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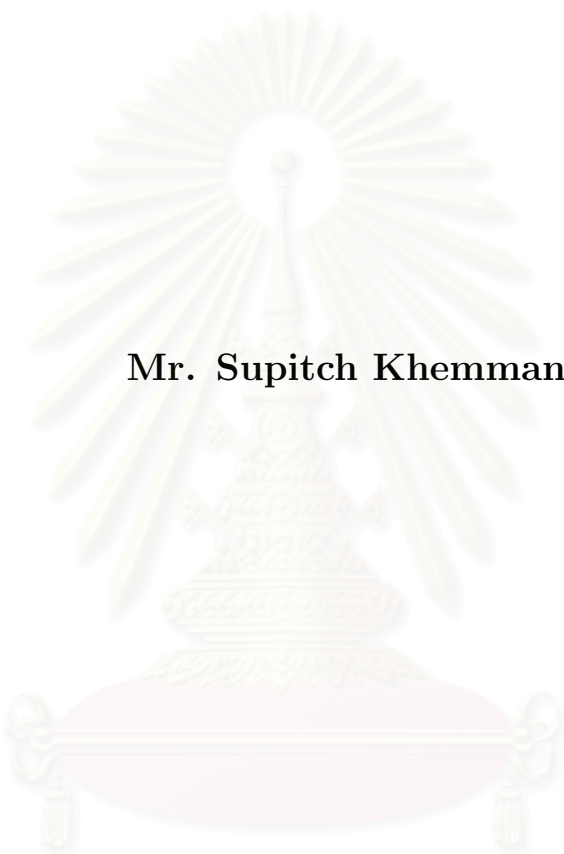
# SEMICLASSICAL LIMIT AND SEPARABILITY
# OF THE JAYNES-CUMMINGS MODEL

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Thesis Title　　　　　Semiclassical Limit and Separability of the

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งานวิจัยนี้ศึกษาในเชิงทฤษฎีเกี่ยวกับแบบจำลอง เจนส์-คัมมิงส์ ของอะตอมสองระดับซึ่งทำอันตรกิริยากับสนามแม่เหล็กไฟฟ้าที่ถูกทำให้เป็นควอนตัม  ประเด็นที่น่าสนใจก็คือ  การที่สนามของแบบจำลองเปลี่ยนไปสู่สนามแม่เหล็กไฟฟ้าคลาสซิก ได้ถูกกำหนดโดยจำนวนโฟตอนที่มีค่าเฉลี่ยสูง  การศึกษาพฤติกรรมของแบบจำลองนี้พบว่า (1) สถานะเริ่มต้นเจาะจงบางแบบทำให้ อะตอมมิคอินเวอร์ชัน ที่ทำนายจากแบบจำลอง เจนส์-คัมมิงส์ สอดคล้องกับที่ทำนายจากแบบจำลองกึ่งคลาสซิก ซึ่งสนามแม่เหล็กไฟฟ้าได้ถูกกำหนดให้เป็นสนามแม่เหล็กไฟฟ้าคลาส-ซิกตั้งแต่เริ่มแรก (2) จากการวิเคราะห์เชิงปริมาณเกี่ยวกับ ระดับขั้นควอนตัมเอนแทงเกิลเมนท์ ของแบบจำลอง เจนส์-คัมมิงส์ พบว่า ที่อุณหภูมิสัมบูรณ์ใดๆที่มากกว่าศูนย์ อะตอมและสนามแม่เหล็กไฟฟ้าจะไม่สามารถแยกกันได้ และระดับขั้นควอนตัมเอนแทงเกิลเมนท์จะมีค่าเข้าสู่ศูนย์ เมื่ออุณหภูมิเข้าสู่ศูนย์และเข้าสู่อนันต์

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SUPITCH KHEMMANI: SEMICLASSICAL LIMIT AND SEPARABIL-ITY OF THE JAYNES-CUMMINGS MODEL. THESIS ADVISOR: PROF. VIRULH SA-YAKANIT, F.D., THESIS CO-ADVISOR: PROF. JOHN S. BRIGGS, Ph.D., 62 pp. ISBN 974-53-1225-8.

The Jaynes-Cummings model (JCM) of a two-level atom interacting with a quantized radiation field has been investigated theoretically. Of particular interest is the transition of a quantized electromagnetic field of the model to a classical one in the limit of very large average photon number. In this thesis, two aspects of the JCM are examined. Firstly, it is shown that, in contrast to the general case, for specific initial atom-field states, there is exact correspondence between the atomic inversion predicted by the JCM and that predicted by the semiclassical model in which the field is treated classically from the outset. Secondly, the degree of quantum entanglement of the JCM is investigated quantitatively and it is shown that, at any finite absolute temperature greater than zero, the atom and field are never separable and the degree of quantum entanglement approaches zero in the high and low temperature limits.

Department of Physics      Student's signature..............................

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# Acknowledgement

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# Chapter I
## Introduction

In 1963, E.T. Jaynes and F.W. Cummings [1] published a paper entitled "comparison of quantum and semiclassical radiation theories, with application to the beam maser". This paper was written at a time when the quantum theory of the laser had not yet been worked out, and presented an attempt at discussing the new behavior that could be expected from treating the field quantum mechanically instead of classically. Jaynes and Cummings began by considering the interaction between a single two-level atom and a near-resonant single field mode, a simple situation which has been known as the *Jaynes-Cummings model* (JCM). This model is a soluble fully quantum mechanical model of an atom in a field which was first used to examine the classical aspects of spontaneous emission and to reveal the existence of Rabi oscillation in the atomic excitation probability for fields with sharply defined energy (or photon number). For field having a statistical distribution of photon number, the oscillations are damped in time and the excitation probability reaches an asymptotic steady value. This is known as the *Cummings collapse* following the work of Cummings in 1965 [2] who studied the long-time behavior of a two-level system interacting with a quantized single-mode field initially in a coherent state. He found that the atomic upper state population, for a resonance situation and for intermediate values of time, collapses such that the peak probability follows a Gaussian envelope and then becomes constant (equal to one for lower state) after an interaction time of the order of the inverse atom-field dipole coupling constant. Further in 1980, Eberly and coworkers [3] found that the Cummings collapse is followed, at much later times, by a sequence of revivals of atomic inversion. This behavior, specifically the phenomenon of revival, represents direct evidence for the discreteness of photons and thus veri-

fies the existence of the field quantization concept. The remarkable fact is that this collapse and revival has been observed experimentally in cavity QED experiments, which were performed first by Rempe *et al.* [4] in 1987 and followed, for example, by Brune *et al.* in 1995 [5]. It was also found that the cavity field, once modified by the JCM interaction with the atom, has statistical properties not found in classical fields. In this thesis, we will study the JCM in two main aspects described in Chapters 2 and 3 respectively.

In Chapter 2, the classical limit of the JCM, in the sense that the single-mode field becomes classical, will be studied and compared closely with the semi-classical model in which the field is treated classically from the outset. At first sight, it is rather surprising that while the coherent state is the most classical state allowed by the uncertainly principle, it leads to a result qualitatively different from the classical Rabi flopping formula. This turns out to be due to the interference of Rabi floppings at different frequencies leading to the collapse. By contrast, the extreme quantum mechanical number state exhibits a nice semi-classical correspondence. However, the direct comparison between the probability expression of the JCM and the semiclassical model leads to the following results. For the coupled atom-field system described by JCM, the classical limit of the field does not bring the JCM in correspondence with the semiclassical one for "general" initial states of the system but here we will show that there are three examples of initial states in which the classical correspondence of JCM can be established. The first one is the initial product state where the field is initially prepared in a state belonging to *coherent class*, the class containing states whose properties are generalized directly from some appropriate properties of the coherent states. The second one is again a product state where the field is initially prepared in a number state. The third one is the specific entangled atom-field state in the two-dimensional subspace of the upper atomic state with $n$ photons in the field and the lower atomic state with $(n + 1)$ photons. The remarkable

behavior of these three initial states is that their classical correspondence have their own specific initial phase only if the two-level atom is prepared in a mixed state.

In the last few years there has been an explosive increase of interest in the question of the entanglement of quantum states. This is in connection with quantum information processing and the possible development of quantum computers. In Chapter 3, we will investigate the separability of the thermal Jaynes-Cummings (JC) state, i.e., the state of the atom-field system (described by JCM) in thermal equilibrium specified by some fixed temperature. By using the definition of separable state (introduced by Werner [6]) and the positive partial transposed criterion (introduced by Peres [7]), we will show that the thermal JC state is never separable, no matter how high the temperature would be, in contrast to the coupled atom-atom and field-field cases where their thermal states are separable at high enough temperature. Moreover, we will use the negativity (as a measure of entanglement), introduced by Vidal and Werner [8], to investigate the degree of entanglement of thermal JC states at various temperatures. We show that the negativity (and thus entanglement) is maximal when their thermal energy is of the same order as the vacuum coupling energy. In addition, as to be expected, the entanglement approaches zero at both low- and high-temperature limits while for any finite temperatures, however large or small, there is always residual entanglement.

# Chapter II

## The Classical Correspondence Rules

## of the Jaynes-Cummings Model

## 2.1 Jaynes-Cummings model

In this section, we will give a review of the celebrated *Jaynes-Cummings model* (JCM) [1] which is the soluble fully quantum mechanical model of an atom (molecule) interacting with the classical electromagnetic (e.m.) field. Before going to that, let us first consider the *semiclassical model* in which an atom is treated quantum mechanically while the e.m. field is a classical one. Without loss of generality (in the frame work of the JCM), consider the classical monochromatic single-mode e.m. field, polarized in $\hat{\mathbf{x}}$-direction, in a cavity of volume $V$ closed by two perfectly reflecting mirrors which are placed orthogonal to the propagating direction $\hat{\mathbf{z}}$ at $z = 0$ and $z = L$. In this case, the electric field has the form [9]

$$\mathbf{E}(z, t) = \hat{\mathbf{x}} q(t) [2\Omega^2 / \epsilon_0 V]^{1/2} \sin kz \ , \tag{2.1}$$

where $q(t) = q_0 \cos(\Omega t - \phi)$ is the measure of the field amplitude ($\phi$ is the initial phase), $k = \Omega/c$ is the wave vector, and $\Omega$ is the single-mode field oscillation frequency. Now, let an atom (molecule) enters the cavity and, of course, it interacts with this e.m. field. In general, the Hamiltonian of this system has the form $H = \sum_{i=1}^{N} \frac{1}{2m_i} |\mathbf{P}_i - q_i \mathbf{A}(\mathbf{r}_i, t)|^2 + V_{\text{Coul}} + \sum_{i=1}^{N} q_i U(\mathbf{r}_i, t)$, where $m_i$, $\mathbf{r}_i$, $\mathbf{p}_i$, and $q_i$ are the mass, position operator, momentum operator, and charge of the $i^{\text{th}}$ electron of the atom respectively. Here, $V_{\text{Coul}}$ is the potential energy due to the Coulomb interactions in an atom, and $\mathbf{A}(\mathbf{r}, t)$ and $U(\mathbf{r}, t)$ are the vector and scalar potential of the e.m. field respectively from which the electric and magnetic field can be derived. In most practical cases, the wavelength of the field is

large compared with the size of an atom, which is typically the order of $10^{-8}$ cm, so the variation of the e.m. field over the atom can be ignored. This would be the case if the field frequency is less than about $10^{18}$ Hz. In this case, the *long-wavelength approximation* can be used in the following ways [10]. Expanding the potentials $\mathbf{A}(\mathbf{r}_i, t)$ and $U(\mathbf{r}_i, t)$ in powers of $\mathbf{r}_i$ around the position of an atom (i.e., the center of mass of an atom, $\mathbf{r}_{cm}$, which is usually set to be the origin of the reference frame) leads to the multipole moments of increasing order for the system of charges with respect to the position of an atom. By keeping the lowest order terms together with the globally neutral condition $\sum_{i=1}^{N} q_i = 0$, the above Hamiltonian becomes $H = \sum_{i=1}^{N} \frac{1}{2m_i} |\mathbf{P}_i - q_i \mathbf{A}(\mathbf{r}_{cm}, t)|^2 + V_{\text{Coul}} + \mathbf{d} \cdot \nabla U(\mathbf{r}_{cm}, t)$, where $\mathbf{P}_i = m_i \dot{\mathbf{r}}_i - q_i \mathbf{A}(\mathbf{r}_{cm}, t)$, and $\mathbf{d} := \sum_{i=1}^{N} q_i \mathbf{r}_i$ is the electric dipole moment operator. By using the Göppert-Mayer transformation [11] (see also [10]), the vector potential term is removed and the above Hamiltonian can be simply written as

$$H = H_{\text{ED}} := H_{\text{atom}} - \mathbf{d} \cdot \mathbf{E}(\mathbf{r}_{cm}, t), \qquad (2.2)$$

where $H_{\text{atom}} := \sum_{i=1}^{N} (|\mathbf{P}'|^2 / 2m_i) + V_{\text{Coul}}$ with $\mathbf{P}' = m_i \dot{\mathbf{r}}_i$. It is clear from the *electric dipole* Hamiltonian (2.2) that, in the long-wavelength approximation, the interaction of an atom with the external e.m. field is simply described in this new representation by a coupling term between the dipole moment operator $\mathbf{d}$ of the atom and the external electric field $\mathbf{E}(\mathbf{r}_{cm}, t)$ evaluated at the position of an atom. Since only the electric dipole interaction appears here, this long-wavelength approximation is also called the *dipole approximation*. Although this derivation is based on the Lagrangian formalism, it can be derived based on the Hamiltonian formalism also by using an appropriate unitary transformation. Moreover, this way of derivation can be generalized to derive the electric dipole Hamiltonian when the field is treated as a quantized field from the beginning [10, 12]. Note that, in the long-wavelength approximation, the electric dipole Hamiltonian is

a good approximation (for describing any atomic transition between states $|\psi_1\rangle$ and $|\psi_2\rangle$ due to the absorption or emission between atom and field) when a transition is *electric-dipole allowed*, i.e., $\langle\psi_1| H_{\text{ED}} |\psi_2\rangle \neq 0$. In case of the *electric-dipole forbidden*, i.e., $\langle\psi_1| H_{\text{ED}} |\psi_2\rangle = 0$ , the Hamiltonian described by electric-quadrupole and magnetic-dipole interaction corresponding to the higher order terms in the expansion may become important [13]. For the JCM and, therefore, in this thesis, the situations are restricted only to the case of electric-dipole allowed so it is enough here to use the electric dipole Hamiltonian $H_{\text{ED}}$ (2.2) in order to describe the interaction of atom and field. Substituting the electric field (2.1) into (2.2), we get the interaction term of (2.2) in the form

$$-\mathbf{d} \cdot \mathbf{E}(\mathbf{r}_{cm}, t) = -d_x E_0 \cos(\Omega t - \phi)\,, \tag{2.3}$$

where $d_x = \mathbf{d}\cdot\hat{\mathbf{x}}$ and $E_0 = q_0 \left[2\Omega^2/\varepsilon_0 V\right]^{1/2} \sin(kz_{\text{cm}})$ with $z_{\text{cm}} = \mathbf{r}_{\text{cm}}\cdot\hat{\mathbf{z}}$. Now, let the set of the energy eigenstates of an atom be $\{|1\rangle, |2\rangle, ..\}$ with the corresponding eigenenergies $\{E_1, E_2, ...\}$ $(E_1 > E_2 > ...)$. Then the first order time-dependent perturbation theory shows that the interaction picture probability amplitude for an atom initially in the state $|i\rangle$ to be found in the state $|n\rangle$ at any time $t > 0$ (assume, without loss of generality, that $E_i < E_n$) is [9]

$$C_n^{(1)}(t) = -\frac{iV_{ni}}{2\hbar} \left[\frac{\mathrm{e}^{i[(\omega_{ni}+\Omega)t-\phi]}}{i(\omega_{ni} + \Omega)} + \frac{\mathrm{e}^{i[(\omega_{ni}-\Omega)t+\phi]}}{i(\omega_{ni} - \Omega)}\right]\,, \tag{2.4}$$

where $V_{ni} = -E_0\langle n|d_x|i\rangle$ and $\omega_{ni} = \omega_n - \omega_i$ $(E_i = \hbar\omega_i)$. Here, the superscript (1) on $C_n(t)$ indicates that the exact $C_n(x)$ is estimated by $C_n^{(1)}(t)$ derived from the "first" order time-dependent perturbation theory. Note that the formula (2.4) is generally obtained for any sinusoidal perturbation of the form $V_0\cos(\Omega t - \phi)$, where $V_0$ in our case is $-d_x E_0$. Notice that the second term of (2.4) with the relatively small denominator $(\omega_{ni} - \Omega)$ is larger than the first term with denominator $(\omega_{ni} + \Omega)$ especially when the near *resonance* condition, i.e., $(\omega_{ni} \approx \Omega)$ is satisfied. So, it is reasonable for this near resonance condition to neglect the

first term in (2.4). The first and second terms in (2.4) are called *antirotating-wave* and *rotating-wave* contribution respectively and neglect of the first term is then called the *rotating-wave approximation*. In a similar way, one may neglect the term corresponding to the transitions $\omega_{mi}$ $(m \neq n)$ in comparison with the term corresponding to $\omega_{ni} \approx \Omega$. Since the antirotating-wave contribution is actually smaller than many nonresonant contributions [9], it follows that the two-level atom approximation is usually only consistent if made simultaneously with the rotating-wave approximation and if one decides to keep the antirotating wave contribution, one must also keep all the nonresonant contributions as well. Since all of the above discussions are based on $C_n^{(1)}(t)$ in (2.4) which, as it is a first order time-dependent perturbation result, is valid only for small $|V_{ni}|$, one may ask what differences may occur if the exact result is used instead of (2.4). Fortunately, the exact result for the transition between two states coupled by the sinusoidal form of interaction is available. In (2.4), notice that the first and second terms clearly come from the antirotating-wave $e^{i(\Omega t - \phi)}$ and rotating-wave $e^{-i(\Omega t - \phi)}$ parts of $\cos(\Omega t - \phi)$ in (2.3) respectively. Now, let us consider first the rotating-wave part $-(d_x E_0/2)e^{-i(\Omega t - \phi)}$ of the interaction in (2.3). In this case, the exact result (at times $t > 0$) is $|C_n(t)|^2 \propto [(\omega_{ni} - \Omega)^2 + 4|V_{ni}|^2/\hbar^2]^{-1}$ (see, e.g., [14]). In contrast, for the antirotating-wave part of interaction, i.e., $-(d_x E_0/2)e^{i(\Omega t - \phi)}$, $|C_n(t)|^2 \propto [(\omega_{ni} + \Omega)^2 + 4|V_{ni}|^2/\hbar^2]^{-1}$. It is clear from these results that besides the nearly resonant condition needed for the rotating-wave and two-level atom approximations, the weak interaction $(|V_{ni}|/\hbar \ll \Omega)$ is also needed. This is why we cannot see this conclusion via (2.4) since the small value of $V_{ni}$ in (2.4) has been assumed from the beginning. Another problem in realistic situations is that the cavity usually contains multimode of the field rather than a single one. However, this problem can be reduced to the case of a two-level atom interacting with the specific single-mode field for the situation that, while $\omega_{ni}$ and $\Omega$ are both varied, there exists only one pair $(\omega_{ni}, \Omega)$ in which the transition from state

$|i\rangle$ to $|n\rangle$ is the most dominating one or, in other words, this pair is much more close to the resonance condition than all other pairs. In this case, an atom can be approximated by a *two-level* atom specified by states $|i\rangle$ and $|n\rangle$ and the field can be approximated by a single-mode with frequency $\Omega$. Moreover, if their interaction is weak enough, the rotating-wave approximation can also be used so the problem of this atom-field interaction can be fully solved analytically. The simple model of these approximations allows us to learn a great deal about the atom-field interaction, and hopefully this knowledge can be generalized to more realistic situations.

For the reasons mentioned above, let us now consider a two-level atom specified by two energy eigenstates $|a\rangle$ (the upper state) and $|b\rangle$ (the lower state) with the corresponding eigenenergies $E_a = \hbar\omega_a$ and $E_b = \hbar\omega_b$ respectively, i.e., $H_{\text{atom}}|a\rangle = E_a|a\rangle$ and $H_{\text{atom}}|b\rangle = E_b|b\rangle$. Let a cavity contains only a single-mode field of the form (2.1) and this single-mode field interacts with a two-level atom by the dipole interaction described by (2.3) which, in the rotating-wave approximation, is $-(d_x E_0/2)\mathrm{e}^{-i(\Omega t - \phi)}$. For this system, the Hamiltonian (2.2) becomes $H_{\text{ED}} = H_{\text{atom}} - (d_x E_0/2)\mathrm{e}^{-i(\Omega t - \phi)}$. This Hamiltonian is written in the two-level atom basis $\{|a\rangle, |b\rangle\}$ as

$$H_{\text{sc}}(t) := H_{\text{ED}}(t) = \frac{\hbar\omega}{2}\sigma_z - \frac{\wp E_0}{2}\left(\sigma_+ \mathrm{e}^{-i(\Omega t - \phi)} + \sigma_- \mathrm{e}^{i(\Omega t - \phi)}\right), \qquad (2.5)$$

where $\omega := \omega_a - \omega_b$ is the frequency difference of the two levels (we have chosen the energy zero to be half way between the upper and lower levels), $\wp := \langle a|d_x|b\rangle = \langle a|\mathbf{d}|b\rangle \cdot \hat{\mathbf{x}}$ is the component of a matrix element of the electric dipole operator along the polarization (we have chosen, without loss of generality, the atomic quantization axis such that $\wp$ is real), $\sigma_z = |a\rangle\langle a| - |b\rangle\langle b|$, $\sigma_+ = |a\rangle\langle b|$, and $\sigma_- = |b\rangle\langle a|$. Here, $H_{\text{ED}}$ in (2.5) is rewritten by $H_{\text{sc}}$. This subscript sc denotes the abbreviated name of the semiclassical which emphasizes that we are considering now the semiclassical model where the e.m. field is treated classically. Note that

$\sigma_z$, $\sigma_+$ and $\sigma_-$, which are exactly the same as the Pauli spin matrices used for the "real" two-level spin$\frac{1}{2}$ system, are sometimes called the *pseudo-spin* operators since there is the correspondence between them and electron (Fermi) operators for an approximated two-level atom [15]. Note also that $\langle a|\mathbf{d}|a\rangle = \langle b|\mathbf{d}|b\rangle = 0$ (which is usually true unless the system has a permanent dipole moment like $H_2O$) have been assumed, without loss of generality, in order to obtain (2.5) since if they do not vanish, then they can be absorbed in the definition of $\omega_a$ and $\omega_b$ leading to a new effective two-level atom without these diagonal terms of interaction. The main quantity which we shall first find from the semiclassical model described by (2.5) is the probability amplitude for finding a two-level atom in state $|a\rangle$ or $|b\rangle$ at time $t > 0$ when the initial conditions at time $t = 0$ are given. Now, let the general state at time $t$, denoted by $|\psi(t)\rangle_{sc} \in \mathcal{H}_a$, the Hilbert space describing a two-level atom, be in the form

$$|\psi(t)\rangle_{sc} = C_a(t)e^{i\delta t/2}e^{-i\omega_a t}|a\rangle + C_b(t)e^{-i\delta t/2}e^{-i\omega_b t}|b\rangle , \qquad (2.6)$$

where $\delta := \omega - \Omega$ is called the *detuning*. Here, $C_a(t)$ and $C_b(t)$ are the probability amplitudes (in the special interaction picture) for finding a two-level atom in the upper state $|a\rangle$ and lower state $|b\rangle$ respectively. Substituting (2.6) into the time-dependent Schrödinger equation $i\hbar\partial|\psi(t)\rangle_{sc}/\partial t = H_{sc}|\psi(t)\rangle_{sc}$, with $H_{sc}$ in (2.5), leads to the matrix equation for determining $C_a(t)$ and $C_b(t)$ from the initial condition $C_a := C_a(0)$ and $C_b := C_b(0)$

$$\begin{bmatrix} C_a(t) \\ C_b(t) \end{bmatrix} = \begin{bmatrix} \cos(Rt/2) - i\delta R^{-1}\sin(Rt/2) & iR_0 R^{-1}e^{i\phi}\sin(Rt/2) \\ iR_0 R^{-1}e^{-i\phi}\sin(Rt/2) & \cos(Rt/2) + i\delta R^{-1}\sin(Rt/2) \end{bmatrix} \begin{bmatrix} C_a \\ C_b \end{bmatrix}, (2.7)$$

where $R := \sqrt{\delta^2 + R_0^2}$ is called the *generalized Rabi flopping frequency*, and $R_0 := \wp E_0/\hbar$ ($|R_0|$ is called the *Rabi flopping frequency*). This flopping frequency ($R \to |R_0|$ as $\delta \to 0$) is used for describing the oscillation of the population between upper and lower levels. For example, for $\delta = 0$ (the resonance case)

and $C_b = 1$ (the initially ground state atom), (2.7) can be easily solved to get $|C_a(t)|^2 = \sin^2(|R_0| t/2)$ and $|C_b(t)|^2 = \cos^2(|R_0| t/2)$. This result clearly shows the oscillation (with frequency $|R_0|$) between upper and lower levels and since $|C_a(t)|^2$ or $|C_b(t)|^2$ is equal to unity for some specific times, the oscillation in this case is called the *complete* oscillation. For $\delta \neq 0$, the oscillation is never complete. The reason for this will become clear when the field is treated quantum mechanically as a quantized field. Note that the name Rabi of the flopping frequency is due to Rabi [16] in 1936 who studied the similar system of a spin$\frac{1}{2}$ magnetic dipole in nuclear magnetic resonance. In general, any two-level system can be viewed geometrically as a spin$\frac{1}{2}$ particle in the magnetic field. This is called the *fictitious spin$\frac{1}{2}$* (see e.g., [12, 17]). From (2.7), the general expression of probability for finding a two-level atom in an upper state $|a\rangle$, denoted by $P_a^{(\mathrm{sc})}(t)$, is

$$
\begin{aligned}
P_a^{(\mathrm{sc})}(t) \; &:= \; |C_a(t)|^2 \\
&= \; \frac{|C_a|^2 \delta^2}{R^2} + \left(\frac{R_0}{R}\right)^2 \left[|C_a|^2 \cos^2(Rt/2) + |C_b|^2 \sin^2(Rt/2)\right] \\
&\quad + \frac{R_0}{R} |C_a| |C_b| \left\{ \sin\left[(\theta_a - \theta_b) - \phi\right] \sin(Rt) + \frac{\delta}{R}\cos\left[(\theta_a - \theta_b) - \phi\right] \right. \\
&\quad \left. \times \cos(Rt) - \frac{\delta}{R}\cos\left[(\theta_a - \theta_b) - \phi\right] \right\},
\end{aligned}
\tag{2.8}
$$

where $\theta_a := Arg(C_a)$ and $\theta_b := Arg(C_b)$. Here, again the superscript (sc) on $P_a(t)$ emphasizes the use of semiclassical model described by (2.5).

Now, let us consider the full quantum case where the e.m. field part is treated quantum mechanically. In quantum mechanics, the classical e.m. field is replaced by its quantized form in the following ways. From (2.1), one can derive the magnetic field from the Maxwell's equations (in vacuum) and then obtain the Hamiltonian (or total energy denoted by $H_{\mathrm{f}}$) in the form $H_{\mathrm{f}} = \frac{1}{2}(p^2 + \Omega^2 q^2)$, where $p(t) = \dot{q}(t)$ with $q(t)$ defined as in (2.1) (see, e.g., [9]). This Hamiltonian is clearly a simple harmonic oscillator with unit mass so the quantization procedure

is simply the replacement of the dynamical variables $q(t)$ and $p(t)$ by the operators $q$ and $p$ respectively. Now, the electric field (2.1) can be quantized by using an operator $q = \sqrt{\hbar/2\Omega}(a + a^\dagger)$ instead of the classical $q(t)$, where $a$ and $a^\dagger$ are the annihilation and creation operators respectively. Substituting this quantized field obtained from the above procedure, i.e., $E(z) = \hat{\mathbf{x}}\varepsilon_\Omega(a + a^\dagger)\sin kz$ (where $\varepsilon_\Omega :=$ $\sqrt{\hbar\Omega/\varepsilon_0 V}$ is the electric field per photon) into (2.2), we obtain the total (time-independent) electric dipole Hamiltonian, denoted by $H_{\mathrm{af}}$, of the full quantum atom-field interaction in the form $H_{\mathrm{af}} = H_{\mathrm{atom}} + \hbar\Omega a^\dagger a - d_x\varepsilon_\Omega \sin(kz_{\mathrm{cm}})(a + a^\dagger)$. Here, the term $\hbar\Omega a^\dagger a$ is the total energy of the free field (without zero point energy) which has been neglected in (2.2) for the "semiclassical" case (where the field was treated classically) since its total energy is just a constant. Again, like what we have done in (2.5), the above $H_{\mathrm{af}}$ can be written in terms of $\sigma_z$, $\sigma_-$, and $\sigma_+$ as

$$H_{\mathrm{af}} = \frac{\hbar\omega}{2}\sigma_z + \hbar\Omega a^\dagger a + \hbar g(\sigma_+ + \sigma_-)(a + a^\dagger) \tag{2.9}$$

where $g = -(\wp E_\Omega/\hbar)\sin(kz_{\mathrm{cm}})$. Notice that this Hamiltonian is still not the one within the rotating-wave approximation since the whole of $q(t)$ (not only $\mathrm{e}^{-i(\Omega t - \phi)}$ part of $\cos(\Omega t - \phi)$) is replaced by an operator $q$. However, doing a rotating-wave approximation of the full quantum Hamiltonian (2.9) is quite similar to one we have done for the semiclassical model. To see that, let us consider the interaction term of (2.9), i.e., $(\sigma_+ + \sigma_-)(a + a^\dagger) = (a\sigma_+ + a^\dagger\sigma_-) + (a\sigma_- + a^\dagger\sigma_+)$. The first term on the right-hand side (called the *rotating-wave* term) couples the atom-field states $|an\rangle$ and $|bn+1\rangle$ for all $n = 0, 1, 2, ...$ and the second term (called the *antirotating-wave* term) couples the atom-field states $|an\rangle$ and $|bn-1\rangle$. Here, $|an\rangle := |a\rangle \otimes |n\rangle$ and $|bn\rangle := |b\rangle \otimes |n\rangle$ (where $\{|n\rangle\}_{n=0}^\infty$ are the energy eigenstates of the quantized field, with the corresponding eigenenergies $n\hbar\Omega$, spanning the Hilbert space of field) so $\{|an\rangle, |bn\rangle\}_{n=0}^\infty$ clearly forms the basis (called the *bare-atom basis*) of the composite quantum system of the atom

and field described by the total Hilbert space $\mathcal{H}_{af} = \mathcal{H}_a \otimes \mathcal{H}_f$. By considering, for example, the antirotating-wave term $(a\sigma_- + a^\dagger \sigma_+)$ for the transition from $|b0\rangle$ to $|a1\rangle$, the transition probability is proportional to $[(\omega + \Omega)^2 + 4g^2]^{-1}$. In contrast, the rotating-wave $(a\sigma_+ + a^\dagger \sigma_-)$ induces a transition from $|a0\rangle$ to $|b1\rangle$ with the probability proportional to $[(\omega - \Omega)^2 + 4g^2]^{-1}$ (see [14]). It is clear from these probabilities that, for $\omega \approx \Omega$ (resonance condition) and weak interaction (instead of condition $|V_{ni}| / \hbar \ll \Omega$ in the semiclassical case, we have here $|g| \sqrt{\langle a^\dagger a \rangle} \ll \Omega$ [18]), the rotating-wave term is (again as in the semiclassical case) the dominating one. Note that one can also see the time-dependent form of both rotating-and antirotating-wave terms, in analogy to the semiclassical one, by considering their time evolution in the interaction picture, i.e., the free evolution $g = 0$ of them [9]. Now, making the rotating-wave approximation (i.e., neglecting the antirotating-wave term $(a\sigma_- + a^\dagger \sigma_+)$ in (2.9)) leads to

$$H_{af} = \frac{\hbar \omega}{2} \sigma_z + \hbar \Omega a^\dagger a + \hbar g (a\sigma_+ + a^\dagger \sigma_-) \ . \tag{2.10}$$

The model specified by this Hamiltonian is called the *Jaynes-Cummings model* (JCM). In fact, neglecting the antirotating-wave term in order to get (2.10) is to ignore the small probability contribution due to the energy non-conserving terms in which emission (absorption) of a photon is accompanied by the transition of the atom from its lower (upper) to its upper (lower) states. Although the antirotating-wave contribution is small, it can profoundly affect the long-time behavior of the system. In particular, the semiclassical version of the JCM (2.5) is perfectly periodic, whereas the inclusion of antirotating-wave term in the semiclassical equation leads to chaotic behavior [19, 20]. Moreover, an error introduced by the rotating-wave approximation in some specific cases was also studied [21]. Note that a more careful treatment of the rotating-wave approximation leads to a diagonal energy shift, which we have ignored in (2.10), in the atomic frequency $\omega$. It is known as the *Bloch-Siegert shift* [22, 18]. Although the JCM (2.10) gives

inaccurate or wrong answers in some situations described above, it is adequate for our analysis of this chapter, because the long-time behavior will not affect the situations in which (as we will discuss in Section 2.4) the average photon number is very large.

## 2.2   Dynamics of the Jaynes-Cummings model

In terms of the bare-atom basis, i.e., $\{|an\rangle, |bn\rangle\}_{n=0}^{\infty}$, the Hamiltonian (2.10) can be written in the form

$$H_{\text{af}} = \left(\sum_{n=0}^{\infty} H_n\right) - \frac{\hbar\omega}{2} |b0\rangle \langle b0|, \tag{2.11}$$

where

$$
\begin{aligned}
H_n &= \left[(n + \frac{1}{2})\hbar\Omega + \frac{\hbar\delta}{2}\right] |an\rangle \langle an| + \left(\hbar g\sqrt{n+1}\right)(|an\rangle \langle bn+1| + |bn+1\rangle \langle an|) \\
&\quad + \left[(n + \frac{1}{2})\hbar\Omega - \frac{\hbar\delta}{2}\right] |bn+1\rangle \langle bn+1|, \ \forall n = 0, 1, 2, ....
\end{aligned}
$$

Notice that $H_n$ couples only the atom-field states $|an\rangle$ and $|bn+1\rangle$ for each $n$. By this reason, the whole Hilbert space $\mathcal{H}_{\text{a}} \otimes \mathcal{H}_{\text{f}}$ should be decomposed into the "mutually orthogonal" two-dimensional subspaces $\mathcal{P}_n$ spanned by $\{|an\rangle, |bn+1\rangle\}$ ($\forall n = 0, 1, 2, \ldots$) and a one-dimensional subspace $\mathcal{L}_0$ spanned by $\{|b0\rangle\}$, i.e., $\mathcal{H}_{\text{a}} \otimes \mathcal{H}_{\text{f}} = \mathcal{L}_0 \oplus (\oplus_{n=0}^{\infty} \mathcal{P}_n)$. Doing in this way, $H_n$ will look like an operator acting only on $\mathcal{P}_n$ in the sense that $H_n(\mathcal{P}_m) = H_n(\mathcal{L}_0) = 0 \, \forall m \neq n$, and the matrix representation of $H_n$ is then simply a $2 \times 2$ matrix which provides us an easy way to calculate its eigenvectors (called the *dressed-atom basis*) and their corresponding eigenenergies (splitting energies). The standard eigenvalue problem for $2 \times 2$ matrices leads to the dressed-atom basis $\{|b0\rangle, |1n\rangle, |2n\rangle\}_{n=0}^{\infty}$, where $|1n\rangle$ and $|2n\rangle$ satisfy $H_n |1n\rangle = E_{1n} |1n\rangle$ and $H_n |2n\rangle = E_{2n} |2n\rangle$ respectively with the energies $E_{1n} := (n + 1/2)\hbar\Omega + \hbar R_n/2$ and $E_{2n} := (n + 1/2)\hbar\Omega - \hbar R_n/2$, where $R_n := \sqrt{\delta^2 + 4g^2(n+1)}$ is called the *quantized generalized Rabi flopping frequency* (see e.g., [9]). The *dressed states* $|1n\rangle$ and $|2n\rangle$ are related to

the *bare states* $|an\rangle$ and $|bn+1\rangle$ by the transformation formulae $|1n\rangle = \sin\theta_n$ $|an\rangle + \cos\theta_n |bn+1\rangle$ and $|2n\rangle = \cos\theta_n |an\rangle - \sin\theta_n |bn+1\rangle$. Here, $\sin\theta_n :=$ $2g\sqrt{n+1}/\sqrt{(R_n - \delta)^2 + 4g^2(n+1)}$. Now, let the general initial state be

$$\begin{aligned}
|\Psi(0)\rangle &= \sum_{n=0}^{\infty} \Big[ C_{an}(0) |an\rangle + C_{bn}(0) |bn\rangle \Big] \\
&= \sum_{n=0}^{\infty} \Big[ C_{an}(0) |an\rangle + C_{bn+1}(0) |bn+1\rangle \Big] + C_{b0}(0) |b0\rangle .
\end{aligned} \quad (2.12)$$

Then the time evolution of this state generated by the Hamiltonian (2.11) can be written as (by using $|\Psi(t)\rangle = \exp(-iH_{\mathrm{af}}t/\hbar) |\Psi(0)\rangle$ and $[H_n, |b0\rangle\langle b0|] = 0 \,\forall n \geq 0$)

$$\begin{aligned}
|\Psi(t)\rangle &= \sum_{n=0}^{\infty} \Big[ C_{an}(t)\mathrm{e}^{i\delta t/2}\mathrm{e}^{-iE_{an}t/\hbar} |an\rangle + C_{bn+1}(t)\mathrm{e}^{-i\delta t/2}\mathrm{e}^{-iE_{bn+1}t/\hbar} |bn+1\rangle \Big] \\
&\quad + C_{b0}(t)\mathrm{e}^{-i\delta t/2}\mathrm{e}^{i\omega t/2} |b0\rangle ,
\end{aligned} \quad (2.13)$$

where $E_{an} = n\hbar\Omega + \hbar\omega/2$ and $E_{bn+1} = (n+1)\hbar\Omega - \hbar\omega/2$ are the unperturbed energy eigenvalues, and $C_{b0}(t) = C_{b0}(0)\mathrm{e}^{i\delta t/2}$. Here, (2.13) is written in the same special interaction picture as in (2.6). Notice that $|C_{b0}(t)|^2 = |C_{b0}(0)|^2$ is time independent, as it should be since $|b0\rangle$ itself is an energy eigenstate of $H_{\mathrm{af}}$. This implies, from the normalization condition of $|\Psi(t)\rangle$ (i.e., $\sum_{n=0}^{\infty}(|C_{an}(t)|^2 + |C_{bn}(t)|^2) = 1 \,\forall t \geq 0$), that the dynamics (Rabi's oscillation between the unperturbed two-level states) must be occurred in $\oplus_{n=0}^{\infty}\mathcal{P}_n$ only. By this reason, we shall consider only the evolution of states in $\oplus_{n=0}^{\infty}\mathcal{P}_n$ (the first two terms on the right-hand side of (2.13)) and write, for convenience, $|\Psi(t)\rangle$ only for this part, i.e.,

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty} C_{an}(t)\mathrm{e}^{i\delta t/2}\mathrm{e}^{-iE_{an}t/\hbar} |an\rangle + C_{bn+1}(t)\mathrm{e}^{-i\delta t/2}\mathrm{e}^{-iE_{bn+1}t/\hbar} |bn+1\rangle . \quad (2.14)$$

Similar to (2.7), (2.14) and the time-dependent Schrödinger equation with Hamiltonian $H_n$ in (2.11) lead to the matrix equation for $C_{an}(t)$ and $C_{bn+1}(t)$,

$$\begin{bmatrix} C_{an}(t) \\ C_{bn+1}(t) \end{bmatrix} =$$
$$\begin{bmatrix} \cos(R_n t/2) - i\delta R_n^{-1}\sin(R_n t/2) & -2igR_n^{-1}\sqrt{n+1}\sin(R_n t/2) \\ -2igR_n^{-1}\sqrt{n+1}\sin(R_n t/2) & \cos(R_n t/2) + i\delta R_n^{-1}\sin(R_n t/2) \end{bmatrix} \begin{bmatrix} C_{an}(0) \\ C_{bn+1}(0) \end{bmatrix} . \quad (2.15)$$

Notice that, for $\delta = 0$ and $C_{bn+1}(0) = 1$ (i.e., a resonant atom initially in the lower level), $|C_{an}(t)|^2 = \sin^2(gt\sqrt{n+1})$ and $|C_{an}(t)|^2 = \cos^2(gt\sqrt{n+1})$ showing the complete oscillation. In contrast, for $\delta \neq 0$, the complete oscillation never occurs. It is clear now that the complete oscillation occurs due to the "complete" exchange of the quantum energy $\hbar\Omega$ with the two-level atom of energy $\hbar\omega$ (which is equal to $\hbar\Omega$). The complete oscillation will be destroyed more and more as the field is further and further from resonance. Similar to (2.8), for the atom-field initial state $|\Psi(0)\rangle = |\phi\rangle \otimes |\varphi\rangle$ (where $|\phi\rangle = C_a|a\rangle + C_b|b\rangle \in \mathcal{H}_a$ and $|\varphi\rangle \in \mathcal{H}_f$), the general expression for the probability of finding a two-level atom in an upper state (at time $t$), denoted for this full quantum case by $P_a(t)$, is

$$P_a(t) := \sum_{n=0}^{\infty} |C_{an}(t)|^2 =$$

$$\sum_{n=0}^{\infty} \left\{ \frac{|C_a|^2 \delta^2 \varphi_n^2}{R_n^2} + \frac{2g^2(n+1)}{R_n^2}(|C_a|^2 \varphi_n^2 - |C_b|^2 \varphi_{n+1}^2)\cos(R_n t) \right.$$

$$+ \frac{2g^2(n+1)}{R_n^2}(|C_a|^2 \varphi_n^2 + |C_b|^2 \varphi_{n+1}^2) + \frac{2\delta g |C_a| |C_b| \varphi_n \varphi_{n+1}\sqrt{n+1}}{R_n^2}[1 - \cos(R_n t)]$$

$$\times \cos\left[(\theta_a - \theta_b) - (\theta_{n+1} - \theta_n)\right] - \frac{2g |C_a| |C_b| \varphi_n \varphi_{n+1}\sqrt{n+1}}{R_n}\sin(R_n t)$$

$$\left. \times \sin\left[(\theta_a - \theta_b) - (\theta_{n+1} - \theta_n)\right] \right\}, \tag{2.16}$$

where $\varphi_n := |\langle n|\varphi\rangle| = \sqrt{P_n}$, $\theta_n := Arg(\langle n|\varphi\rangle)$, $\theta_a := Arg(C_a)$, and $\theta_b := Arg(C_b)$.

The difference between the quantum $P_a(t)$ (2.16) and the semiclassical (2.8) is that in the quantum case, the probability of finding a two-level atom in an upper state $|a\rangle$ has contributions (expressed in (2.16) via the discrete sum) from many number states $|n\rangle$, $n = 0, 1, 2, ...$ while the semiclassical probability (2.8) does not have it. This contribution leads (in contrast to the semiclassical perfect periodicity) to the *collapse* of $P_a(t)$ due to the interference of Rabi flopping at different frequencies. This behavior will be explored in the next section.

# 2.3 Cummings collapse and its generalization

From the previous section, it is clear that the mathematics is simplest when one considers a single atom of precisely known energy (i.e., a particular upper or lower level) brought suddenly into a cavity with precisely known energy (i.e., a definite number of photons). However, this situation is not a realistic one. Usually, one can only specify the statistical properties of the cavity single-mode field. Over the years, there have been studies of the JCM with a great variety of initial single-mode fields. These studies have shown that there can be remarkable differences in the behavior of the system with different initial conditions. One of the important initial fields, which is experimentally the most prepared one, is the field expressed as a coherent superposition of the number states called the *coherent state* and denoted by $|\alpha\rangle$. It is also called the *quasi-classical* state since the expectation values (with respect to $|\alpha\rangle$) of position, momentum, and thus energy reproduce the results from the classical harmonic oscillator as the average energy $\bar{n}\hbar\Omega$ ($\bar{n} = |\alpha|^2$) reaches the classical energy limit. Note that the coherent state $|\alpha\rangle$ has many interesting properties, such as (1) it is an eigenstate of the annihilation operator, i.e., $a|\alpha\rangle = \alpha|\alpha\rangle$, (2) it satisfies the minimum uncertainty for all time of evolution, (3) $\langle n|\alpha\rangle = e^{-|\alpha|^2/2}\alpha^n/\sqrt{n!}$ so $|\langle n|\alpha\rangle|^2 = \bar{n}^n e^{-\bar{n}}/n!$ is the Poisson type of distribution with the average $\bar{n} = |\alpha|^2$, and (4) the evolution of $|\alpha_0\rangle$ at any time $t$ is still a coherent state $|\alpha(t)\rangle$ with $\alpha(t) = \alpha_0 e^{-i\Omega t} = |\alpha_0| e^{-(i\Omega t - \phi)}$, where $\phi = Arg(\alpha_0)$ is the initial phase of the corresponding classical oscillator. In our context, $\phi$ here is the initial phase of the electric field (2.1) whose amplitude is simply described by $q(t) = q_0 \cos(\Omega t - \phi)$. For atom and field initially prepared (independently) in an upper state $|a\rangle$ and a coherent state $|\alpha\rangle$ respectively (i.e., $|\Psi(0)\rangle = |a\rangle \otimes |\alpha\rangle$), the probability $P_a(t)$ (for $\delta = 0$) in (2.16) reduces to the form $P_a(t) = e^{-|\alpha|^2} \sum_{n=0}^{\infty} (|\alpha|^2/n!) \cos^2\left(gt\sqrt{n+1}\right)$. For sufficiently "intense field" and

"short enough time" $(t \ll |\alpha|/g)$, this sum can be shown to reduce to (see [9])

$$P_a(t) = \frac{1}{2} + \frac{1}{2}\cos(2|\alpha|gt)e^{-(gt)^2/2}. \tag{2.17}$$

Similarly, for a two-level atom initially in the lower state, we have (see [23])

$$P_a(t) = \frac{1}{2} - \frac{1}{2}\cos(2|\alpha|gt)e^{-(gt)^2/2}. \tag{2.18}$$

The above forms of $P_a(t)$ show that the Rabi oscillations are damped with a Gaussian envelope whose width is independent of the photon number $\bar{n} = |\alpha|^2$, a result sometimes called the *Cummings collapse* due to Cummings who first showed this behavior in 1965 [2]. Note that Eberly and coworkers [3] and also other works such as [24, 25, 26] found that the Cummings collapse is followed, at much later time, by a sequence of revivals. However, although this revival is a very important phenomenon that represents direct evidence for the discreteness of photons, we will not concern ourselves with it. This is because the classical limit behavior of the JCM, which we will discuss in the next section, needs only the (intermediate) collapse formula when the collapse time $(\sim g^{-1})$ tends to infinity in the limit of large photon numbers. Note that besides $P_a(t)$, another quantity frequently used to describe this collapse behavior in many works is the so-called *atomic inversion* denoted by $w(t)$. It is defined by $w(t) := P_a(t) - P_b(t) = 2P_a(t) - 1$ $(= \langle\sigma_z(t)\rangle)$ which clearly behaves in the same way as $P_a(t)$. To derive (2.17) or (2.18), one may use the Poisson summation formula (see, e.g., [27, 23]) or directly use the Taylor expansion. One of the very important properties leading to (2.17) is that, for sufficiently intense field $(|\alpha|^2 = \bar{n} \gg 1)$, $\left[(\bar{n} + \Delta n)^{1/2} - (\bar{n} - \Delta n)^{1/2}\right] \approx 1$ for a Poisson distribution (i.e., for a coherent state of the initial field) where $\Delta n = \sqrt{\bar{n}} = |\alpha|$. This property can be immediately generalized to $\left[(\bar{n} + \Delta n)^{1/2} - (\bar{n} - \Delta n)^{1/2}\right] \approx \Delta n/\sqrt{\bar{n}}$ for sufficiently large $\bar{n}$ such that $\bar{n}^3 \gg (\Delta n)^4$. This means that any initial state of the field, which is not necessarily a coherent state, having large average photon number $\bar{n} \gg 1$ (compared with the deviation $\Delta n$ in such a way that $\bar{n}^3/(\Delta n)^4 \gg 1$)

may lead to the similar behavior of collapse described by (2.17). However, since the coherent state has the specific form of phase, i.e., $\langle n|\,\alpha\rangle = \left(\mathrm{e}^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}\right)\mathrm{e}^{in\phi}$, we will define the class of states as follows.

**Definition:** Let $\mathcal{N} := \{0, 1, 2, ...\}$ and $\beta \in [0, 2\pi]$. Let $\left\{\,|\gamma\rangle\,\middle|\,\langle n|\gamma\rangle = \sqrt{P_n}\,\mathrm{e}^{in\beta}\right.$ $\forall n \in \mathcal{N}$, where the probability $P_n$ has the property that $\bar{n}^3 \gg (\Delta n)^4\big\}$ be denoted by $\mathcal{C}oh(\beta)$. We shall call $\mathcal{C}oh(\beta)$ the *coherent class* of $\beta$.

Note that the coherent state $|\alpha\rangle$ with $\bar{n} \gg 1$ belongs to $\mathcal{C}oh(\beta)$, where $\beta = Arg(\alpha) = \phi$, the initial phase of the classical field. Physically, the corresponding probability of the state belonging to $\mathcal{C}oh(\beta)$ moves back and forth in the harmonic well like the coherent state with a little shape distortion. The phase $\beta$ plays a similar role to the classical initial phase $\phi$ of the coherent state. To see what form of $P_a(t)$ for the initial field belongs to the class, let that initial state of the atom-field be $|\Psi(0)\rangle = |\phi\rangle \otimes |\psi\rangle$, where $|\phi\rangle = C_a\,|a\rangle + C_b\,|b\rangle$ and $|\psi\rangle \in \mathcal{C}oh(\beta)$. Here, the initial state of the two-level atom is a general linear combination of upper and lower states rather than exclusively the upper or lower state, as in the simple previous setting for obtaining (2.17) or (2.18). This linear combination will lead to the classical correspondence rules concerning the initial phase of the classical field which will be described in the next section. Now, for sufficiently large $\bar{n}$ such that $\bar{n} + 1 \approx \bar{n}$ and $\delta^2/g^4\bar{n}^2 \ll 1$ (this relation is automatically satisfied for the near resonance case, i.e., $\delta \approx 0$) and sufficiently small time such that $t \ll \sqrt{\bar{n}}/g$, one can use the properties of $\mathcal{C}oh(\beta)$ to show that (i) each term of the summand (excluding $\varphi_n^2$, and $\varphi_n\varphi_{n+1}$) in (2.16) are the slowly varying functions of $n$ between $\bar{n} - \Delta n$ and $\bar{n} + \Delta n$ around $\bar{n}$, (ii) $P_n = \varphi_n^2$ has a main contribution around $\bar{n}$ from $\bar{n} - \Delta n$ to $\bar{n} + \Delta n$ and, around this $\bar{n}$, $P_{n+1} \approx P_n$ and $\varphi_n\varphi_{n+1} \approx P_n$, (iii) $\theta_{n+1} - \theta_n = \beta$. From these facts, one can do a Taylor's expansion of each summand (excluding $P_n$) in (2.16) around $\bar{n}$ up to the second order of $(n - \bar{n})$ (ex-

pansion up to zeroth or first order is not enough to show the behavior of collapse which will be discussed later). After this expansion, some of the terms can be neglected compared to the others and by the use of $\sum_{n=0}^{\infty} P_n = 1$, $\sum_{n=0}^{\infty} n P_n = \bar{n}$, and $\sum_{n=0}^{\infty} (n - \bar{n})^2 P_n = (\Delta n)^2$, we finally obtain

$$
\begin{aligned}
P_a(t) \approx{} & C_1(|C_a|, \bar{n}, (\Delta n)^2, \delta) + C_2(|C_a|, \bar{n}, \delta, \Delta\theta, \beta) + \Big[ C_3(|C_a|, \bar{n}, \delta, \Delta\theta, \beta) \\
& \times \sin(R_{\bar{n}} t) + C_4(|C_a|, \bar{n}, \delta, \Delta\theta, \beta) \cos(R_{\bar{n}} t) \Big] \\
& \times \exp\left(-C_5(\bar{n}, (\Delta n)^2, \delta) t^2\right),
\end{aligned}
\tag{2.19}
$$

where

$$
C_1(|C_a|, \bar{n}, (\Delta n)^2, \delta) = \frac{2g^2 \bar{n} + \delta^2 |C_a|^2}{R_{\bar{n}}^2} + \frac{8g^4 \delta^2 (\Delta n)^2 (2|C_a|^2 - 1)}{R_{\bar{n}}^6},
$$

$$
C_2(|C_a|, \bar{n}, \delta, \Delta\theta, \beta) = \frac{2g\delta\sqrt{\bar{n}}}{R_{\bar{n}}^2} |C_a| \sqrt{1 - |C_a|^2} \cos(\Delta\theta - \beta),
$$

$$
C_3(|C_a|, \bar{n}, \delta, \Delta\theta, \beta) = \frac{-2g\sqrt{\bar{n}}}{R_{\bar{n}}} |C_a| \sqrt{1 - |C_a|^2} \sin(\Delta\theta - \beta),
$$

$$
C_4(|C_a|, \bar{n}, \delta, \Delta\theta, \beta) = \frac{2g\sqrt{\bar{n}}}{R_{\bar{n}}^2} \Bigg[ g\sqrt{\bar{n}}(2|C_a|^2 - 1) \\
- \delta |C_a| \sqrt{1 - |C_a|^2} \cos(\Delta\theta - \beta) \Bigg],
$$

and

$$
C_5(\bar{n}, (\Delta n)^2, \delta) = \frac{2g^4 (\Delta n)^2}{R_{\bar{n}}^2} \geq 0.
$$

Here, $\Delta\theta = \theta_a - \theta_b$ and $R_{\bar{n}} = \sqrt{\delta^2 + 4g^2 \bar{n}}$. Note that the collapse terminated by the Gaussian envelope, $\exp(-C_5 t^2)$, in (2.19) comes from the approximation of the parabolic one, i.e., $(1 - C_5 t^2)$ when time $t$ is small enough. So, in the case that time is not small enough for this approximation (but still must be small enough for the condition $t \ll \sqrt{\bar{n}}/g$), one can use the parabolic envelope instead of the Gaussian one in order to get the more accurate result. One can verify

the expression (2.19) in the special case when $|\varphi\rangle = |\alpha\rangle$, the coherent state, and $C_a = 1$ or $C_a = 0$ ($C_b = 1$) which means that a two-level atom is initially prepared in the upper or lower level respectively. In this special case, the expression (2.19) reduces to the simple one with $C_5 = 2\bar{n}g^4/(\delta^2 + 4g^2\bar{n})$ which is the same form of the exponent of (3) in [3]. In the more special case, i.e., the previous case together with $\delta = 0$ (resonance condition), the expression (2.19) reduces, for $C_a = 1$, to (2.17) or, for $C_a = 0$, to (2.18). It is clear now that the evolution of atom-field systems when the initial field states belong to $\mathcal{C}oh(\beta)$ (which contains the coherent states) exhibits the behavior of Cummings collapse described by (2.19). This equation may be called the *generalized collapse formula*.

## 2.4   The classical correspondence rules of the Jaynes-Cummings model

One of the interesting questions about the JCM is whether or not it reduces to the semiclassical model in the limit of large photon number. There are some methods which show directly how the Hamiltonian (2.10) reduces to the semiclassical one (2.5). For example, (1) by replacing the annihilation operator $a$ by $\langle a \rangle \mathrm{e}^{-i\Omega t}$ [18] (where $\langle a \rangle$ is the expectation value of $a$ with respect to the initial state of the field), (2.10) then reduces to (2.5) with the amplitude of the field proportional to $\langle a \rangle$, and (2) by assuming that the state of the atom-field is separable at all time, i.e., $|\Psi(t)\rangle \approx |\phi(t)\rangle \otimes |\psi(t)\rangle$ [14], (2.10) then again reduces to (2.5) with the amplitude of the field proportional to $\langle a \rangle = \langle \psi(0)|a|\psi(0)\rangle$. Notice that the resulting (2.5) from (2.10) described above appears correct if $\langle a \rangle \neq 0$ especially when the initial field state is the coherent state $|\alpha\rangle$. However, these methods fail to describe the classical correspondence if the initial state of the field is the specific number state $|n\rangle$ since $\langle a \rangle = \langle n|a|n\rangle = 0$ for all $n = 0, 1, 2, \ldots$. This zero of the expectation value leads to (2.5) describing the evolution of a two-level atom

alone so there are no Rabi oscillations in the system dynamics. This is in contrast with our results which will be shown below (see Case 2). In our method, the classical correspondence will be investigate by direct comparison between $P_a^{\text{sc}}(t)$ in (2.8) and $P_a(t)$ in (2.16) instead of dealing with the direct reduction of the Hamiltonian (2.10) to (2.5). We will show that there is a *classical correspondence* from the JCM to the semiclassical model if $P_a(t) \to P_a^{\text{sc}}(t)$ as the average photon number of the initial field $\bar{n} \to \infty$ such that the average energy $\bar{n}\hbar\Omega$ reaches the order of magnitude of classical energies. In general, any initial state of the field cannot make this classical correspondence possible, e.g., when the field cavity is maintained at a finite temperature $T$ so that the photon number distribution is that of one mode black-body radiation. In other words, the initial state $|\psi(0)\rangle$ of the field is the one which satisfies $|\langle n|\psi(0)\rangle|^2 = (1 + \bar{n})^{-1}\left[\bar{n}/(1 + \bar{n})\right]^n$, where $\bar{n} = (e^{\beta\hbar\Omega} - 1)^{-1}$. In this case, as $n$ increases, the probability of finding a two-level atom in an upper state stays "constant" very quickly after the short collapse instead of flopping up and down (Rabi oscillation) predicted by the semiclassical model (see [25, 28]). Now, we will give three cases of the initial state of atom-field in which the classical correspondence will be established. However, before going to that, let us first consider an important remark.

**Remark:** Let $\bar{n} \geq 0$ be the average photon number. Then, we have

$$-2g\sqrt{\bar{n}} = R_0 \quad \text{if and only if} \quad \bar{n}\hbar\Omega = E_{\text{cl}}, \qquad (2.20)$$

where $E_{\text{cl}}$ is the energy of the classical field. This remark can be proved from (2.1) (which implies that $E_0 = q_0\sqrt{2\Omega^2/\epsilon_0 V}\sin kz_{\text{cm}}$), and the relation $E_{\text{cl}} = (\Omega q_0)^2/2$. Note that the relation $n\hbar\Omega \approx E_{\text{cl}}$ ($\approx$ sign is used because $n$ here is the integral part of $\bar{n}$) is also equivalent to $\langle |\mathbf{E}|^2 \rangle_n \approx \langle |\mathbf{E}_{\text{cl}}(t)|^2 \rangle_t (2n + 1 \approx 2n$, for large $n$, is used), where $\langle \rangle_n$ and $\langle \rangle_t$ denote the quantum average with respect to number

state $|n\rangle$ and cycle average respectively. Here, $\mathbf{E}_{\text{cl}}$ and means $\mathbf{E}$ defined respectively in (2.1) and the quantized field $\mathbf{E} = \hat{\mathbf{x}}\varepsilon_{\Omega}(a + a^{\dagger})\sin kz$.

**Case 1:** $|\Psi(0)\rangle = |\phi\rangle \otimes |\varphi\rangle$, where $|\phi\rangle = C_a |a\rangle + C_b |b\rangle$, and $|\varphi\rangle \in \mathcal{C}oh(\beta)$.

With this initial condition, we will now try to find $P_a(t)$ where $\bar{n} \to \infty$ such that $\bar{n}\hbar\Omega = E_{\text{cl}}$. For some fixed Rabi flopping frequency, $R_0$, relevant to the semiclassical situation which is a two-level atom interacting with the classical e.m. field, (2.20) implies that $g$ must be very small, say $g \to 0$. Combining this $g \to 0$ and the expression $-2g\sqrt{\bar{n}} = R_0$ obtained from (2.20), the classical limit $\bar{n}\hbar\Omega = E_{\text{cl}}$ implies that the probability $P_a(t)$ in (2.19) can be expressed as

$$
\begin{aligned}
P_a(t) \;:=\; & |C_a(t)|^2 \\
=\; & \frac{|C_a|^2 \delta^2}{R^2} + \left(\frac{R_0}{R}\right)^2 \left[|C_a|^2 \cos^2(Rt/2) + |C_b|^2 \sin^2(Rt/2)\right] \\
& + \frac{R_0}{R} |C_a| |C_b| \left\{ \sin\left[(\theta_a - \theta_b) - \beta\right]\sin(Rt) + \frac{\delta}{R}\cos\left[(\theta_a - \theta_b) - \beta\right] \right. \\
& \left. \times \cos(Rt) - \frac{\delta}{R}\cos\left[(\theta_a - \theta_b) - \beta\right] \right\},
\end{aligned}
$$

where $R = \sqrt{\delta^2 + R_0^2}$, reduces to $P_a^{(\text{sc})}(t)$ in (2.8) (with $\phi = \beta$) which is the probability obtained from the semiclassical approach where the field is treated classically. In other words, if the total system is initially prepared in the state $|\phi\rangle \otimes |\varphi\rangle$, where $|\varphi\rangle \in \mathcal{C}oh(\beta)$ is any coherent-like state, the classical limit of the field ($\bar{n}\hbar\Omega = E_{\text{cl}}$) leads to the semiclassical result (where the initial phase of classical field is equal to $\beta$) while $|\varphi\rangle$ itself has no classical field as its classical limit. Hence, not only the exact coherent state $|\alpha\rangle$ (which has, by itself, the classical field of initial phase $\phi = Arg(\alpha)$ as its classical limit) satisfies this classical correspondence but also the states in $\mathcal{C}oh(\beta) \,\forall \beta \in [0, 2\pi]$ (which have the same phase structure (of their overlap with the number states) as the coherent state and $\Delta n$ small compared to $\bar{n}$ (here, in the way that $\bar{n}^3 \gg (\Delta n)^4$)), where

$P_n$ is not necessarily the Poisson type of distributions. In contrast, from (2.16) and (2.20), it seems to be impossible that the states, which has no properties of $\mathcal{C}oh(\beta)$ especially the phase structure, will satisfy this classical correspondence. These are the reasons why we call $\mathcal{C}oh(\beta)$ the coherent class of $\beta$. Note that since $g \to 0$ for the classical limit of the field, the (small time) collapse formula (2.19) can be used for all time in this situation, since the collapse time $t_c \sim g^{-1} \to \infty$ . This is not a surprise since we get the Rabi oscillation, as a result, in the classical limit of the field.

**Case 2:** $|\Psi(0)\rangle = |\phi\rangle \otimes |N\rangle$, where $|\phi\rangle = C_a |a\rangle + C_b |b\rangle$, and $N \in \mathcal{N}$ is arbitrary. Note that $|N\rangle \notin \mathcal{C}oh(\beta) \, \forall \beta \in [0, 2\pi]$ (even for large enough $N$), because $\forall n \neq N$, $\langle n|N\rangle = 0$ which is the complex number of modulus zero with arbitrary argument. This is why we have to consider this initial condition as another case. By using this initial condition, one can derive (like in (2.16)) from (2.12) and (2.15) that

$$
\begin{aligned}
P_a(t) &= |C_a|^2 \left[ \cos^2(R_N t/2) + \left( \frac{\delta}{R_N} \right)^2 \sin^2(R_N t/2) \right] \\
&\quad + \frac{4g^2 |C_b|^2 N}{R_{N-1}^2} \sin^2(R_{N-1} t/2).
\end{aligned}
\tag{2.21}
$$

Notice that there is no collapse in this case since the fixed number state $|N\rangle$ cannot provide the interference (of Rabi flopping at different frequencies) which leads to the collapse like in Case 1. In the classical limit of field, i.e., $N\hbar\Omega = E_{\mathrm{cl}}$, (2.21) reduces, by the use of $N + 1 \approx N$ and (2.20), to

$$
P_a(t) = \frac{|C_a|^2 \delta^2}{R^2} + \left( \frac{R_0}{R} \right)^2 \left[ |C_a|^2 \cos^2(Rt/2) + |C_b|^2 \sin^2(Rt/2) \right].
\tag{2.22}
$$

Comparing (2.22) to (2.8), one can see that $P_a(t) \approx P_a^{(\mathrm{sc})}(t)$ if and only if (here, $R_0 \neq 0$ ) (i) $C_a = 0$ or $C_b = 0$, or (ii) $\delta = 0$ with $\phi = (\theta_a - \theta_b) \pm n\pi, n \in \mathcal{N}$. These mean that the system of a two-level atom, initially prepared in the state $|\phi\rangle$, interacting with the classical e.m. field of initial phase $\phi$ has a classical correspondence from the full quantum approach, where the state of the total system

is initially prepared in $|\phi\rangle \otimes |N\rangle$, if and only if (i) or (ii) above is satisfied. Note that if (i) and (ii) (only $\delta = 0$ is enough) are both satisfied, the situation is simply the complete oscillation between the upper and lower states. Note also that if the two-level atom is prepared in the mixed state, then the classical correspondence cannot be established if the resonance condition ($\delta = 0$) is not satisfied.

**Case 3:** $|\Psi(0)\rangle \in \mathcal{P}_N$, where $N \in \mathcal{N}$ is arbitrary.

Since $\mathcal{P}_N$ is a two-dimensional subspace spanned by $\{|aN\rangle, |bN+1\rangle\}$, $|\Psi(0)\rangle$ here is simply $|\Psi(0)\rangle = C_{aN}(0)|aN\rangle + C_{bN+1}(0)|bN+1\rangle$ with the probability amplitudes $C_{aN}(0) = C_a$ and $C_{bN+1}(0) = C_b$ because there is only one number state possible for each upper and lower state of a two-level atom, i.e., $C_{an}(0) = C_{bn+1}(0) = 0 \,\forall n \neq N$. Now, from ( 2.15), we get the matrix equation

$$\begin{bmatrix} C_{aN}(t) \\ C_{bN+1}(t) \end{bmatrix} =$$
$$\begin{bmatrix} \cos(R_N t/2) - i\delta R_N^{-1}\sin(R_N t/2) & -2ig R_N^{-1}\sqrt{N+1}\sin(R_N t/2) \\ -2ig R_N^{-1}\sqrt{N+1}\sin(R_N t/2) & \cos(R_N t/2) + i\delta R_N^{-1}\sin(R_N t/2) \end{bmatrix} \begin{bmatrix} C_a \\ C_b \end{bmatrix}.$$
$$(2.23)$$

In the classical limit of field, i.e., $N\hbar\Omega = E_{\text{cl}}$, comparing (2.23) to (2.7) tells us (by the use of $N+1 \approx N$, (2.8), and (2.20)) that the probability $P_a(t) \approx P_a^{(\text{sc})}(t)$ (here, $P_a(t) := |C_{aN}|^2$) if and only if (i) $C_a = 0$ or $C_b = 0$, or (ii) $\phi = \pm 2n\pi, n \in \mathcal{N}$. This condition (i) is just a verification of condition (i) in Case 2 because of the fact that both cases (for this condition) have the same form of $|\Psi(0)\rangle$ which leads to the same result in the classical limit of field. However, condition (ii) for both cases are not the same. Condition (ii) in this case shows that the system of a two-level atom interacting with the classical e.m. field of initial phase $\phi$ has a classical correspondence from the full quantum approach, where $|\Psi(0)\rangle \in \mathcal{P}_N$, if and only if initially the classical e.m. field has "only the electric field component" (since $\phi = \pm 2n\pi$).

From these three cases, one can see that the initial mixture of the two-level atom states is very important for the rules of classical correspondence. If a two-level atom is initially prepared in the mixed state (between upper and lower states), the initial phase $\phi$ of the classical field plays an important roles. In contrast, if it is initially prepared in the upper or lower state, the initial phase $\phi$ is not important and we can group all these three cases into a more compact form by defining the similar but bigger class $\mathcal{Coh} := \left\{ |\gamma\rangle \mid |\langle n|\gamma\rangle|^2 = P_n, \text{ where the probability } P_n \text{ has the property that } \bar{n}^3 \gg (\Delta n)^4 \right\}$, which is the same definition of $\mathcal{Coh}(\beta)$ without the restriction of phase structure. Now, $\mathcal{Coh}(\beta) \subset \mathcal{Coh} \; \forall \beta \in [0, 2\pi]$ and the number state $|n\rangle \in \mathcal{Coh}$ for large enough $n$. By this definition, we can conclude that if a two-level atom is initially prepared in the upper or lower state, then the probability $P_a(t)$ can be calculated directly from (2.19) for all initial states $|\Psi(0)\rangle = |\phi\rangle \otimes |\varphi\rangle$ (where $|\phi\rangle = |a\rangle$ or $|b\rangle$ and $|\varphi\rangle \in \mathcal{Coh}$) and $P_a(t)$ can be approximated by $P_a^{(\text{sc})}(t)$ in the classical limit of field, i.e., $P_a(t) \approx P_a^{(\text{sc})}(t)$ when $\bar{n}\hbar\Omega = E_{\text{cl}}$. Here $P_a^{(\text{sc})}(t)$ is the probability obtained from the semiclassical approach where the initial phase of the classical e.m. field is "arbitrary". Note that, for $|\varphi\rangle = |n\rangle$, we have $\bar{n} = n$, $\Delta n = 0$ which implies $C_5 = 0$ (no collapse), and $\bar{n}^3 \gg (\Delta n)^4$, i.e., $n^3 \gg 0$ which is true even for small $n$. Moreover, the small time condition $t \ll \sqrt{\bar{n}}/g$ for (2.19) can be neglected for $|\varphi\rangle = |n\rangle$ or any $|\varphi\rangle \in \mathcal{Coh}$ which the interference of Rabi flopping frequencies does not lead to the collapse.

# Chapter III

## Inseparability of Light and Matter

## 3.1 Definition of the separable state

A composite quantum system consisting of two subsystems, described by the Hilbert space $\mathcal{H}_1$ and $\mathcal{H}_2$, is physically described by Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. If these subsystems 1 and 2 are prepared respectively in the states described by the density operators $\rho_1$ and $\rho_2$ which are prepared by devices for each subsystem that function independently, then it is clear from the physical point of view that the density operator of the composite quantum system (denoted by $\rho$ ) can be written in the product form $\rho = \rho_1 \otimes \rho_2$. In this case, the density operator $\rho$ is called *uncorrelated state*. In general, an operator $A$ acting on $\mathcal{H}$ is called the *state* if $Tr A = 1$ and $A$ is positive (denoted by $A \geq 0$ ), i.e., $Tr(AP) \geq 0$ for any projector $P$. In other words, an operator $A$ (where $A$ is Hermitian) is the state if all of its eigenvalues are positive and its trace is equal to unity. From this definition, it is clear that any density operator is the state and we will use the term *state* instead of density operator or density matrix throughout this thesis unless specified.

In contrast to the uncorrelated state, one may expect that any correlated states, which produce nonclassical phenomena of the composite quantum system, are originated only from the nature of quantum mechanics. This is not true. Werner [6] showed that the correlated states in which their statistical properties can be reproduced by a classical mechanism can exist in the following way. Suppose that (1) each of two preparing devices (for each subsystem) has a switch with setting $n = 1, ..., k$ and that, (2) with setting $n$ of the device $i$, produces system in the state $\rho_i^{(n)}$, and (3) we have also a random generator which produces

numbers $n = 1, ..., k$ with probability $p_n$, we can then combine these three devices into a new preparing apparatus by first, in each individual experiment, drawing a random number $n \in \{1, ..., k\}$ and then setting the switches of two preparing devices according to the result. In short, this kind of correlated states can be prepared by two distant observers who receive instructions from a common source by a classical communication channel. It is clear from this preparation processes that the expectation value of the joint measurement of observables $A_1$ (of subsystem 1) and $A_2$ (of subsystem 2) is $Tr(\rho A_1 \otimes A_2) = \sum_{n=1}^{k} p_n Tr(\rho_1^{(n)} A_1) Tr(\rho_2^{(n)} A_2)$, which implies that the correlated state $\rho$ is in the form

$$\rho = \sum_{n=1}^{k} p_n \rho_1^{(n)} \otimes \rho_2^{(n)}. \tag{3.1}$$

Werner [6] called the states, which can be written or approximated in the trace norm by the form (3.1), the *classically correlated states* and called any states, which are not classically correlated states, the *EPR correlated states* in order to emphasize the crucial role of such states in the Einstein-Podolsky-Rosen (EPR) paradox [29]. From many subsequent works, the classically and EPR correlated states are also called the *separable* and *inseparable* states respectively and the inseparable state is sometimes called the *entangled* state.

## 3.2 Positive partial transposed criterion and negativity

One of the main problems about the composite quantum system is how we can tell whether or not it is separable, i.e., it is in the separable state expressed by (3.1). It has been shown also by Werner [6] that the separable states always admit hidden-variable models [30] (where the sample space of the hidden-variable space is, in fact, the common source $\{1, ..., k\}$), and hence satisfied the Bell's inequalities [31]. In other words, Bell's inequalities are the necessary conditions for separability, i.e., every state violating Bell's inequalities must be inseparable.

Note that this necessary condition is not sufficient since there are inseparable states satisfying Bell's inequalities, i.e., admitting a hidden-variable model [6]. So far, only some necessary conditions of separability in general systems have been found [32, 33, 34, 35] and the important step is due to Peres [7], who has provided a very strong condition. He started with (3.1) and noticed that the partial transposition of $\rho$ (denoted by $\rho^{T_1}$), defined by $\langle m|\langle\mu|\rho^{T_1}|n\rangle|\nu\rangle := \langle n|\langle\mu|\rho|m\rangle|\nu\rangle$ for all $n, m, \mu, \nu$ (where $\{|n\rangle|\nu\rangle\}$ is the basis set of $\mathcal{H}$), is also a (separable) state. Note that this conclusion is independent of what subsystems are chosen to be transposed, i.e., one can use $\rho^{T_1}$ or $\rho^{T_2}$ in order to obtain the same criterion. This criterion can be simply verified as follows. From (3.1), $\rho^{T_1} = \sum_{n=1}^{k} p_n(\rho_1^{(n)})^T \otimes \rho_2^{(n)}$ and $(\rho_1^{(n)})^T = (\rho_1^{(n)})^*$ since $\rho_1^{(n)}$ is Hermitian. Let $\{\lambda_i^{(n)}\}$ be the set of eigenvalues of $\rho_i^{(n)}$. Then, since $\lambda_i^{(n)} \geqslant 0$ is the positive real number for all $i$, $\{\lambda_i^{(n)}\}$ is also the set of eigenvalues of $(\rho_1^{(n)})^*$ which implies that $(\rho_1^{(n)})^T = (\rho_1^{(n)})^* \geqslant 0$. Moreover, it is clear that since $\rho_1^{(n)}$ is Hermitian for all $n$, $Tr(\rho_1^{(n)})^T = Tr(\rho_1^{(n)}) = 1$ $\forall n$. Now, we have $(\rho_1^{(n)})^T \geqslant 0$ and $Tr(\rho_1^{(n)}) = 1$ which implies that $(\rho_1^{(n)})^T$ is a state and, hence, $\rho^{T_1}$ is a (separable) state. Since $\rho^{T_1}$ is a state for separable state $\rho$, the *necessary condition* for separability is that $\rho^{T_1}$ has only *non-negative eigenvalues*. In other words, if there exists even a single negative eigenvalue of $\rho^{T_1}$, then the composite quantum system is inseparable or entangled. This criterion is called the *positive partial transposed (ppt) criterion*. Peres [7, 36] also showed that this criterion is more sensitive than Bell's inequalities for detecting quantum inseparability. Note that, in contrast to Peres's conjecture, i.e., this necessary condition is also sufficient, it was shown by Horodeckis [37] that it is true only for some specific systems such as $2 \times 2$ and $2 \times 3$ systems but this ceases to be true for general larger composite systems.

Suppose now we know from the ppt criterion that the composite quantum system is inseparable or entangled (i.e., the negative eigenvalues of $\rho^{T_1}$ are found), this criterion still does not tell us about the degree of entanglement, i.e., how

much or how strong this entangled state is. The problem of defining a quantity capable of measuring a *degree of entanglement* is a subject of several recent studies [38, 39, 40, 41]. The one which is suitable for us to compute the degree of entanglement of the thermal Jaynes-Cummings state (which will be done in the next section) is called the *negativity*. It was introduced by Vidal and Werner [8] based on the trace norm of the partial transpose $\rho^{T_1}$, denoted by $\left\|\rho^{T_1}\right\|_1 := Tr\sqrt{(\rho^{T_1})^\dagger \rho^{T_1}}$. It essentially measures the degree to which $\rho^{T_1}$ fails to be positive and therefore it can be regarded as a quantitative version of Peres's ppt criterion for separability. To see how it is defined, notice that, since $\rho^{T_1}$ satisfies $Tr(\rho^{T_1}) = 1$ while it may have negative eigenvalues $\mu_k < 0$ ($\mu_k := \lambda_k < 0$, where $\{\lambda_i\}$ is the set of eigenvalues of $\rho^{T_1}$), one can obtain

$$\left\|\rho^{T_1}\right\|_1 := Tr\sqrt{(\rho^{T_1})^\dagger \rho^{T_1}} = \sum_i |\lambda_i| = 1 + 2\left|\sum_k \mu_k\right| =: 1 + 2N(\rho) \qquad (3.2)$$

where $N(\rho) := |\sum_k \mu_k| = \sum_k |\mu_k|$ is called the *negativity* which is the absolute value of the sum of negative eigenvalues of $\rho^{T_1}$. Note that, in (3.2), $\sum_i$ is a finite (infinite) sum for finite(infinite) dimensional Hilbert space $\mathcal{H}$ and $\sum_k$ is clearly a finite sum for finite dimensional $\mathcal{H}$ while it can be both finite or infinite sum for infinite dimensional $\mathcal{H}$ depending on each specific problem. It is clear that any separable state implies $N(\rho) = 0$ (since its partial transpose is also a state) but, as the Peres's necessary condition is not sufficient, the converse is not true. A simple example of this negativity is the entangled state belonging to a $2 \times 2$ system [42]. In this case, all separable states have $N(\rho) = 0$ while the maximally entangled state, which has the set of eigenvalues $\{-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$ of its partial transpose, leads to $N(\rho) = 1/2$.

## 3.3 Inseparability of the thermal Jaynes-Cummings state

From (3.1), one can see that the unitary time evolution of the composite quantum system, under which all separable initial states remain separable, necessarily factorizes into a product of two separate time evolutions, i.e., the subsystems must have no interaction with each other. This means that an interaction time evolution automatically generates inseparable states at (almost) later times. It may happen that the state is separable at some specific times, e.g., the evolution of the initial separable state $|an\rangle$ in the JCM is a linear combination of $|an\rangle$ and $|bn+1\rangle$ (see section 2.2) so in the resonance case the complete Rabi oscillation ensures that, at some specific times, the state is in the separable state $|an\rangle$ or $|bn+1\rangle$ which their corresponding density operators are clearly uncorrelated, i.e., the most simple case of separable state. However, one may expect in general that the composite quantum system in nature should be (mostly) inseparable due to the interactions between subsystems. This should be true if the composite quantum system of interest is an *isolated* system. In real life, there is no such (perfect) isolated system or even in the laboratories. To be precise, all composite quantum systems should be associated with some temperature $T$ due to contact with a heat reservoir. In contrast to the case of an isolated system that the inseparable state will always be generated by the interaction time evolution, the temperature effect brings the evolution of states into stable thermal equilibrium state which will be separated at large enough temperature. For finite systems described by finite dimensional Hilbert spaces, this result is clearly expected since the thermal state or Gibbs state becomes proportional to the product of identity operator as the temperature approaches infinity. Thus, by continuity, any finite quantum system is separable at large enough temperature. For example, consider the thermal state of two coupled (atomic) two-level systems or qubits (by a dipole-dipole

interaction) described by the Hamiltonian

$$H_{\mathrm{aa}} = \frac{\hbar\omega_1}{2}\sigma_z^{(1)} + \frac{\hbar\omega_2}{2}\sigma_z^{(2)} + \hbar g(\sigma_-^{(1)}\sigma_+^{(2)}\sigma_+^{(1)}\sigma_-^{(2)}) \tag{3.3}$$

Here, the operators superscripted by (1) and (2) are the extended operators [17] of subsystems (1) and (2) respectively. Applying the ppt criterion (be reminded that from Horodecki's work [37] ppt criterion is also sufficient for this $2 \times 2$ case) to the thermal state $\rho_{\mathrm{aa}} = \mathrm{e}^{-\beta H_{\mathrm{aa}}}/Z$ with $\beta = (kT)^{-1}$ shows that $\rho_{\mathrm{aa}}$ is separable for all temperatures large enough such that the inequality $\sinh(\hbar g/kT) < (\delta^2 + 4g^2)/4g^2$ holds [43], where $\delta = \omega_1 - \omega_2$ is the detuning between the two-level energies. Roughly speaking, we see that the two-coupled qubits are in a separable state as soon as the thermal energy is much larger than the interaction energy, and entangled at lower energies (Zero temperature is singular as the ground state is separable). For continuous system described by the infinite dimensional Hilbert spaces, Simon [44] showed that the ppt criterion may also be applied successfully to continuous variable Gaussian states, i.e., the states whose Wigner function is a Gaussian in phase space. For the subclass of bipartite Gaussian state with a single oscillator for subsystem 1 and an arbitrary number for subsystem 2, the ppt criterion is again necessary and sufficient for separability, as shown by Werner and Wolf [45]. A simple example for this class is the two linearly coupled field modes (with frequencies $\Omega_1$ and $\Omega_2$ ) described by the Hamiltonian

$$H_{\mathrm{ff}} = \hbar\Omega_1 a^\dagger a + \hbar\Omega_2 b^\dagger b + \hbar g(ab^\dagger + a^\dagger b) \tag{3.4}$$

The thermal state $\rho_{\mathrm{ff}} = \mathrm{e}^{-\beta H_{\mathrm{ff}}}/Z$ is obviously Gaussian and the application of ppt criterion reveals that $\rho_{\mathrm{ff}}$ is separable for any choice of parameters and for all temperatures [43]. Besides these two examples, another interesting one is the correlation in quantum Brownian motion. It was shown by Eisert an Plenio [46] that there exists a large class of initial states (i.e., Gaussian states of the whole system with some appropriate parameters including environment temperature)

for which no entanglement will be created at all times between the system of interest and the environment. These states and also $\rho_{\text{ff}}$ above are rather special. As shown by Clifton and Halvorson [47], one expects bipartite mixed states of infinite-dimensional systems to be generically inseparable: the set of inseparable states is dense in the set of all states (with respect to trace norm topology).

Complementing the known results about the coupled atom-atom (3.3) and field-field (3.4) model, we will apply the ppt criterion to a composite quantum system consisting of *atomic* two-level system and a continuous variable single-mode quantized *field* based on the JCM which has been described in Chapter 2. To be precise, we will use the ppt criterion to investigate the entanglement of the coupled atom-field in thermal equilibrium where the interaction of atom and field in the cavity is described by the JCM with Hamiltonian

$$H_{\text{af}} = \frac{\hbar\omega}{2}\sigma_z + \hbar\Omega a^\dagger a + \hbar g(a\sigma_+ + a^\dagger\sigma_-) \ , \tag{3.5}$$

which has been defined in (2.10). Now, the *thermal Jaynes-Cummings* (JC) *state* (denoted by $\rho_{\text{af}}$) corresponding to Hamiltonian (3.5), which will be investigated by the ppt criterion, is

$$\rho_{\text{af}} = \frac{\mathrm{e}^{-\beta H_{\text{af}}}}{Z} \ , \tag{3.6}$$

where $Z = Tr(\mathrm{e}^{-\beta H_{\text{af}}})$ is the partition function, and $\beta = 1/kT$ is the inverse temperature. The next step is to find its partial transpose. To do that first inserting (2.11) into (3.6), we get

$$\rho_{\text{af}} = \frac{1}{Z}\mathrm{e}^{\frac{\beta\hbar\omega}{2}|b0\rangle\langle b0|}\prod_{n=0}^{\infty}\mathrm{e}^{-\beta H_n} \ , \tag{3.7}$$

since $[|b0\rangle\langle b0|, H_n] = [H_m, H_n] = 0 \,\forall m \neq n$. Let $l \in \mathcal{N} := \{0, 1, 2, 3, ...\}$. Then an operator $\mathrm{e}^{-\beta H_l}$ can be written in the dressed-atom basis of $\mathcal{H}_{\text{a}} \otimes \mathcal{H}_{\text{f}}$, i.e., $\{|b0\rangle, |1n\rangle, |2n\rangle\}_{n=0}^{\infty}$ (with the completeness relation $|b0\rangle\langle b0| + \sum_{i=1}^{2}\sum_{n=0}^{\infty}|in\rangle\langle in| = 1$) in the form

$$\mathrm{e}^{-\beta H_l} = \sum_{i=1}^{2}\left[\mathrm{e}^{-\beta E_{il}}|il\rangle\langle il|\right] + |b0\rangle\langle b0| + 1'_l \ , \tag{3.8}$$

where $1'_l := \sum_{i=1}^{2} \sum_{n \neq l} |in\rangle \langle in|$ is an identity operator of the subspace $\oplus_{n \neq l} \mathcal{P}_n \subset$ $\mathcal{H}_a \otimes \mathcal{H}_f$ (see the definitions of $\mathcal{P}_n, \mathcal{H}_a$ and $\mathcal{H}_f$ in Section 2.2 of Chapter 2). To obtain (3.8), we have used the fact that $\mathcal{L}_0 \perp \mathcal{P}_n \forall n$, $\mathcal{P}_n \perp \mathcal{P}_m \forall n \neq m$, $H_n(\mathcal{P}_n) = 0 \; \forall n \neq m$ (see Section 2.2), and $H_n |in\rangle = E_{in} |in\rangle$ ($i = 1, 2$), where $E_{in}$ have been defined below (2.11). Next, consider $\mathrm{e}^{-\beta H_{l-1}} 1'_l$ (for $l \geqslant 1$) which is, from (3.8), $\mathrm{e}^{-\beta H_{l-1}} 1'_l = \left[ \sum_{i=1}^{2} (\mathrm{e}^{-\beta E_{il-1}} |il - 1\rangle \langle il - 1|) + |b0\rangle \langle b0| + 1'_{l-1} \right] 1'_l$. From the definition of $1'_l$ defined below (3.8), the above equation becomes

$$\mathrm{e}^{-\beta H_{l-1}} 1'_l = \sum_{i=1}^{2} \left[ \mathrm{e}^{-\beta E_{il-1}} |il - 1\rangle \langle il - 1| \right] + 1'_{l-1,l} , \qquad (3.9)$$

where $1'_{l-1,l} := \sum_{i=1}^{2} \sum_{n \neq l, l-1} |in\rangle \langle in|$ is an identity operator in the subspace $\oplus_{n \neq l, l-1} \mathcal{P}_n$. Operating $\mathrm{e}^{-\beta H_{l-1}}$ on (3.8) and using (3.9), we get

$$\mathrm{e}^{-\beta H_{l-1}} \mathrm{e}^{-\beta H_l} = \sum_{i=1}^{2} \left[ \mathrm{e}^{-\beta E_{il}} |il\rangle \langle il| + \mathrm{e}^{-\beta E_{il-1}} |il - 1\rangle \langle il - 1| \right]$$
$$+ |b0\rangle \langle b0| + 1'_{l-1,l} .$$

Successively doing this for $l - 2, l - 3, ..., 0$, we obtain

$$\prod_{n=0}^{\infty} \mathrm{e}^{-\beta H_n} = \lim_{l \to \infty} \prod_{n=0}^{l} \mathrm{e}^{-\beta H_n} = \sum_{i=1}^{2} \sum_{n=0}^{\infty} \left[ \mathrm{e}^{-\beta E_{in}} |in\rangle \langle in| \right] + |b0\rangle \langle b0| , \qquad (3.10)$$

since $\lim_{l \to \infty} 1'_{0,1,2,...,l} = 0$. Substituting (3.10) into (3.7), we get

$$\rho_{\mathrm{af}} = \frac{1}{Z} \left\{ \mathrm{e}^{\beta \hbar \omega / 2} |b0\rangle \langle b0| + \sum_{i=1}^{2} \sum_{n=0}^{\infty} \left[ \mathrm{e}^{-\beta E_{in}} |in\rangle \langle in| \right] \right\} .$$

By using the transformation formula between bare-atom and dressed-atom basis defined below (2.11), the above equation becomes

$$\rho_{\mathrm{af}} = \frac{1}{Z} \sum_{n=0}^{\infty} \left[ A_n |an\rangle \langle an| + C_n (|an\rangle \langle bn + 1| + |bn + 1\rangle \langle an|) + B_n |bn\rangle \langle bn| \right],$$
$$(3.11)$$

where

$$B_0 := \mathrm{e}^{\beta \hbar \omega / 2}, \; B_{n+1} = \mathrm{e}^{-\beta E_{1n}} \cos^2 \theta_n + \mathrm{e}^{-\beta E_{2n}} \sin^2 \theta_n ,$$
$$A_n := \mathrm{e}^{-\beta E_{1n}} \sin^2 \theta_n + \mathrm{e}^{-\beta E_{2n}} \cos^2 \theta_n ,$$
$$C_n := \frac{1}{2} \sin(2\theta_n)(\mathrm{e}^{-\beta E_{1n}} - \mathrm{e}^{-\beta E_{2n}}) .$$

From the definition of partial transpose defined in Section 3.2 (where the two-level atom and field parts are identified with subsystem 1 and 2 respectively), i.e., $\langle an|\rho_{\mathrm{af}}^{\mathrm{T_a}}|bm\rangle := \langle bn|\rho_{\mathrm{af}}|am\rangle \, \forall \, n, m$, the partial transposition of $\rho_{\mathrm{af}}$ in (3.11) can be written as

$$\rho_{\mathrm{af}}^{\mathrm{T_a}} = \frac{1}{Z}\left[A_0 \, |a0\rangle \, \langle a0| + \sum_{n=0}^{\infty} \rho_n^{\mathrm{T_a}}\right], \tag{3.12}$$

where

$$\rho_n^{\mathrm{T_a}} := B_n \, |bn\rangle \, \langle bn| + C_n(|bn\rangle \, \langle an+1| + |an+1\rangle \, \langle bn|)$$
$$+ A_{n+1} \, |an+1\rangle \, \langle an+1| \, .$$

Here, the superscript $T_a$ on both $\rho_{af}$ and $\rho_n$ indicates that a two-level atom part is chosen to be transposed.

At this point, we will use (3.12), in contrast to (3.3) and (3.4), to show that the negative eigenvalues of $\rho_{\mathrm{af}}^{\mathrm{T_a}}$ always exist at any temperature $T > 0$ which implies by the ppt criterion that the thermal Jaynes-Cummings state is always inseparable or entangled. It is clear from (3.12) that the matrix representation of $\rho_{af}^{Ta}$ is a block diagonal matrix, one block being a $1 \times 1$ matrix with eigenvalue $A_0/Z > 0$, and the others, which are the matrix representation of $\rho_n^{\mathrm{T_a}} \, /Z$ for $n = 0, 1, 2, ...$, are the $2 \times 2$ matrices.

From the definition of $\rho_n^{\mathrm{T_a}}$ defined below (3.12), it is clear that its matrix representation is

$$\rho_n^{\mathrm{T_a}} = \begin{pmatrix} B_n & C_n \\ C_n & A_{n+1} \end{pmatrix} \tag{3.13}$$

with the simple eigenvalues

$$\lambda_n^{(\pm)} = \frac{(B_n + A_{n+1}) \pm \sqrt{(B_n + A_{n+1})^2 - 4(B_n A_{n+1} - C_n^2)}}{2Z} \, . \tag{3.14}$$

Since our aim is to check for negative eigenvalues, the only relevant one is $\lambda_n^{(-)}$ and it is clear that $\lambda_n^{(-)} < 0$ if $B_n A_{n+1} - C_n^2 < 0$ . Note that, up to now, there are no theorems proving that the ppt criterion for this case is also sufficient for

separability, i.e., we cannot make any conclusions (about separability) for the situations where a positive eigenvalue is found. However, as we will show that the negative eigenvalues always exist at any temperature, only the necessary condition is enough for our case. Now, for further convenience, we introduce the scaled dimensionless parameters

$$x := \beta\hbar\,|g| = \frac{\hbar\,|g|}{kT}, \quad d := \frac{\Omega}{|g|}, \quad \text{and} \quad \Delta := \frac{\delta}{|g|} \tag{3.15}$$

so that $x$ is the ratio of (vacuum) coupling energy to thermal energy inversely proportional to the absolute temperature $T$. The parameters $d$ and $\Delta$ are scaled field mode frequency and detuning respectively. Note that, for most realistic settings (for instance in quantum optics), not only $d > 1$ but, in fact, $d \gg 1$. Therefore, we will consider the case $d > 1$ only in this thesis. From the definitions of $B_n$, $A_n$, and $C_n$ defined below (3.11), one obtains

$$
\begin{aligned}
B_n A_{n+1} - C_n^2 \;=\; & \mathrm{e}^{-(2n+1)xd}\Bigg[ \cosh\left(\frac{x}{2}\sqrt{\Delta^2 + 4n}\right) \cosh\left(\frac{x}{2}\sqrt{\Delta^2 + 4(n+2)}\right) \\
& + \frac{\Delta \sinh\left(\frac{x}{2}\sqrt{\Delta^2 + 4n}\right) \cosh\left(\frac{x}{2}\sqrt{\Delta^2 + 4(n+2)}\right)}{\sqrt{\Delta^2 + 4n}} \\
& - \frac{\Delta \sinh\left(\frac{x}{2}\sqrt{\Delta^2 + 4(n+2)}\right) \cosh\left(\frac{x}{2}\sqrt{\Delta^2 + 4n}\right)}{\sqrt{\Delta^2 + 4(n+2)}} \\
& - \frac{\Delta^2 \sinh\left(\frac{x}{2}\sqrt{\Delta^2 + 4(n+2)}\right) \sinh\left(\frac{x}{2}\sqrt{\Delta^2 + 4n}\right)}{\sqrt{[\Delta^2 + 4(n+2)][\Delta^2 + 4n]}} \\
& - \frac{4(n+1)\sinh^2\left(\frac{x}{2}\sqrt{\Delta^2 + 4(n+1)}\right)}{\Delta^2 + 4(n+1)} \Bigg].
\end{aligned}
\tag{3.16}
$$

At any given $x$, one can see that the relevant terms in (3.16) are only the first and the last term so that, for sufficiently large $n$, we have $(B_n A_{n+1} - C_n^2) \sim \mathrm{e}^{-(2n+1)xd} f_n(x)$, where

$$f_n(x) := \cosh(x\sqrt{n})\cosh(x\sqrt{n+2}) - \sinh^2(x\sqrt{n+1}). \tag{3.17}$$

One can see that, since $e^{-(2n+1)xd} > 0$, $B_n A_{n+1} - C_n^2 < 0$ if $f_n(x) < 0$. From (3.17), the systematic expansion of $f_n(x)$ (for fixed $x$) reveals that asymptotically

$$f_n(x) \sim f_n^{as}(x) := 1 - \frac{x e^{2x\sqrt{n+1}}}{16 n^{3/2}} \tag{3.18}$$

for large $n$ (see Appendix). It is clear from this expression that if $n$ is large enough, $f_n(x)$ will be negative and will stay negative for greater values of $n$. Therefore, based on the ppt criterion, we can state that, for all parameters of our model (i.e., the frequency pair of atom-field $(\omega, \Omega)$, the coupling constant $g$, and the inverse temperature $\beta = 1/kT$), *the thermal Jaynes-Cummings state is always entangled*. Seen in the light of the work of Clifton and Halvorson [47], this result confirms the generic expectation of inseparability for an infinite-dimensional bipartite system. Note that since the JCM is the simplest model of the coupled matter-light systems, *it seems that (at any temperature) the coupled systems of light and matter, in general, are never in a separable state.*

## 3.4 Negativity of the thermal Jaynes-Cummings state

By the definition of negativity (3.2), the negativity (at given $x$) for the thermal Jaynes-Cummings state is $N(\rho_{\mathrm{af}}) = \sum_{n \in I} |\lambda_n^{(-)}(x)|$, where the index set $I$ is one which $\lambda_n^{(-)}(x) < 0 \ \forall n \in I$ at $x$. Here, $\lambda_n^{(-)}$ in (3.14) is rewritten by $\lambda_n^{(-)}(x)$ to stress its dependence on $x \propto 1/T$. To proceed, we shall simplify the following considerations by restricting ourselves to the case of resonance, i.e., $\delta = \omega - \Omega = 0$. In this case, $B_n A_{n+1} - C_n^2$ in (3.16) has the exact simple form

$$B_n A_{n+1} - C_n^2 = e^{-(2n+1)xd} f_n(x), \tag{3.19}$$

with $f_n(x)$ from (3.17) and $d$ here can either be $\Omega/|g|$ or $\omega/|g|$ since we are considering $\delta = 0$ case. Now, in order to see what index $I$ described above is,
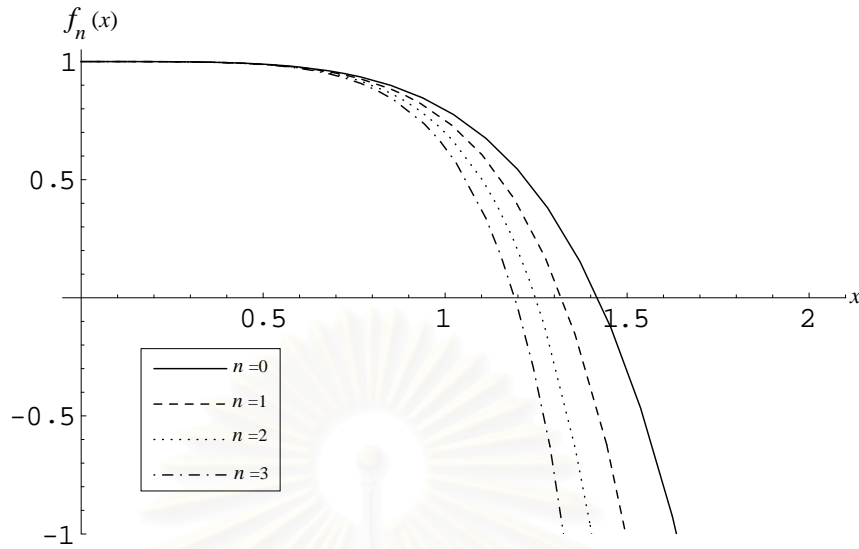
Figure 3.1: The strictly decreasing functions $f_n(x)$ as a function of $x$ for $n = 0$, 1, 2, 3. The sequence of their zeros, i.e., $x_0(0), x_0(1), x_0(2), ...$, where $x_0(0) > x_0(1) > x_0(2) > ...$, is clearly a decreasing sequence (converging to zero).

notice that (at each $x$) $\lambda_n^{(-)}(x) < 0$ if $B_n A_{n+1} - C_n^2 < 0$ which, by (3.19), is true if $f_n(x) < 0$.

In Fig. 3.1, it is clear that $f_n(x)$ is a strictly decreasing function of $x$ (for each $n$) and its zero (denoted by $x_0(n)$) has the property that $x_0(n) \to 0$ as $n \to \infty$. This property can be (formally) proved by combining the strictly decreasing property of $f_n(x)$ (which may be proved, for example, by Taylor's series method) and its asymptotic form (3.18) showing that there always exists $n$ such that $f_n(x) < 0$ at any given $x$ (even when $x \to 0$). From that property of $x_0(n)$, one can see that, at each $x$, there exists the *smallest* integer $n_-(x)$ such that $f_n(x) < 0 \, \forall n \geq n_-(x) (f_n(x) > 0 \forall n < n_-(x))$ which implies that $\lambda_n^{(-)}(x) < 0 \, \forall n \geq n_-(x) \, (\lambda_n^{(-)}(x) > 0 \forall n < n_-(x))$. Now, it is clear that the index set $I$ is $I = \{n \, | \, n \geq n_-(x)\}$ so the negativity of the thermal Jaynes-Cummings state (at given $x$) is

$$N(\rho_{\text{af}}) = \sum_{n=n_-(x)}^{\infty} \left| \lambda_n^{(-)}(x) \right| . \tag{3.20}$$

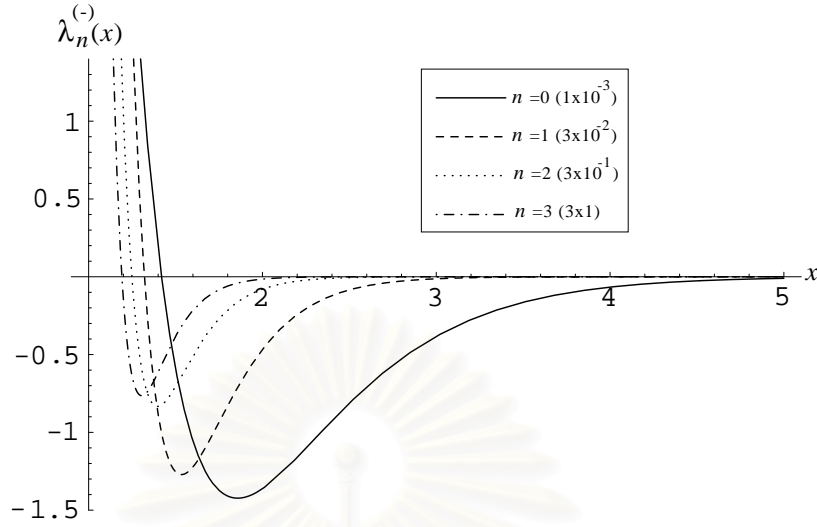Before computing this sum, let us consider the graphs of $\lambda_n^{(-)}(x)$ at various $n$ in

Figure 3.2: The eigenvalues $\lambda_n^{(-)}(x)$ as a function of $x$ for $d = 2$ and $n = 0, 1, 2, 3$. Each $\lambda_n^{(-)}(x)$ is plotted with their own scale (see text) and its zero is, in fact, the zero of $f_n(x)$ i.e., $x_0(n)$ (see Fig. 3.1).

order to have a feeling with it. From Fig. 3.2 ($d = 2$ is set), it is clear that (at each $x$) $\lambda_n^{(-)}(x)$ quickly goes to zero as $n$ ($\geq n_-(x)$) increases. This behavior suggests that this infinite sum should converge very quickly after a few $n$ have summed so the easiest approximation of the negativity is $N(\rho_{\mathrm{af}}) \approx |\lambda_{n_-(x)}^{(-)}(x)|$ at each $x$. Note that since $\lambda_n^{(-)}(x) < 0 (= 0)$ if and only if $f_n(x) < 0 (= 0)$ implies $\lambda_n^{(-)}(x) < 0 \, \forall x > x_0(0)$, we have $n_-(x) = 0 \, \forall x > x_0(0)$ so the negativity has a simple approximation (for large $x$), i.e., $N(\rho_{\mathrm{af}}) \approx |\lambda_0^{(-)}(x)| \, \forall x > x_0(0)$.

Moreover, Fig. 3.2 shows that this approximation is more and more accurate for both small $x$ ($x \to 0$) and large $x$ ($x \to \infty$) where the negativity tends to zero because, in these regions, $\lambda_n^{(-)}(x) \to 0$ more faster than that in the moderate region. In both these regions, the asymptotic formula of $N(\rho_{\mathrm{af}})$ can be derived as follows. First, consider the large $x$ ($x \to \infty$) case. In this case we have, from the above discussions, $N(\rho_{\mathrm{af}}) \approx |\lambda_0^{(-)}(x)|$ . From the definition of $A_n$ and $B_n$ defined below (3.11), we have (for $\delta = 0$) $B_0 = \mathrm{e}^{xd/2}$, $A_1 = \mathrm{e}^{-3xd/2} \cos(x\sqrt{2})$ and, from (3.19), $B_0 A_1 - C_0^2 = \mathrm{e}^{-xd} f_0(x)$. Notice that $B_0 + A_1 =$

$e^{xd/2} + e^{-3xd/2}\cos(x\sqrt{2}) = e^{xd/2} + [e^{(\sqrt{2}-3d/2)x} + e^{-(\sqrt{2}+3d/2)x}]/2 \sim e^{xd/2}$ for $x \to \infty$

(since $d > 1$), and $-4(B_0 A_1 - C_0^2) = -4e^{-xd} f_0(x) \sim e^{(2-d)x}$ since (for fixed

$n) f_n(x) \sim -e^{2x\sqrt{n+1}}/4$ (see Appendix) for $x \to \infty$. Substituting these formulae

into (3.14), we obtain (for $x \to \infty$)

$$N(\rho_{\text{af}}) \approx \left|\lambda_0^{(-)}(x)\right| \sim \frac{e^{xd/2}\left|1 - \sqrt{1 + e^{2(1-d)x}}\right|}{2Z}$$

$$\approx \frac{e^{xd/2}e^{2(1-d)x}}{4Z} \ . \tag{3.21}$$

To obtain (3.21), we have applied the formula $\sqrt{1+t} \approx 1 + t/2$ for $t \approx 0$ to the

square root term where $e^{2(1-d)x} \approx 0$ for $x \to \infty$. Next, we have to determine $z$

for $x \to \infty$. From (3.11), it is clear that

$$Z = Tr(\rho_{\text{af}}) = \sum_{n=0}^{\infty}(A_n + B_n)$$

$$= e^{xd/2}[-1 + 2\sum_{n=0}^{\infty} e^{-nxd}\cos(x\sqrt{n})] \ . \tag{3.22}$$

For $x \to \infty$, only $n = 0$ contributes to the sum in (3.22) so we get $Z \sim e^{xd/2}$.

Substituting this asymptotic form of $Z$ to (3.21), we obtain $N(\rho_{\text{af}}) \approx e^{2(1-d)x}/4$

or

$$\log[N(\rho_{\text{af}})] \approx -2x(d-1) - \log(4) \ \text{ for } x \to \infty. \tag{3.23}$$

Second, consider the small $x(x \to 0)$ care. In this case, similar to $x \to \infty$, we have

again $N(\rho_{\text{af}}) \approx \left|\lambda_{n_-(x)}^{(-)}\right|$. From the property of $x_0(n)$, i.e., $x_0(n) \to 0$ as $n \to \infty$,

we have $n_-(x) \to \infty$ as $x \to 0$. Applying large $n_-(x)$ to $B_n$, one gets $B_{n_-(x)} =$

$\left(e^{-[(n_-(x)-1/2)d-\sqrt{n_-(x)}]x} + e^{-[(n_-(x)-1/2)d+\sqrt{n_-(x)}]x}\right)/2 \sim e^{-n_-(x)xd}$ for large $n_-(x)$.

Similarly, $A_{n_-(x)+1} \sim e^{-n_-(x)xd}$. From ( 3.19), we have $B_{n_-(x)}A_{n_-(x)+1} - C_{n_-(x)}^2 \sim$

$e^{-2n_-(x)xd}f_{n_-(x)}^{\text{as}}$. Substituting these formulae to (3.14), we obtain (for $x \to 0$)

$$N(\rho_{\text{af}}) \approx \left|\lambda_{n_-(x)}^{(-)}(x)\right| \sim \frac{e^{-n_-(x)xd}}{Z}\left|1 - \sqrt{1 - f_{n_-(x)}^{\text{as}}(x)}\right| \ . \tag{3.24}$$

To proceed, consider (for given $x$) $f_y(x)$, where $y$ is the continuous version of $n$

in (3.18). For any $y \gg 1$, $f_y^{\text{as}}(x) \approx 1 + x\frac{e^{2x\sqrt{y}}}{16}y^{3/2}$. Differentiating with respect

to $y$ leads to $\partial_y f_y^{\text{as}}(x) \approx x(f_y^{\text{as}}(x) - 1)/(16)^2\sqrt{y}$. For given $x(\to 0)$, let $y_0(x)(\to \infty)$ satisfy $f_{y_0(x)}^{\text{as}}(x) = 0$. Then $\partial_y \, f_{y_0(x)}^{\text{as}}(x) \approx -x/(16)^2\sqrt{y_0(x)} \approx 0$. Since $x_0(n) \to 0$ as $n \to \infty$ implies $|x_0(n-1) - x_0(n)| \to 0$ as $n \to \infty$, (for give $x$) $x_0(y_0(x)) \in [x_0(n_-(x)), x_0(n_-(x)) - 1]$ implies $|y_0(x) - n_-(x)| \to 0$ as $n \to \infty$ (since $n_-(x) \to \infty$ as $x \to 0$). This means that, for given $x \to 0$, $n_-(x)$ is very close to $y_0(x)$ so, together with the above $\partial_y f_{y_0(x)}^{\text{as}}(x) \approx 0$, we can conclude that $f_{n_-(x)}^{\text{as}}(x) \approx 0$ (but still less than zero). From this approximation, applying $\sqrt{1 + t} \approx 1 + t/2$ for $t \approx 0$ to the square root term in (3.24), we get

$$N(\rho_{\text{af}}) \approx \frac{\mathrm{e}^{-n_-(x)xd}\left|f_{n_-(x)}^{\text{as}}(x)\right|}{2Z} \ . \tag{3.25}$$

For the partition function $Z$, the derivation of its asymptotic form is not so simple as in the previous $x \to \infty$ case. Notice that, for given $x \to 0$, there exists $n' \gg 1$ such that $xn' \ll 1$ but still large enough so that $\mathrm{e}^{-n'xd}\cosh(x\sqrt{n'}) = \left[\mathrm{e}^{-(n'd-\sqrt{n'})x} + \mathrm{e}^{-(n'd+\sqrt{n'})x}\right]/2 \sim \mathrm{e}^{-n'xd}$. Substituting this formula in (3.22), one obtains (for $x \to 0$)

$$
\begin{aligned}
Z &= \mathrm{e}^{-xd/2}\left[-1 + 2\left(\sum_{n=0}^{n'}\mathrm{e}^{-nxd}\cosh(x\sqrt{n}) + \sum_{n=n'}^{\infty}\mathrm{e}^{-nxd}\cosh(x\sqrt{n})\right)\right] \\
&\approx -1 + 2\left[(n'+1) - \sum_{n=0}^{n'-1}\mathrm{e}^{-nxd} + \sum_{n=0}^{\infty}\mathrm{e}^{-nxd}\right] \\
&\approx -1 + 2\left[(n'+1) - n' + \sum_{n=0}^{\infty}(\mathrm{e}^{-xd})^n\right] \\
&= 1 + \frac{2}{1 - \mathrm{e}^{-xd}} \approx \frac{2}{xd} \ ,
\end{aligned}
$$

since $\mathrm{e}^{-xd} \approx 1 - xd$ for $x \to 0$. Substituting this $Z$ into (3.25) and using (3.18), we get

$$
\begin{aligned}
N(\rho_{\text{af}}) &= \frac{xd\mathrm{e}^{-n_-(x)xd}}{4}\left|1 - \frac{x\mathrm{e}^{2x\sqrt{n_-(x)}}}{16n_-^{3/2}(x)}\right| \\
&\approx \frac{xd\mathrm{e}^{-n_-(x)xd}}{4}\left|1 - \frac{1}{16n_-^{3/2}(x)}\right| \\
&\approx \frac{xd\mathrm{e}^{-n_-(x)xd}}{4} \ ,
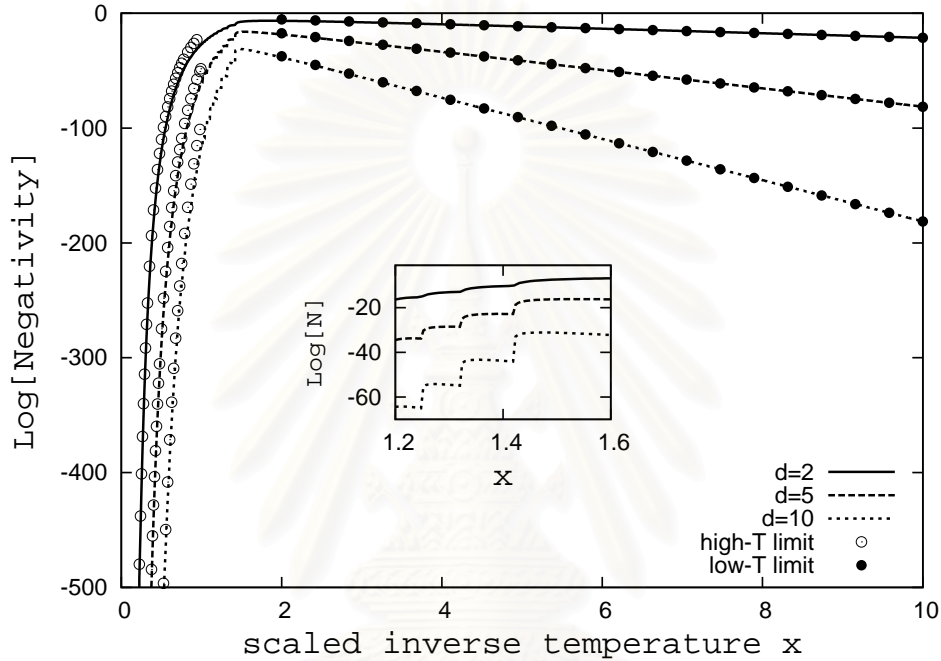\end{aligned}
$$

Figure 3.3: Numerical computation of the logarithm of the negativity $N[\rho_{\mathrm{af}}]$ of the thermal state for the JCM as a function of scaled inverse temperature $x = \hbar|g|/kT$ for the case of resonance $\delta = 0$ at three different scaled frequencies $d = \Omega/|g| = 2, 5, 10$. Negativity is maximal for thermal energies close to the (vacuum) coupling energy, i.e. $x \approx 1.5$. In both the high-temperature ($x \to 0$) and low-temperature ($x \to \infty$) limit the negativity approaches zero. Both these limiting cases are well described by simple analytical expressions (empty circle for small $x$ and full circles for large $x$), see text. Note that near the maximum the negativity displays step-type behavior near $x = 1.42, x = 1.32, x = 1.25, \ldots$ which reflects the discrete nature of the quantized field (see inset and text).

since $n_-(x) \to \infty$ as $x \to 0$. Taking *log* to this formula leads to $\log[N(\rho_{\mathrm{af}})] \approx -n_-(x)xd + \log(xd) - \log(4)$ for $x \to 0$. As we know that $n_-(x) \to \infty$ as $x \to 0$, it happens that $n_-(x)$ increases very rapidly (as $x$ goes to zero) so that $-n_-(x)xd$ is a dominant term in $\log[N(\rho_{\mathrm{af}})]$ and $\log[N(\rho_{\mathrm{af}})]$ is then enough to be written as

$$\log[N(\rho_{\mathrm{af}})] \approx -n_-(x)xd \quad \text{for } x \to 0. \tag{3.26}$$

Unlike the simple formula (3.23) for the large $x$ case in (3.26), where $n_-(x)$ can be approximated by $y_0(x)$ for small $x$, in this case $n_-(x)$ has to be determined from the transcendental equation $f^{\mathrm{as}}_{n_-(x)}(x) = 0$. To see how precise (3.23) and (3.26) are, let us look at Fig. 3.3, a numerical computation of (the logarithm of) the negativity $N(\rho_{\mathrm{af}})$ is displayed as a function of scaled inverse temperature $x$ for three different values of scaled frequency $d$. The solid line represents $d = 2$, the dashed line is $d = 5$, and the dotted line $d = 10$. Qualitatively, the three curves follow the same pattern. Negativity (and thus entanglement) is maximal for values of $x \approx 1.5$, i.e., when the thermal energy is of the same order as the (vacuum) coupling energy. In both high ($x \to 0$) - and low ($x \to \infty$) - temperature limit, negativity approaches zero. The high-temperature case displays a more drastic loss of negativity. This decay of the negativity (in these both temperature limits) can be described as follows. For the low-temperature ($x \to \infty$) limit, the lower the temperature, the smaller is the Boltzmann weight of those eigenvalues which implies that, eventually, when the temperature approaches zero, only the (separable) ground state $|b0\rangle$ is populated and the negativity approaches zero as $x \to \infty$. For high temperature ($x \to 0$) limit, the decay of the negativity has quite a different origin. Here, fewer and fewer eigenvalues $\lambda_n^{(-)}(x)$ contribute to $N(\rho_{\mathrm{af}})$ so that despite a growing thermal occupation of the levels, the negativity due to $\left|\lambda_n^{(-)}(x)\right|$ only occurs for larger and larger values of $n$. Both asymptotic behaviors described by (3.23) for the low-temperature limit and (3.26) for the high-temperature limit are plotted in Fig 3.3 for a few values of $x$ and $d$ shown by *full circles* and *empty*

*circles* respectively. It is clear that both asymptotic formulae are in very good agreement with fully numerical computation of $\log[N(\rho_{\text{af}})]$. Note that, for the empty circles, we used the asymptotic equation $f^{\text{as}}_{n_-(x)}(x) = 0$ from (3.18) to determine $n_-(x)$ numerically. Note also that both asymptotic formulae are more accurate for bigger $d$ so, unlike (3.26) where $N(\rho_{\text{af}})$ rapidly goes to zero, we keep $\log(4)$ term in (3.23) in order to get an accurate enough result for the $d = 2$ case. Of course, for the case of $d = 5, 10$ or higher alone, the $\log(4)$ term can also be neglected (with an acceptable accuracy) as we have done for obtaining (3.26).

Another interesting behavior at the negativity is the *step-type* behavior which occurs near the maximum $x \approx 1.5$ (see inset of Fig. 3.3). In order to understand it, note first that $x_0(0) \approx 1.42$ is the first zero (starting from large $x$) for $f_n(x)$, $n = 0, 1, 2, ...$ (see Fig 3.1). For $x > x_0(0) \approx 1.42$, all eigenvalues $\lambda_n^{(-)}(x)$ ($n = 0, 1, 2, ...$) are all negative and contribute to the negativity. For $x < x_0(0)$, the first eigenvalue becomes positive (since $f_0(x) > 0$ implies $B_0 A_1 - C_0^2 > 0$ so $\lambda_0^{(-)}(x) > 0$ )and no longer contributes to negativity $N(\rho_{\text{af}})$ which explains the abrupt change near $x = x_0(0)$. By the same argument, at the second zero $x_0(1) \approx 1.32$, the second eigenvalue $\lambda_1^{(-)}(x)$ turns from negative to positive and no longer contributes to $N(\rho_{\text{af}})$ which again explains the abrupt change near $x = x_0(1)$. The other abrupt changes are clearly explained by the same mechanism applied near the third zero $x_0(2) \approx 1.25$, the fourth zero $x_0(3) \approx 1.19$, and so on. Since this step-type behavior comes directly from the discreteness of the eigenvalues, it clearly reflects the discrete nature of the quantized field.

# Chapter IV
## Conclusion and Discussion

In this thesis, we have examined various aspects of the Jaynes-Cummings model (JCM). In particular, we have obtained the following results:

1. For three initial atom-field states, we have shown, in contrast to the general case of black-body radiation, there is a direct correspondence between the large average photon-number limit and the semiclassical approximation in which the field is treated classically and only the atom treated by quantum mechanics. These three initial states are:

   (a) The initial state $|\Psi(0)\rangle = |\phi\rangle \otimes |\varphi\rangle$, where $|\phi\rangle = C_a |a\rangle + C_b |b\rangle$ is the general linear combination atom state and $|\varphi\rangle \in \mathcal{C}oh(\beta)$ is any field state belonging to the class of coherent-like states called *coherent class* denoted by $\mathcal{C}oh(\beta) := \left\{ |\gamma\rangle \mid \langle n|\gamma\rangle = \sqrt{P_n}\, e^{in\beta} \,\forall n \in \mathcal{N}, \text{ where the probability } P_n \text{ has the property that } \bar{n}^3 \gg (\Delta n)^4 \right\}$. In the limit of very large average photon number (i.e., $\bar{n}\hbar\Omega = E_{\mathrm{cl}}$), $P_a(t)$ calculated from this $|\Psi(0)\rangle$ goes to semiclassical probability $P_a^{(\mathrm{sc})}(t)$ obtained from the semiclassical model with the initial phase of the classical electric field equal to $\beta$;

   (b) The initial state $|\Psi(0)\rangle = |\phi\rangle \otimes |N\rangle$, where $|\phi\rangle = C_a |a\rangle + C_b |b\rangle$ and $|N\rangle$ is the number state with fixed $N$ photons. In the limit of large photon number (i.e., $N\hbar\Omega = E_{\mathrm{cl}}$), $P_a(t) \approx P_a^{(\mathrm{sc})}(t)$ if (i) $C_a = 0$ or $C_b = 0$, or (ii) $\delta = 0$ with the initial phase $\phi = (\theta_a - \theta_b) \pm n\pi$ $(n = 0, 1, 2, \ldots)$, where $\theta_a = Arg(C_a)$ and $\theta_b = Arg(C_b)$;

   (c) The initial state $|\Psi(0)\rangle = C_{aN}(0) |aN\rangle + C_{bN+1}(0) |bN+1\rangle$ in the two-dimensional subspace $\mathcal{P}_n$. For $\bar{n}\hbar\Omega = E_{\mathrm{cl}}$, $P_a(t) \approx P_a^{\mathrm{sc}}(t)$ if (i) $C_a = 0$

or $C_b = 0$, or (ii) the initial phase $\phi = \pm 2n\pi$ ($n = 0, 1, 2, \ldots$), i.e., initially the classical e.m. field has only the electric field component.

This three initial states show that the initial mixture of the two-level atom states is very important for the rules of classical correspondence. If a two-level atom is initially prepared in the mixed state (between upper and lower states), the initial phase $\phi$ of the classical field plays an important roles. In contrast, if it is initially prepared in the upper or lower state, the initial phase $\phi$ is not important and we can group all these three initial states into a more compact form by defining the similar but bigger class $\mathcal{C}oh := \left\{ |\gamma\rangle \mid |\langle n|\gamma\rangle|^2 = P_n \right.$, where the probability $P_n$ has the property that $\bar{n}^3 \gg (\Delta n)^4 \left.\right\}$, which is the same definition of $\mathcal{C}oh(\beta)$ without the restriction of phase structure. By this definition, the correspondence rules between the JCM and semiclassical model is simple. Let the initial state be $|\Psi(0)\rangle = |\phi\rangle \otimes |\varphi\rangle$, where $|\phi\rangle = |a\rangle$ or $|b\rangle$, and $|\varphi\rangle \in \mathcal{C}oh$. Then, independent of the classical initial phase $\phi$, $P_a(t) \approx P_a^{\text{sc}}(t)$ when $\bar{n}\hbar\Omega = E_{\text{cl}}$. It would be interesting to extend or generalize these case studies in order to find the general theorems of this correspondence.

2. By calculating the thermal atom-field density matrix the ppt-criterion and negativity showed that, even though the entanglement goes to zero in the limits of zero and infinite temperature, at any finite temperature the atom (matter) and the e.m. field (light) are always entangled, i.e., the classical separable behavior is not attained.

It would be interesting to extend these investigations to more general states in the framework of the JCM and also to include dynamics. In particular, it is an open question whether ppt entangled state exist in the context of coupled few-level and ("Gaussian") continuous variable systems.

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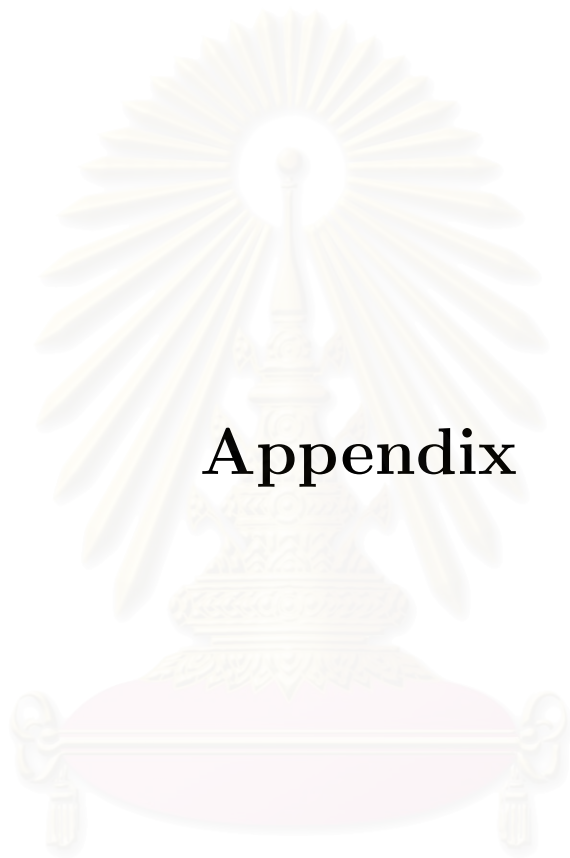
# Appendix

# Appendix

## Asymptotic Formulae of $f_n(x)$

In this appendix, we derive the asymptotic formulae of $f_n(x)$ introduced in Chapter 3. Applying $\cosh^2(x\sqrt{n+1}) - \sinh^2(x\sqrt{n+2}) = 1$ to (3.17), we get

$$f_n(x) = 1 + [\cos(x\sqrt{n})\cosh(x\sqrt{n+2}) - \cosh^2(x\sqrt{n+1})]. \qquad (1)$$

For fixed $x$ and large (enough) $n$, $\cosh(x\sqrt{n}) \sim \frac{e^{x\sqrt{n}}}{2}$, $\cos h(x\sqrt{n+2}) \sim \frac{e^{x\sqrt{n+2}}}{2}$, and $\cosh^2(x\sqrt{n+1}) \sim \frac{e^{2x\sqrt{n+1}}}{4}$ so we have

$$\cosh(x\sqrt{n})\cosh(x\sqrt{n+2}) - \cosh^2(x\sqrt{n+1})$$

$$\sim \frac{e^{x(\sqrt{n+2}+\sqrt{n})} - e^{2x\sqrt{n+1}}}{4}$$

$$= -\frac{e^{2x\sqrt{n+1}}}{4}\left[1 - e^{x(\sqrt{n+2}+\sqrt{n}-2\sqrt{n+1})}\right]. \qquad (2)$$

Consider

$$\sqrt{n+2} + \sqrt{n} - 2\sqrt{n+1}$$

$$= \sqrt{n}\left(1 + \sqrt{1+\frac{2}{n}} - 2\sqrt{1+\frac{1}{n}}\right)$$

$$= \sqrt{n}\left[1\left(1 + \frac{1}{n} - \frac{1}{2n^2} + \dots\right) - 2\left(1 + \frac{1}{2n} - \frac{1}{8n^2} + \dots\right)\right]$$

$$= -\frac{1}{4n^{3/2}} + O(1/n^{5/2}) \approx -\frac{1}{4n^{3/2}} \qquad (3)$$

for large $n$. Substituting (3) into (2) and then (2) into (1), we obtain

$$f_n(x) \sim 1 - \frac{e^{2x\sqrt{n+1}}}{4}(1 - e^{-x/4n^{3/2}}).$$

Since, for large (enough) $n$, $e^{-x/4n^{3/2}} \approx 1 - \frac{x}{4n^{3/2}}$, the above equation becomes

$$f_n(x) \sim f_n^{\mathrm{as}}(x) := 1 - \frac{xe^{2x\sqrt{n+1}}}{16n^{3/2}} \qquad (4)$$

for fixed $x$ and large (enough) $n$. Note that the above $n + 1$ in the square root in the exponent of (4) can also be approximated by $n$ but we keep it in order to get a more accurate result (than without keeping it) if $f_n^{\text{as}}(x)$ is supposed to be used for not so large $n$. Next, in order to derive the asymptotic form of $f_n(x)$ for fixed $n$ and large (enough) $x$, notice that $\sqrt{n + 2} + \sqrt{n} - 2\sqrt{n + 1} < 0$ for all $n = 0, 1, 2, \ldots$. Hence, it is clear now from (2) and (1) that

$$f_n(x) \sim -\frac{e^{2x\sqrt{n+1}}}{4} \tag{5}$$

for fixed $n$ and large (enough) $x$.

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