CHAPTER III

ON THE FUNCTIONS THAT PRESERVE HARMONICITY IN THE EUCLIDEAN SPACE

1. Introduction and results

Let m be a positive integer, r be a non-negative integer, and Ω be a domain in the euclidean space R^m. We denote by $H(\Omega)$ the space of all real harmonic functions on Ω. Let Ω' be another domain in the euclidean space R^n (n is a positive integer that may not be equal to m.) and let $C^r(\Omega,\Omega')$ be the space of continuous maps from Ω into Ω' for which each of its component functions has continuous partial derivatives up to order r. In the special case, when $r = 0$, we shall simply write $C(\Omega,\Omega')$ instead of $C^0(\Omega,\Omega')$ and also $C(\Omega)$ for $C^0(\Omega,R)$.

Now, let $\varphi \in C(\Omega)$ and $f \in C(\Omega,\Omega')$ with $\varphi \not\equiv 0$, i.e. φ is not identically zero. We wish to characterize the couple (φ,f) such that

(1) for all $h \in H(\Omega')$ we have $\varphi(h \circ f) \in H(\Omega)$.

Such a couple (φ,f) is said to preserve harmonicity. The behaviour of this function f has been studied by Nualtaranee ([18]) in the case $m = n = 2$ and $\varphi \equiv 1$; and he showed that either f or $\bar{f}$ must be holomorphic where $\bar{f}$ is the complex conjugate of f.

Some properties of the couple (φ,f) can be easily derived as follows: First putting $h \equiv 1$ in (1), one gets $\varphi \in H(\Omega)$. Moreover, if we write $f = (f_1,\dots,f_n)$ and put $h(y_1,\dots,y_n) = y_j$, the j th coordinate, we get $\varphi f_j \in H(\Omega)$. Hence φf_j is analytic; In other words φ and φf_j are necessarily real analytic. A more delicate characterization of the couple (φ,f) is contained in the following theorem.

<u>Theorem 3.1</u> Let Ω and Ω' be two domains in R^m and R^n respectively. Let $\varphi \in C(\Omega)$ with $\varphi \not\equiv 0$ and $f \in C^2(\Omega,\Omega')$ with $f = (f_1,\dots,f_n)$. Then a necessary and sufficient condition that the couple (φ,f) satisfies (1) are the following:

(i) φ is harmonic in Ω,

(ii) for each $j = 1,\dots,n$, we have

$$\varphi\Delta f_j + 2\nabla\varphi \cdot \nabla f_j \equiv 0 \quad (\Delta - \text{Laplacian}, \ \nabla - \text{gradient}),$$

(iii) for each $j, k \in \{1,\dots,n\}$ with $j \neq k$, we have

$$\nabla f_j \cdot \nabla f_k \equiv 0,$$

(iv) $|\nabla f_1| = \dots = |\nabla f_n|$ in Ω.

This theorem can be regarded as the key theorem to this chapter. It will be proved in §2. Some important consequences of Theorem 3.1 are contained in the following theorem.

<u>Theorem 3.2</u> Let Ω and Ω' be two domains in R^m and R^n respectively. Let $\varphi \in C(\Omega)$ with $\varphi \not\equiv 0$ and $f \in C^2(\Omega,\Omega')$ with $f = (f_1,\dots,f_n)$. Assume that the couple (φ,f) satisfies the condition (1). Then

(i) if $m = n$, the map f must be conformal (i.e. preserve angles) at all points $p_0 \in \Omega$ such that $\nabla f_1(p_0) \neq 0$,

(ii) if $m < n$, the map f must be constant.

The proof of Theorem 3.2 is given in §3.

2. Proof of Theorem 3.1

Assume that the function $g = \varphi(h \circ f)$ is harmonic for all harmonic function h on Ω'. We try to show that the conditions (i), (ii), (iii), and (iv) of Theorem 3.1 hold. Let us write $g = \varphi(h \circ f)$ in term of its coordinates

$$
\begin{aligned}
g(x_1,\ldots,x_m) &= \varphi(x_1,\ldots,x_m)h(f_1(x_1,\ldots,x_m),\ldots,f_n(x_1,\ldots,x_m)) \\
&= \varphi(x_1,\ldots,x_m)h(y_1,\ldots,y_n),
\end{aligned}
$$

where $y_j = f_j(x_1,\ldots,x_m)$ for all $j = 1,\ldots,n$.

To prove (i), put $h \equiv 1$ in (1) then $\varphi \in H(\Omega)$. For (ii), let $j \in \{1,\ldots,n\}$ and take $h(y_1,\ldots,y_n) = y_j$.

Hence $g = \varphi f_j$ and $0 \equiv \Delta g = \varphi \Delta f_j + (\Delta\varphi) f_j + 2\nabla\varphi \cdot \nabla f_j$.

Since $\varphi \in H(\Omega)$, then $\Delta\varphi = 0$ and hence we get

$$\varphi \Delta f_j + 2\nabla\varphi \cdot \nabla f_j \equiv 0 \text{ in } \Omega.$$

To prove (iii), let $j, k \in \{1,\ldots,n\}$ with $j \neq k$ and take $h(y_1,\ldots,y_n) = y_j y_k$. Then $g = \varphi(f_j f_k)$ and

$$0 \equiv \Delta g = \varphi \Delta(f_j f_k) + \Delta\varphi(f_j f_k) + 2\nabla\varphi \cdot \nabla(f_j f_k). \tag{2}$$

Since $\Delta\varphi = 0$, $\Delta(f_j f_k) = f_j \Delta f_k + (\Delta f_j) f_k + 2\nabla f_j \cdot \nabla f_k$, and $\nabla(f_j f_k) = f_j \nabla f_k + (\nabla f_j) f_k$, then (2) implies

$$(3) \quad 0 \equiv \Delta g = f_j(\varphi\Delta f_k) + f_k(\varphi\Delta f_j) + 2f_j(\nabla\varphi \cdot \nabla f_k) + 2f_k(\nabla\varphi \cdot \nabla f_j) + 2\varphi(\nabla f_j \cdot \nabla f_k).$$

Applying the result from (ii) to the equation (3), we are left with

$$(4) \qquad \varphi(\nabla f_j \cdot \nabla f_k) \equiv 0 \quad \text{in } \Omega.$$

It follows from (4) that at every point $p \in \Omega$ where $\varphi(p) \neq 0$ we have $\nabla f_j \cdot \nabla f_k = 0$. Take now a point $p_0 \in \Omega$ such that $\varphi(p_0) = 0$; as we have already pointed out φ is harmonic in Ω and since φ is not identically to zero in Ω then φ is not identically zero in any ball $B(p_0,\delta)$ centered at p_0 and of radius $\delta > 0$. Thus there exists a sequence (p_l) of points in Ω such that $p_l \to p_0$ as $l \to +\infty$ and $\varphi(p_l) \neq 0$ for all l. Hence $\nabla f_j(p_l) \cdot \nabla f_k(p_l) = 0$ for all l. Using the continuity of $\nabla f_j \cdot \nabla f_k$ in Ω, we now get $\nabla f_j(p_0) \cdot \nabla f_k(p_0) = 0$. Thus

$$(5) \qquad \nabla f_j \cdot \nabla f_k \equiv 0 \quad \text{in } \Omega.$$

This proves (iii). Finally, we prove (iv). Let $j, k \in \{1,\dots,n\}$ and take $h(y_1,\dots,y_n) = y_j^2 - y_k^2$. So $g = \varphi \times (h \circ f) = \varphi(f_j^2 - f_k^2)$, hence

$$\begin{aligned} 0 \equiv \Delta g &= f_j^2\Delta\varphi + \varphi(2f_j\Delta f_j + 2|\nabla f_j|^2) + 4f_j\nabla\varphi \cdot \nabla f_j \\ &\quad - f_k^2\Delta\varphi - \varphi(2f_k\Delta f_k + 2|\nabla f_k|^2) - 4f_k\nabla\varphi \cdot \nabla f_k \\ &= 2\varphi(|\nabla f_j|^2 - |\nabla f_k|^2) + 0 - 0, \end{aligned}$$

and by the same argument as in (ii), we get $|\nabla f_j|^2 \equiv |\nabla f_k|^2$. Thus $|\nabla f_j| = |\nabla f_k|$ in Ω. This completes the proof of the necessity in Theorem 3.1.

Conversely, assume that the four properties (results of Theorem 3.1) hold. We shall prove that the couple (φ,f) satisfies (1). Let h be a harmonic function on Ω' then $g = \varphi \times (h \circ f)$ clearly belong to $C^2(\Omega)$. Its Laplacian with respect to the variables $x_1,\ldots,x_m$ is determined as follows:

$$
\begin{aligned}
(6) \qquad \Delta g &= \varphi\Delta(h \circ f) + (h \circ f)\Delta\varphi + 2\nabla\varphi \cdot \nabla(h \circ f) \\
&= \varphi\Delta(h \circ f) + 2\nabla\varphi \cdot \nabla(h \circ f)
\end{aligned}
$$

because φ is harmonic. Now, for each $i \in \{1,\ldots,m\}$, we have

$$(7) \qquad \frac{\partial(hof)}{\partial x_i} = \frac{\partial h}{\partial y_1}\frac{\partial f_1}{\partial x_i} + \frac{\partial h}{\partial y_2}\frac{\partial f_2}{\partial x_i} + \cdots + \frac{\partial h}{\partial y_n}\frac{\partial f_n}{\partial x_i} .$$

Therefore

$$
\begin{aligned}
\nabla\varphi\cdot\nabla(hof) &= \left(\frac{\partial\varphi}{\partial x_1},\ldots,\frac{\partial\varphi}{\partial x_m}\right).\left(\frac{\partial(hof)}{\partial x_1},\ldots,\frac{\partial(hof)}{\partial x_m}\right) \\
&= \sum_{i=1}^{m} \frac{\partial\varphi}{\partial x_i}\frac{\partial(hof)}{\partial x_i} \\
&= \sum_{i=1}^{m} \frac{\partial\varphi}{\partial x_i}\left(\sum_{j=1}^{n} \frac{\partial h}{\partial y_j}\frac{\partial f_j}{\partial x_i}\right) \\
&= \sum_{j=1}^{n} \frac{\partial h}{\partial y_j}\left(\sum_{i=1}^{m} \frac{\partial\varphi}{\partial x_i}\frac{\partial f_j}{\partial x_i}\right) \\
&= \sum_{j=1}^{n} \frac{\partial h}{\partial y_j}(\nabla\varphi\cdot\nabla f_j).
\end{aligned}
$$

Hence

$$(8) \quad 2\nabla\varphi\cdot\nabla(hof) = 2\sum_{j=1}^{n} \frac{\partial h}{\partial y_j}(\nabla\varphi\cdot\nabla f_j).$$

Moreover, we get from (7) that

$$(9)\qquad \frac{\partial^2(hof)}{\partial x_i^2} = \frac{\partial h}{\partial y_1}\frac{\partial^2 f_1}{\partial x_i^2} + \frac{\partial h}{\partial y_2}\frac{\partial^2 f_2}{\partial x_i^2} + \dots + \frac{\partial h}{\partial y_n}\frac{\partial^2 f_n}{\partial x_i^2}$$

$$+ \frac{\partial f_1}{\partial x_i}\left(\frac{\partial^2 h}{\partial y_1^2}\frac{\partial f_1}{\partial x_i} + \frac{\partial^2 h}{\partial y_1 \partial y_2}\frac{\partial f_2}{\partial x_i} + \dots + \frac{\partial^2 h}{\partial y_1 \partial y_n}\frac{\partial f_n}{\partial x_i}\right)$$

$$+ \frac{\partial f_2}{\partial x_i}\left(\frac{\partial^2 h}{\partial y_2 \partial y_1}\frac{\partial f_1}{\partial x_i} + \frac{\partial^2 h}{\partial y_2^2}\frac{\partial f_2}{\partial x_i} + \dots + \frac{\partial^2 h}{\partial y_2 \partial y_n}\frac{\partial f_n}{\partial x_i}\right)$$

$$+ \dots$$

$$+ \frac{\partial f_n}{\partial x_i}\left(\frac{\partial^2 h}{\partial y_n \partial y_1}\frac{\partial f_1}{\partial x_i} + \frac{\partial^2 h}{\partial y_n \partial y_2}\frac{\partial f_2}{\partial x_i} + \dots + \frac{\partial^2 h}{\partial y_n^2}\frac{\partial f_n}{\partial x_i}\right).$$

Formula (9) allows us to compute

$$(10)\qquad \Delta(hof) = \frac{\partial^2}{\partial x_1^2}(hof) + \frac{\partial^2}{\partial x_2^2}(hof) + \dots + \frac{\partial^2}{\partial x_m^2}(hof)$$

$$= \frac{\partial h}{\partial y_1}\Delta f_1 + \frac{\partial h}{\partial y_2}\Delta f_2 + \dots + \frac{\partial h}{\partial y_n}\Delta f_n$$

$$+ \frac{\partial^2 h}{\partial y_1^2}|\nabla f_1|^2 + \frac{\partial^2 h}{\partial y_2^2}|\nabla f_2|^2 + \dots + \frac{\partial^2 h}{\partial y_n^2}|\nabla f_n|^2$$

$$+ 2\sum_{j=2}^{n}\frac{\partial^2 h}{\partial y_1 \partial y_j}(\nabla f_1 \cdot \nabla f_j) + \sum_{j=3}^{n}\frac{\partial^2 h}{\partial y_2 \partial y_j}(\nabla f_2 \cdot \nabla f_j)$$

$$+ \dots + \sum_{j=n}^{n}\frac{\partial^2 h}{\partial y_{n-1}\partial y_j}(\nabla f_{n-1} \cdot \nabla f_j).$$

By using the assumptions (i),(iii),and (iv), (10) reduces to

$$(11)\qquad \Delta(hof) = \frac{\partial h}{\partial y_1}\Delta f_1 + \frac{\partial h}{\partial y_2}\Delta f_2 + \dots + \frac{\partial h}{\partial y_n}\Delta f_n .$$

Now multiply (11) with φ and add it to (8), we get

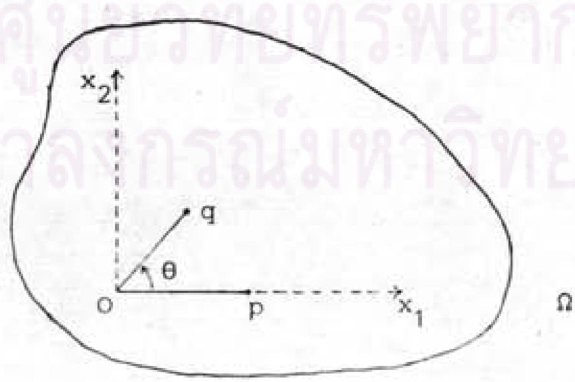
$$(12) \qquad \Delta g = \sum_{j=1}^{n} \varphi \frac{\partial h}{\partial y_j} \Delta f_j + \sum_{j=1}^{n} 2 \frac{\partial h}{\partial y_j} (\nabla\varphi \cdot \nabla f_j)$$

$$= \sum_{j=1}^{n} \frac{\partial h}{\partial y_j} (\varphi\, \Delta f_j + 2\nabla\varphi \cdot \nabla f_j).$$

Applying assumption (ii) to equation (12), we get $\Delta g \equiv 0$. Hence g is harmonic in Ω. This completes the proof of Theorem 3.1.

3. <u>Proof of Theorem 3.2</u>

For the proof of (i), let p_0 be any point of Ω where $\nabla f_1(p_0) \neq 0$ (and also $\nabla f_2(p_0) \neq 0, \ldots, \nabla f_n(p_0) \neq 0$). Choose the origin of axes O in Ω at p_0 and take two directions from O at angle θ. Without loss of generality, take one direction on the x_1-axis. the other in the x_1, x_2-plane and take points $p = (\varepsilon, 0, \ldots, 0)$, $q = (\varepsilon \cos\theta, \varepsilon \sin\theta, 0, \ldots, 0)$ on these directions with $\varepsilon > 0$ (see the figure below). Let $A = f(O)$, $B = f(p)$, and $C = f(q)$.



Thus

$A = (f_1(0,\dots,0),\dots,f_n(0,\dots,0)) \in \Omega',$

$B = (f_1(\varepsilon,0,\dots,0),\dots,f_n(\varepsilon,0,\dots,0)) \in \Omega',$

$C = (f_1(\varepsilon\cos\theta,\varepsilon\sin\theta,0,\dots,0),\dots,f_n(\varepsilon\cos\theta,\varepsilon\sin\theta,0,\dots,0)) \in \Omega'.$

By using the Mean-Value Theorem, the vectors AB and AC, with its end points given as above, are determined as follows:

$$AB = (f_1(\varepsilon,0,\dots,0)-f_1(0,\dots,0),\dots,f_n(\varepsilon,0,\dots,0)-f_n(0,\dots,0))$$

$$= (\varepsilon\frac{\partial f_1}{\partial x_1}(\eta_1,0,\dots,0),\dots,\ \varepsilon\frac{\partial f_n}{\partial x_1}(\eta_n,0,\dots,0)),$$

$$AC = (f_1(\varepsilon\cos\theta,\ \varepsilon\sin\theta,\ 0,\dots,0)-f_1(0,\dots,0),\dots,$$

$$f_n(\varepsilon\cos\theta,\ \varepsilon\sin\theta,\ 0,\dots,0)-f_n(0,\dots,0))$$

$$= (\varepsilon\ (\cos\theta\ \frac{\partial f_1}{\partial x_1}(\tau_1\cos\theta,\tau_1\sin\theta,0,\dots,0)+\sin\theta\ \frac{\partial f_1}{\partial x_2}(\tau_1\cos\theta,\tau_1\sin\theta,0,$$

$$\dots,0)),\dots,\varepsilon(\cos\theta\ \frac{\partial f_n}{\partial x_1}(\tau_n\cos\theta,\tau_n\sin\theta,0,\dots,0)+$$

$$\sin\theta\ \frac{\partial f_n}{\partial x_2}(\tau_n\cos\theta,\tau_n\sin\theta,0,\dots,0))),$$

where η_j and τ_k $(j = 1,\dots,n,\ k = 1,\dots,n)$ are some positive real numbers such that $\eta_j,\ \tau_k \to 0$ as $\varepsilon \to 0$. Now, if τ is the angle between vectors AB and AC, then

$$\cos\tau = \frac{AB\cdot AC}{|AB|\,|AC|}$$

$$= \frac{\cos\theta\left(\sum_{j=1}^{n}\left(\frac{\partial f_j}{\partial x_1}(0)\right)^2\right) + \sin\theta\left(\sum_{j=1}^{n}\frac{\partial f_j}{\partial x_1}(0)\,\frac{\partial f_j}{\partial x_2}(0)\right)}{\sqrt{\sum_{j=1}^{n}\left(\frac{\partial f_j}{\partial x_1}(0)\right)^2}\ \sqrt{\sum_{j=1}^{n}\left(\cos\theta\,\frac{\partial f_j}{\partial x_1}(0) + \sin\theta\,\frac{\partial f_j}{\partial x_2}(0)\right)^2}} + o(\varepsilon)$$

Using the fact that if the row vectors of the matrix $A = [a_{ij}]_{n\times n}$ are orthonomal, then so are the column vectors, we let

$$a_{ij} = \frac{1}{|\nabla f_1(0)|}\ \frac{\partial f_i}{\partial x_j}(0) \qquad (i,j = 1,\ldots,n).$$

Then we get that

$$\sum_{j=1}^{n}\left(\frac{\partial f_j}{\partial x_1}(0)\right)^2 = |\nabla f_1(0)|^2,\quad \sum_{j=1}^{n}\frac{\partial f_j}{\partial x_1}(0)\ \frac{\partial f_j}{\partial x_2}(0) = 0\ ,$$

and

$$\sum_{j=1}^{n}\left(\cos\theta\,\frac{\partial f_j}{\partial x_1}(0) + \sin\theta\,\frac{\partial f_j}{\partial x_2}(0)\right)^2 = |\nabla f_1(0)|^2\ .$$

This implies that $\cos\tau \to \cos\theta$ as $\varepsilon \to 0$, i.e. θ_0 = limit of angle τ is θ in cosine form. Ignoring $\pm$ sign for angles (as we should), we get conformality, This proves (i) of Theorem 3.2.

We note that (ii) follows immediately from Theorem 2.1(iii). In fact, since there are n orthogonal vectors in R^m with $m < n$, then there exists $j_0 \in \{1,\ldots,n\}$ such that $\nabla f_{j_0} = 0$ in Ω. By Theorem 3.1 (iv), we get $|\nabla f_1| = \ldots = |\nabla f_n| = 0$ in Ω. Hence f is constant in Ω as we wish to prove. This completes the proof of Theorem 3.2.