CHAPTER II

ON HYPERPLANE MEANS OF NON-NEGATIVE SUBHARMONIC FUNCTIONS

1. Introduction

Let m be a positive integer and R^{m+1} be the euclidean space of dimension m+1; O denotes the origin, and an arbitrary point of R^{m+1} is represented by

$$M = (X,y) = (x_1,\dots,x_m,y).$$

We put

$$X^2 = |X|^2 = x_1^2 + \dots + x_m^2, \quad dX = dx_1 \dots dx_m.$$

For each real number a, let D_a be the open half-space

$$D_a = \{M \in R^{m+1} \mid y > a\}.$$

When a = 0, D_a is simply replaced by D.

In D, the Poisson kernel P is given by

$$P(X,y) = 2s_{m+1}^{-1}\, y(X^2+y^2)^{-\frac{1}{2}(m+1)}$$

where s_{m+1} denotes the surface-area of the unit sphere in R^{m+1}. It is well-known that

$$\int_{R^m} P(X,y)dX = 1.$$

Let f be a function defined at least on the hyperplane given by the equation y = a, such that the function

$$X \mapsto (X^2+1)^{-\frac{1}{2}(m+1)} f(X,a)$$

is Lebesgue-integrable in R^m. The Poisson integral I_f^a in D_a of the restriction of f to the hyperplane is given by

$$I_f^a(X,y) = \int_{R^m} P(X-Z, y-a) f(Z,a)\, dZ$$

for all (X,y) belonging to D_a. We note that I_f^a is harmonic in D_a (Flett, [6], Theorem 6).

Let u be a non-negative subharmonic function in D, we define a function $M(u,\cdot)$ on $(0,+\infty)$ by writing

$$M(u,y) = \int_{R^m} u(X,y)\, dX$$

the integral being taken in the sense of Lebesgue. Thus $M(u,\cdot)$ exists and is non-negative (the value $+\infty$ being permitted) in $(0,+\infty)$.

Recently, Rippon ([19], Theorem 2) proved.

<u>Theorem A.</u> If u is a positive subharmonic function and has a harmonic majorant in D, if $M(u,1) < +\infty$, then either $M(u,\cdot)$ is decreasing in $[1,+\infty)$ or

$$(1) \qquad \int_1^{+\infty} \min\,[1, (y/M(u,y))^{1/m}]\, dy < +\infty.$$

Here we immediately add the following property (which will be proved in §3).

<u>Theorem 2.1</u> With the assumptions of Theorem A, if $M(u,\cdot)$ is not decreasing in $[1,+\infty)$ then

(2) $$\lim_{y\to+\infty} \sup \frac{M(u,y)}{y^{m+1}} = +\infty.$$

We now give some results on Green's potential, which are necessary when we investigate Theorem A in the case that u^p has a harmonic majorant $(p > 1)$.

Let τ denote the fundamental superharmonic function in R^{m+1}, that is, for any M and N belonging to R^{m+1},

$$\tau(M,N) = \begin{cases} -\log (MN) & (m = 1,\ M \neq N), \\ (MN)^{1-m} & (m \geqslant 2,\ M \neq N), \\ +\infty & (M = N), \end{cases}$$

MN being the distance between M and N. It is well known that the function $M \to \tau(M,N)$ is harmonic in $R^{m+1} \setminus \{N\}$ and superharmonic in R^{m+1}. The distribution $\Delta\tau$ of τ at 0 is related to the Dirac measure δ by $\Delta\tau = -\lambda_{m+1}\delta$, where $\lambda_2 = 2\pi$ and $\lambda_{m+1} = (m-1)s_{m+1}$, when $m \geqslant 2$ (Brelot, [1], Chapter 4, §2).

Let M and N be two points in D and M^* the reflexion of M with respect to the hyperplane $y = 0$. The Green's kernel is defined by

$$G(M,N) = \tau(M,N) - \tau(M^*,N)$$

Similarly, we can define G_a in D_a with M^* understood as the reflexion of M with respect to the hyperplane $y = a$.

If μ is a positive Radon measure on D_a, the function G_a^μ, defined by

$$G_a^\mu(M) = \int_{D_a} G_a(M,N)d\mu(N) \qquad (M \in D_a),$$

is called the <u>Green's potential</u> of μ in D_a. Either $G_a^\mu \equiv +\infty$ or G_a^μ is superharmonic in D_a (Helm, [9], Chapter 6). If G_a^μ is superharmonic, it is called a <u>potential</u> (of a measure). The use of the word "potential" is due to the fact that the greatest harmonic minorant of a positive superharmonic is identical to zero if and only if it is a potential.

The "only if" is a consequence of the following Riesz decomposition theorem for half-spaces.

<u>Theorem B</u>. Let u be subharmonic in D and μ the measure distribution $\Delta u/\lambda_{m+1}$. Then in order that G^μ be a potential, it is necessary and sufficient that u has a harmonic majorant in D. In that case

$$u(M) = h_u(M) - G^\mu(M) \qquad (M \in D),$$

where h_u denotes the least harmonic majorant of u in D.

Theorem B (Brelot,[2], Theorem 8) also holds for D_a provided that D and G are replaced by D_a and G_a.

Now, we shall investigate Theorem A in the case that u^p has a harmonic majorant for some positive real number p such that $p > 1$. We get the following results which will be proved in §3.

Theorem 2.2 If u is a non-negative subharmonic function and u^p has a harmonic majorant in D, if G^μ is positive and $M(G^\mu,1) < +\infty$, then either $M(u,\cdot)$ is decreasing, convex and continuous in $[1,+\infty)$ or $M(u,\cdot)$ is identically $+\infty$ in $[1,+\infty)$.

The second part of this chapter is devoted to study Theorem A in the case that u has no harmonic majorant in D. We get the following result inspired by Armitage.

Theorem 2.3 Suppose that

(i) u is a non-negative subharmonic function in D,

(ii) u has no harmonic majorant in $R^n \times (1,+\infty)$,

(iii) $M(u,\cdot)$ is locally bounded in $(0,+\infty)$.

Then

(3) $$\lim_{y\to+\infty} \frac{M(u,y)}{y^{m+1}} = +\infty .$$

We shall close this chapter by giving two examples. The first example will show that the condition $M(G^\mu,1) < +\infty$ is necessary in Theorem 2.2; and the second example will show that the condition (3) is best possible in the sense that for each real number $\varepsilon > 0$ we can find u satisfy (i),(ii), and (iii) such that

(4) $$\lim_{y\to+\infty} \frac{M(u,y)}{y^{m+1+\varepsilon}} < +\infty .$$

2. Some preliminary results

In this section, we shall prove two lemmas that will be used in proving Theorem 2.2.

Lemma 2.1 If u is a non-negative subharmonic function, $u^p (p > 1)$ has a harmonic majorant in D, and $M(u,1) < +\infty$, then $M(u;\cdot)$ is decreasing convex and continuous in $[1,+\infty)$.

Proof First, we shall prove that u also has a harmonic majorant in in D. By using the following identity

$$\int_{R^m} \{x^2+(y+1)^2\}^{-\frac{1}{2}(m+1)} dX = \frac{1}{2} s_{m+1}(y+1)^{-1},$$

(where s_{m+1} is the surface area of the unit sphere in R^{m+1}) and Holder's inequality, we obtain the following inequality:

$$\int_{R^m} \{x^2+(y+1)^2\}^{-\frac{1}{2}(m+1)} u(X,y)\, dX$$

$$= \int_{R^m} \{x^2+(y+1)^2\}^{-\frac{1}{2p}(m+1)} \{x^2+(y+1)^2\}^{-\frac{1}{2p}(p-1)(m+1)} u(X,y)dX$$

$$\leqslant \left(\int_{R^m} \{x^2+(y+1)^2\}^{-\frac{1}{2}(m+1)} u^p(X,y)dX\right)^{\frac{1}{p}} \left(\int_{R^m} \{x^2+(y+1)^2\}^{-\frac{1}{2}(m+1)} dX\right)^{1-\frac{1}{p}}$$

$$= \left(\int_{R^m} \{x^2+(y+1)^2\}^{-\frac{1}{2}(m+1)} u^p(X,y)dX\right)^{\frac{1}{p}} \left(\frac{1}{2} s_{m+1}(y+1)^{-1}\right)^{1-\frac{1}{p}}$$

If we let $K(u,y) = \int_{R^m} \{x^2+(y+1)^2\}^{-\frac{1}{2}(m+1)} u(X,y)dX$, then the above inequality can be written in terms of K as follows:

(5) $$K(u,y) \leqslant \{K(u^p,y)\}^{\frac{1}{p}} \{\tfrac{1}{2}s_{m+1}(y+1)^{-1}\}^{1-\frac{1}{p}}.$$

Hence (Nualtaranee, [17], Theorem B) the inequality (5) implies that u has a harmonic majorant in D.

By making a translation, we may suppose that u is non-negative and subharmonic in a neighborhood, Ω say, of $\bar{D}$ with

$$M(u,0) = \int_{R^m} u(X,0)dX < +\infty$$

and that u has a harmonic majorant in Ω. With these assumptions we can write (Nualtaranee, [17], Theorem A), for $P = (X,y)$ in D,

(6) $$u(P) = cy + \frac{2y}{s_{m+1}} \int_{R^m} \frac{u(Z,0)dZ}{|P-Z|^{m+1}} - \int_D G(P,M)d\mu(M)$$

where the value of c is given by (Nualtaranee, [17], page 253)

$$c = 2s_{m+1}^{-1} \lim_{y\to+\infty} K(u,y).$$

Since u^p has a harmonic majorant in D, we can conclude from (5) that $c = 0$. So (6) reduces to

(7) $$u(P) = \frac{2y}{s_{m+1}} \int_{R^m} \frac{u(Z,0)dZ}{|P-Z|^{m+1}} - \int_D G(P,M)d\mu(M).$$

But the Green's potential of μ is a non-negative function, then

$$u(P) \leqslant \frac{2y}{s_{m+1}} \int_{R^m} \frac{u(Z,0)dZ}{|P-Z|^{m+1}}$$

Hence

$$M(u,y) \leqslant \frac{2y}{s_{m+1}} \int_{R^m} \left[\int_{R^m} \frac{u(Z,0)dZ}{|P-Z|^{m+1}} \right] dX$$

$$= \int_{R^m} \left[\frac{2y}{s_{m+1}} \int_{R^m} \frac{dX}{|P-Z|^{m+1}} \right] u(Z,0)dZ$$

$$= M(u,0).$$

This implies that $M(u,y) \leqslant M(u,0)$ for all y in $[0,+\infty)$. Thus (Nualtaranee, [16], Theorem A) $M(u,y)$ is real-valued decreasing convex and continuous in $[1,+\infty)$. This completes the proof of Lemma 2.1.

We need the following theorem which is a weak form of a result of Brawn ([3], Theorem 2.2) to prove Lemma 2.2.

<u>Theorem C</u>. Let u be a positive superharmonic function in D. If $M(u,\cdot)$ is finite on $(0,+\infty)$, then $M(u,\cdot)$ is continuous and concave on $(0,+\infty)$.

<u>Lemma 2.2</u> Let u be a positive superharmonic function in $R^m \times (0,+\infty)$. Then either $M(u,\cdot)$ is finite on $(0,+\infty)$ or $M(u,\cdot)$ is identical to $+\infty$ on $(0,+\infty)$.

<u>Proof</u> Let h be a positive harmonic function in $R^m \times (0,+\infty)$ such that $M(h,\cdot)$ is finite on $(0,+\infty)$. (We could, for example, take $h(X,y) = y(X^2 + y^2)^{-\frac{1}{2}(m+1)}$ then $M(h,\cdot) = \frac{1}{2} s_{m+1}$). For each positive integer n define v_n in $R^m \times (0,+\infty)$ by writing $v_n = \min(u,nh)$.

Then each v_n is positive and superharmonic in $R^m \times (0,+\infty)$, and we have

$$M(v_n,\cdot) \leqslant nM(h,\cdot) < +\infty$$

on $(0,+\infty)$. By Theorem C, $M(v_n,\cdot)$ is continuous and concave on $(0,+\infty)$.

Since the sequence (v_n) increases to the limit u in $R^m \times (0,+\infty)$, it follows from Lebesgue's monotone convergence theorem that

$$\lim_{n\to+\infty} M(v_n,y) = M(u,y)$$

for all $y \in (0,+\infty)$, Hence $M(u,\cdot) \equiv +\infty$ on $(0,+\infty)$ or $M(u,\cdot)$ is real-valued on $(0,+\infty)$. This proves the lemma.

3. Proof of Theorem 2.1, 2.2, and 2.3.

Proof of Theorem 2.1 We work by contradiction. Suppose that

$$\limsup_{y\to+\infty} y^{-m-1} M(u,y) \leqslant c$$

for some $c > 0$. Then there exists $y_0 > 0$ such that

$$M(u,y) \leqslant cy^{m+1} \qquad (y \geqslant y_0)$$

Thus

$$\frac{y}{M(u,y)} \geqslant \frac{1}{cy^m} \qquad (y \geqslant y_0)$$

and then

$$\int_{y_0}^{+\infty} \left(\frac{y}{M(u,y)}\right)^{\frac{1}{m}} dy \geq c^{-\frac{1}{m}} \int_{y_0}^{+\infty} \frac{1}{y}\, dy = +\infty .$$

This contradicts (1). Therefore

$$\lim_{y \to +\infty} \sup \frac{M(u,y)}{y^{m+1}} = +\infty$$

Thus Theorem 2.1 is proved.

Proof of Theorem 2.2. We shall prove by cases.

Case 1. Assume that $M(u,1) < +\infty$. Then, by Lemma 2.1, we can conclude that $M(u,\cdot)$ is decreasing convex and continuous on $[1,+\infty)$.

Case 2. Assume that $M(u,1) = +\infty$. By making a translation, we may suppose that u is non-negative and subharmonic in a neighborhood, Ω say, of $\bar{D}$ with

$$(8) \qquad M(u,0) = \int_{R^n} u(X,0)\, dX = +\infty ,$$

$$M(G^{\mu},0) < +\infty ,$$

and that u^p has a harmonic majorant in Ω. With these assumption, we can write (Nualtaranee, [17], Theorem A), for $P = (X,y)$ in D,

$$(9) \qquad u(P) = \frac{2y}{s_{m+1}} \int_{R^m} \frac{u(Z,0)dZ}{|P-Z|^{m+1}} - \int_D G(P,M)d\mu(M).$$

With the assumption that $M(G^{\mu},0) < +\infty$ then, by Lemma 2.2, $M(G^{\mu},\cdot) < +\infty$ on $(0,+\infty)$. Thus (9) and Fubini's Theorem imply

(10) $$M(u,y) = M(u,0) - M(G^{\mu},y).$$

Hence (8) and (10) implies that $M(u,y) = +\infty$ on $(0,+\infty)$. This proves Theorem 2.2.

<u>Proof of Theorem 2.3.</u> Let $a > 1$. We prove first that

(11) $$\lim_{\substack{|(X,y)| \to +\infty \\ (X,y) \in R^m \times [1,a]}} u(X,y) = 0.$$

Suppose (11) is not true. Then there exists $\varepsilon > 0$ and a sequence of points $(P_n) = ((X_n,y_n))$ in $R^m \times [1,a]$ such that $|P_n| \to +\infty$ as $n \to +\infty$ and $u(P_n) \geq \varepsilon$ for all n. By working with a subsequence, if necessary, we may also suppose that $B(P_n,\frac{1}{2}) \cap B(P_k,\frac{1}{2}) = \phi$ whenever $n \neq k$ (where $B(P,r)$ is the open ball of center P and of radius r). By the volume mean-value inequality, for each $n \in N$, we have

$$\int_{B(P_n,\frac{1}{2})} u(X,y)dXdy \geq v_{m+1}(\tfrac{1}{2})^{m+1}u(P_n) \geq v_{m+1}(\tfrac{1}{2})^{m+1}\varepsilon$$

where v_{m+1} is the volume of $B(0,1)$ in R^{m+1}. Hence

$$\int_{R^m \times (\frac{1}{2},a+\frac{1}{2})} u(X,y)dXdy \geq \int_{\bigcup_{n=1}^{\infty} B(P_n,\frac{1}{2})} u(X,y)dXdy = \sum_{n=1}^{\infty} \int_{B(P_n,\frac{1}{2})} u(X,y)dXdy$$

$$= +\infty.$$

On the other hand, since $M(u,\cdot)$ is locally bounded in $(0,+\infty)$, we have

$$\int_{R^m \times (\frac{1}{2},a+\frac{1}{2})} u(X,y)dXdy = \int_{\frac{1}{2}}^{a+\frac{1}{2}} M(u,y)dy < +\infty.$$

Thus we have a contradiction. So (11) is true.

Now let $\eta > 0$. (For the moment, think of η as fixed.) Define I_u in $R^m \times (1,+\infty)$ by

$$I_u(X,y) = \frac{2(y-1)}{s_{m+1}} \int_{R^m} \frac{u(Z,1)dZ}{\{|X-Z|^2+(y-1)^2\}^{(m+1)/2}}$$

and define J_u in $R^m \times (-\infty, 2\eta+1)$ by

$$J_u(X,y) = \frac{2(2\eta+1-y)}{s_{m+1}} \int_{R^m} \frac{u(Z,2\eta+1)\,dZ}{\{|X-Z|^2+(2\eta+1-y)^2\}^{(m+1)/2}}.$$

Thus I_u and J_u are half-space Poisson integrals and $I_u + J_u$ is harmonic in $R^m \times (1,2\eta+1) = \Omega_\eta$ say. We next aim to prove that

(12) $$u \leqslant I_u + J_u \quad \text{in } \Omega_\eta$$

It follows from (11) and the upper semi-continuity of u that u is bounded on $R^m \times \{1\}$ and $R^m \times \{2\eta+1\}$. Hence there are two decreasing sequences (f_n) and (g_n) of bounded real-valued continuous functions on R^m such that

$$\lim_{n\to+\infty} f_n(Z) = u(Z,1) \text{ and } \lim_{n\to+\infty} g_n(Z) = u(Z,2\eta+1)$$

for all Z in R^m. Then, for each n, I_{f_n} is harmonic in $R^m \times (1,+\infty)$ and J_{g_n} is harmonic in $R^m \times (-\infty,2\eta+1)$. Also

$$\lim_{n\to+\infty} (I_{f_n} + J_{g_n}) = I_u + J_u \quad \text{in } \Omega_\eta .$$

Hence, to prove (12), it is enough to show

(13) $$u \leqslant I_{f_n} + J_{g_n} \quad \text{in } \Omega_\eta$$

We have

(14) $$\lim_{\substack{(X,y)\to(Z,1) \\ (X,y)\in\Omega_\eta}} I_{f_n}(X,y) + J_{g_n}(X,y) = f_n(Z,1) + J_{g_n}(Z,1)$$

$$\geqslant f_n(Z,1) \geqslant u(Z,1).$$

Similarly, we have

(15) $$\lim_{\substack{(X,y)\to(Z,2\eta+1) \\ (X,y)\in\Omega_\eta}} (I_{f_n}(X,y) + J_{g_n}(X,y)) \geqslant u(Z,2\eta+1)\ .$$

Also, by (11),

(16) $$\lim_{\substack{|(X,y)|\to+\infty \\ (X,y)\in\Omega_\eta}} \inf(I_{f_n}(X,y) + J_{g_n}(X,y)) \geqslant 0 = \lim_{\substack{|(X,y)|\to+\infty \\ (X,y)\in\Omega_\eta}} u(X,y).$$

(13) now follows from (14), (15), (16), and the maximum principle. Hence (12) is true. In particular, for all $X \in R^m$

$$u(X,\eta+1) \leqslant \frac{2\eta}{s_{m+1}} \int_{R^m} \frac{u(Z,1)\,dZ}{\{|X-Z|^2 + \eta^2\}^{(m+1)/2}} + \frac{2\eta}{s_{m+1}} \int_{R^m} \frac{u(Z,2\eta+1)\,dZ}{\{|X-Z|^2 + \eta^2\}^{(m+1)/2}}$$

$$\leqslant \frac{2}{s_{m+1}}\, \eta^{-m}\, (M(u,1) + M(u,2\eta+1))$$

$$= C_2, \text{ say.}$$

Also there is a constant C_1 such that $u(X,1) \leqslant C_1$ for all $X \in R^m$. Now put

$$H_\eta(X,y) = \frac{\eta + 1 - y}{\eta} C_1 + \frac{y - 1}{\eta} C_2\ ,$$

where $(X,y) \in R^m \times [1,\eta+1]$. Then H_η is harmonic in $R^m \times (1,\eta+1)$. Also, $H_\eta = C_1 \geqslant u$ on $R^m \times \{1\}$, $H_\eta = C_2 \geqslant u$ on $R^m \times \{\eta+1\}$, and by (11)

$$\lim_{\substack{|(X,y)| \to +\infty \\ (X,y) \in R^m \times (1,\eta+1)}} \inf H_\eta(X,y) \geqslant 0 = \lim_{\substack{|(X,y)| \to +\infty \\ (X,y) \in R^m \times (1,\eta+1)}} u(X,y).$$

Thus

$$H_\eta \geqslant u \quad \text{in } R^m \times (1,\eta+1). \tag{17}$$

Let h_η be the least harmonic majorant of u in $R^m \times (1,\eta+1)$. If $\eta > 1$, we have

$$\begin{aligned} h_\eta(0,\ldots,0,2) &\leqslant H_\eta(0,\ldots,0,2) \\ &= \frac{\eta-1}{\eta} C_1 + \frac{1}{\eta} C_2 \qquad (18) \\ &= O(1) + \frac{2}{s_{m+1}} \eta^{-m-1} M(u,2\eta+1) \text{ as } \eta \to +\infty. \end{aligned}$$

Now $\lim_{\eta \to +\infty} H_\eta$ is either harmonic in $R^m \times (1,+\infty)$ or identical to $+\infty$ in $R^m \times (1,+\infty)$. Since u has no harmonic majorant in $R^m \times (1,+\infty)$, we must have $\lim_{\eta \to +\infty} h_\eta \to +\infty$. In particular, $h_\eta(0,\ldots,0,2) \to +\infty$ as $\eta \to +\infty$, and (18) now implies that $M(u,2\eta+1)/\eta^{m+1} \to +\infty$ as $\eta \to +\infty$. This proves Theorem 2.3.

4. Examples

In this section we shall present two examples. The first example will show that we cannot relax the condition $M(G^\mu,1) < +\infty$ in Theorem 2.2 to obtain such results. The second example will assert that the condition (3) of Theorem 2.3 is best possible in the sense that for each $\varepsilon > 0$, there exists a non-negative subharmonic function u in D which has no harmonic majorant in D and

$$\lim_{y\to+\infty} \frac{M(u,y)}{y^{m+1+\varepsilon}} < +\infty.$$

Example 1. Let $d\mu(X,y) = dX\, d\delta_1(y)$ for all (X,y) in D, where δ_1 is the Dirac measure placed at 1 in the real line. Then the support of μ is the hyperplane $y = 1$ and

$$G^{\mu}(X,y) = \min\{1,y\} \qquad (y > 0).$$

To see this, we note that

$$G^{\mu}(X,y) = \int_{R^m} G((X,y),(Z,1))dZ$$

$$= \frac{1}{2}(y+1-|y-1|)$$

$$= \min\{1,y\}$$

where the second equality is due to equation (8) in ([13]). Hence $M(G^{\mu},y) = +\infty$ for all $y > 0$. Now, let

$$u(X,y) = y - G^{\mu}(X,y).$$

Then u is a non-negative subharmonic function in D with

$$M(u,y) = \begin{cases} 0 & (0 < y \leqslant 1), \\ +\infty & (y > 1). \end{cases}$$

Example 2. Let $\varepsilon > 0$ be given. Then, by Theorem 2.2 in [16], there exists a non-negative subharmonic function u in D such that

$$M(u,y) = y^{n+1+\varepsilon/2} .$$

For this subharmonic function u, we have

$$\frac{M(u,y)}{y^{n+1+\varepsilon}} = y^{-\varepsilon/2} \to 0 \quad \text{as } y \to +\infty.$$

It remains to show that u has no harmonic majorant in D. In fact, such a function u is given by

$$u(X,y) = \begin{cases} \rho^k F_{m,k}(\theta) & (0 \leqslant \theta < \theta_0) \\ 0 & (\theta_0 \leqslant \theta < \frac{1}{2}\pi), \end{cases}$$

where $k = 1 + \varepsilon/2$, $F_{m,k}(\theta) = F(-k, m+k-1; \frac{1}{2}m, \frac{1-\cos\theta}{2})$, θ_0 is the smallest positive zero of $F_{m,k}$, $-\pi < \theta < \pi$, and F is the hypergeometric function of x $(-1 < x < 1)$ with parameter $-k$, $m+k-1$, $\frac{1}{2}m$. We let s_m denote the surface area of the unit sphere in R^m; and $d\sigma$ denote the surface area element on the sphere $|X| = r$ in R^m, where $r > 0$, Then

$$\int_{R^m} \frac{u(X,y)dX}{(X^2+y^2)^{\frac{m+1}{2}}} = \int_0^{+\infty}\int_{|X|=r} \frac{u(X,y)}{(X^2+y^2)^{\frac{m+1}{2}}} d\sigma(X)dr.$$

Since, for each r and y fixed, the function u is constant on the sphere $|X| = r$, we have, by using $\rho = y \sec\theta$ and the change of variable $r = y\tan\theta$, that

$$\int_{R^m} \frac{u(X,y)dX}{(X^2+y^2)^{\frac{m+1}{2}}} = \frac{1}{\rho^{m+1}} \int_0^{+\infty} [s_m r^{m-1}(\rho^k F_{m,k}(\theta))]dr$$

$$= s_m \int_0^{\theta_0} [y^{m-1}\tan^{m-1}\theta\ \rho^{k-m-1} F_{m,k}(\theta) y \sec^2\theta] d\theta$$

$$= s_m \int_0^{\theta_0} [y^m \tan^{m-1}\theta\ y^{k-m-1} \sec^{k-m-1}\theta\ F_{m,k}(\theta) \sec^2\theta] d\theta$$

$$= s_m \int_0^{\theta_0} [y^{k-1}\tan^{m-1}\theta\ \sec^{k-m+1}\theta\ F_{m,k}(\theta)] d\theta$$

$$= s_m y^{k-1} \int_0^{\theta_0} \tan^{m-1}\theta\ \sec^{k-m-1}\theta\ F_{m,k}(\theta) d\theta.$$

Here the integral is a positive constant since, by Theorem 4 in [16], the integrand is positive, real-valued and continuous in $(0,\theta_0)$. Hence (Kuran, [14], Theorem 4) u has no harmonic majorant in D.