$(21) \qquad \Big\| \sup_{|x - x_0| < \delta} \{f(x)^+\} \Big\| \leqslant \varepsilon \|e\|.$$

For each fixed n, we always have

$$f(x) - n|x - x_0|e \leqslant f(x) \leqslant f(x)^+ \qquad (x \in \Omega).$$

Thus

$$(22) \qquad \sup_{|x-x_0|<\delta} \{f(x) - n|x - x_0|e\} \leqslant \sup_{|x-x_0|<\delta} \{f(x)^+\}.$$

Let $c \in R_+$ be so large that $\omega \leqslant ce$. Let $x \in \Omega$ be such that $|x_0 - x| \geqslant \delta$. Let $n \in N$ be so large that $n > 1/\delta$, then

$$n|x_0 - x| > (1/\delta)\cdot\delta = 1;$$

and thus

$$n|x_0 - x|e > e.$$

Let $n_0 \in N$ be so large that $n_0|x_0 - x|e > ce$. This implies that $- n_0|x - x_0|e < - ce$ and thus we have

$$(23) \qquad f(x) - n_0|x_0 - x|e < f(x) - ce \leqslant \omega - ce \leqslant 0.$$

It is obvious that (23) is true for all $x \in \Omega$ which lies outside the ball $B(x_0, \delta)$ in R^m. Hence (22) and (23) imply

$$\begin{aligned} 0 = f(x_0) \leqslant f_{n_0}(x_0) &= \sup_{x\in\Omega} \{f(x) - n_0|x_0 - x|e\} \\ &= \sup_{|x-x_0|<\delta} \{f(x) - n_0|x_0 - x|e\} \vee \\ &\quad \sup_{|x-x_0|\geqslant\delta} \{f(x) - n_0|x_0 - x|e\} \\ &\leqslant \sup_{|x-x_0|<\delta} \{f(x)^+\} \vee 0. \end{aligned}$$

Therefore

$$0 \leqslant f_{n_0}(x_0) \leqslant \sup_{|x-x_0|<\delta} \{f(x)^+\}.$$

This last inequality together with (21) imply that

(24) $$\|f_{n_0}(x_0)\| < \varepsilon \|e\|.$$

This proves that $f_n(x_0) \to f(x_0)$ as $n \to +\infty$.

<u>case</u> 2. Assume that $f(x_0) = -\infty$. We shall use Definition E to prove that $f_n(x_0) \to f(x_0) = -\infty$ as $n \to +\infty$, i.e. we must verify that

(25) $$\exists M \in R_+, \forall p \in E_+, \exists n_0 \in N, \forall n \geqslant n_0, \exists q \in E\ [f_n(x_0) \leqslant -p+q, \text{ where } \|q\| \leqslant M].$$

Since E has a unit e, then for each $p \in E_+$ we can find $c \in R_+$ such that $p \leqslant ce$. Thus (25) is reduced to

(26) $$\forall c \in R_+, \exists n_0 \in N, \forall n \geqslant n_0\ [f_n(x_0) \leqslant -ce].$$

Let $c > 0$ be given, by (ii) of Theorem 4.8, and since E has a unit, we can find $\delta > 0$ such that, for all $x \in B(x_0, \delta) \subset \Omega$,

(27) $$f(x) \leqslant -ce.$$

It follows from (27) that

(28) $$\sup_{|x-x_0| < \delta} \{f(x)\} \leqslant -ce.$$

Now, let $n_0 \in N$ be so large that

(29) $$n_0 \delta e \geqslant ce + \omega,$$

where $\omega = \sup_{x \in \Omega} \{f(x)\}$ Let $x \in \Omega$ be such that $|x_0 - x| \geqslant \delta$, then (29) implies

$$(30) \qquad f(x) - n_0|x_0 - x|e \leqslant \omega - n_0\delta e \leqslant -ce.$$

It is obvious that (30) is true for all $x \in \Omega$ which lies outside the ball $B(x_0, \delta)$ in R^m. Hence (28) and (30) imply

$$\begin{aligned} f_{n_0}(x_0) &= \sup_{|x-x_0| < \delta} \{f(x) - n_0|x - x|e\} \vee \\ &\quad \sup_{|x-x_0| \geqslant \delta} \{f(x) - n_0|x_0 - x|e\} \\ &\leqslant (-ce) \vee (-ce) = -ce. \end{aligned}$$

Since $(f_n(x_0))$ is a decreasing sequence, then we have $f_n(x_0) \leqslant -ce$ for all $n \geqslant n_0$. Thus (26) holds. This proves Part 1 of Theorem 4.9.

<u>Part</u> 2. We shall extend our results from Part 1 to the more general case, i.e. the image of f may not be majorized. We employ a useful device called a partition of unity.

Let $\{U_j \mid j = 1,2,\ldots\}$ be a locally finite open cover of Ω such that $\bar{U}_j$ is a compact subset of Ω. Let $j \in N$ and $x \in U_j$ Since f is u.s.c. at x, by assumption (i) or (ii) (depending on f(x) is either finite or $-\infty$ respectively), $\|f(x)^+\|$ must be bounded in some open neighborhoods of x in Ω. Do this at all points of $\bar{U}_j$ and noting that $\bar{U}_j$ is compact, we can conclude that $\|f(x)^+\|$ is also bounded in $\bar{U}_j$ and hence in U_j. This implies that f(x) must be majorized on U_j. (Remember that E has a unit.) But, we always have

$$f(x) \leqslant f(x)^+ \qquad (x \in \Omega).$$

Hence $f(U_j)$ is majorized. By Part 1, there is a sequence (f_n^j) of continuous functions on U_j such that

$$(31) \qquad f_n^j(x) \searrow f(x) \text{ as } n \to +\infty \qquad (x \in U_j).$$

Choose a continuous partition of unity $\{\varphi_j \mid j = 1,2,\ldots\}$ such that $\operatorname{supp} \varphi_j \subset U_j$ for all j. Put

$$f_n(x) = \sum_{j=1}^{\infty} \varphi_j(x) f_n^j(x) \qquad (n = 1,2,\ldots \text{and } x \in \Omega).$$

Then each f_n is continuous on Ω. By (31), we have

$$f_1(x) \geqslant f_2(x) \geqslant \ldots \qquad (x \in \Omega)$$

and

$$f_n(x) \to \sum_{j=1}^{\infty} \varphi_j(x) f(x) = f(x) \text{ as } n \to +\infty \qquad (x \in \Omega).$$

This completes the proof of Theorem 4.9.

4. Integration of functions with values in extended Banach lattices

In the next section we shall give a definition and some properties of vector-valued subharmonic functions. To do this, we need some basic idea of the Bochner integral in an extended Banach lattice which is a slight modification of the usual Bochner integral that appeared in the book of Diestel and Uhr ([4], page 41-52). The basis of this material is an extended Banach lattice $\bar{E}$ and a σ-finite measure space $(\Omega, \mathcal{B}, \mu)$, for instance Ω is a subset of the euclidean space R^m and μ is the Lebesgue or surface area measure on R^m.

A function $s : \Omega \longrightarrow E$ is called simple if there exist $y_1, \ldots, y_n \in E$ and $B_1, \ldots, B_n \in \mathcal{B}$ such that $s = \Sigma_{i=1}^{n} y_i \chi_{B_i}$ where $\chi_{B_i}(x) = 1$ if $x \in B_i$ and $\chi_{B_i}(x) = 0$ if $x \notin B_i$. A function $f : \Omega \longrightarrow \bar{E}$ is called μ-measurable if there exists a sequence of simple functions (s_n) on Ω such that

$$s_n(x) \rightarrow f(x) \text{ as } n \rightarrow +\infty .$$

(μ -almost everywhere), in norm if $f(x) \in E$, otherwise as in Definition E. Moreover, a function $f: \Omega \longrightarrow E$ is called measurable if for every open set V in E the set $f^{-1}(V)$ is in $\mathcal{B}$.

We note some properties of μ measurable funcfions.

Theorem 4.10. Let $f,g: \Omega \rightarrow E \cup \{-\infty\} \subset \bar{E}$ be μ-measurable, then

(i) $f + g$ is μ-measurable,

(ii) λf is μ-measurable for all $\lambda \in R$.

(iii) Assume either Ω is a compact metric space and μ is a positive Borel measure on Ω or Ω is an open subset of R^m and μ is the Lebesgue measure on Ω . Let (f_n) be a suquence of continuous functions from Ω into E and supppose that $f_n(x) \rightarrow f(x)$ as $n \rightarrow +\infty$ for all $x \in \Omega$ (the value of f is in $\bar{E}$). Then f is μ-measurable.

Proof. For (i), let (s_n) and (s_n') be two sequences of simple functions on Ω such that

(1) $s_n(x) \rightarrow f(x)$ as $n \rightarrow +\infty$ (pointwise on $\Omega \setminus Z$)

and

(2) $s'_n(x) \rightarrow g(x)$ as $n \rightarrow +\infty$ (pointwise on $\Omega \setminus Z'$),

where Z and Z' are two measurable subsets of Ω such that $\mu(Z) = \mu(Z') = 0$. We shall prove that

(3) $(s_n + s'_n)(x) \rightarrow (f + g)(x)$ as $n \rightarrow +\infty$ (pointwise on $\Omega \setminus Z \cup Z'$).

Let $x \in \Omega \setminus Z \cup Z'$. If $f(x)$ and $g(x) \in E$. then it is obvious from (1) and (2) that (3) holds. Now, suppose that $f(x) = -\infty$ and $g(x) \in E$. It follows from (2) that for each $\varepsilon > 0$ we can find $n_0 \in N$ such that

$$\|s'_n(x) - g(x)\| < \varepsilon \quad (n \geqslant n_0).$$

Thus

$$\|s'_n(x)^+\| \leqslant \|s'_n(x)\| \leqslant \|g(x)\| + \varepsilon \quad (n \geqslant n_0).$$

This proves that $\{\|s_n(x)^+\|\}$ is bounded. Hence, by Theorem 4.5(ii), we obtain (3). Finally, we assume that $f(x) = -\infty$ and $g(x) = -\infty$. Then it is obvious from (1), (2), and Definition E that (3) holds. This proof (i). The proof of (ii) is obvious from the definition of μ-measurability.

Finally, we shall prove (iii). We divide the proof of (iii) into two parts.

Part 1. We assume that (Ω, ρ) is a compact metric space and μ is a positive Borel measure on Ω. Let (f_n) be a sequence of continuous functions from Ω into E such that $f_n(x) \to f(x)$ as $n \to +\infty$ for all $x \in \Omega$. Let $n \in N$ be fixed. Since f_n is uniformly continuous on Ω, there exists $\delta(n) > 0$ such that, for all $x, y \in \Omega$,

$$\rho(x,y) < \delta(n) \Longrightarrow \|f_n(x) - f_n(y)\| < 1/n.$$

It is obvious that $\Omega = \bigcup_{x \in \Omega} B(x, \delta(n))$ and since Ω is compact, we can find $r \in N$ such that

$$(4) \qquad \Omega = \bigcup_{j=1}^{r} B(x_j, \delta(n)).$$

Put $U_1 = B(x_1, \delta(n))$ and $U_j = B(x_j, \delta(n)) \setminus (U_1 \cup \ldots \cup U_{j-1})$ $(j = 1,\ldots,r)$. Then Ω is a disjoint union of subsets U_j. We define a simple function s_n on Ω as follows:

$$s_n = \sum_{j=1}^{r} f_n(x_j)\chi_{U_j}$$

where χ_{U_j} is the characteristic function on U_j. Let $x \in \Omega$, then $x \in U_j \subset B(x_j, \delta(n))$ for exactly one $j \in \{1,2,\ldots,r\}$. Hence

$$\|f_n(x) - s_n(x)\| = \|f_n(x) - f_n(x_j)\| < 1/n.$$

Since x is arbitrary, we get

$$(5) \qquad \|f_n - s_n\|_{sup} \leqslant 1/n.$$

As $f_n \to f$ pointsise on Ω, then $s_n \to f$ pointwise on Ω. To prove this, let $x \in \Omega$ and we consider three cases.

Case 1. Let $f(x) \in E$. Let $\varepsilon > 0$ be given, by using the assumption that $f_n(x) \to f(x)$ as $n \to +\infty$ and (5), we can find $n_0 \in N$ such that

$$\|f_n(x) - f(x)\| < \varepsilon/2 \text{ and } \|f_n(x) - s_n(x)\| < \varepsilon/2 \qquad (n \geqslant n_0).$$

Thus we have $\|f(x) - s_n(x)\| < \varepsilon$ for all $n \geqslant n_0$.

Case 2. Let $f(x) = -\infty$. Since $f_n(x) \to f(x) = -\infty$ as $n \to +\infty$, then, by Definition E, there exists $M' \in R_+$ such that for each $p \in E_+$ we can find $n_1 \in N$ such that

$$(6) \qquad \forall n \geqslant n_1, \exists q \in E \ [f_n(x) \leqslant -p + q, \text{ where } \|q\| \leqslant M'].$$

Let $\varepsilon > 0$ be given, by (5), we can find $n_2 \in N$ such that

$$(7) \qquad \|s_n(x) - f_n(x)\| < \varepsilon \qquad (n \geqslant n_2).$$

It follows from (7) that

$$(8) \qquad s_n(x) - f_n(x) \leqslant q(\varepsilon)$$

for some $q(\varepsilon) \in E$ such that $\|q(\varepsilon)\| \leqslant \varepsilon$. Let $n_0 = \max\{n_1, n_2\}$ and $M = M' + \varepsilon$, then we obtain from (6) and (8) that

$$s_n(x) \leqslant -p + q + q(\varepsilon) \qquad (n \geqslant n_0),$$

where $q + q(\varepsilon) \in E$ and $\|q + q(\varepsilon)\| \leqslant M$. This proves that $s_n(x) \to f(x) = -\infty$ as $n \to +\infty$.

Case 3. If $f(x) = +\infty$, the proof is similar to case 2.

Combining all three cases, we can conclude that $s_n(x) \to f(x)$ as $n \to +\infty$.

This proves (iii) in the case that Ω is a compact metric space.

Part 2. Assume that Ω is an open subset of R^m and μ is the Lebesgue measure on Ω. We may assume that Ω is a countable union of closed balls $\bar{B}(x_k,\delta)$ for some $\delta > 0$ and $k = 1,2,\ldots$. We put

$$\Omega_1 = \bar{B}_1(x_1, \delta),$$

$$\Omega_k = \bar{B}_k(x_k, \delta) \setminus (\Omega_1^o \cup \ldots \cup \Omega_{k-1}^o) \qquad (k = 2,3,\ldots),$$

where Ω_k^o means the interior of Ω_k. Thus Ω is a countable unions of compact set Ω_k $(k = 1,2,\ldots)$ and $\Omega_k \cap \Omega_j$ is a set of measure zero for each $k \neq j$. It follows from Part 1 that the restriction $f|_{\Omega_k}$ of f to each Ω_k is μ-measurable, and for each k there is a sequence $(s_j^{(k)})$ $(j = 1,2,\ldots)$ of simple functions on Ω_k which converges pointwise to $f|_{\Omega_k}$. We define s_n by the following values:

$$s_n = s_n^{(k)} \qquad \text{on } \Omega_k^o \quad (\text{for } k = 1,2,\ldots n),$$

$$s_n = 0 \qquad \text{if } x \notin \Omega_1^o \cup \ldots \cup \Omega_n^o.$$

Then each s_n is a simple function, and the sequence (s_n) converges almost everywhere to f. This proves Part 2. Thus (iii) is proved. The proof of Theorem 4.10 is now complete.

It follows immediately from Theorem 4.10(iii) that every vector-valued upper semi-continuous function on sets Ω as in Theorem 4.10 (iii) is μ-measurable. Next, we shall define Bochner integral which is a straight forward abstraction of the Lebesgue integral with absolute value signs are replaced by norm signs.

A μ-measurable function $f:\Omega \to \bar{E}$ is called Bochner integrable if there exists a sequence of simple functions (s_n) such that

$$(9) \qquad \lim_{n\to+\infty} \int_\Omega \|s_n - f\| d\mu = 0.$$

(We define $\|+\infty\| = \|-\infty\| = +\infty$.) In this case, we define

$$(10) \qquad \int_\Omega f d\mu = \lim_{n\to+\infty} \int_\Omega s_n d\mu$$

where $\int_\Omega s_n d\mu$ is defined in the natural way.

We note that, for each $m,n \in N$,

$$(11) \qquad \left\| \int_\Omega s_n d\mu - \int_\Omega s_m d\mu \right\| \leqslant \int_\Omega \|s_n - s_m\| d\mu$$

$$\leqslant \int_\Omega \|s_n - f\| d\mu + \int_\Omega \|s_m - f\| d\mu$$

Hence, if f is Bochner integrable then (9) and (11) imply that the sequence $(\int_\Omega s_n d\mu)$ is a Cauchy sequence in E and therefore it must converge to some point of E, $\int_\Omega f d\mu$ say. Moreover, let (s_n') be another sequence of simple functions such that

$$(12) \qquad \lim_{n\to+\infty} \int_\Omega \|s_n' - f\| d\mu = 0.$$

By (9) and (12), we get

$$(13) \qquad \lim_{n\to+\infty} \int_\Omega \|s_n - s_n'\| d\mu = 0.$$

Since, for each $n \in N$, we have

$$(14) \qquad \left\| \int_\Omega s_n d\mu - \int_\Omega s_n' d\mu \right\| \leqslant \int_\Omega \|s_n - s_n'\| d\mu,$$

It follows from (13) and (14) that the integral in (10) is independent of the defining sequence (s_n). This proves that our definition in (10) is well-defined.

For a μ-measurable function $f\colon \Omega \to E_+$ which is not Bochner integrable, we define

(15) $$\int_\Omega f d\mu = +\infty.$$

Finally for a μ-measurable function $f\colon \Omega \to \bar{E}$, we define

(16) $$\int_\Omega f d\mu = \int_\Omega f^+ d\mu - \int_\Omega f^- d\mu$$

provided that either one of the integrals on the right-hand side of (16) is finite.

A concise characterization of Bochner integrable functions is given next.

<u>Theorem</u> A. Assume $\mu(\Omega) < +\infty$. A μ-measurable function $f\colon \Omega \to \bar{E}$ is Bochner integrable iff $\int_\Omega \|f\| d\mu < +\infty$.

We then note some elementary properties of the Bochner integral.

<u>Theorem</u> B. If f is μ-Bochner integrable, then

(i) $\int_{A \cup B} f d\mu = \int_A f d\mu + \int_B f d\mu$ $(A, B \in \mathcal{B}$ and $A \cap B = \phi)$,

(ii) $\|\int_\Omega f d\mu\| \leqslant \int_\Omega \|f\| d\mu$,

(iii) for each $e' \in E'$, $e'f := e' \circ f$ is μ-integrable and

$$e'\left(\int_{\Omega} f d\mu \right) = \int_{\Omega} e' f d\mu.$$

The proof of Theorem A and B can be found in ([4]) page 45-46 (with some minor modification). It follows from Theorem A that a continuous function on a compact set is μ-Bochner integrable.

Theorem 4.11. Let $f: \Omega \rightarrow \bar{E}_+$ be μ-Bochner integrable, then

(i) $\int_{\Omega} f d\mu \geqslant 0$,

(ii) $A \subset B$ implies $\int_A f d\mu \leqslant \int_B f d\mu \quad (A, B \in \mathcal{B})$.

Proof. To prove (i), let (s_n) be a sequence of simple functions on Ω such that (9) holds. Thus, for each $n \in N$, we have

$$0 \leqslant \left| |s_n(x)| - |f(x)| \right| \leqslant |s_n(x) - f(x)|$$

and hence

$$0 \leqslant \left\| |s_n(x)| - f(x) \right\| \leqslant \|s_n(x) - f(x)\|,$$

μ-almost everywhere. Therefore (9) implies

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \left\| |s_n| - f \right\| d\mu = 0.$$

By the definition of Bochner integral,

$$\int_{\Omega} f d\mu = \lim_{n \rightarrow +\infty} \int_{\Omega} |s_n| d\mu \geqslant 0.$$

Thus (i) is proved. Part (ii) follows immediately from (i) via a suitable characteristic function. This proves Theorem 4.11.

Next we shall state and prove two theorems concerning monotone convergence in a Banach lattice.

Theorem 4.12. Suppose $f_n : \Omega \rightarrow \bar{E}_+$ is μ-measurable for $n = 1,2,\ldots$, $f_1 \geqslant f_2 \geqslant \ldots \geqslant f \geqslant 0$, $f_n(x) \rightarrow f(x)$ as $n \rightarrow +\infty$ for every $x \in \Omega$, and f_1 is μ-Bochner integrable. Then f is μ-Bochner integrable and

$$(17) \qquad \lim_{n \to +\infty} \int_\Omega f_n d\mu = \int_\Omega f d\mu.$$

Proof. By assumption, we have

$$\lim_{n \to +\infty} \|f_n(x) - f(x)\| = 0 \qquad (\text{μ-almost everywhere})$$

and, for each $n \in N$,

$$\|f_n(x) - f(x)\| \leqslant \|f_n(x)\| + \|f(x)\| \leqslant 2\|f_1(x)\|$$

(μ-almost everywhere). Thus, by the scalar Diminated Convergence Theorem, we get

$$(18) \qquad \lim_{n \to +\infty} \int_\Omega \|f_n - f\| d\mu = 0.$$

Hence f is μ-Bochner integrable and then (17) follows immediately form (18) and Theorem B(ii). This completes the proof of Theorem 4.12.

Theorem 4.13. Let (f_n) be a sequence of μ-Bochner integrable function on Ω and suppose that

(i) $0 \leqslant f_1(x) \leqslant f_2(x) \leqslant \ldots \leqslant f(x) \leqslant +\infty \quad (x \in \Omega)$.

(ii) $f_n(x) \rightarrow f(x)$ as $n \rightarrow +\infty \quad (x \in \Omega)$.

(iii) There exists a positive real number C such that, for

for each n,

$$\int_\Omega \|f_n\| \, d\mu \leqslant C.$$

Then f is μ-Bochner integrable and

$$\lim_{n \to +\infty} \int_\Omega f_n d\mu = \int_\Omega f d\mu.$$

Proof. By assumption (i), $\|f_n(x)\| \nearrow \|f(x)\|$ as $n \to +\infty$ for each $x \in \Omega$. The condition (iii) and the scalar Monotone Convergence Theorem imply $\|f\|$ is μ-integrable. Since

$$\|f_n - f\| \leqslant 2\|f\| \qquad (n = 1,2,\ldots, \text{ and } \mu\text{-a.e.})$$

and

$$\lim_{n \to +\infty} \|f_n - f\| = 0 \qquad (\mu\text{-a.e.}),$$

then, by the scalar Dominated Convergence Theorem, we have

$$\lim_{n \to +\infty} \int_\Omega \|f_n - f\| \, d\mu = 0.$$

This proves Theorem 4.13.

Proposition 4.1. Assume E is an AL-space (see Appendix for its definition). Let $f: \Omega \to E_+$ be μ-Bochner integrable, then

$$\left\| \int_\Omega f d\mu \right\| = \int_\Omega \|f\| \, d\mu.$$

Proof. Since f is positive (this means that $f(x) \geqslant 0$ for all $x \in \Omega$), there exists a sequence of positive simple functions (s_n) on Ω such that

$$\lim_{n \to +\infty} \int_\Omega \|s_n - f\| \, d\mu = 0.$$

We note that

$$| \, \|s_n\| - \|f\| \, | \leqslant \|s_n - f\|$$

whence

$$0 \leqslant \lim_{n \to +\infty} \int_\Omega | \, \|s_n\| - \|f\| \, | d\mu \leqslant \lim_{n \to +\infty} \int_\Omega \|s_n - f\| d\mu = 0.$$

This proves that

$$\int_\Omega \|f\| \, d\mu = \lim_{n \to +\infty} \int_\Omega \|s_n\| \, d\mu.$$

Since E is an AL-space, we have

$$\int_\Omega \|s_n\| \, d\mu = \| \int_\Omega s_n d\mu \|$$

Hence

$$\begin{aligned} \int_\Omega \|f\| \, d\mu &= \lim_{n \to +\infty} \| \int_\Omega s_n d\mu \| \\ &= \| \lim_{n \to +\infty} \int_\Omega s_n d\mu \| \\ &= \| \int_\Omega f d\mu \| . \end{aligned}$$

This proves Proposition 4.1.

5. Subharmonic function

Definition 4.4. Let Ω be a domain in R^m and $\bar{E}$ an extended Banach lattice. A function $f: \Omega \to \bar{E}$ is said to be (vector-valued) subharmonic (s.h.) in Ω if

(i) f is u.s.c. in Ω .

(ii) If $x_0 \in \Omega$ and $r > 0$ satisfies $\bar{B}(x_0,r) \subset \Omega$, then

$$f(x_0) \leqslant s_m^{-1} r^{-m+1} \int_{\partial B(x_0,r)} f(x)d\sigma(x), \tag{19}$$

where $d\sigma(x)$ denotes the surface area element on $\partial B(x_0,r)$.

Remark 4.2 Here and subsequently $s_m = 2\pi^{m/2}/\Gamma(m/2)$ = surface area of the unit sphere in R^m. We admit the function which is identically $-\infty$ to be subharmonic.

Let us discuss the existence of the integral on the right-hand side of (19). First, we observe that f is u.s.c. on $\partial B(x_0,r)$. Then there exists a decreasing sequence (f_n) of continuous functions on $\partial B(x_0,r)$ such that

$$f_n(x) \to f(x) \text{ as } n \to +\infty \quad (x \in \Omega).$$

This implies that

$$f_1^+(x) \geqslant f_2^+(x) \geqslant \ldots \geqslant f^+(x) \geqslant 0,$$

and

$$f_n^+(x) \to f^+(x) \text{ as } n \to +\infty \quad (x \in \Omega).$$

So f^+ is σ-Bochner integrable since f_1 is, cf. Theorem 4.12. Thus

$$0 \leqslant \int_{\partial B(x_0,r)} f^+(x)d\sigma(x) < +\infty.$$

Moreover, by lower semi-continuity of f^-, we get that f^- is

σ-measurable on $\partial B(x_0,r)$ and

$$0 \leqslant \int_{\partial B(x_0,r)} f^-(x)d\sigma(x) \leqslant +\infty.$$

Thus

$$\int_{\partial B(x_0,r)} f(x)d\sigma(x) = \int_{\partial B(x_0,r)} f^+(x)d\sigma(x) - \int_{\partial B(x_0,r)} f^-(x)d\sigma(x)$$

may be finite or $-\infty$. So the integral on the right-hand side of (19) always exists and its value may be finite or $-\infty$.

Definition 4.5. Let Ω be a domain in R^m and $\bar{E}$ an extended Banach lattice. A function $f:\Omega \to \bar{E}$ is called weakly (vector-valued) subharmonic (w.s.h.) in Ω if

(i) f is u.s.c. in Ω.

(ii) If $x_0 \in \Omega$ and $r > 0$ satisfies $\bar{B}(x_0,r) \subset \Omega$, then

(20) $$\forall e' \in E'_+[e'(f(x_0)) \leqslant s_m^{-1} r^{-m+1} \int_{\partial B(x_0,r)} e'(f(x)d\sigma(x)].$$

(We define $e'(-\infty) = -\infty$, where $e' \in E'_+$.)

It is obvious that if f is s.h. in Ω, then f is w.s.h. in Ω. The converse is true if f is continuous or E is an AL-space. To prove this, we need a lemma whose proof is given in the Appendix.

Lemma 4.1. Let E be a Banach lattice and $e_0 \in E$. If $e'(e_0) \geqslant 0$ for all $e' \in E'_+$, then $e_0 \geqslant 0$.

Theorem 4.14. Let $f:\Omega \to E$ be weakly subharmonic. Then f is subharmonic if f is continuous or E is an AL-space.

Proof. We begin by assuming that f is continuous on Ω. Let $x_0 \in \Omega$ and assume that $\bar{B}(x_0,r) \subset \Omega$. Since f is continuous on the compact set $\partial B(x_0,r)$, f is σ-Bochner integrable. Hence (20) and Theorem B(iii) imply that

$$e'(f(x_0)) \leq s_m^{-1} r^{-m+1} \int_{\partial B(x_0,r)} e'(f(x))d\sigma(x)$$

$$= e'(s_m^{-1} r^{-m+1} \int_{\partial B(x_0,r)} f(x)d\sigma(x))$$

for all $e' \in E'_+$. Thus, by Lemma 4.1, we get (19).

On the other hand, suppose that (20) holds and E is an AL-space. We wish to prove that (19) holds. If $f(x_0) = -\infty$, then (19) is trivially true. Now assume that $f(x_0) > -\infty$. We shall show that

$$(21) \qquad \int_{\partial B(x_0,r)} f(x)d\sigma(x) > -\infty.$$

Let (f_n) be a decreasing sequence of continuous functions on Ω such that $f_n(x) \to f(x)$ as $n \to +\infty$ for each $x \in \Omega$. Hence, by (20) and Theorem B(iii), we get that, for each n,

$$(22) \qquad e'(f(x_0)) \leq s_m^{-1} r^{-m+1} \int_{\partial B(x_0,r)} (e'f)(x)d\sigma(x)$$

$$\leq s_m^{-1} r^{-m+1} \int_{\partial B(x_0,r)} (e'f_n)(x)d\sigma(x)$$

$$= e'(s_m^{-1} r^{-m+1} \int_{\partial B(x_0,r)} f_n(x)d\sigma(x)).$$

Thus we get from Lemma 4.1 that

$$f(x_0) \leq s_m^{-1} r^{-m+1} \int_{\partial B(x_0,r)} f_n(x) d\sigma(x)$$

and so

$$f(x_0)^- \geq s_m^{-1} r^{-m+1} \int_{\partial B(x_0,r)} f_n(x)^- d\sigma(x).$$

Moreover,

$$(23) \qquad \left\| \int_{\partial B(x_0,r)} f_n^-(x) d\sigma(x) \right\| \leq s_m r^{m-1} \|f(x_0)^-\| := C.$$

Since E is an AL-space, by Proposition 4.1, we have

$$(24) \qquad \int_{\partial B(x_0,r)} \|f_n^-(x)\| \, d\sigma(x) = \left\| \int_{\partial B(x_0,r)} f_n^-(x) d\sigma(x) \right\| .$$

Thus (23) and (24) yield

$$\int_{\partial B(x_0,r)} \|f_n^-(x)\| \, d\sigma(x) \leq C.$$

Hence, by Theorem 4.13, we get (21). This proves that f is σ-Bochner integrable. Then (20) and Lemma 4.1 imply (19). This proves Theorem 4.14.

We note some properties of subharmonic functions.

<u>Theorem</u> 4.15.

(i) If $f_1,\dots,f_k$ are s.h. in Ω and $t_1,\dots,t_k$ are non-negative real numbers, then $f = \sum_{n=1}^{k} t_n f_n$ is s.h.

(ii) If $f_1,\dots,f_k$ are s.h. in Ω then so is $f(x) = \sup_{n=1 \text{ to } k} f_n(x)$.

(iii) If $f \in C^2$ in Ω, then f is s.h. in Ω iff $\Delta f \geq 0$ in Ω.

<u>Proof</u>. (i) is obvious from Definition 4.4 and Theorem 4.6. To prove (ii), we note that $f_1 \vee f_2$ is u.s.c. by Theorem 4.7. Suppose $\bar{B}(x_0,r) \subset \Omega$. Then, by Theorem 4.11(i),

$$f_1(x_0) \leqslant s_m^{-1} r^{-m+1} \int_{\partial B(x_0,r)} f_1(x) d\sigma(x)$$

$$\leqslant s_m^{-1} r^{-m+1} \int_{\partial B(x_0,r)} (f_1 \vee f_2)(x) d\sigma(x);$$

$$f_2(x_0) \leqslant s_m^{-1} r^{-m+1} \int_{\partial B(x_0,r)} f_2(x) d\sigma(x)$$

$$\leqslant s_m^{-1} r^{-m+1} \int_{\partial B(x_0,r)} (f_1 \vee f_2)(x) d\sigma(x).$$

Therefore

$$(f_1 \vee f_2)(x_0) \leqslant s_m^{-1} r^{-m+1} \int_{\partial B(x_0,r)} (f_1 \vee f_2)(x) d\sigma(x).$$

The general case follows by induction.

Finally, we prove (iii). Let $f \in C^2(\Omega)$ and $e' \in E'_+$. Thus the functional $e' \circ f$ is also C^2 in Ω. By a scalar property of s.h., we have

$$e' \circ f \text{ is s.h. in } \Omega \text{ iff } \Delta(e' \circ f) = e'(\Delta f) \geqslant 0.$$

Hence, by Lemma 4.1 and Theorem 4.14, we can conclude that

$$f \text{ is s.h. in } \Omega \text{ iff } \Delta f \geqslant 0.$$

This prove (iii).

Next, we proceed to study vector-valued harmonic functions. Let Ω be a domain in R^m and E a Banach lattice. Let $f:\Omega \longrightarrow E$ be a function. If f is C^2 in Ω and $\Delta f = 0$ there, then f is said to be <u>(vector-valued) harmonic</u> in Ω where the Laplacian operator Δ is defined to be

$$\Delta = \frac{\partial^2}{\partial x_1^{\,2}} + \dots + \frac{\partial^2}{\partial x_m^{\,2}}.$$

Every harmonic function is subharmonic since it is continuous and weakly subharmonic (see also Theorem 4.14). Let $x_0 \in \Omega$ and suppose that $\bar{B}(x_0,r) \subset \Omega$. The Poisson Kernel $K(x,\xi)$ in $B(x_0,r)$ is given by

$$(25) \qquad K(x,\xi) = \frac{1}{s_m}\frac{r^2 - |\xi - x_0|^2}{r|\xi - x|^m} \qquad ((x,\xi) \in \partial B(x_0,r) \times B(x_0,r)).$$

It is well-known that $K(x,\xi)$ is harmonic in Ω for each fixed x. Now we shall prove the Poisson integral formula for harmonic functions.

<u>Theorem</u> 4.16. It f is harmonic in $B(x_0,r)$ and continuous in $\bar{B}(x_0,r)$, then for each $\xi \in B(x_0,r)$ we have

$$(26) \qquad f(\xi) = \int_{\partial B(x_0,r)} K(x,\xi) f(x)\, d\sigma(x).$$

<u>Proof</u>. The integral on the right-hand side of (26) exists for each $\xi \in B(x_0,r)$ since the integrand is continuous on a compact set. One can see that this integral is equal to $f(\xi)$ by utilizing a corollary of the Hahn-Banach Theorem and the scalar Poisson integral formula. In fact, for each $e' \in E'_+$, we have

$$e'(f(\xi)) = (e' \circ f)(\xi)$$
$$= \int_{\partial B(x_0,r)} K(x,\xi)(e' \circ f)(x)d\sigma(x)$$
$$= \int_{\partial B(x_0,r)} e'(K(x,\xi)f(x))d\sigma(x)$$
$$= e'\left(\int_{\partial B(x_0,r)} K(x,\xi)f(x)d\sigma(x)\right)$$

Hence, by a corollary to the Hahn-Banach Theorem, we get (26). This proves Theorem 4.16.

We proceed to discuss that the Poisson integral can be used to solve the problem of Dirichlet for a ball. We have more precisely

Theorem 4.17. Suppose that the function $f(x)$ is continuous on $\partial B(x_0,r)$, and for $\xi \in B(x_0,r)$, let

$$u(\xi) = \int_{\partial B(x_0,r)} K(x,\xi)f(x)d\sigma(x).$$

Then $u(\xi)$ solves the problem of Dirichlet for $B(x_0,r)$, i.e. u is harmonic in $B(x_0,r)$ and continuous on $\bar{B}(x_0,r)$, with boundary values $f(x)$.

Proof. The proof of this theorem is the same as in the scalar case. It can be found, for example, in Hayman ([8], page 32).

Now, we shall state some equivalences for scalar-valued subharmonic functions. The proof of this theorem can be found in Krantz ([10], page 71-72).

Theorem C. Let $\Omega \subset R^m$ be a domain and $f: \Omega \to R \cup \{-\infty\}$ be u.s.c. Then the following statements are equivalent.

(i′) f is subharmonic in Ω.

(ii′) For every x_0 and $r > 0$ satisfying $\bar{B}(x_0,r) \subset \Omega$ and for each real-valued continuous h on $\bar{B}(x_0,r)$ that is harmonic on $B(x_0,r)$ and satisfies $h \geqslant f$ on $\partial B(x_0,r)$, we have $h \geqslant f$ on $B(x_0,r)$, too.

(iii′) If $K \subset \Omega$ is compact and $h \in C(K)$ is harmonic on the interior of K and majorizes f on ∂K, then $h \geqslant f$ on K.

(iv′) f is the limit of decreasing sequence of subharmonic functions.

(v′) If $\mathrm{dist}(x_0, \Omega^c) > \delta > 0$ and μ is any positive Borel measure on $[0,\delta]$ then

$$(*) \quad f(x_0) \int_0^\delta d\mu(r) \leqslant \frac{1}{s_m} \int_0^\delta \int_{\partial B(0,1)} f(x_0 + r\xi) d\sigma(\xi) d\mu(r).$$

(vi′) For each $\delta > 0$ and $x_0 \in \Omega$ with $\mathrm{dist}(x_0, \Omega^c) > \delta$, there exists one positive Borel measure μ on $[0,\delta]$ with $(\mathrm{supp}\ \mu) \cap (0,\delta] \neq \phi$ so that (*) holds.

(vii′) If $\bar{B}(x_0,r) \subset \Omega$ and $K(x,\xi)$ is the Poisson kernel for $B(x_0,r)$ then

$$f(\xi) \leqslant \int_{\partial B(x_0,r)} K(x,\xi) f(x) d\sigma(x), \quad (\forall\, \xi \in B(x_0,r)).$$

(viii′) If $\bar{B}(x_0,r) \subset \Omega$ then

$$f(x_0) \leqslant \frac{1}{r^m v_m} \int_{B(x_0,r)} f(x) dx$$

where v_m denotes the volume of the unit ball in R^m.

The following theorem is a generalization of Theorem C to the vector-valued case. All statements are almost the same as of Theorem C and its proof will be based on it.

Theorem 4.18. Let $\Omega \subset R^m$ be a domain and E a Banach lattice. Let $f:\Omega \to \bar{E}$ be u.s.c. Assume further that either f is continuous or E is an AL-space. Then the following statements are equivalent:

(i) f is subharmonic on Ω.

(ii) For every $x_0 \in \Omega$ and $r > 0$ satisfying $\bar{B}(x_0,r) \subset \Omega$ and for every vector-valued continuous function h on $\bar{B}(x_0,r)$ that is harmonic on $B(x_0,r)$ and satisfies $h \geqslant f$ on $\partial B(x_0,r)$, we have $h \geqslant f$ on $B(x_0,r)$, too.

(iii) If $K \subset \Omega$ is compact and $h \in C(K,E)$ is harmonic in the interior of K and majorizes f on ∂K then $h \geqslant f$ on K.

(iv) f is the limit of a decreasing sequence of subharmonic functions.

(v) If $\operatorname{dist}(x_0,\Omega^c) > \delta > 0$ and μ is any positive Borel measure on $[0,\delta]$ then

$$(*) \quad \int_0^\delta d\mu(r)f(x_0) \leqslant \frac{1}{s_m} \int_0^\delta \int_{\partial B(0,1)} f(x_0+r\xi)d\mu(\xi)d\mu(r).$$

(vi) For each $\delta > 0$ and $x_0 \in \Omega$ with $\operatorname{dist}(x_0, \Omega^c) > 0$, there exists one positive Borel measure μ on $[0,\delta]$ with $(\operatorname{supp} \mu) \cap (0,\delta] \neq \phi$ such that $(*)$ holds.

(vii) If $\bar{B}(x_0,r) \subset \Omega$ and $K(x,\xi)$ is the Poisson kernel for $B(x_0,r)$ then

$$f(\xi) \leqslant \int_{\partial B(x_0,r)} K(x,\xi)f(x)d\sigma(x)$$

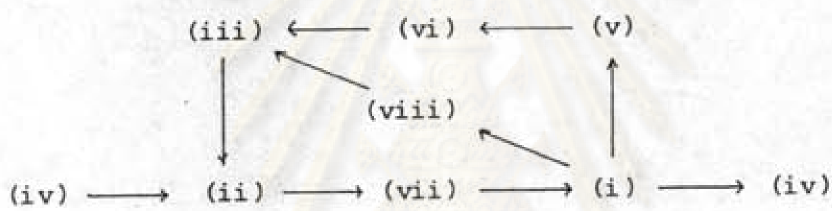
for all $\xi \in B(x_0,r)$.

(viii) If $\bar{B}(x_0,r) \subset \Omega$, then

$$f(x_0) \leqslant \frac{1}{r^m v_m} \int_{B(x_0,r)} f(x)dx$$

where v_m denotes the volume of the unit ball in R^m.

Proof. The scheme of the proof is the following:



(v) $\Rightarrow$ (vi). Trivial. Choose μ to be the Lebesgue measure on $[0,\delta]$.

(vi) $\Rightarrow$ (iii). Let K be a compact subset of Ω. Let $h \in C(K,E)$ which is harmonic on the interior of K and majorizes f on ∂K. Let $e' \in E'_+$ and apply e' to both side of (*) of Theorem 4.18. Then, by Theorem C ((vi$'$) $\Rightarrow$ (iii$'$)), we have

(27) $$e' \circ h \geqslant e' \circ f \quad \text{on } K.$$

Since (27) is true for all $e' \in E'_+$ then, by Lemma 4.1, $h \geqslant f$ on K.

(iii) $\Rightarrow$ (ii). Trivial.

(ii) $\Rightarrow$ (viii). Assume (ii)holds. Let f_n on $\bar{B}(x_0,r)$ be continuous with $f_n \searrow f$ (Definition 4.1). Let h_n be the solution to the Dirichlet problem on $B(x_0,r)$ with boundary data $f_n|_{\partial B(x_0,r)}$.

Then, for $\xi \in B(x_0,r)$,

(28) $$f(\xi) \leqslant h_n(\xi) = \int_{\partial B(x_0,r)} K(x,\xi)f_n(x)d\sigma(x).$$

The assumption that f is continuous or E is an AL-space implies

(29) $$\int_{\partial B(x_0,r)} K(x,\xi)f(x)d\sigma(x) > -\infty.$$

(See the proof of Theorem 4.14.) Hence, by Theorem 4.12 and 4.13, the right hand side of (28) tends to (29) as $n \to +\infty$.

(vii) $\Rightarrow$ (i). Set $\xi = x_0$ in (vii).

(i) $\Rightarrow$ (iv). For each $n \in N$, define $f_n = f$.

(iv) $\Rightarrow$ (ii). Let $h \geqslant f$ on $\partial B(x_0,r)$, $\bar{B}(x_0,r) \subset \Omega$, and h harmonic. Let (f_n) be a decreasing sequence of subharmonic functions that converge pointwise to f. Let $e' \in E'_+$, then $(e'\circ f_n)$ is also a decreasing sequence of s.h. functions on Ω such that $(e'\circ f_n) \to (e'\circ f)$ as $n \to +\infty$ pointwise. Hence, by Theorem C ((iv′) $\Rightarrow$ (ii′)),

(30) $$e'\circ h \geqslant e'\circ f \quad \text{on } B(x_0,r).$$

Since (30) is true for all $e' \in E'_+$, then Lemma 4.1 yields

$$h \geqslant f \quad \text{on } B(x_0,r).$$

(i) $\Rightarrow$ (viii). Suppose $\bar{B}(x_0,\delta) \subset \Omega$. Since

$$f(x_0) \leqslant \frac{1}{s_m r^{m-1}} \int_{\partial B(x_0,r)} f(x)d\sigma(x) \qquad (0 < r < \delta),$$

then

$$\int_0^{\delta} s_m r^{m-1} f(x_0)dr \leqslant \int_0^{\delta} \int_{\partial B(x_0,r)} f(x)d\sigma(x)dr$$

and

$$\frac{s_m r^m}{m} f(x_0) \leqslant \int_{B(x_0,\delta)} f(x)dx,$$

Hence

$$f(x_0) \leqslant \frac{m}{s_m r^m} \int_{B(x_0,\delta)} f(x)dx = \frac{1}{v_m r^m} \int_{B(x_0,\delta)} f(x)dx.$$

(viii) $\Longrightarrow$ (iii). Same as (vi) $\Longrightarrow$ (iii).

(i) $\Longrightarrow$ (v). Integrate out (i) against μ. This completes the proof of Theorem 4.18.

Remark 4.3. We use the assumption that f is continuous or E is an Al-space only in the proof of the implication (ii) $\Longrightarrow$ (viii). The other implications of Theorem 4.18 are valid without using this assumption.

Theorem 4.19. Let $f:\Omega \to E \cup \{-\infty\}$ be subharmonic and not identically $-\infty$. Then f is locally Bochner integrable on Ω. In particular, $P_f = \{x \in \Omega | \ f(x) = -\infty\}$ has Lebesgue measure zero.

Proof. Let $U = \{x \in \Omega | \ f$ is Bochner integrable on a neighborhood of $x\}$. Let $x_0 \in \Omega \setminus P_f$. Then, for r sufficiently small,

$$(31) \qquad -\infty < f(x_0) \leqslant \frac{1}{s_m r^{m-1}} \int_{\partial B(x_0,r)} f(x)d\sigma(x) < +\infty$$

where the third inequality comes from the fact that the Bochner integral of u.s.c. functions on a compact set is always finite or $-\infty$. By integrating (31) with respect to r, we get

$$(32) \qquad -\infty < f(x_0) \leqslant \frac{1}{v_m r^m} \int_{B(x_0,r)} f(x)dx < +\infty.$$

So we know that U is nonempty, open, and $U^c \subset P_f$. To see that $U^c \subset P_f$, let $y \in U^c$ and suppose that $f(y) > -\infty$. Then there is a neighborhood of y such that (32) holds. This contradicts to the definition of y.

Next, we shall show that U^c is open. If $U^c = \phi$ we are done. For, suppose not, let $x_0 \in U^c$. Let $r > 0$ be such that $\bar{B}(x_0,r) \subset \Omega$. We shall prove that there exists $0 < r_0 < r$ such that $f(x) \equiv -\infty$ on $B(x_0,r_0)$. Suppose that we can't find such an r_0 then there exists a sequence (x_n) of points in $B(x_0,r)$ such that $x_n \longrightarrow x_0$ as $n \rightarrow +\infty$ and $f(x_n) > -\infty$ for all n. Let n_0 be so large that

$$x_0 \in B(x_{n_0},r_1) \subset B(x_0,r),$$

for some $r_1 < r$. Let $r_2 > 0$ be so small that

$$B(x_0,r_2) \subset B(x_{n_0},r_1) \subset B(x_0,r).$$

Hence

$$+\infty = \int_{B(x_0,r_2)} \|f(x)\| \, dx \leqslant \int_{B(x_{n_0},r_1)} \|f(x)\| \, dx < +\infty,$$

where the second inequality follows from the fact that $f(x_{n_0}) > -\infty$ and f is subharmonic at x_0. This is a contradiction. Hence $f(x) \equiv -\infty$ in some neighborhood of x_0. So U^c is open. Since Ω is connected, $U^c = \phi$. Then f is locally Bochner integrable on Ω. This proves Theorem 4.19.

Theorem D. Let $f:\Omega \to R \cup \{-\infty\}$ be subharmonic and not identically $-\infty$, and $\varphi \in C_c^\infty(\Omega)$, $\varphi \geqslant 0$; then

$$\int_\Omega f\Delta\varphi dx \geqslant 0.$$

The proof of Theorem D can be found in ([10]) page 74. We shall use it to prove

Theorem 4.20. Let $f:\Omega \to E \cup \{-\infty\}$ be subharmonic and not identically $-\infty$, and $\varphi \in C_c^\infty(\Omega)$, $\varphi \geqslant 0$; then

$$\int_\Omega f\Delta\varphi dx \geqslant 0.$$

This theorm generalizes Theorem 4.15(iii) since it means that if f is subharmonic then $\Delta f \geqslant 0$ in the distribution sense.

Proof. Let K be the support of φ. Then, by Theorem 4.19,

$$-\infty < \int_\Omega f\Delta\varphi dx = \int_K f\Delta\varphi dx < +\infty.$$

Let $e' \in E'_+$, then

(33) $$e'(\int_\Omega f\Delta\varphi dx) = \int_\Omega (e' \circ f)\Delta\varphi dx \geqslant 0$$

where the last inequality comes from Theorem D. Since (33) is true for all $e' \in E'_+$, Lemma 4.1 implies

$$\int_\Omega f\Delta\varphi dx \geqslant 0.$$

This proves Theorem 4.20.

<u>Theorem</u> 4.21. If f is harmonic on Ω, then $\|f\|^p$ is s.h. for all $p \geqslant 1$.

<u>Proof</u>. Let $x_0 \in \Omega$ and suppose that $\bar{B}(x_0,r) \subset \Omega$. Then

$$f(x_0) = \frac{1}{s_m r^{m-1}} \int_{\partial B(x_0,r)} f(x)d\sigma(x).$$

Hence, by Theorem B(ii),

$$\|f(x_0)\| \leqslant \frac{1}{s_m r^{m-1}} \int_{\partial B(x_0,r)} \|f(x)\| d\sigma(x).$$

This proves that $\|f\|$ is s.h. on Ω. Thus, by the scalar Jensen's inequality, we get $\|f\|^p$ is s.h.

<u>Theorem</u> 4.22. If $f:\Omega \subset \mathbb{C} \rightarrow E$ is holomorphic, then $|f|$ and $\|f\|^p$ is subharmonic for all $p \geqslant 1$.

<u>Proof</u>. Let $e' \in E'_+$. Then $e' \circ f$ is holomorphic. Hence $e' \circ f$ is harmonic. Since the relation

$$\Delta(e' \circ f) = e'(\Delta f) = 0,$$

is true for all $e' \in E'_+$. Then f is harmonic. It follows from Theorem 4.15(ii) that $|f| = f \vee (-f)$ is s.h. and then Theorem 4.21 implies that $\|f\|^p$ is s.h. for all $p \geqslant 1$. This proves Theorem 4.22.

6. The hyperplane means of positive vector-valued subharmonic functions

Let D denote the half-space $R^m \times (0,+\infty)$, for $m = 1,2,\ldots,$ a typical point in D being of the form

$$P = (X,y) = (x_1,\ldots,x_m,y), \quad y > 0.$$

We put

$$X^2 = |X|^2 = x_1^{\,2} + \ldots + x_m^{\,2}, \quad dX = dx_1\ldots dx_m.$$

Let E be a Banach lattice and $u: D \rightarrow E_+$ be a positive vector-valued subharmonic function. We define a function $M(u,\cdot)$ on $(0,+\infty)$ by writing

$$M(u,y) = \int_{R^m} u(X,y)dX,$$

where the integral being taken in the sense of Bochner with respect to the Lebesgue measure. Thus $M(u,y)$ exists and its value is an element of $E_+ \cup \{+\infty\}$.

The behaviour of these hyperplane means has been studied by a number of authors (see, for instance, [3,11,16,19]) but only in the scalar case, i.e. the Banach lattice E is the real field R. In this section, we shall study the behaviour of these hyperplane means in a more general setting, i.e. in the vector-valued case. The following results on the behaviour of $M(u,\cdot)$ (for a given non-negative real-valued subharmonic $u:D \rightarrow R_+$) are known.

Theorem E. If $M(u,\cdot)$ is bounded on $(0,+\infty)$, it is decreasing and convex (Kuran, [11], Theorem 4).

Theorem F. If $M(u,\cdot)$ is bounded on $(0,+\infty)$, and in addition u^r is subharmonic for some r satisfying $0 < r < 1$, then $M(u,y) \to 0$ as $y \to +\infty$ (Kuran, [11], Theorem 7).

Theorem G. If u is (non-negative and) harmonic on D and $M(u,y)$ is finite everywhere then $M(u,\cdot)$ is constant (Kuran, [11], Theorem 6).

Theorem H. If $M(u,\cdot)$ is constant and finite, then u is harmonic on D (Kuran, [12]).

Theorem I. If $M(u,\cdot)$ is bounded on each interval of the form $(0,a]$ and $M(u,y) = o(y^n)$ as $y \to +\infty$, then $M(u,\cdot)$ is decreasing and convex (Flett, [5], Theorem 2).

We note some results concerning the behaviour of $M(u,\cdot)$ for a given positive vector-valued subharmonic $u:D \to E_+$, which are similar to those scalar cases and our proofs will be based on them.

Theorem 4.23. If $M(u,\cdot)$ is norm bounded on $(0,+\infty)$, it is decreasing and convex.

Proof. Since $M(u,y)$ is norm bounded on $(0,+\infty)$, then u is Bochner integrable in R^m for each fixed y on $(0,+\infty)$. Let $e' \in E'_+$, by Theorem B(iii) of §4, we have

$$M(e' \circ u, y) = e'(M(u,y)) \quad \text{for all } y \in (0,+\infty).$$

Hence $M(e' \circ u, \cdot)$ is also bounded in $(0, +\infty)$. By Theorem E, we get $M(e' \circ u, \cdot)$ is decreasing and convex. Since e' is any positive element of E', then, by Lemma 4.1, $M(u,y)$ is also decreasing and convex. This proves Theorem 4.23.

<u>Theorem</u> 4.24. Let $M(u,\cdot)$ be norm bounded on $(0,+\infty)$ and in addition $(e' \circ u)^r$ be subharmonic for some r satisfying $0 < r < 1$ and for all $e' \in E'$. If $M(u,y) \to e_0 \in E$ as $y \to +\infty$, then $e_0 = 0$.

<u>Proof</u>. Let $e' \in E'$, then $M(e' \circ u, \cdot) = e'M(u,\cdot)$ is bounded on $(0,+\infty)$ and, by assumption, there exists $r \in (0,1)$ such that $(e' \circ u)^r$ is subharmonic. Thus, by Theorem F,

(1) $$M(e' \circ u, y) \to 0 \quad \text{as } y \to +\infty.$$

It follows from (1), Theorem B(iii) of §4, and the continuity of e' that

(2) $$0 = \lim_{y \to +\infty} M(e' \circ u, y) = \lim_{y \to +\infty} e'(M(u,y)) = e'(\lim_{y \to +\infty} M(u,y)).$$

Since (2) is true for all $e' \in E'$, then

$$\lim_{y \to +\infty} M(u,y) = e_0 = 0.$$

This proves Theorem 4.24.

<u>Theorem</u> 4.25. If u is (positive) harmonic on D and $M(u,y)$ is finite everywhere, then $M(u,\cdot)$ is constant.

<u>Proof</u>. Let $e' \in E'$, then, by assumption, $e' \circ u$ is harmonic on D

and $M(e' \circ u, y)$ is finite everywhere. Hence, Theorem G implies

(3) $$M(e' \circ u, \cdot) = e'(M(u,\cdot)) = \text{constant}.$$

Since (3) is true for all $e' \in E'$, then $M(u,\cdot)$ is constant.

Theorem 4.26. If $M(u,\cdot)$ is constant and finite then u is harmonic on D.

Proof. Let $e' \in E'$, then, by assumption, $M(e' \circ u,\cdot)$ is constant and finite. Hence, by Theorem H, $e' \circ u$ is harmonic on D. Since e' is arbitrary, $e' \circ u$ is harmonic on D for all $e' \in E'$. Thus, by [7] (Chapter II, §3.3), u is harmonic on D.

Theorem 4.27. If $M(u,\cdot)$ is norm bounded on each interval of the form $(0,a]$ and $\|M(u,y)\| = o(y^n)$ as $y \to +\infty$. THen $M(u,\cdot)$ is decreasing and convex.

Proof. Just apply a positive linear functional to u and use Theorem I together with Lemma 4.1 of §5, we get the required results.

We now give some results on the Poisson integral, which are necessary for the proof of our next result, i.e. Theorem 4.28. In D, the Poisson kernel P is given by

$$P(X,y) = 2s_{m+1}^{-1}y(X^2 + y^2)^{-\frac{1}{2}(m+1)},$$

where s_{m+1} denotes the surface area of the unit sphere in R^{m+1}. It is well-known that

$$\int_{R^m} P(X,y)dX = 1.$$

Let $f: D \rightarrow E$ be a function defined at least on the hyperplane of the equation $y = a$ such that the function

$$(4) \qquad X \mapsto (X^2 + 1)^{-\frac{1}{2}(m+1)} f(X,a),$$

is Bochner integrable with respect to the Lebesgue measure in R^m. The Poisson integral I_f^a in $D_a := R^m \times (a, +\infty)$ of the restriction of f to the hyperplane is given by

$$(5) \qquad I_f^a(X,y) = \int_{R^m} P(X - Z, y - a) f(Z,a) dZ$$

for all $(X,y) \in D_a$. It follows from (4) that $I_f^a(X,y)$ is an element of E for all $(X,y) \in D_a$. We note that I_f^a is harmonic in D_a. To see this, let $e' \in E'$, by Theorem B(iii) of §4, we have

$$(6) \qquad e'(I_f^a(X,y)) = \int_{R^m} P(X - Z, y - a) e'f(Z,a) dZ.$$

The right hand side of (6) is harmonic in D_a (Flett, [6], Theorem 6). Hence $e'I_f^a$ is harmonic in D_a for all $e' \in E'$. Thus I_f^a is also harmonic in D_a ([7], Chapter II, §3.3).

A Banach lattice E is said to be order continuous if every order bounded increasing sequence in E converges in the norm topology of E. It can be proved that if (x_n) is a decreasing sequence in an order continuous Banach lattice E with $\inf_n \{x_n\} = 0$, then $\lim_{n \rightarrow +\infty} x_n = 0$ ([15], page 7). A simple example of an order continuous Banach lattice is the space $L_p(\mu)$ with $1 \leqslant p < +\infty$. We need the concept of an order continuous Banach lattice in the following theorem.

Theorem 4.28. Let E be an order continuous Banach lattice and $u: D \to E$ be continuous function. Suppose that

(i) u is a positive subharmonic function in D,

(ii) u has no harmonic majorant in $R^m \times (1, +\infty)$,

(iii) $M(u, \cdot)$ is locally norm bounded in $(0, +\infty)$.

Then the set

$$\{M(u,y)/y^{m+1} \mid y \in (0,+\infty)\}$$

is not majorized.

We need the following three lemmas for proving Theorem 4.28.

Lemma 4.2. Suppose that $u: B(x_0,r) \subset R^m \to E_+$ is harmonic. Then for $|\xi - x_0| = \rho < r$ we have

$$(7) \qquad \frac{(r-\rho)r^{m-2}}{(r+\rho)^{m+1}} u(x_0) \leqslant u(\xi) \leqslant \frac{(r+\rho)r^{m-2}}{(r-\rho)^{m-1}} u(x_0).$$

We have, by Theorem 4.16 in §5 with $\rho < R < r$,

$$(8) \qquad u(\xi) = \int_{\partial B(x_0,R)} K(x,\xi)u(x)d\sigma(x)$$

where $K(x,\xi)$ is given by

$$(9) \qquad K(x,\xi) = \frac{1}{s_m} \frac{R^2 - |\xi - x_0|^2}{R|\xi - x|^m}.$$

Hence (9) implies that

$$(10) \qquad \frac{R^2 - \rho^2}{R(R+\rho)^m} \leqslant s_m K(x,\xi) \leqslant \frac{R^2 - \rho^2}{R(R-\rho)^m}.$$

Also setting $\rho = 0$ in (10) and using (8) we obtain

$$u(x_0) = \frac{1}{s_m R^{m-1}} \int_{\partial B(x_0,R)} u(x)\,d\sigma(x). \tag{11}$$

By multiplying (10) by $u(x)$, integrating (10) over $\partial B(x_0,R)$, and using (8) and (11), we get (7) with R instead of r. We obtain (7) by allowing R tend to r from below.

Lemma 4.3. Let E be an order continuous Banach lattice. Suppose that (u_n) is an increasing sequence of vector-valued positive harmonic functions from a domain $\Omega \subset R^m$ into E. Then either the sequence $(u_n(x))$ is not majorized for each $x \in \Omega$ or

$$u_n(x) \longrightarrow u(x) \text{ as } n \to +\infty$$

uniformly in every compact subset of Ω and u is harmonic in Ω.

Proof. Let $x_0 \in \Omega$ and suppose that the sequence $(u_n(x_0))$ is majorized. Since E is order continuous, then $(u_n(x_0))$ must converge (in norm) to some point of E, $u(x_0)$ say. Then we have for $n > k > M_0(\varepsilon)$

$$\|u_n(x_0) - u_k(x_0)\| < \varepsilon.$$

Suppose $B(x_0,r) \subset \Omega$. We note that $u_n - u_k$ is a positive harmonic function then, by Lemma 4.2, we have for $|x - x_0| = \rho < r$

$$0 < \|u_n(x) - u_k(x)\| \leqslant \frac{\varepsilon(r+\rho)r^{m-2}}{(r-\rho)^{m-1}} \qquad (n > k > M_0(\varepsilon)),$$

Thus $u_n(x)$ converges uniformly in $\bar{B}(x_0,\rho)$ for $\rho < r$ to a limit $u(x)$. Thus $u(x)$ is finite and continuous in $\bar{B}(x_0,\rho)$.

Similarly from the left hand inequality of Lemma 4.2 we see that if $(u_n(x_0))$ is not majorized then $(u_n(x))$ is not majorized in $B(x_0,r)$. Thus the sets where $(u_n(x))$ is majorized and $(u_n(x))$ is not majorized are both open in Ω and so one of these sets must be empty.

If the sequence $(u_n(x))$ is majorized for each $x \in \Omega$, we put

$$u(x) = \lim_{n\to+\infty} u_n(x) \qquad (x \in \Omega).$$

Since the convergence is uniformly on every compact subset of Ω, then u is continuous. We aim to prove that u is harmonic in Ω. Let $x_0 \in \Omega$ and suppose that $\bar{B}(x_0,r) \subset \Omega$. Then, by Theorem 4.16, for each $n \in N$ and each $\xi \in B(x_0,r)$ we have

$$u_n(\xi) = \int_{\partial B(x_0,r)} K(x,\xi)u_n(x)d\sigma(x),$$

where $K(x,\xi)$ is the Poisson kernel given by equation (9). Letting n tend to infinity and applying Theorem 4.13, we deduce that

$$u(\xi) = \int_{\partial B(x_0,r)} K(x,\xi)u(x)d\sigma(x)$$

and in view of Theorem 4.17 we deduce that $u(\xi)$ is harmonic on $B(x_0,r)$. Since x_0 is arbitrary, then $u(x)$ is harmonic in Ω. This proves Lemma 4.3.

Let $f(x)$ be a vector-valued continuous function on $\partial B(x_0,r)$. For each $\xi \in B(x_0,r) := B$, we put

$$PI(f,B)(\xi) = \int_{\partial B} K(x,\xi)f(x)d\sigma(x).$$

Then, by Theorem 4.17, PI(f,B) is harmonic in $B(x_0,r)$ and continuous on $\bar{B}(x_0,r)$ with boundary values f(x). The following results concerning PI(f,B) are known (Helms, [9], page 69).

<u>Theorem</u> J. Let $f: \Omega \subset R^m \to R$ be a continuous subharmonic function and let $\bar{B}(x_0,r) \subset \Omega$. If $B = B(x_0,r)$, define

$$g = \begin{cases} PI(f,B) & \text{on } B, \\ f & \text{on } \Omega \setminus B. \end{cases}$$

Then (i) $f \leqslant g$ on Ω, (ii) g is harmonic on B, and (iii) g is subharmonic on Ω.

It should be noted that Theorem J is also true in the vector-valued case. By using functionals, one can see that (i) follows from Lemma 4.1, (ii) follows from ([7], Chapter II, §3.3), and (iii) follows from Theorem 4.14 (g is continuous since f is).

<u>Lemma</u> 4.4. Let $\Omega \subset R^m$ be a domain, E an order continuous Banach lattice, and let $u: \Omega \to E_+$ be continuous subharmonic function. If u has a harmonic majorant in Ω, then u has a least harmonic majorant in Ω.

<u>Proof</u>. Let (B_n) be sequence of open balls in R^m such that

(i) $\bar{B}_n \subset \Omega$ for all n,

(ii) $\bigcup_{n=1}^{\infty} B_n = \Omega$,

(iii) for each n, B_n occurs infinitely often in the sequence (B_n).

Define

$$u_1 = \begin{cases} u & \text{on } \Omega \setminus B_1, \\ PI(u,B_1) & \text{on } B_1, \end{cases}$$

and, inductively,

$$u_n = \begin{cases} u_{n-1} & \text{on } \Omega \setminus B_n, \\ PI(u_{n-1},B_n) & \text{on } B_n. \end{cases}$$

By the remark at the end of Theorem J, we have, by induction, $u_{n-1} \leqslant u_n$ on Ω, u_n is harmonic on B_n, and u_n is subharmonic on Ω. Let h be a harmonic function on Ω such that $u \leqslant h$ on Ω. On B_1,

$$u_1 = PI(u,B_1) \leqslant PI(h,B_1) = h.$$

Moreover, $u_1 = u \leqslant h$ on $\Omega \setminus B_1$. Thus $u_1 \leqslant h$ on Ω and by induction, $u_n \leqslant h$ on Ω for all n. So the sequence $(u_n(x))$ is majorized for each $x \in \Omega$. Furthermore, since the sequence (u_n) is increasing and E is order continuous, we can define

$$u_\infty(x) = \lim_{n \to +\infty} u_n(x).$$

We note that $u_\infty(x) \in E$ for all $x \in \Omega$. Next we shall prove that u_∞ is harmonic on Ω. Consider a fixed n. Since B_n occurs infinitely often in the sequence of balls, there is a subsequence (B_{n_k}) such that $B_n = B_{n_k}$ for all $k \geqslant 1$. On B_n, we have $u_\infty = \lim_{k \to +\infty} u_{n_k}$. Since u_{n_k} is harmonic on $B_{n_k} = B_n$ and the sequence $(u_n(x))$ is majorized for each $x \in \Omega$, u_∞ is also harmonic on B_n by Lemma 4.3.

Therefore u_∞ is harmonic on Ω since it is harmonic on each balls B_n.

Finally, we shall prove that u_∞ is the least harmonic majorant of u in Ω. Let $h \geqslant u$ be a harmonic function on Ω. We have already proved that $u_n \leqslant h$ on Ω for all n. Since E is order continuous and $u_n \nearrow u_\infty$, $u_\infty \leqslant h$ on Ω. This proves Lemma 4.4.

Now we are ready to prove Theorem 4.28 by using the above three lemmas.

<u>Proof of Theorem</u> 4.28. Let $\eta > 0$. (For the moment, think of η as fixed.) Define I_u in $R^m \times (1,+\infty)$ by

$$I_u(X,y) = \frac{2(y-1)}{s_{m+1}} \int_{R^m} \frac{u(Z,1)dZ}{\{|X-Z|^2+(y-1)^2\}^{(m+1)/2}}$$

and define J_u in $R^m \times (-\infty,2\eta+1)$ by

$$J_u(X,y) = \frac{2(2\eta+1-y)}{s_{m+1}} \int_{R^m} \frac{u(Z,2\eta+1)dZ}{\{|X-Z|^2+(2\eta+1-y)^2\}^{(m+1)/2}}$$

where s_{m+1} denotes the surface area of the unit sphere in R^{m+1}. Thus I_u and J_u are half-space Poisson integrals, and $I_u + J_u$ is harmonic in $R^m \times (1,2\eta+1) = \Omega_\eta$, say. We next aim to prove that

(12) $$u \leqslant I_u + J_u \quad \text{in } \Omega_\eta.$$

Let $e' \in E'_+$. It is obvious that $e' \circ u$ satisfies the conditions (i) and (iii) of Theorem 2.3 in Chapter II. Hence, by relation (12) of Chapter II, we get

(13) $$e' \circ u \leqslant e' \circ I_u + e' \circ J_u \quad \text{in } \Omega_\eta.$$

Since (13) is true for all $e' \in E'_+$, by Lemma 4.1, (12) is true. In particular, for $X \in R^m$

$$u(X,\eta+1) \leqslant \frac{2\eta}{s_{m+1}} \cdot \int_{R^m} \frac{u(Z,1)dZ}{\{|X-Z|^2+\eta^2\}^{(m+1)/2}} + \frac{2\eta}{s_{m+1}} \int_{R^m} \frac{u(Z,2\eta+1)dZ}{\{|X-Z|^2+\eta^2\}^{(m+1)/2}}$$

$$\leqslant \frac{2}{s_{m+1}} \eta^{-m} \left(M(u,1) + M(u,2\eta+1)\right)$$

$$= C_2, \text{ say.} \quad (C_2 \text{ depends on } \eta.)$$

Also there is a constant C_1 such that $u(X,1) \leqslant C_1$ for all $X \in R^m$. Now put

$$H_\eta(X,y) = \frac{\eta+1-y}{\eta} C_1 + \frac{(y-1)}{\eta} C_2,$$

where $(X,y) \in R^m \times [1,\eta+1]$. Then H_η is harmonic $R^m \times (1,\eta+1)$. Also $H_\eta = C_1 \geqslant u$ on $R^m \times \{1\}$ and $H_\eta = C_2 \geqslant u$ on $R^m \times \{\eta+1\}$. Let $e' \in E'_+$. Then $e'H_\eta = e'(C_1) \geqslant e'(u)$ on $R^m \times \{1\}$ and $e'H_\eta = e'(C_2) \geqslant e'(u)$ on $R^m \times \{\eta+1\}$. Since $e' \circ u$ satisfies (i) and (iii) of Theorem 2.3 in Chapter II, then, by relation (11) of Chapter II, we get

$$\lim_{\substack{|(X,y)| \to +\infty \\ (X,y) \in R^m \times (1,\eta+1)}} \inf e'H_\eta(X,y) \geqslant 0 = \lim_{\substack{|(X,y)| \to +\infty \\ (X,y) \in R^m \times (1,\eta+1)}} e'u(X,y).$$

Hence, by the relation (17) of Chapter II, we obtain

(14) $$e'H_\eta \geqslant e'u \quad \text{in } R^m \times (1,\eta+1).$$

Since (14) is true for all $e' \in E'_+$, we get from Lemma 4.1 that

(15) $$H_\eta \geqslant u \quad \text{in } R^m \times (1,\eta+1).$$

Let h_η be the least harmonic majorant of u in $R^m \times (1,\eta+1)$. (Such a function h_η exists by Lemma 4.4.) If $\eta > 1$, we have

$$
\begin{aligned}
h_\eta(0,\ldots,0,2) &\leqslant H_\eta(0,\ldots,2) \\
&= \frac{\eta-1}{\eta} C_1 + \frac{1}{\eta} C_2 \qquad (16)\\
&= O(1) + \frac{2}{s_{m+1}} \eta^{-m-1} M(u,2\eta+1) \quad \text{as } \eta \to +\infty.
\end{aligned}
$$

Now, by Lemma 4.3, the sequence $(h_n(X,y))$ is either not majorized for each $(X,y) \in R^m \times (1,+\infty)$ or

$$h_n(X,y) \to h(X,y) \quad \text{as } n \to +\infty,$$

for all $(X,y) \in R^m \times (1,+\infty)$ where h is a harmonic function in $R^m \times (1,+\infty)$. Since u has no harmonic majorant in $R^m \times (1,+\infty)$, we must have the sequence $(h_n(X,y))$ is not majorized for each $(X,y) \in R^m \times (1,+\infty)$.

In particular, the sequence $(h_n(0,\ldots,0,2))$ is not majorized. It then follows from (16) that the set

$$\left\{\frac{M(u,2n+1)}{n^{m+1}} \;\middle|\; n = 2,3,\ldots\right\} \qquad (17)$$

is not majorized. Thus (17) implies that the set

$$\left\{\frac{M(u,2y+1)}{y^{m+1}} \;\middle|\; y \in (0,+\infty)\right\}$$

is not majorized. This completes the proof of Theorem 4.28.