Case 1. K is a seminear-field with a' as a category II special element. Let $x \in S \setminus \{a\}$. Then $a^2 = xa = a$. If $f(a) = a'$, then $f(xa) = f(x)f(a) = f(x)a' = f(x)$. Therefore $x = xa = a$, a contradiction. Hence $f(a) \neq a'$.

Subcase 1.1. $f(x) \neq a'$. Then $f(a)f(a) = f(a)f(x)$, so $f(a) = f(x)$. Hence $a = x$, a contradiction.

Subcase 1.2. $f(x) = a'$. Let $y \in S \setminus \{a,x\}$. Then $f(y) \neq f(x) = a'$ and $aa = ay$. Therefore $f(a)f(a) = f(a)f(y)$ which implies that $f(a) = f(y)$. Thus $a = y$, a contradiction.

Case 2. K is a seminear-field with a category III, IV or V special element. Then $|K| = 2$. But $|S| > 2$, so f is not an injection which is a contradiction.

Case 3. K is a seminear-field with a' as a category VI special element. Then $(a')^2 \neq a'$. If $f(a) = a'$, then $f(a)f(a) = a'a' \neq a' = f(a)$. Thus $a^2 \neq a$, a contradiction. Hence $f(a) \neq a'$. Since $|S \setminus \{a\}| \geq 2$ and f is an injection, there exists an $x \in S \setminus \{a\}$ such that $f(x) \neq a'$. Therefore $f(a)f(x) = f(a)f(a)$ which implies that $f(x) = f(a)$ and hence $x = a$. This is a contradiction.

#

Theorem 3.2. Let S be a Classification B seminear-ring w.r.t. a. Assume that there exists a $b \in S \setminus \{a\}$ such that $bx = xb = x$ for all $x \in S \setminus \{a\}$ and $ax = xa = x$ for all $x \in S$ and $x+y \neq a$ for all $x,y \in S$. If $(S \setminus \{a\}, \cdot)$ satisfies the right [left] Ore condition, then S can be embedded into a seminear-field with a category II special element and not into any other category of seminear-fields.

Proof. By assumption, $S \setminus \{a\}$ is an M.C. seminear-ring satisfying the right [left] Ore condition. By Theorem 1.45, $Q(S\setminus\{a\})$ exists. Let e' be the multiplicative identity of $Q(S\setminus\{a\})$. Let $f : S\setminus\{a\} \to Q(S\setminus\{a\})$ be the natural embedding, that is, $f(x) = [(x^2,x)]$ for all $x \in S\setminus\{a\}$. Let a' be a symbol not representing any element of $Q(S\setminus\{a\})$. Extend the binary operation of $Q(S\setminus\{a\})$ to $K = Q(S\setminus\{a\}) \cup \{a'\}$ by defining $a'\alpha = \alpha a' = \alpha$, $a'+\alpha = e'+\alpha$ and $\alpha+a' = \alpha+e'$ for all $\alpha \in K$. Then we can show that K is a seminear-field with a category II special element. Extend $f : S\setminus\{a\} \to Q(S\setminus\{a\})$ to $f : S \to K$ by defining $f(a) = a'$. Clearly, f is an injection and $f(b)$ is the multiplicative identity of $Q(S\setminus\{a\})$, that is, $f(b) = e'$.

Claim that $a+a = b+b$, $a+y = b+y$ and $y+a = y+b$ for all $y \in S$. Since $(b+b)b = bb+bb = b+b = ab+ab = (a+a)b$, $a+a = b+b$. Let $y \in S$. Then $(a+y)b = ab+yb = bb+yb = (b+y)b$. Thus $a+y = b+y$. Similarly, we can show that $y+a = y+b$. So we have the claim.

To show that f is a homomorphism, let $x,y \in S$.

Case 1. $x \neq a$, $y \neq a$. This case is clear.

Case 2. $x = y = a$. Then $f(x+y) = f(a+a) = f(b+b) = f(b)+f(b) = e'+e' = a'+a' = f(a)+f(a) = f(x)+f(y)$ and $f(xy) = f(aa) = f(a) = a'f(a) = f(a)f(a) = f(x)f(y)$.

Case 3. $x = a$, $y \neq a$. Then $f(x+y) = f(a+y) = f(b+y) = f(b)+f(y) = e'+f(y) = a'+f(y) = f(a)+f(y) = f(x)+f(y)$ and $f(xy) = f(ay) = f(y) = a'f(y) = f(a)f(y) = f(x)f(y)$.

Case 4. $x \neq a$, $y = a$. The proof is the same as in Case 3.

Since $|S| > 2$, S cannot be embedded into a seminear-field with a category III, IV or V special element.

Next, suppose that there is a monomorphism $f : S \to K$ where K is a seminear-field with a' as a category I special element. If f(a) = a', then f(a)f(b) = a'f(b) = a' = f(a). Hence ab = a, a contradiction. Thus $f(a) \neq a'$. Similarly, we can show that $f(b) \neq a'$. Since f(a)f(b) = f(ab) = f(bb) = f(b)f(b), f(a) = f(b). Hence a = b, a contradiction.

Assume that there is a monomorphism $f : S \to K$ where K is a seminear-field with a' as a category VI special element. Then $xy \neq a'$ for all $x,y \in K$. If f(a) = a', then $a'a' = f(a)f(a) = f(a^2) = f(a) = a'$ which is a contradiction. Hence $f(a) \neq a'$. Also, if f(b) = a', then f(a)a' = f(a)f(b) = f(ab) = f(b) = a', a contradiction. Thus $f(b) \neq a'$. Since f(a)f(b) = f(ab) = f(bb) = f(b)f(b), f(a) = f(b). Hence a = b, a contradiction.

#

Example 3.3. $(Z^+,+,\cdot)$ is an M.C. seminear-ring where x+y = maximum of x,y for all $x,y \in Z^+$ and $\cdot$ is the usual multiplication. Let a be a symbol not representing any element of Z^+. Define + and $\cdot$ on $S = Z^+ \cup \{a\}$ by defining ax = xa = x for all $x \in S$, a+x = x+a = x for all $x \in Z^+$ and a+a = 1. Then we can show that S is a Classification B seminear-ring w.r.t. a. We see that $1 \in S \setminus \{a\}$ is a multiplicative identity, ax = xa = x, $x+y \neq a$ for all $x,y \in S$ and $(S \setminus \{a\},\cdot)$ satisfies the right Ore condition. Hence this is an example of a seminear-ring satisfying the hypotheses of Theorem 3.2.

Theorem 3.4. Let S be an M.C. Classification B, C or D seminear-ring. If $(S,\cdot)$ satisfies the right [left] Ore condition, then S can be embedded into a 0-seminear-field.

Proof. By Theorem 1.45, S can be embedded into Q(S).

By Proposition 1.26, Q(S) can be embedded into a 0-seminear-field and hence so can S.

#

Remark. Let K be the 0-seminear-field and $f : S \to K$ the embedding given by the construction used in Theorem 3.4. Then $K = Q(S) \cup \{a'\}$ where a' is a 0-special element and $f : S \to K$ is given by

$f(x) = [(x^2,x)]$ for all $x \in S$.

Theorem 3.5. Let S be an M.C. Classification B, C or D seminear-ring. If $(S,\cdot)$ satisfies the right [left] Ore condition, then S can be embedded into an ∞-seminear-field.

Proof. The proof is similar to the proof of Theorem 3.4, using Proposition 1.27.

#

Remark. Let K be the ∞-seminear-field and $f : S \to K$ the embedding given by the construction used in Theorem 3.5. Then $K = Q(S) \cup \{a'\}$ where a' is an ∞-special element and $f : S \to K$ is given by

$f(x) = [(x^2,x)]$ for all $x \in S$.

Lemma 3.6. Let S be an M.C. seminear-ring such that $(S,\cdot)$ satisfies the right [left] Ore condition. Then the following statements hold:

(i) If $(S,+)$ is a left zero semigroup, then $(Q(S),+)$ is a left zero semigroup.

(ii) If $(S,+)$ is a right zero semigroup, then $(Q(S),+)$ is a right zero semigroup.

Proof. Let $[(a,b)],[(c,d)] \in Q(S)$. Then there exist $u,v \in S$ such that $bu = dv$. Thus $[(a,b)]+[(c,d)] = [(au+cv,bu)]$.

(i) Since $x+y = x$ for all $x,y \in S$, $[(a,b)]+[(c,d)] = [(au,bu)] = [(a,b)]$.

(ii) Since $y+x = x$ for all $x,y \in S$, $[(a,b)]+[(c,d)] = [(cv,bu)] = [(cv,dv)] = [(c,d)]$.

#

Theorem 3.7. Let S be an M.C. Classification B, C or D seminear-ring such that $(S,+)$ is a left zero semigroup. If $(S,\cdot)$ satisfies the right [left] Ore condition, then S can be embedded into an additive left zero seminear-field with a category I special element.

Proof. By Lemma 3.6, $(Q(S),+)$ is a left zero semigroup. By Theorem 1.45, S can be embedded into Q(S). By Proposition 1.28, Q(S) can be embedded into an additive left zero seminear-field with a category I special element and hence so can S.

#

Remark. Let K be the additive left zero seminear-field with a category I special element and $f : S \to K$ the embedding given by the construction used in Theorem 3.7. Then $K = Q(S) \cup \{a'\}$ where a' is a symbol not representing any element of Q(S) such that $a'x = xa' = a'$, $a'+x = a'$ and $x+a' = x$ for all $x \in K$ and $f : S \to K$ is given by $f(x) = [(x^2,x)]$ for all $x \in S$.

Theorem 3.8. Let S be an M.C. Classification B, C or D seminear-ring such that $(S,+)$ is a right zero semigroup. If $(S,\cdot)$ satisfies the right [left] Ore condition, then S can be embedded into an additive right zero seminear-field with a category I special element.

Proof. The proof is similar to the proof of Theorem 3.7, using Proposition 1.29.

#

Remark. Let K be the additive right zero seminear-field with a category I special element and $f : S \to K$ the embedding given by the construction used in Theorem 3.8. Then $K = Q(S) \cup \{a'\}$ where a' is a symbol not representing any element of Q(S) such that $a'x = xa' = a'$, $a'+x = x$ and $x+a' = a'$ for all $x \in K$ and $f : S \to K$ is given by $f(x) = [(x^2,x)]$ for all $x \in S$.

We shall now give examples of Classification B, C and D seminear-rings $(S,+,\cdot)$ such that $(S,+)$ is a left or a right zero semigroup.

Example 3.9. Define $\oplus$ on $\mathbb{Z}^+$ by $x \oplus y = x [x \oplus y = y]$ for all $x,y \in \mathbb{Z}^+$. Then $(\mathbb{Z}^+,\oplus)$ and $(\mathbb{Z}^+ \smallsetminus \{1\},\oplus)$ are left [right] zero semigroups. Furthermore,

(1) $(\mathbb{Z}^+,\oplus,\cdot)$ is a Classification B seminear-ring w.r.t.1,

(2) $(\mathbb{Z}^+,\oplus,\cdot)$ is a Classification C seminear-ring w.r.t.2

and (3) $(\mathbb{Z}^+ \smallsetminus \{1\},\oplus,\cdot)$ is a Classification D seminear-ring w.r.t.2

where $\cdot$ is the usual multiplication.

Theorem 3.10. Let S be an M.C. Classification B, C or D seminear-ring. If $(S,\cdot)$ satisfies the right [left] Ore condition, then S can be embedded into a seminear-field with a category II special element.

Proof. By Theorem 1.45, S can be embedded into Q(S). By Proposition 1.30, Q(S) can be embedded into a seminear-field with a category II special element and hence so can S.

#

Remark. Let K be the seminear-field with a category II special element and $f : S \to K$ the embedding given by the construction used in Theorem 3.10. Let e be the identity of $(Q(S),\cdot)$. Then $K = Q(S) \cup \{a'\}$ where a' is a symbol not representing any element of Q(S) such that $a'\alpha = \alpha a' = \alpha$ for all $\alpha \in K$, $\alpha+a' = \alpha+e'$, $a'+\alpha = e'+\alpha$ for all $\alpha \in Q(S)$ and

$$a'+a' = \begin{cases} a' \text{ or } e' & \text{if } \alpha+\alpha = \alpha \text{ for all } \alpha \in Q(S) \\ e'+e' & \text{; otherwise} \end{cases}$$

and $f : S \to K$ is given by $f(x) = [(x^2,x)]$ for all $x \in S$.

Theorem 3.11. Let S be an M.C. Classification B, C or D seminear-ring. If $(S,\cdot)$ satisfies the right [left] Ore condition, then S can be embedded into a seminear-field with a category VI special element.

Proof. The proof is similar to the proof of Theorem 3.10, using Proposition 1.31.

#

Remark. Let K be the seminear-field with a category VI special element and $f : S \to K$ the embedding given by the construction used in Theorem 3.11. Let $d' \in Q(S)$. Then $K = Q(S) \cup \{a'\}$ where a' is a symbol not representing any element of Q(S) such that

$$(a')^2 = (d')^2,\ \alpha a' = \alpha d' \text{ and } a'\alpha = d'\alpha \text{ for all } \alpha \in Q(S)$$

$$\alpha+a' = \alpha+d' \text{ and } a'+\alpha = d'+\alpha \text{ for all } \alpha \in Q(S),$$

$$a'+a' = \begin{cases} a' \text{ or } d' & \text{if } \alpha+\alpha = \alpha \text{ for all } \alpha \in Q(S) \\ d'+d' & \text{; otherwise} \end{cases}$$

and $f : S \to K$ is given by $f(x) = [(x^2,x)]$ for all $x \in S$.

Theorem 3.12. Let S be a Classification D seminear-ring. If S is not L.M.C., then S cannot be embedded into a seminear-field with a category I, II, III, IV or V special element.

Proof. Assume that S is a Classification D seminear-ring w.r.t. a. Since S is not L.M.C., there exists a $z \in S$ such that z is not L.M.C. in S. Therefore there exist $x,y \in S$ such that $zx = zy$ but $x \neq y$.

Case 1. $z = a$. Then $ax = ay$. Clearly, $x = a$ or $y = a$. Without loss of generality, assume that $x = a$. Then $y \neq a$ and $aa = ay$. Assume that there exists a monomorphism $f : S \to K$ where K is a seminear-field with a category I, II, III, IV or V special element.

Subcase 1.1. K is a seminear-field with a' as a category I special element. Then $a'x = xa' = a'$ for all $x \in K$. If $f(a) = a'$, then $f(ay) = f(a)f(y) = a'f(y) = a' = f(a)$. Therefore $ay = a$, a contradiction. Thus $f(a) \neq a'$. Similarly, we can show that $f(y) \neq a'$. Since $f(a)f(a) = f(a)f(y)$, $f(a) = f(y)$. Hence $a = y$, a contradiction.

Subcase 1.2. K is a seminear-field with a' as a category II special element. Then $a'x = xa' = x$ for all $x \in K$. If $f(a) = a'$, then $f(a)f(a) = f(a)$. Thus $a^2 = a$, a contradiction. Therefore $f(a) \neq a'$. Similarly, we can show that $f(y) \neq a'$. Since $f(a)f(a) = f(a)f(y)$, $f(a) = f(y)$. Hence $a = y$, a contradiction.

Subcase 1.3. K is a seminear-field with a category III, IV or V special element. Then $|K| = 2$. But $|S| > 2$, a contradiction.

Case 2. $z \neq a$. Clearly, $x = a$ or $y = a$. Without loss of generality, assume that $x = a$. Then $y \neq a$ and $za = zy$.

Subcase 2.1. K is a seminear-field with a' as a category I or II special element. Clearly, $f(a) \neq a'$, $f(z) \neq a'$ and $f(y) \neq a'$. Since $f(z)f(a) = f(z)f(y)$, $f(a) = f(y)$. Hence $a = y$, a contradiction.

<u>Subcase 2.2</u>. K is a seminear-field with a category III, IV or V special element. Using the same proof as in Subcase 1.3 we can get a contradiction.

#

<u>Corollary 3.13</u>. Let S be a Classification D seminear-ring w.r.t. a. If a is not L.M.C. in S, then S cannot be embedded into a seminear-field with a category I, II, III, IV or V special element.

<u>Theorem 3.14</u>. Let S be a Classification E seminear-ring such that $|S| > 2$. Then S cannot be embedded into a seminear-field with a category I, II, III, IV or V speical element.

<u>Proof</u>. Assume that S is a Classification E seminear-ring w.r.t. a. Then the proof of this proposition is similar to the proof of Case 1 in Theorem 3.12 (substitute a^2 for y).

#

<u>Theorem 3.15</u>. Let S be a Classification D seminear-ring w.r.t. a such that a is not L.M.C. in S. If $xa \neq a$ for all $x \in S \smallsetminus \{a\}$, $x+y \neq a$ for all $x,y \in S$ and $(S \smallsetminus \{a\},\cdot)$ satisfies the right [left] Ore condition, then S can be embedded into a seminear-field with a category VI special element and not into any other category of seminear-fields.

<u>Proof</u>. Let $d \in S \smallsetminus \{a\}$ be such that $ax = dx$ for all $x \in S \smallsetminus \{a\}$. Since $xa \neq a$ for all $x \in S \smallsetminus \{a\}$, $xa = xd$ for all $x \in S \smallsetminus \{a\}$ and $ad = da$. By Proposition 2.43, $a+a = d+d$, $a+x = d+x$ and $x+a = x+d$ for all $x \in S$. Since $x+y \neq a$ for all $x,y \in S$ $\{a\}$, $S \smallsetminus \{a\}$ is an M.C. seminear-ring. Hence $Q(S \smallsetminus \{a\})$ exists. Let $f : S \smallsetminus \{a\} \to Q(S \smallsetminus \{a\})$ be the natural

embedding, that is, $f(x) = [(x^2,x)]$ for all $x \in S \setminus \{a\}$. Let a' be a symbol not representing any element of $Q(S \setminus \{a\})$. Extend the binary operation of $Q(S \setminus \{a\})$ to $K = Q(S \setminus \{a\}) \cup \{a'\}$ by defining $a'\alpha = f(d)\alpha$, $\alpha a' = \alpha f(d)$, $a'+\alpha = f(d)+\alpha$ and $\alpha+a' = \alpha+f(d)$ for all $\alpha \in K$. Then we can show that K is a seminear-field with a category VI special element. Extend $f : S \setminus \{a\} \to Q(S \setminus \{a\})$ to $f : S \to K$ by defining $f(a) = a'$. Clearly, f is an injection.

To show that f is a homomorphism, let $x,y \in S$.

Case 1. $x = y = a$. Then $f(x+y) = f(a+a) = f(d+d) = f(d)+f(d) = a'+a' = f(a)+f(a) = f(x)+f(y)$. Similarly, we can show that $f(xy) = f(x)f(y)$.

Case 2. $x = a$, $y \neq a$. Then $f(x+y) = f(a+y) = f(d+y) = f(d)+f(y) = a'+f(y) = f(a)+f(y) = f(x)+f(y)$. Similarly, we can show that $f(xy) = f(x)f(y)$.

Case 3. $x \neq a$, $y = a$. The proof is similar to the proof of Case 2.

Case 4. $x \neq a$, $y \neq a$. This case is clear.

Hence f is a homomorphism and by Theorem 3.12, we are done.

#

Theorem 3.16. Let S be a Classification E seminear-ring w.r.t. a such that $|S| > 2$. If $x+y \neq a$ for all $x,y \in S$ and $(S \setminus \{a\},\cdot)$ satisfies the right [left] Ore condition, then S can be embedded into a seminear-field with a category VI special element and not into any other category of seminear-fields.

Proof. The proof is similar to the proof of Theorem 3.15 (substitute a^2 for d).

#

Theorem 3.17. Let S be a Classification D seminear-ring w.r.t. a such that a is not L.M.C. in S. Let $d \in S \setminus \{a\}$ be such that $ax = dx$ for all $x \in S \setminus \{a\}$. If there exist $x,y \in S \setminus \{a\}$ such that $x+y = a$ and there exist $u,v \in S \setminus \{d\}$ such that $u+v = d$, then S cannot be embedded into a seminear-field with a category VI special element.

Proof. Suppose that there exists a monomorphism $f : S \to K$ where K is a seminear-field with a' as a category VI special element. Let e be the identity of $(K \setminus \{a'\}, \cdot)$.

Claim that $f(a) \neq a'$. To prove this, suppose not. Then $f(x), f(y) \in K \setminus \{a'\}$. Thus $a' = f(a) = f(x+y) = f(x)+f(y) = f(x)e+f(y)e = (f(x)+f(y))e = a'e$, a contradiction. Hence we have the claim. Similarly, we can show that $f(d) \neq a'$. Since $f(a)f(d) = f(d)f(d)$, $f(a) = f(d)$. Hence $a = d$, a contradiction. #

Example 3.18. $Z^+ \setminus \{1,3\}$ with the usual addition and multiplication is an M.C. seminear-ring. Let a and b be symbols not representing any element of $Z^+ \setminus \{1,3\}$. Extend + and · from $Z^+ \setminus \{1,3\}$ to $S = (Z^+ \setminus \{1,3\}) \cup \{a,b\}$ by defining

$$a^2 = 36,\ b^2 = 9,\ ab = ba = 18,$$

$$ax = xa = 6x \text{ and } bx = xb = 3x \quad \text{for all } x \in Z^+ \setminus \{1,3\},$$

$$a+a = 12,\ b+b = a,\ a+b = b+a = 9,$$

$$a+x = x+a = 6+x \text{ and } b+x = x+b = 3+x \quad \text{for all } x \in Z^+ \setminus \{1,3\}.$$

Then we can show that S is a Classification D seminear-ring w.r.t. a. We see that $ax = 6x$ for all $x \in S \setminus \{a\}$, $b \in S \setminus \{a\}$ is such that $b+b = a$ and $2,4 \in S \setminus \{b\}$ are such that $2+4 = 6$. Hence this example satisfies the hypotheses of Theorem 3.17.

Theorem 3.19. Let S be a Classification E seminear-ring w.r.t. a. If there exist $x,y \in S \setminus \{a\}$ such that $x+y = a$ and there exist $u,v \in S \setminus \{a^2\}$ such that $u+v = a^2$, then S cannot be embedded into a seminear-field with a category VI special element.

Proof. The proof is similar to the proof of Theorem 3.17 (substitute a^2 for d).
#

Theorem 3.20. Let S be a Classification D seminear-ring w.r.t. a such that a is not L.M.C. in S. Let $d \in S \setminus \{a\}$ be such that $ax = dx$ for all $x \in S \setminus \{a\}$. If there exist $x,y \in S \setminus \{a\}$ such that $x+y = a$ and there exist $z,w \in S$ such that $zw = d$ and $u+v \neq d$ for all $u,v \in S$, then S cannot be embedded into a seminear-field with a category VI speical element.

Proof. Suppose that there exists a monomorphism $f : S \to K$ where K is a seminear-field with a' as a category VI special element. Let e be the identity of $(K \setminus \{a'\}, \cdot)$. If $f(a) = a'$, then $f(x) \neq a'$ and $f(y) \neq a'$. Thus $a' = f(a) = f(x+y) = f(x)+f(y) = f(x)e+f(y)e = (f(x)+f(y))e = a'e$, a contradiction. Thus $f(a) \neq a'$. By Proposition 1.33, $f(z)f(w) \neq a'$. Thus $f(d) = f(z)f(w) \neq a'$. Since $f(a)f(d) = f(d)f(d)$, $f(a) = f(d)$. Hence $a = d$, a contradiction.
#

Example 3.21. $Z^+ \setminus \{1,2\}$ with the usual addition and multiplication is an M.C. seminear-ring. Let a and b be symbols not representing any element of $Z^+ \setminus \{1,2\}$. Extend + and $\cdot$ from $Z^+ \setminus \{1,2\}$ to $S = (Z^+ \setminus \{1,2\}) \cup \{a,b\}$ by defining

$$a^2 = 16,\ b^2 = 4,\ ab = ba = 8,$$

$$ax = xa = 4x \text{ and } bx = xb = 2x \quad \text{for all } x \in Z^+ \setminus \{1,2\},$$

$$a+a = 8,\ b+b = a,\ a+b, b+a = 6,$$

$$a+x = x+a = 4+x \text{ and } b+x = x+b = 2+x \quad \text{for all } x \in Z^+ \setminus \{1,2\}.$$

Then we can show that $(S,+,\cdot)$ is a Classification D seminear-ring w.r.t. a. We see that $ax = 4x$ for all $x \in S \setminus \{a\}$, $b \in S \setminus \{a\}$ is such that $b+b = a$, $b^2 = 4$ and $x+y \neq 4$ for all $x,y \in S$. Hence this is an example of a seminear-ring satisfying the hypotheses of Theorem 3.20.

<u>Theorem 3.22</u>. Let S be a Classification E seminear-ring w.r.t. a. If there exist $x,y \in S \setminus \{a\}$ such that $x+y = a$ and there exist $z,w \in S$ such that $zw = a^2$ and $u+v \neq a^2$ for all $u,v \in S$, then S cannot be embedded into a seminear-field with a category VI special element.

<u>Proof</u>. The proof is similar to the proof of Theorem 3.20 (substitute a^2 for d).
#

<u>Theorem 3.23</u>. Let S be a Classification D seminear-ring w.r.t. a such that a is not L.M.C. in S. Let $d \in S \setminus \{a\}$ be such that $ax = dx$ for all $x \in S \setminus \{a\}$. Assume that $xa \neq a$ for all $x \in S \setminus \{a\}$ and $u+v \neq d$ for all $u,v \in S$ and $uv \neq d$ for all $u,v \in S \setminus \{d\}$. If $(S \setminus \{d\},\cdot)$ satisfies the right [left] Ore condition, then S can be embedded into a seminear-field with a category VI special element and not into any other category of seminear-fields.

<u>Proof</u>. By Proposition 2.26, $xa = xd$ for all $x \in S \setminus \{a\}$ and $ad = da$. By Proposition 2.49, $S \setminus \{d\}$ is M.C.. By assumption, $S \setminus \{d\}$ is a seminear-ring. Thus $Q(S \setminus \{d\})$ exists. Let $f : S \setminus \{d\} \to Q(S \setminus \{d\})$

be the natural embedding, that is, $f(x) = [(x^2,x)]$ for all $x \in S\setminus\{d\}$. Let a' be a symbol not representing any element of $Q(S\setminus\{d\})$. Extend $+$ and $\cdot$ on $Q(S\setminus\{d\})$ to $K = Q(S\setminus\{d\}) \cup \{a'\}$ by defining $a'\alpha = f(a)\alpha$, $\alpha a' = \alpha f(a)$, $a'+\alpha = f(a)+\alpha$ and $\alpha+a' = \alpha+f(a)$ for all $\alpha \in K$. Then we can show that K is a seminear-field with a category VI special element. Extend $f : S\setminus\{d\} \to Q(S\setminus\{d\})$ to $f : S \to K$ by defining $f(d) = a'$. Clearly, f is an injection.

To show that f is a homomorphism, let $x,y \in S$.

Case 1. $x \neq d$, $y \neq d$. By definition, f is a homomorphism.

Case 2. $x = y = d$. Then $f(x+y) = f(d+d) = f(a+a) = f(a)+f(a) = a'+a' = f(d)+f(d) = f(x)+f(y)$. Similarly, we can show that $f(xy) = f(x)f(y)$.

Case 3. $x = d$, $y \neq d$. Then $f(x+y) = f(d+y) = f(a+y) = f(a)+f(y) = a'+f(y) = f(d)+f(y) = f(x)+f(y)$. To show that $f(xy) = f(x)f(y)$, we shall consider two subcases.

Subcase 3.1. $y = a$. Then $f(xy) = f(da) = f(ad) = f(d^2) = f(a^2) = f(a)f(a) = a'f(a) = f(d)f(a) = f(x)f(y)$.

Subcase 3.2. $y \neq a$. Then $f(xy) = f(dy) = f(ay) = f(a)f(y) = a'f(y) = f(d)f(y) = f(x)f(y)$.

Case 4. $x \neq d$, $y = d$. The proof is similar to the proof of Case 3.

Hence f is a homomorphism and by Theorem 3.12 we are done.

#

Theorem 3.24. Let S be a Classification E seminear-ring w.r.t. a such that $|S| > 2$. If $u+v \neq a^2$ for all $u,v \in S$, $uv \neq a^2$ for all $u,v \in S\setminus\{a^2\}$ and $(S\setminus\{a^2\},\cdot)$ satisfies the right [left] Ore condition, then S can be embedded into a seminear-field with a category VI special element and not into any other category of seminear-fields.

Proof. The proof is similar to the proof of Theorem 3.23 (substitute a^2 for d).
#

Theorem 3.25. Let S be a Classification A seminear-ring such that $(S,\cdot)$ satisfies the right [left] Ore condition. Let $\mathscr{K}_I$ be a category whose objects are seminear-fields with a category I special element. Let K be an object in $\mathscr{K}_I$ and $f : S \to K$ the embedding given by the construction immediately following Theorem 1.38. Then (S,f,K) is a quotient seminear-field of S w.r.t. $\mathscr{K}_I$.

Proof. Let K' be any seminear-field with a category I special element and $i : S \to K'$ a homomorphism. Define $g : K \to K'$ as follows : for $\alpha \in K$, choose $(c,d) \in \alpha$. Define $g(\alpha) = i(c)i(d)^{-1}$.

Let $(c',d') \in \alpha$. Then $(c,d) \sim (c',d')$. There exist $x,y \in S\setminus\{a\}$ such that $cx = c'y$ and $dx = d'y$. Thus $i(c)i(x) = i(c')i(y)$ and $i(d)i(x) = i(d')i(y)$. Therefore $i(c) = i(c')i(y)i(x)^{-1}$ and $i(d')^{-1}i(d) = i(y)i(x)^{-1}$, so $i(c) = i(c')i(d')^{-1}i(d)$. Therefore $i(c)i(d)^{-1} = i(c')i(d')^{-1}$ and hence g is well-defined.

To show that g is a homomorphism, let $\alpha,\beta \in K$. Choose $(x,y) \in \alpha$ and $(z,w) \in \beta$. There exist $u \in S$ and $v \in S\setminus\{a\}$ such that $yu = zv$. Thus $\alpha\beta = [(xu,wv)]$ and $i(u) = i(y)^{-1}i(z)i(v)$. Hence $g(\alpha\beta) = i(xu)i(wv)^{-1} = i(x)i(u)i(v)^{-1}i(w)^{-1} = i(x)i(y)^{-1}i(z)i(w)^{-1} = g(\alpha)g(\beta)$. There exist $p,q \in S\setminus\{a\}$ such that $yp = wq$. Therefore $\alpha+\beta = [(xp+zq,yp)]$ and $g(\alpha+\beta) = i(xp+zq)i(yp)^{-1} = i(xp)i(yp)^{-1}+i(zq)i(yp)^{-1} = i(x)i(y)^{-1}+i(z)i(u)^{-1} =$

$g(\alpha)+g(\beta)$.

To show that $g\circ f = i$, let $x \in S$. If $x = 0$, then $(g\circ f)(x) = g(f(0)) = g(0) = 0 = i(0)$. Assume that $x \neq 0$. Then $(g\circ f)(x) = g([(x^2,x)]) = i(x^2)i(x)^{-1} = i(x)$. Hence $g\circ f = i$.

Suppose that there exists a homomorphism $h : K \to K'$ such that $h\circ f = i$. Let $\alpha \in K$ and choose $(x,y) \in \alpha$. Then $g(\alpha) = i(x)i(y)^{-1} = ((h\circ f)(x))((h\circ f)(y))^{-1} = h([(x^2,x)])h([(y^2,y)])^{-1} = h([(x^2,x)])h([(y,y^2)]) = h([(x^2,x)][(y,y^2)]) = h([(x,y)]) = h(\alpha)$. Thus $g = h$.

#

Definition 3.26. Let K be a seminear-field with a as a special element. Then K is called almost full w.r.t. a if $a+x \neq a$ and $x+a \neq a$ for all $x \in K\setminus\{a\}$. K is called full w.r.t. a if $a+x \neq a$ and $x+a \neq a$ for all $x \in K$.

Example 3.27.

(1) (x) in the proof of Theorem 2.9 is an example of a seminear-field which is almost full w.r.t. a' but not full w.r.t. a'.

(2) (xi) in the proof of Theorem 2.9 is an example of a seminear-field which is not almost full w.r.t. a'.

(3) (xii) in the proof of Theorem 2.9 is an example of a seminear-field which is full w.r.t. a'.

Theorem 3.28. Let S be a Classification B seminear-ring w.r.t. a. Assume that there exists an element $b \in S\setminus\{a\}$ such that $bx = xb = x$ for all $x \in S\setminus\{a\}$ and $ax = xa = x$ for all $x \in S$ and $x+y \neq a$ for all $x,y \in S$ and $(S\setminus\{a\},\cdot)$ satisfies the right [left] Ore condition. Let K be the seminear-field with a category II special element and $f : S \to K$ the embedding given by the construction in Theorem 3.2.

Let $\bar{K}$ be any seminear-field with $\bar{a}$ as a category II special element and $i : S \to \bar{K}$ a monomorphism. Then the following statements hold:

(i) If there are $x,y \in S\smallsetminus\{a\}$ such that $\bar{a} = \bar{a}+i(x)i(y)^{-1}$, then there is no monomorphism $g : K \to \bar{K}$ such that $g\circ f = i$.

(ii) If $\bar{K}$ is almost full w.r.t. $\bar{a}$, then there is a unique monomorphism $g : K \to \bar{K}$ such that $g\circ f = i$.

<u>Proof</u>. First, we shall show that $i(b) \neq \bar{a}$, suppose not. Then $i(b) = i(ab) = i(a)i(b) = i(a)\bar{a} = i(a)$. Thus $a = b$, a contradiction. Hence $i(b) \neq \bar{a}$. Since $b = b^2$, $i(b) = \bar{e}$ where $\bar{e}$ is the identity of $(\bar{K}\smallsetminus\{\bar{a}\},\cdot)$.

Claim that $i(a) = \bar{a}$. To prove this, suppose not. Since $i(b)i(a) = i(b)i(b)$, $i(a) = i(b)$. Thus $a = b$, a contradiction. Hence we have the claim.

(i) Assume that there exists a monomorphism $g : K \to \bar{K}$ such that $g\circ f = i$. Then $g(a') = g(f(a)) = (g\circ f)(a) = i(a) = \bar{a}$ and $g([(b,b)]) = g(f(b)) = (g\circ f)(b) = i(b) = \bar{e}$. Since $\bar{a} = \bar{a}+i(x)i(y)^{-1}$, $y = y+x$. Thus $g([(x,y)]) = g([(x^2,x)][(y,y^2)]) = ((g\circ f)(x))((g\circ f)(y))^{-1} = i(x)i(y)^{-1}$. Therefore $\bar{a} = \bar{a}+i(x)i(y)^{-1} = \bar{a}+g([(x,y)]) = g(a'+[(x,y)]) = g(e'+[(x,y)]) = g([(y+x,y)]) = \bar{e}$, a contradiction.

(ii) Since $a+a \neq a$, $\bar{a}+\bar{a} \neq \bar{a}$. Since $(\bar{a}+\bar{a})\bar{e} = (\bar{e}+\bar{e})\bar{e}$ and $\bar{K}$ is almost full w.r.t. $\bar{a}$, $\bar{a}+\bar{a} = \bar{e}+\bar{e}$. Let $\alpha \in K\smallsetminus\{a'\}$ and choose $(c,d) \in \alpha$. Define $g(\alpha) = i(c)i(d)^{-1}$ and $g(a') = \bar{a}$. Using a proof similar to the proof of Theorem 3.25, we obtain that g is well-defined.

To show that g is an injection, let $\alpha,\beta \in K$ be such that $g(\alpha) = g(\beta)$. If $\alpha = a'$, then $\beta = a'$. Suppose that $\alpha \neq a'$. Then $\beta \neq a'$. Choose $(x,y) \in \alpha$ and $(z,w) \in \beta$. Then there exist $u,v \in S\setminus\{a\}$ such that $yu = wv$. Thus $i(u) = i(y)^{-1}i(w)i(v)$ and $i(x)i(u)i(v)^{-1}i(w)^{-1} = i(x)i(y)^{-1}i(w)i(v)i(v)^{-1}i(w)^{-1} = i(x)i(y)^{-1} = g(\alpha) = g(\beta) = i(z)i(w)^{-1}$. Therefore $i(xu) = i(x)i(u) = i(z)i(v) = i(zv)$. Since i is an injection, $xu = zv$. Thus $(x,y) \sim (z,w)$ and hence $\alpha = \beta$.

Claim that $\bar{a}+u = \bar{e}+u$ and $u+\bar{a} = u+\bar{e}$ for all $u \in \bar{K}\setminus\{\bar{a}\}$. Let $u \in \bar{K}\setminus\{\bar{a}\}$. Since $(\bar{a}+u)u = \bar{a}u+uu = u+uu = \bar{e}u+uu = (\bar{e}+u)u$ and $\bar{K}$ is almost full w.r.t. $\bar{a}$, $\bar{a}+u = \bar{e}+u$. Similarly, we can show that $u+\bar{a} = u+\bar{e}$. So we have the claim.

To show that g is a homomorphism, let $\alpha,\beta \in K$.

Case 1. $\alpha = \beta = a'$. Then $g(\alpha+\beta) = g(a'+a') = g([(b,b)]+[(b,b)]) = g([(b+b,b)]) = i(b+b)i(b)^{-1} = \bar{e}+\bar{e} = \bar{a}+\bar{a} = g(a')+g(a') = g(\alpha)+g(\beta)$ and $g(\alpha\beta) = g(a'a') = g(a') = \bar{a} = \bar{a}\bar{a} = g(a')g(a') = g(\alpha)g(\beta)$.

Case 2. $\alpha = a'$, $\beta \neq a'$. Choose $(z,w) \in \beta$. Then $\alpha+\beta = a'+[(z,w)] = [(b,b)]+[(z,w)]$. There exist $u,v \in S$ such that $bu = wv$. Thus $\alpha+\beta = [(bu+zv,bu)]$, so $g(\alpha+\beta) = i(bu+zv)i(bu)^{-1} = i(bu)i(bu)^{-1}+i(zv)i(wv)^{-1} = \bar{e}+i(z)i(w)^{-1} = \bar{a}+i(z)i(w)^{-1} = g(a')+g(\beta) = g(\alpha)+g(\beta)$ and $g(\alpha\beta) = g(a'\beta) = g(\beta) = \bar{a}g(\beta) = g(a')g(\beta) = g(\alpha)g(\beta)$.

Case 3. $\alpha \neq a'$, $\beta = a'$. The proof is similar to the proof of Case 2.

Case 4. $\alpha \neq a'$, $\beta \neq a'$. Choose $(x,y) \in \alpha$ and $(z,w) \in \beta$. There exist $p,q \in S$ such that $yp = wq$. Thus $g(\alpha+\beta) = i(xp+zq)i(yp)^{-1} =$

$i(xp)i(yp)^{-1}+i(zq)i(wq)^{-1} = i(x)i(y)^{-1}+i(z)i(w)^{-1} = g(\alpha)+g(\beta)$. There exist $u,v \in S$ such that $yu = zv$. Thus $\alpha\beta = [(xu,wv)]$ and $i(y)i(u) = i(z)i(v)$. Therefore $g(\alpha\beta) = i(xu)i(wv)^{-1} = i(x)i(u)i(v)^{-1}i(w)^{-1} = i(x)i(y)^{-1}i(z)i(w)^{-1} = g(\alpha)g(\beta)$.

Let $x \in S$. If $x = a$, then $(g\circ f)(x) = (g\circ f)(a) = g(f(a)) = g(a') = \bar{a} = i(a)$. Assume that $x \neq a$. Then $(g\circ f)(x) = g(f(x)) = g([(x^2,x)]) = i(x^2)i(x)^{-1} = i(x)$. Hence $g\circ f = i$.

Let $h : K \to \bar{K}$ be a monomorphism such that $h\circ f = i$. Let $\alpha \in K$. If $\alpha = a'$, then $g(\alpha) = g(a') = \bar{a} = i(a) = (h\circ f)(a) = h(f(a)) = h(a') = h(\alpha)$. Suppose that $\alpha \neq a'$. Choose $(x,y) \in \alpha$. Then $g(\alpha) = g([(x,y)]) = i(x)i(y)^{-1} = ((h\circ f)(x))((h\circ f)(y))^{-1} = h([(x^2,x)])h([(y,y^2)]) = h([(x^2,x)][(y,y^2)]) = h([(x,y)]) = h(\alpha)$. Thus $g = h$.

#

Theorem 3.29. Let S be an M.C. Classification B(C,D) seminear-ring such that $(S,\cdot)$ satisfies the right [left] Ore condition. Let $\mathcal{K}_0$ be the category whose objects are 0-seminear-fields. Let K be the 0-seminear-field and $f : S \to K$ the embedding given by the construction in the remark immediately following Theorem 3.4. Then (S,f,K) is a quotient seminear-field of S w.r.t. $\mathcal{K}_0$.

Proof. Let $\bar{K}$ be any 0-seminear-field and $i : S \to \bar{K}$ a homomorphism. By the construction of K, $K = Q(S) \cup \{a'\}$ where a' is a 0-special element of K. Let $\bar{a}$ be a 0-special element of $\bar{K}$.

Claim that $i(x) \neq \bar{a}$ for all $x \in S$. To prove this, suppose not. Then there exists an $x \in S$ such that $i(x) = \bar{a}$. Let $y \in S\setminus\{x\}$. Then

$i(xx) = i(x)i(x) = \bar{a}\bar{a} = \bar{a} = \bar{a}i(y) = i(x)i(y) = i(xy)$, so $xx = xy$. Since S is M.C., $x = y$ which is a contradiction. So we have the claim.

Define $g : K \to \bar{K}$ as follows : for $\alpha \in K \setminus \{a'\}$, choose $(x,y) \in \alpha$. Define $g(\alpha) = i(x)i(y)^{-1}$ and $g(a') = \bar{a}$. Using the same proof as in Theorem 3.25, we can show that g is well-defined.

To show that g is a homomorphism, let $\alpha,\beta \in K$.

Case 1. $\alpha = \beta = a'$. Then $g(\alpha+\beta) = g(a'+a') = g(a') = \bar{a} = \bar{a}+\bar{a} = g(a')+g(a') = g(\alpha)+g(\beta)$. Similarly, we can show that $g(\alpha\beta) = g(\alpha)g(\beta)$.

Case 2. $\alpha = a'$, $\beta \neq a'$. Then $g(\alpha+\beta) = g(a'+\beta) = g(\beta) = \bar{a}+g(\beta) = g(a')+g(\beta) = g(\alpha)+g(\beta)$ and $g(\alpha\beta) = g(a'\beta) = g(a') = \bar{a} = \bar{a}g(\beta) = g(a')g(\beta) = g(\alpha)g(\beta)$.

Case 3. $\alpha \neq a'$, $\beta = a'$. The proof is similar to the proof of Case 2.

Case 4. $\alpha \neq a'$, $\beta \neq a'$. The proof is similar to the proof of Case 4 in Theorem 3.28(ii).

Hence g is a homomorphism. Using the same proof as in Theorem 3.28, we get that g is the unique homomorphism such that $g \circ f = i$. #

Theorem 3.30. Let S be an M.C. Classification B(C,D) seminear-ring such that $(S,\cdot)$ satisfies the right [left] Ore condition. Let $\mathcal{K}_\infty$ be the category whose objects are ∞-seminear-fields. Let K be the ∞-seminear-field and $f : S \to K$ the embedding given by the construction in the remark immediately following Theorem 3.5. Then (S,f,K) is a quotient seminear-field of S w.r.t. $\mathcal{K}_\infty$.

Proof. Let $\bar{K}$ be any ∞-seminear-field and $i : S \to \bar{K}$ a homomorphism. By the construction of K, $K = Q(S) \cup \{a'\}$ where a' is an ∞-special element of K. Let $\bar{a}$ be an ∞-special element of $\bar{K}$. Using the same proof as in Theorem 3.29, we can show that $i(x) \neq \bar{a}$ for all $x \in S$.

Define $g : K \to \bar{K}$ as follows : for $\alpha \in K \setminus \{a'\}$, choose $(x,y) \in \alpha$. Define $g(\alpha) = i(x)i(y)^{-1}$ and $g(a') = \bar{a}$. Using the same proofs as in Theorem 3.25 and Theorem 3.29, we can show that g is well-defined and $g(\alpha\beta) = g(\alpha)g(\beta)$ for all $\alpha,\beta \in K$.

To show that $g(\alpha+\beta) = g(\alpha)+g(\beta)$ for all $\alpha,\beta \in K$, let $\alpha,\beta \in K$.

<u>Case 1</u>. $\alpha = \beta = a'$. Then $g(\alpha+\beta) = g(a'+a') = g(a') = \bar{a} = \bar{a}+\bar{a} = g(a')+g(a') = g(\alpha)+g(\beta)$.

<u>Case 2</u>. $\alpha = a'$, $\beta \neq a'$. Then $g(\alpha+\beta) = g(a'+\beta) = g(a') = \bar{a} = \bar{a}+g(\beta) = g(a')+g(\beta) = g(\alpha)+g(\beta)$.

<u>Case 3</u>. $\alpha \neq a'$, $\beta = a'$. The proof is similar to the proof of Case 2.

<u>Case 4</u>. $\alpha \neq a'$, $\beta \neq a'$. The proof is similar to the proof of Case 4 in Theorem 3.28(ii).

Hence g is a homomorphism. Using the same proof as in Theorem 3.28, we get that g is the unique homomorphism such that $g \circ f = i$. #

<u>Theorem 3.31</u>. Let S be an M.C. Classification B(C,D) seminear-ring such that $(S,+)$ is a left zero semigroup and $(S,\cdot)$ satisfies the right [left] Ore condition. Let $\mathscr{H}_L$ be the category whose objects are additive left zero seminear-fields with a category I special element. Let K be the object in $\mathscr{H}_L$ and $f : S \to K$ the embedding given by the

construction in the remark immediately following Theorem 3.7. Then (S,f,K) is a quotient seminear-field of S w.r.t. $\mathcal{H}_L$.

Proof. The proof of this theorem is similar to the proofs of Theorem 3.29 and Theorem 3.30.

#

Theorem 3.32. Let S be an M.C. Classification B(C,D) seminear-ring such that (S,+) is a right zero semigroup and (S,·) satisfies the right [left] Ore condition. Let $\mathcal{H}_R$ be the category whose objects are additive right zero seminear-fields with a category I special element. Let K be the object in $\mathcal{H}_R$ and f : S → K the embedding given by the construction in the remark immediately following Theorem 3.8. Then (S,f,K) is a quotient seminear-field of S w.r.t. $\mathcal{H}_R$.

Proof. The proof of this theorem is similar to the proofs of Theorem 3.29 and Theorem 3.30.

#

Theorem 3.33. Let S be an M.C. Classification C seminear-ring w.r.t. a. Let b ε S∖{a} be such that ab = a. Assume that (S,·) satisfies the right [left] Ore condition. Let K be the seminear-field with a category II special element and f : S → K the embedding given by the construction in the remark immediately following Theorem 3.10. Let $\bar{K}$ be any seminear-field with $\bar{a}$ as a category II special element and i : S → $\bar{K}$ a monomorphism. Then the following statements hold:

(i) If i(b) = $\bar{a}$, then there is no monomorphism g : K → $\bar{K}$ such that g∘f = i.

(ii) If i(b) ≠ $\bar{a}$ and $\bar{K}$ is full w.r.t. $\bar{a}$, then there is a unique monomorphism $g : K \to \bar{K}$ such that $g \circ f = i$.

<u>Proof</u>. Since S is M.C., a is L.M.C. in S. By Proposition 2.23, $ba = a$. Let $a' \in K$ be such that $(K \setminus \{a'\}, \cdot)$ is a group.

(i) Assume that $i(b) = \bar{a}$ and there is a monomorphism $g : K \to \bar{K}$ such that $g \circ f = i$. Then $\bar{a} = i(b) = (g \circ f)(b) = g(f(b)) = g([(b^2,b)] = g([(b,b)])$. Thus $g(a') = g(a')\bar{a} = g(a')g([(b,b)]) = g(a'[(b,b)]) = g([(b,b)])$. Hence $a' = [(b,b)]$, a contradiction.

(ii) First, claim that $i(x) \neq \bar{a}$ for all $x \in S$. Assume that there is an $x \in S$ such that $i(x) = \bar{a}$. Then $i(b) \neq \bar{a} = i(x) = i(xb) = i(x)i(b) = \bar{a}i(b) = i(b)$, a contradiction. So we have the claim. Define $g : K \to \bar{K}$ as follows : for $\alpha \in K \setminus \{a'\}$, choose $(x,y) \in \alpha$. Define $g(\alpha) = i(x)i(y)^{-1}$ and $g(a') = \bar{a}$. Using the same proofs as in Theorem 3.25 and Theorem 3.28, we get that g is well-defined and g is the unique monomorphism such that $g \circ f = i$.

#

<u>Proposition 3.34</u>. Let K be any seminear-field with a as a category VI special element. Let $d \in K \setminus \{a\}$ be such that $ax = dx$ and $xa = xd$ for all $x \in K$. Let $\bar{K}$ be any seminear-field with $\bar{a}$ as a category VI special element. Let $\bar{d} \in \bar{K} \setminus \{\bar{a}\}$ be such that $\bar{a}x = \bar{d}x$ and $x\bar{a} = x\bar{d}$ for all $x \in \bar{K}$. If there is a monomorphism $g : K \to \bar{K}$, then $g(a) = \bar{a}$ and $g(d) = \bar{d}$.

<u>Proof</u>. Let e and $\bar{e}$ be the identities of $(K \setminus \{a\}, \cdot)$ and $(\bar{K} \setminus \{\bar{a}\}, \cdot)$, respectively. Then $g(d) = g(de) = g(d)g(e) = g(d)\bar{e} \neq \bar{a}$. Therefore

$g(a)g(a) = g(aa) = g(ad) = g(a)g(d)$. If $g(a) \neq \bar{a}$, then $g(a) = g(d)$. Hence $a = d$, a contradiction. Thus $g(a) = \bar{a}$. Since $g(d)g(d) = g(a)g(d)$, $g(d) = g(a)\bar{e} = \bar{a}\bar{e} = \bar{d}\bar{e} = \bar{d}$.

#

<u>Theorem 3.35</u>. Let S be an M.C. Classification C seminear-ring such that $(S,\cdot)$ satisfies the right [left] Ore condition. Let K be the seminear-field with a' as a category VI special element and $f : S \to K$ the embedding given by the construction in the remark immediately following Theorem 3.11. Suppose that there is an element $[(d,d_2)] \in K \setminus \{a'\}$ such that

$$a'\alpha = [(d_1,d_2)]\alpha, \quad \alpha a' = \alpha[(d_1,d_2)],$$

$$a'+\alpha = [(d_1,d_2)]+\alpha \text{ and } \alpha+a' = \alpha+[(d_1,d_2)]$$

for all $\alpha \in K$. Let $\bar{K}$ be any seminear-field with $\bar{a}$ as a category VI special element. Let $\bar{d} \in \bar{K}\setminus\{\bar{a}\}$ be such that $\bar{a}x = \bar{d}x$ and $x\bar{a} = x\bar{d}$ for all $x \in \bar{K}$ and let $i : S \to \bar{K}$ be a monomorphism. Then the following hold :

(i) If there is a $y \in S$ such that $i(y) = \bar{d}$ but $f(x) \neq [(d_1,d_2)]$ for all $x \in S$, then there is no monomorphism $g : K \to \bar{K}$ such that $g \circ f = i$.

(ii) If there are $y,u \in S$ such that $y \neq u$ and $i(y) = \bar{d}$ and $f(u) = [(d_1,d_2)]$, then there is no monomorphism $g : K \to \bar{K}$ such that $g \circ f = i$.

(iii) If there is a $u \in S$ such that $f(u) = [(d_1,d_2)]$ but $i(y) \neq \bar{d}$ for all $y \in S$, then there is no monomorphism $g : K \to \bar{K}$ such that $g \circ f = i$.

(iv) If $i(x) \neq \bar{d}$ and $f(x) \neq [(d_1,d_2)]$ for all $x \in S$ and $i(d_1)i(d_2)^{-1} = \bar{d}$ and $\bar{K}$ is full w.r.t. $\bar{a}$, then there is a unique monomorphism $g : K \to \bar{K}$ such that $g \circ f = i$.

Proof. Let $b \in S \setminus \{a\}$ be such that $ab = a$. Then $yb = y$ for all $y \in S$. If there exists an $x \in S$ such that $i(x) = \bar{a}$, then $\bar{a} = i(x) = i(xb) = i(x)i(b) = \bar{a}i(b)$. This is a contradiction, so $i(x) \neq \bar{a}$ for all $x \in S$. Since $\bar{K}$ is full w.r.t. $\bar{a}$, $\bar{a}+\bar{a} = \bar{d}+\bar{d}$ and $\bar{a}+x = \bar{d}+x$ and $x+\bar{a} = x+\bar{d}$ for all $x \in \bar{K}$.

(i) Assume that there is a monomorphism $g : K \to \bar{K}$ such that $g \circ f = i$. Then $g(f(y)) = (g \circ f)(y) = i(y) = \bar{d} = g([(d_1,d_2)])$, by Proposition 3.34. Thus $f(y) = [(d_1,d_2)]$, a contradiction.

(ii) Assume that there is a monomorphism $g : K \to \bar{K}$ such that $g \circ f = i$. Then $i(u) = (g \circ f)(u) = g(f(u)) = g([(d_1,d_2)]) = \bar{d}$, by Proposition 3.34. Thus $i(u) = \bar{d} = i(y)$. Hence $u = y$, a contradiction.

(iii) Assume that there is a monomorphism $g : K \to \bar{K}$ such that $g \circ f = i$. Then $i(u) = (g \circ f)(u) = g(f(u)) = g([(d_1,d_2)]) = \bar{d}$, a contradiction.

(iv) Define $g : K \to \bar{K}$ as follows : for $\alpha \in K \setminus \{a'\}$, choose $(x,y) \in \alpha$. Define $g(\alpha) = i(x)i(y)^{-1}$ and $g(a') = \bar{a}$. Using the same proofs as in Theorem 3.25 and Theorem 3.28, we can show that g is well-defined and an injection.

To show that g is a homomorphism, let $\alpha,\beta \in K$.

Case 1. $\alpha = \beta = a'$. Then $g(\alpha+\beta) = g(a'+a') = g([(d_1,d_2)]+[(d_1,d_2)]) =$

$g([(d_1d_2+d_1d_2,d_2d_2)] = g([(d_1+d_1,d_2)]) = i(d_1+d_1)i(d_2)^{-1} = \bar{d}+\bar{d} =$ $\bar{a}+\bar{a} = g(a')+g(a')$. There are $x,y \in S$ such that $d_2x = d_1y$. Then $[(d_1,d_2)][(d_1,d_2)] = [(d_1x,d_2y)]$ and $i(x) = i(d_2)^{-1}i(d_1)i(y)$. Thus $g(\alpha\beta) = i(d_1x)i(d_2y)^{-1} = i(d_1)i(d_2)^{-1}i(d_1)i(d_2)^{-1} = \bar{d}\bar{d} = \bar{a}\bar{a} = g(a')g(a')$.

<u>Case 2</u>. $\alpha = a'$, $\beta \neq a'$. Choose $(z,w) \in \beta$. Then $g(\alpha+\beta) = g(a'+\beta) =$ $g([(d_1,d_2)]+\beta) = g([(d_1,d_2)]+[(z,w)])$. There are $u,v \in S$ such that $d_2u = wv$. Thus $g(\alpha+\beta) = g([(d_1u+zv,d_2u)]) = i(d_1u+zv)i(d_2u)^{-1} =$ $i(d_1)i(d_2)^{-1}+i(z)i(w)^{-1} = \bar{d}+g(\beta) = \bar{a}+g(\beta) = g(a')+g(\beta)$. There are $x,y \in S$ such that $d_2x = zy$. Then $g(\alpha\beta) = g(a'\beta) = g([(d_1,d_2)]\beta) =$ $g([(d_1,d_2)][(z,w)]) = g([(d_1x,wy)]) = i(d_1x)i(wy)^{-1} =$ $i(d_1)i(d_2)^{-1}i(z)i(w)^{-1} = \bar{d}g(\beta) = \bar{a}g(\beta) = g(a')g(\beta) = g(\alpha)g(\beta)$.

<u>Case 3</u>. $\alpha \neq a'$, $\beta = a'$. The proof is similar to the proof of Case 2.

<u>Case 4</u>. $\alpha \neq a'$, $\beta \neq a'$. The proof is similar to the proof of Case 4 in Theorem 3.28(ii).

Let $x \in S$. Then $(g\circ f)(x) = g(f(x)) = g([(x^2,x)]) = i(x^2)i(x)^{-1} =$ $i(x)$, so $g\circ f = i$.

Using a proof similar to the one used in Theorem 3.28, we can show that $g : K \to \bar{K}$ is the unique monomorphism such that $g\circ f = i$. #

<u>Theorem 3.36</u>. Let S be an M.C. Classification D seminear-ring such that $(S,\cdot)$ satisfies the right [left] Ore condition. Let K be the seminear-field with a category II special element and $f : S \to K$ the embedding given by the construction in the remark immediately

following Theorem 3.10. Let $\bar{K}$ be a seminear-field with $\bar{a}$ as a category II special element such that $\bar{K}$ is full w.r.t. $\bar{a}$ and $i : S \to \bar{K}$ a monomorphism. Then there is a unique monomorphism $g : K \to \bar{K}$ such that $g \circ f = i$.

Proof. First, claim that $i(x) \neq \bar{a}$ for all $x \in S$. Assume that there exists an $x \in S$ such that $i(x) = \bar{a}$. Then $i(x) = \bar{a} = \bar{a}\bar{a} = i(x)i(x) = i(xx)$ which implies that $xx = x$. Let $a \in S$ be such that $(S \setminus \{a\}, \cdot)$ is a cancellative semigroup. Then $ax \neq a$. Since $axx = ax$ and S is M.C., $ax = a$ which is a contradiction. So we have the claim.

Let $a' \in K$ be such that $(K \setminus \{a'\}, \cdot)$ is a group. Define $g : K \to \bar{K}$ as follows : for $\alpha \in K \setminus \{a'\}$, choose $(x,y) \in \alpha$. Define $g(\alpha) = i(x)i(y)^{-1}$ and $g(a') = \bar{a}$. Using a proof similar to the one used in Theorem 3.28, we get that g is the unique monomorphism such that $g \circ f = i$.

#

Theorem 3.37. Let S be an M.C. Classification D seminear-ring such that $(S, \cdot)$ satisfies the right [left] Ore condition. Let K be the seminear-field with a' as a category VI special element and $f : S \to K$ the embedding given by the construction in the remark immediately following Theorem 3.11. Suppose that there is an element $[(d_1,d_2)] \in K \setminus \{a'\}$ such that

$$a'\alpha = [(d_1,d_2)]\alpha \quad , \quad \alpha a' = \alpha[(d_1,d_2)]$$

$$a'+\alpha = [(d_1,d_2)]+\alpha \quad \text{and} \quad \alpha+a' = \alpha+[(d_1,d_2)]$$

for all $\alpha \in K$. Let $\bar{K}$ be any seminear-field with $\bar{a}$ as a category VI special element. Let $\bar{d} \in \bar{K} \setminus \{\bar{a}\}$ be such that $\bar{a}x = \bar{d}x$ and $x\bar{a} = x\bar{d}$ for all $x \in \bar{K}$. Let $i : S \to \bar{K}$ be a monomorphism. If there is an $x \in S$

such that $i(x) = \bar{a}$, then there is no monomorphism $g : K \to \bar{K}$ such that $g \circ f = i$. Furthermore, if $i(x) \neq \bar{a}$ for all $x \in S$, then the following hold :

(i) If there is a $y \in S$ such that $i(y) = \bar{d}$ and $f(x) \neq [(d_1,d_2)]$ for all $x \in S$, then there is no monomorphism $g : K \to \bar{K}$ such that $g \circ f = i$.

(ii) If there is a $y \in S$ such that $i(y) = \bar{d}$ and there is a $u \in S$ such that $f(u) = [(d_1,d_2)]$ where $u \neq y$, then there is no monomorphism $g : K \to \bar{K}$ such that $g \circ f = i$.

(iii) If $i(y) \neq \bar{d}$ for all $y \in S$ and there is a $u \in S$ such that $f(u) = [(d_1,d_2)]$, then there is no monomorphism $g : K \to \bar{K}$ such that $g \circ f = i$.

(iv) If $i(y) \neq \bar{d}$ for all $y \in S$ and $f(y) \neq [(d_1,d_2)]$ for all $y \in S$ and $i(d_1)i(d_2)^{-1} = \bar{d}$, then there is a unique monomorphism $g : K \to \bar{K}$ such that $g \circ f = i$.

<u>Proof</u>. Assume that there is a monomorphism $g : K \to \bar{K}$ such that $g \circ f = i$. By Proposition 3.34, $g(f(x)) = (g \circ f)(x) = i(x) = \bar{a} = g(a')$. Thus $f(x) = a'$, a contradiction.

(i), (ii), (iii) and (iv) are proven in a similar way to the proof in Theorem 3.35.

#

<u>Proposition 3.38</u>. Let S be a Classification D seminear-ring w.r.t. a such that a is not L.M.C. in S. Let $\bar{K}$ be a seminear-field with $\bar{a}$ as a category VI special element and let $\bar{d} \in \bar{K} \setminus \{\bar{a}\}$ be such that $\bar{a}x = \bar{d}x$

and $x\bar{a} = x\bar{d}$ for all $x \in \bar{K}$. If there is a monomorphism $i : S \to \bar{K}$, then either $i(a) = \bar{d}$ or $i(a) = \bar{a}$.

<u>Proof</u>. Let $d \in S\smallsetminus\{a\}$ be such that $ax = dx$ for all $x \in S\smallsetminus\{a\}$. Assume that $i(a) \neq \bar{d}$ and $i(a) \neq \bar{a}$. Let $\bar{e}$ be the identity of $(\bar{K}\smallsetminus\{\bar{a}\},\cdot)$.

<u>Case 1</u>. $i(x) \neq \bar{a}$ for all $x \in S$. Since $i(a)i(d) = i(ad) = i(dd) = i(d)i(d)$, $i(a) = i(d)$. Thus $a = d$, a contradiction.

<u>Case 2</u>. There is an $x \in S$ such that $i(x) = \bar{a}$. Then $x \neq a$ and $ax = dx$. Therefore $i(a)\bar{d} = i(a)\bar{a} = i(a)i(x) = i(ax) = i(dx) = i(d)i(x) = i(d)\bar{a} = i(d)\bar{d}$. Thus $i(a) = i(d)\bar{e}$.

<u>Subcase 2.1</u>. $i(d) \neq \bar{a}$. Then $i(a) = i(d)$. Thus $a = d$, a contradiction.

<u>Subcase 2.2</u>. $i(d) = \bar{a}$. Then $i(a) = \bar{a}\bar{e} = \bar{d}\bar{e} = \bar{d}$, a contradiction.

Hence either $i(a) = \bar{d}$ or $i(a) = \bar{a}$.

#

<u>Proposition 3.39</u>. Let S be a Classification E seminear-ring w.r.t. a. Let $\bar{K}$ be a seminear-field with $\bar{a}$ as a category VI special element and let $\bar{d} \in \bar{K}\smallsetminus\{\bar{a}\}$ be such that $\bar{a}x = \bar{d}x$ and $x\bar{a} = x\bar{d}$ for all $x \in \bar{K}$. If there is a monomorphism $i : S \to \bar{K}$, then either $i(a) = \bar{d}$ or $i(a) = \bar{a}$.

<u>Proof</u>. This proof is similar to the proof of Proposition 3.38 (substitute a^2 for d).

#

<u>Theorem 3.40</u>. Let S be a Classification D seminear-ring w.r.t. a such that a is not L.M.C. in S. Assume that $xa \neq a$ for all $x \in S \smallsetminus \{a\}$ and $x+y \neq a$ for all $x,y \in S$ and $(S \smallsetminus \{a\}, \cdot)$ satisfies the right [left] Ore condition. Let K be the seminear-field with a category VI special element and $f : S \to K$ the embedding given by the construction in Theorem 3.15. Let $\bar{K}$ be any seminear-field with $\bar{a}$ as a category VI special element, let $\bar{d} \in \bar{K} \smallsetminus \{\bar{a}\}$ be such that $\bar{a}x = \bar{d}x$ and $x\bar{a} = x\bar{d}$ for all $x \in \bar{K}$ and let $i : S \to \bar{K}$ be a monomorphism. Then the following hold :

(i) If $i(a) = \bar{d}$, then there is no monomorphism $g : K \to \bar{K}$ such that $g \circ f = i$.

(ii) If $i(a) = \bar{a}$ and $\bar{K}$ is full w.r.t. $\bar{a}$, then there is a unique monomorphism $g : K \to \bar{K}$ such that $g \circ f = i$.

<u>Proof</u>. Let $d \in S \smallsetminus \{a\}$ be such that $ax = dx$ for all $x \in S \smallsetminus \{a\}$.

(i) Assume that $i(a) = \bar{d}$ and there is a monomorphism $g : K \to \bar{K}$ such that $g \circ f = i$. Claim that $i(d) = \bar{a}$. To prove this, suppose not. Since $i(a)i(a) = i(a)i(d)$ and $i(a) = \bar{d} \neq \bar{a}$, $i(a) = i(d)$. Thus $a = d$, a contradiction, so we have the claim. By Proposition 3.34, $g(f(d)) = (g \circ f)(d) = i(d) = \bar{a} = g(a')$ where $a' \in K$ is such that $(K \smallsetminus \{a'\}, \cdot)$ is a group. Therefore $f(d) = a' = f(a)$. Hence $d = a$, a contradiction.

(ii) Since $i(a) = \bar{a}$ and $i(d)i(d) = i(a)i(d)$, $i(d) = i(a)\bar{e} = \bar{a}\bar{e} = \bar{d}\bar{e} = \bar{d}$. Define $g : K \to \bar{K}$ as follows : for $\alpha \in K \smallsetminus \{a'\}$, choose $(x,y) \in \alpha$. Define $g(\alpha) = i(x)i(y)^{-1}$ and $g(a') = \bar{a}$. Using a proof similar to the one used in Theorem 3.35(iv) (substitute f(d) for $[(d_1,d_2)]$), we get that g is the unique monomorphism such that $g \circ f = i$. #

Theorem 3.41. Let S be a Classification E seminear-ring w.r.t. a such that $|S| > 2$. Assume that $x+y \neq a$ for all $x,y \in S$ and $(S\setminus\{a\},\cdot)$ satisfies the right [left] Ore condition. Let K be the seminear-field with a category VI special element and $f : S \to K$ the embedding given by the construction in Theorem 3.15 (substitute a^2 for d). Let $\bar{K}$ be any seminear-field with $\bar{a}$ as a category VI special element and let $\bar{d} \in \bar{K}\setminus\{\bar{a}\}$ be such that $\bar{a}x = \bar{d}x$ and $x\bar{a} = x\bar{d}$ for all $x \in \bar{K}$ and let $i : S \to \bar{K}$ be a monomorphism. Then the following hold :

(i) If $i(a) = \bar{d}$, then there is no monomorphism $g : K \to \bar{K}$ such that $g\circ f = i$.

(ii) If $i(a) = \bar{a}$ and $\bar{K}$ is full w.r.t. $\bar{a}$, then there is a unique monomorphism $g : K \to \bar{K}$ such that $g\circ f = i$.

Proof. This proof is similar to the proof of Theorem 3.40 (substitute a^2 for d).

#

Proposition 3.42. Let S be a Classification D seminear-ring w.r.t. a such that a is not L.M.C. in S. Let $d \in S\setminus\{a\}$ be such that $ax = dx$ for all $x \in S\setminus\{a\}$. If there is a monomorphism $i : S \to \bar{K}$ where $\bar{K}$ is a seminear-field with $\bar{a}$ as a category VI special element, then either $i(d) = \bar{a}$ or $i(d) = \bar{d}$ where $\bar{d} \in \bar{K}\setminus\{\bar{a}\}$ is such that $\bar{a}x = \bar{d}x$ and $x\bar{a} = x\bar{d}$ for all $x \in \bar{K}$.

Proof. Let $\bar{e}$ be the identity of $(\bar{K}\setminus\{\bar{a}\},\cdot)$. Assume that $i(d) \neq \bar{a}$. Claim that $i(a) = \bar{a}$. To prove this, suppose not. Since

$i(a)i(a) = i(a)i(d)$, $i(a) = i(d)$. Thus $a = d$, a contradiction. So we have the claim. Since $\bar{d}i(a) = \bar{a}i(a) = i(a)i(a) = i(a)i(d) = \bar{a}i(d) = \bar{d}i(d)$, $i(d) = \bar{e}i(a)$. Thus $i(d) = \bar{e}\bar{a} = \bar{e}\bar{d} = \bar{d}$.

#

Proposition 3.43. Let S be a Classification E seminear-ring w.r.t. a. If there is a monomorphism $i : S \to \bar{K}$ where $\bar{K}$ is a seminear-field with $\bar{a}$ as a category VI special element, then either $i(a^2) = \bar{a}$ or $i(a^2) = \bar{d}$ where $\bar{d} \in \bar{K} \setminus \{\bar{a}\}$ is such that $\bar{a}x = \bar{d}x$ and $x\bar{a} = x\bar{d}$ for all $x \in \bar{K}$.

Proof. This proof is similar to the proof of Theorem 3.42 (substitute a^2 for d).

#

Theorem 3.44. Let S be a Classification D seminear-ring w.r.t. a such that a is not L.M.C. in S. Let $d \in S \setminus \{a\}$ be such that $ax = dx$ for all $x \in S \setminus \{a\}$. Assume that $xa \neq a$ for all $x \in S \setminus \{a\}$ and $u+v \neq d$ for all $u,v \in S$ and $uv \neq d$ for all $u,v \in S \setminus \{d\}$ and $(S \setminus \{d\}, \cdot)$ satisfies the right [left] Ore condition. Let K be the seminear-field with a category VI special element and $f : S \to K$ the embedding given by the construction in Theorem 3.23. Let $\bar{K}$ be any seminear-field with $\bar{a}$ as a category VI special element and let $\bar{d} \in \bar{K} \setminus \{\bar{a}\}$ be such that $\bar{a}x = \bar{d}x$ and $x\bar{a} = x\bar{d}$ for all $x \in \bar{K}$. If $i : S \to \bar{K}$ is a monomorphism, then the following hold :

(i) If $i(d) = \bar{d}$, then there is no monomorphism $g : K \to \bar{K}$ such that $g \circ f = i$.

(ii) If $i(d) = \bar{a}$ and $\bar{K}$ is full w.r.t. $\bar{a}$, then there is a unique monomorphism $g : K \to \bar{K}$ such that $g \circ f = i$.

Proof. (i) Let a' be a special element of K. Claim that $i(a) = \bar{a}$. To prove this, suppose not. Since $i(a)i(a) = i(a)i(d)$ and $i(d) = \bar{d} \neq \bar{a}$, $i(a) = i(d)$. Thus $a = d$ which is a contradiction, so we have the claim. Assume that there is a monomorphism $g : K \to \bar{K}$ such that $g \circ f = i$. By Proposition 3.34, $g(f(a)) = (g \circ f)(a) = i(a) = \bar{a} = g(a')$. Thus $f(a) = a' = f(d)$. Hence $a = d$, a contradiction.

(ii) Assume that $i(d) = \bar{a}$ and $\bar{K}$ is full w.r.t. $\bar{a}$. Then $i(a) \neq \bar{a}$. Since $i(a)i(a) = i(a)i(d)$, $i(a) = \bar{e}i(d) = \bar{e}\bar{a} = \bar{e}\bar{d} = \bar{d}$. Thus $i(a) = \bar{d}$. Define $g : K \to \bar{K}$ as follows : for $\alpha \in K \setminus \{a'\}$, choose $(x,y) \in \alpha$. Define $g(\alpha) = i(x)i(y)^{-1}$ and $g(a') = \bar{a}$. Using a proof similar to the one used in Theorem 3.35(iv) (substitute f(a) for $[(d_1,d_2)]$), we get that g is the unique monomorphism such that $g \circ f = i$.

#

Theorem 3.45. Let S be a Classification E seminear-ring w.r.t. a such that $|S| > 2$. Assume that $u+v \neq a^2$ for all $u,v \in S$ and $uv \neq a^2$ for all $u,v \in S \setminus \{a^2\}$ and $(S \setminus \{a^2\}, \cdot)$ satisfies the right [left] Ore condition. Let K be a seminear-field with a category VI special element and $f : S \to K$ the embedding given by the construction in Theorem 3.23 (substitute a^2 for d). Let $\bar{K}$ be any seminear-field with $\bar{a}$ as a category VI special element and let $\bar{d} \in \bar{K} \setminus \{\bar{a}\}$ be such that $\bar{a}x = \bar{d}x$ and $x\bar{a} = x\bar{d}$ for all $x \in \bar{K}$. If $i : S \to \bar{K}$ is a monomorphism, then the following hold :

(i) If $i(a^2) = \bar{d}$, then there is no monomorphism $g : K \to \bar{K}$ such that $g \circ f = i$.

(ii) If $i(a^2) = \bar{a}$ and $\bar{K}$ is full w.r.t. $\bar{a}$, then there is a unique monomorphism $g : K \to \bar{K}$ such that $g \circ f = i$.

<u>Proof</u>. This proof is similar to the proof of Theorem 3.44 (substitute a^2 for d).

#