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SEGAL-BARGMANN TRANSFORM ON SPHERES

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เ ราศึกษาการแปลงซีกัล-บาร์กมันน์บนทรงกลม d มิติ S^d ซึ่งส่งค่าจาก $L^2(S^d)$ ไปยัง ฟังก์ชันโฮโลโมฟิกบน ทรงกลมเชิงซ้อน $S^d_{\mathbb{C}}$ ในการวิจัยครั้งนี้ เราได้พิสูจน์ว่าการแปลงซีกัล-บาร์กมันน์เป็นฟังก์ชันสมมติจาก $L^2(S^d)$ ไปทั่วถึง ปริภูมิของฟังก์ชันโฮโลโมฟิกบน $S^d_{\mathbb{C}}$ ซึ่งสามารถอินทิเกรตกำลังสองเทียบกับเมเชอร์บางชนิดได้

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We study the Segal-Bargmann transform on the d-dimensional sphere S^d, mapping $L^2(S^d)$ into holomorphic functions on the complexification $S^d_{\mathbb{C}}$. In this work, we prove that the Segal-Bargmann transform is an isometry from $L^2(S^d)$ onto the space of holomorphic functions on $S^d_{\mathbb{C}}$ which are square integrable with respect to a certain measure.

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Chapter 1

Introduction

The Segal-Bargmann transform is a transform which is widely studied by physicists. It is used for describing the wave-particle phenomena in quantum field theory. The Segal-Bargmann transform for $\mathbb{R}^d$ is a unitary map $C_t : L^2(\mathbb{R}^d) \to \mathcal{H}L^2(\mathbb{C}^d, \nu_t)$ defined by

$$C_t f(z) = \int_{\mathbb{R}^d} (2\pi t)^{-\frac{d}{2}} e^{-(z-x)^2/2t} f(x)\, dx, \quad z \in \mathbb{C}^d,$$

where $(z-x)^2 = (z_1 - x_1)^2 + (z_2 - x_2)^2 + \cdots + (z_d - x_d)^2$ and $\mathcal{H}L^2(\mathbb{C}^d, \nu_t)$ denotes the space of holomorphic functions that are square integrable with respect to the measure

$$\nu_t(z) = (2\pi t)^{-\frac{d}{2}} e^{-|\mathrm{Im} z|^2/2t}.$$

There are a lot of generalizations of the Segal-Bargmann transform on $\mathbb{R}^d$ to more general settings. In 1993, Hall ([3]) has obtained a generalization of the Segal-Bargmann transform on a compact Lie group which is geometric in nature and keeps more of a structure of the original Segal-Bargmann transform. In his work, the space $\mathbb{R}^d$ is replaced by a connected compact Lie group K and $\mathbb{C}^d$ is replaced by the complexification $K_{\mathbb{C}}$ of K. Later, the Segal-Bargmann transform was also extended by Stenzel ([7]) to the case of compact symmetric spaces . His proof relies on heavy machinery in theory of symmetric spaces. In this thesis we study the Segal-Bargmann transform on the d-dimensional sphere S^d, which is a special case of a compact symmetric space. However, we define and prove everything explicitly using only elementary methods. Indeed, we follow the outline already given by Hall and Mitchell ([4]). We provide necessary backgrounds and give the proofs in complete detail.

Chapter 2

Complex sphere

The d-dimensional sphere is the subset of $\mathbb{R}^{d+1}$ given by

$$S^d = \{x \in \mathbb{R}^{d+1} \mid x_1^2 + \cdots + x_{d+1}^2 = 1\}.$$

We naturally define the complexified sphere $S^d_{\mathbb{C}}$ to be

$$S^d_{\mathbb{C}} = \{z \in \mathbb{C}^{d+1} \mid z_1^2 + \cdots + z_{d+1}^2 = 1\}.$$

Then $S^d_{\mathbb{C}}$ is a d-dimensional complex manifold. We will show that we can identify $S^d_{\mathbb{C}}$ with the cotangent bundle $T^*(S^d)$ of S^d, which is defined by

$$T^*(S^d) = \{(\mathbf{x}, \mathbf{p}) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \mid |\mathbf{x}| = 1, \mathbf{x} \cdot \mathbf{p} = 0\}.$$

Notice that $T^*(S^d)$ is a $2d$-dimensional real manifold. If we view $S^d_{\mathbb{C}}$ as a $2d$-dimensional real manifold, then we can identify these two manifolds together via the following map

$$\mathbf{a}(\mathbf{x}, \mathbf{p}) = (\cosh p)\mathbf{x} + i\frac{\sinh p}{p}\mathbf{p}$$

where $(\mathbf{x}, \mathbf{p}) \in T^*(S^d)$ and $p = |\mathbf{p}|$. Since $\lim_{p\to 0} \frac{\sinh p}{p} = 1$, it is well-defined when $p = 0$. First note that

$$\begin{aligned}
\mathbf{a}(\mathbf{x}, \mathbf{p}) \cdot \mathbf{a}(\mathbf{x}, \mathbf{p}) &= \sum_{k=1}^{d+1} \left(x_k^2 \cosh^2 p - \frac{p_k^2}{p^2} \sinh^2 p \right) + i \left(\frac{2 p_k x_k}{p} \sinh p \cosh p \right) \\
&= \cosh^2 p - \sinh^2 p \\
&= 1.
\end{aligned}$$

This shows that $\mathbf{a}$ maps $T^*(S^d)$ into $S^d_{\mathbb{C}}$. To prove that it is injective, let $(\mathbf{x}, \mathbf{p})$, $(\mathbf{y}, \mathbf{q}) \in T^*(S^d)$ be such that

$$\mathbf{a}(\mathbf{x}, \mathbf{p}) = \mathbf{a}(\mathbf{y}, \mathbf{q}).$$

Then

$$x_k \cosh p + i p_k \frac{\sinh p}{p} = y_k \cosh q + i q_k \frac{\sinh q}{q},$$

i.e.,

$$x_k \cosh p = y_k \cosh q \qquad \text{and} \qquad p_k \frac{\sinh p}{p} = q_k \frac{\sinh q}{q}$$

for $k = 1, \ldots, d + 1$. This implies that

$$\mathbf{x}^2 \cosh^2 p = \mathbf{y}^2 \cosh^2 q \qquad \text{and} \qquad \sinh^2 p = \sinh^2 q.$$

Consequently $\sinh p = \pm \sinh q$. Since $p, q \geq 0$, it follows that $\sinh p = \sinh q$ and that $p = q$. Hence $x_k = y_k$ and $p_k = q_k$ for every k.

Next, we show that $\mathbf{a}$ is surjective. Let $\mathbf{z} \in S^d_{\mathbb{C}}$ and write $\mathbf{z} = \mathbf{r} + i\mathbf{s}$, where $\mathbf{r}, \mathbf{s} \in \mathbb{R}^{d+1}$. Choose

$$\mathbf{p} = \frac{\sinh^{-1} |\mathbf{s}|}{|\mathbf{s}|} \mathbf{s} \qquad \text{and} \qquad \mathbf{x} = \frac{\mathbf{r}}{\cosh p}. \tag{2.1}$$

Then

$$\frac{\sinh p}{p}\mathbf{p} = \frac{\sinh(\sinh^{-1}|\mathbf{s}|)}{\sinh^{-1}|\mathbf{s}|}\ \sinh^{-1}|\mathbf{s}|\ \frac{\mathbf{s}}{|\mathbf{s}|} = \mathbf{s} \quad \text{and} \quad \cosh p\ \mathbf{x} = \mathbf{r}.$$

Hence $\mathbf{a}(\mathbf{x}, \mathbf{p}) = \mathbf{z}$. From (2.1), we can see that

$$\mathbf{a}^{-1}(\mathbf{r} + i\mathbf{s}) = \Big(\frac{\mathbf{r}}{\cosh p}, \frac{\sinh^{-1}|\mathbf{s}|}{|\mathbf{s}|}\mathbf{s}\Big).$$

It is clear that $\mathbf{a}$ and $\mathbf{a}^{-1}$ are smooth functions. Hence $\mathbf{a}$ is a diffeomorphism.

Next we will show that the smooth manifold S^d is diffeomorphic to the homogeneous manifold $SO(d+1)/SO(d)$. Let us recall some definitions and theorems about homogeneous manifolds first.

Definition 2.1. Let $\eta : G \times M \to M$ be an action of G on M on the left. As usual, we write

$$\eta(g, m) = g \cdot m.$$

The action is called *transitive* if whenever m and n belong to M there exists a g in G such that $g \cdot m = n$. For $x_0 \in M$, the set

$$G_{x_0} = \{g \in G \mid g \cdot x_0 = x_0\}$$

forms a closed subgroup of G called the *isotropy group at* x_0.

If G is a Lie group and H is a closed subgroup of G, then we can define a differentiable structure on the quotient space G/H so that it is a smooth manifold, called a *homogeneous manifold.* Moreover, there is a natural transtive left-action of G on G/H. Conversely, if M is a smooth manifold and there is a transitive left-action by a Lie group G on M, then M can be identified with the quotient manifold G/G_{x_0}, where x_0 is a point in M. This is summarized in the following theorem.

Theorem 2.2 ([9], Theorem 3.62). *Let $\eta : G \times M \to M$ be a transitive left-action of the Lie group G on the manifold M. Let $x_0 \in M$, and let H be the isotropy group at x_0. Define a mapping $\beta : G/H \to M$ by $\beta(gH) = g \cdot x_0$. Then β is a diffeomorphism.*

Let $\mathbb{F}$ be the field $\mathbb{R}$ and $\mathbb{C}$. For $d \geq 1$, we define the *special orthogonal group* $SO(d, \mathbb{F})$ to be the set of $d \times d$ matrices A such that $A \cdot A^t = I$ and $\det A = 1$. Equivalently, $SO(d, \mathbb{F})$ is the set of $d \times d$ matrices A such that $\det A = 1$ and $[Ax, Ay] = [x, y]$ for all $x, y \in \mathbb{F}$, where $[x, y] = \sum_{i=1}^{d} x_i y_i$ for $x, y \in \mathbb{F}^d$.
In case $\mathbb{F} = \mathbb{R}$, we may simply write $SO(d, \mathbb{R})$ as $SO(d)$.

Proposition 2.3. *The manifold S^d is diffeomorphic to $SO(d+1)/SO(d)$.*

Proof. Let $\{e_i |\ i = 1, \ldots, d+1\}$ be the canonical basis of $\mathbb{R}^{d+1}$ where e_i is the $d+1$-tuple consisting of all zeroes except for a 1 in the i-th spot. Define an action $\eta : SO(d+1) \times S^d \to S^d$ by multiplying $A \in SO(d+1)$ to a vector in S^d :

$$\eta(A, x) = Ax.$$

It is obvious that this action is transitive and the isotropy group at e_{d+1} is the set of matrices in $SO(d+1)$ of the form

$$\mathbf{X} = \begin{pmatrix} & & 0 \\ & B & \vdots \\ & & \\ 0 & \ldots & 1 \end{pmatrix}.$$

The matrix B occuring in this subgroup is precisely the matrix in $SO(d)$. Hence, we identify this isotropy group with $SO(d)$. It follows from Theorem 2.2 that the homogeneous manifold $SO(d+1)/SO(d)$ is diffeomorphic to S^d. $\square$

Similarly, we can show that $S^d_{\mathbb{C}}$ is diffeomorphic to $SO(d+1,\mathbb{C})/SO(d,\mathbb{C})$ as complex manifolds.

Chapter 3

Spherical harmonics

We would like to represent a function defined on the surface of the unit sphere by an expansion similar to a Fourier series by a class of functions called the *spherical harmonics*. For more details, a reader is referred to [1] and [8].

A function f defined on $\mathbb{R}^d$ is said to be *homogeneous of degree* k if

$$f(ax) = a^k f(x) \qquad \text{for any } x \in \mathbb{R}^d \text{ and any } a > 0.$$

Let $\mathcal{P}_k(\mathbb{R}^d)$ be the set of all homogeneous polynomials of degree k on $\mathbb{R}^d$. If $P \in \mathcal{P}_k(\mathbb{R}^d)$, then it can be written in the form

$$P(x) = \sum_{|\alpha|=k} c_\alpha x^\alpha,$$

where α denotes a d-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_d)$ of nonnegative integers, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$, $x^\alpha = x_1^{\alpha_1} \ldots x_d^{\alpha_d}$ and $c_\alpha \in \mathbb{C}$.

It is clear that the set of monomials $\{x^\alpha : |\alpha| = k\}$ is a basis for this space. Then $\dim \mathcal{P}_k(\mathbb{R}^d)$ equals the number of distinct multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ with

$|\alpha| = k$. Hence

$$\dim \mathcal{P}_k(\mathbb{R}^d) = \binom{d+k-1}{d-1}.$$

We introduce an inner product $\langle P, Q\rangle$ on $\mathcal{P}_k$ by letting $\langle P, Q\rangle = P(D)\overline{Q}$ for all P, Q in $\mathcal{P}_k$, where $P(D)$ is the differential operator in which we replace $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_d^{\alpha_d}$ by $\dfrac{\partial^{\alpha_1+\cdots+\alpha_d}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_d^{\alpha_d}}$. Since P and Q are homogeneous polynomials of the same degree, $\langle P, Q\rangle$ is scalar-valued. It is clearly linear in the first variable, conjugate linear in the second variable and hermitian symmetric. To verify that it is an inner product, it is enough to show that $\langle P, P\rangle \geq 0$, with equality only if $P = 0$. If $\alpha \neq \beta$, then

$$\Big(\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}\Big) x_1^{\beta_1} x_2^{\beta_2} \ldots x_d^{\beta_d} = 0.$$

When $\alpha = \beta$, this derivative equals $\alpha_1!\alpha_2!\ldots\alpha_d! = \alpha!$. Consequently, if $P(x) = \sum_{|\alpha|=k} c_\alpha x^\alpha$, then $\langle P, P\rangle = \sum_{|\alpha|=k} |c_\alpha|^2 \alpha!$. But this last expression vanishes if and only if all the coefficients c_α are 0.

Theorem 3.1. *If $P \in \mathcal{P}_k$, then*

$$P(x) = P_0(x) + |x|^2 P_1(x) + \cdots + |x|^{2l} P_l(x),$$

where P_j is a homogeneous harmonic polynomial of degree $k-2j$, for $j = 0, 1, \ldots, l$.

Proof. Any polynomial of degree less than 2 is harmonic. Thus we may assume that $k \geq 2$. Consider the linear mapping $\varphi : \mathcal{P}_k \to \mathcal{P}_{k-2}$ defined by letting $\varphi(P) = \Delta P$ for $P \in \mathcal{P}_k$, where Δ is the Laplace operator on $\mathbb{R}^d$. We first show that φ maps $\mathcal{P}_k$ onto $\mathcal{P}_{k-2}$. If this were not the case we could find a nonzero $Q \in \mathcal{P}_{k-2}$ that is orthogonal to Range(φ). That is

$$\overline{\langle \Delta P, Q\rangle} = \langle Q, \Delta P\rangle = 0, \quad \text{for all } P \in \mathcal{P}_k.$$

In particular, this must be true for $P(x) = |x|^2 Q(x)$. Thus

$$0 = \langle Q, \Delta P \rangle = Q(D)\overline{\Delta P} = \Delta Q(D)\overline{P} = P(D)\overline{P} = \langle P, P \rangle .$$

But this is impossible since $P \neq 0$.

Let $\mathcal{A}_j \subseteq \mathcal{P}_j, j \geq 2$, be the class of all harmonic polynomials in $\mathcal{P}_j$ and $\mathcal{B}_j = |x|^2 \mathcal{P}_{j-2}$.

We claim that $\mathcal{P}_j$ is the orthogonal direct sum of $\mathcal{A}_j$ and $\mathcal{B}_j$.

$$\begin{aligned} \left\langle |x|^2 Q, P \right\rangle \text{ for all } Q \in \mathcal{P}_{j-2} &\Leftrightarrow Q(D)\Delta\overline{P} = 0 \text{ for all } Q \in \mathcal{P}_{j-2} \\ &\Leftrightarrow \langle Q, \Delta P \rangle = 0 \text{ for all } Q \in \mathcal{P}_{j-2} \\ &\Leftrightarrow \Delta P = 0. \end{aligned}$$

In particular, for $j = k$ and $P \in \mathcal{P}_k$ we have $P(x) = P_0(x) + |x|^2 Q(x)$ with P_0 harmonic and $Q \in \mathcal{P}_{k-2}$. It is clear that the desired statement follows by induction.

□

Corollary 3.2. *The restriction to the unit sphere S^{d-1} of any polynomial of d-variables is a sum of restrictions to S^{d-1} of harmonic polynomials.*

The restriction to the unit sphere S^{d-1} of a homogeneous harmonic polynomial of degree k is called a *spherical harmonic of degree k*. We let $\mathcal{H}_k$ denote the space of spherical harmonics of degree k.

Let $\varphi : \mathcal{A}_k \to \mathcal{H}_k$ be defined by

$$\varphi(P) = P|_{S^{d-1}}.$$

It is evident that this map has a trivial kernel. If $Y \in \mathcal{H}_k$, we can choose $P(x) = x^k Y(x/|x|)$ for $x \neq 0$. Then φ is an isomorphism of $\mathcal{A}_k$ onto $\mathcal{H}_k$. Hence,

$$\dim \mathcal{H}_k = \dim \mathcal{A}_k = \dim \mathcal{P}_k - \dim \mathcal{P}_{k-2} = \frac{(d+2k-2)}{k} \binom{d+k-3}{k-1}. \tag{3.1}$$

To prove the next proposition, let us recall Green's theorem.

Theorem 3.3 (Green's theorem). *Let $u, v \in C^2(\overline{U})$, where U is bounded subset of $\mathbb{R}^d$. Then*

$$\int_U (u\Delta v - v\Delta u)\, dV = \int_{\partial U} (u\partial_n v - v\partial_n u)\, ds$$

where ∂_n denotes differentiation with respect to the outward unit normal vector.

Proposition 3.4. *If Y_k and Y_l are spherical harmonics of degree k and l, with $k \neq l$, then*

$$\int_{S^{d-1}} Y_k(x)Y_l(x)\, dx = 0.$$

Proof. By Green's theorem,

$$\int_{S^{d-1}} Y_k\, \partial_n Y_l - Y_l\, \partial_n Y_k\, ds = \int_{|x|\leq 1} (Y_k \Delta Y_l - Y_l \Delta Y_k)\, dx = 0.$$

But then for each $x \in S^{d-1}$,

$$\begin{aligned}(\partial_n Y_k)(x) &= \frac{d}{dr} Y_k(rx)|_{r=1} \\ &= \frac{d}{dr}(r^k Y_k(x))|_{r=1} \\ &= kY_k(x).\end{aligned}$$

Similarly, $\partial_n Y_l = lY_l$. Thus

$$(l-k)\int_{S^{d-1}} Y_k(x)Y_l(x)\, dx = 0.$$

Since $l \neq k$, the last integral vanishes, as desired. □

Define $L^2(S^{d-1})$ to be the Hilbert space of square-integrable functions on the $(d-1)$-dimensional sphere S^{d-1} with respect to surface measure dx. Then each $\mathcal{H}_k$ is a subspace of $L^2(S^{d-1})$. Moreover, we have

Theorem 3.5. $L^2(S^{d-1}) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$.

Proof. $L^2(S^{d-1}) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$ is true when the following three conditions are satisfied:

1. $\mathcal{H}_k$ is a closed subspace of $L^2(S^{d-1})$ for each k;
2. $\mathcal{H}_k$ is orthogonal to $\mathcal{H}_l$ for $k \neq l$;
3. For every $f \in L^2(S^{d-1})$, there exists a sequence (f_m), where $f_m \in \mathcal{H}_m$ for each m, such that

$$f = f_0 + f_1 + \ldots,$$

   where the sum converges in the norm of $L^2(S^{d-1})$.

Condition 1 above holds because each $\mathcal{H}_k$ is finite dimensional and hence is closed in $L^2(S^{d-1})$. Condition 2 follows from the Proposition 3.4.

To prove condition 3, we only need to show that the linear span of $\cup_{k=0}^{\infty}\mathcal{H}_k$ is dense in $L^2(S^{d-1})$. As we have already noted from the Corollary 3.2 that if P is a polynomial on $\mathbb{R}^d$, then $P|_{S^{d-1}}$ can be written as a finite sum of elements of $\cup_{k=0}^{\infty}\mathcal{H}_k$. By the Stone-Weierstrass theorem, the set of restrictions $P|_{S^{d-1}}$, as P ranges over all polynomials on $\mathbb{R}^d$, is dense in $C(S^{d-1})$ with respect to the supremum norm. Because $C(S^{d-1})$ is dense in $L^2(S^{d-1})$ and the L^2-norm is less than or equal to the supremum norm on S^{d-1}, this implies that the linear span of $\cup_{k=0}^{\infty}\mathcal{H}_k$ is dense in $L^2(S^{d-1})$ as desired. □

If $\{Y_{1,k}, \ldots, Y_{N_k,k}\}$ is an orthonormal basis of $\mathcal{H}_k$, then it follows from Theorem 3.5 that the collection $\cup_{k=0}^{\infty}\{Y_{1,k}, \ldots, Y_{N_k,k}\}$ is an orthonormal basis of $L^2(S^{d-1})$. Thus, if $f \in L^2(S^{d-1})$ then there exists a unique representation,

$$f = \sum_{k=0}^{\infty}\sum_{n=1}^{N_k}\langle f, Y_{n,k}\rangle Y_{n,k}$$

where the series on the right converges to f in the L^2 norm. Let us fix a point x in $S_{\mathbb{C}}^{d-1}$ and consider the linear functional L on $\mathcal{H}_k$ that assigns to each Y in $\mathcal{H}_k$ the value $Y(x)$. By Riesz representation theorem, there exists a unique spherical harmonic $Z_x^{(k)}$ such that

$$L(Y) = Y(x) = \int_{S^{d-1}} Y(t) Z_x^{(k)}(t)\, dt$$

for any $Y \in \mathcal{H}_k$. This function $Z_x^{(k)}$ is called the *zonal harmonic of degree* k *with pole* x.

Lemma 3.6. *Let* $\{Y_1, Y_2, \ldots, Y_{N_k}\}$ *be an orthonormal basis of* $\mathcal{H}_k$. *Then*

(i) $Z_x^{(k)}(t) = \sum_{m=1}^{N_k} \overline{Y_m(x)} Y_m(t)$;

(ii) *If* ρ *is a rotation ,then* $Z_{\rho x}^{(k)}(\rho t) = Z_x^{(k)}(t)$;

(iii) $\sum_{m=1}^{N_k} |Y_m(x)|^2 = \frac{N_k}{C_{d-1}}$ *where* $C_{d-1} = \int_{S^{d-1}} dx$ *is the total volume of* S^{d-1};

(iv) *If* $Y \in \mathcal{H}_k$, *then* $|Y(x)|^2 \le \frac{N_k}{C_{d-1}} \|Y\|_2^2$, *where* $N_k = \dim \mathcal{H}_k$.

Proof. Since $\{Y_1, Y_2, \ldots, Y_{N_k}\}$ is an orthonormal basis of $\mathcal{H}_k$,

$$Z_x^{(k)} = \sum_{m=1}^{N_k} \langle Z_x^{(k)}, Y_m \rangle Y_m.$$

But by the defining property of zonal harmonics,

$$\langle Z_x^{(k)}, Y_m \rangle = \int_{S^{d-1}} \overline{Y_m(t)} Z_x^{(k)}(t)\, dt = \overline{Y_m(x)}.$$

Then $Z_x^{(k)}(t) = \sum_{m=1}^{N_k} \overline{Y_m(x)} Y_m(t)$. To verify (ii), let $w = \rho t$. We have

$$\int_{S^{d-1}} Z_{\rho x}^{(k)}(\rho t) Y(t)\, dt = \int_{S^{d-1}} Z_{\rho x}^{(k)}(w) Y(\rho^{-1} w)\, dw = Y(\rho^{-1}\rho x) = Y(x).$$

By the uniqueness of zonal harmonic, we have $Z_{\rho x}^{(k)}(\rho t) = Z_x^{(k)}(t)$. To verify (iii), suppose x_1 and x_2 are in S^{d-1}. We can find a rotation ρ in $SO(d)$ such that $\rho x_1 = x_2$. Then

$$Z_{x_2}^{(k)}(x_2) = Z_{x_1}^{(k)}(x_1).$$

Consequently, $Z_x^{(k)}(x)$ is a constant, say c. From (i), we have $c = Z_x^{(k)}(x) = \sum_{m=1}^{N_k} |Y_m(x)|^2$. Since $\|Y_m\|_2 = 1$ for all m,

$$N_k = \sum_{m=1}^{N_k} \int_{S^{d-1}} |Y_m(x)|^2\, dx = \int_{S^{d-1}} \sum_{m=1}^{N_k} |Y_m(x)|^2\, dx = \int_{S^{d-1}} c\, dx = c\, C_{d-1}.$$

Hence $\sum_{m=1}^{N_k} |Y_m(x)|^2 = c = N_k/C_{d-1}$, so (iii) follows. From (i) it implies that

$$\|Z_x^{(k)}\|_2^2 = \int_{S^{d-1}} |Z_x^{(k)}(t)|^2\, dt = \sum_{m=1}^{N_k} |Y_m(x)|^2 = \frac{N_k}{C_{d-1}}.$$

Let $Y \in \mathcal{H}_k$. Then $Y(x) = \int_{S^{d-1}} Y(t) Z_x^{(k)}(t)\, dt$. By Schwarz's inequality,

$$|Y(x)|^2 \leq \|Z_x^{(k)}\|_2^2\, \|Y\|_2^2 \leq \frac{N_k}{C_{d-1}}\, \|Y\|_2^2\,.$$

This establishes (iv). □

Chapter 4

Laplacian on a sphere

We define the spherical Laplacian on S^d to be an operator defined on S^d given by the formula:

$$\Delta_{S^d} = \sum_{k<l} \left(x_k \frac{\partial}{\partial x_l} - x_l \frac{\partial}{\partial x_k} \right)^2.$$

Notice that expressions like $\dfrac{\partial}{\partial x_k}$ do not make sense when applied to a function that is defined only on the sphere S^d. For a smooth function f on S^d, extend f smoothly to a neighborhood of S^d, then apply $x_l \dfrac{\partial}{\partial x_k} - x_k \dfrac{\partial}{\partial x_l}$, and then restrict again to S^d. Note that

$$\left(x_l \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_l} \right) |x|^2 = 2x_l x_k - 2x_k x_l = 0.$$

Thus these derivatives are all in directions tangent to S^d. Therefore the values of the operator on S^d is independent of the choice of extension.

We choose the domain of Δ_{S^d} to be the space H of linear combinations of spherical harmonics on S^d. Since this space is dense in $L^2(S^d)$, it follows that Δ_{S^d} is a densely-defined operator on $L^2(S^d)$. Moreover, Δ_{S^d} is essentially self-adjoint, which we prove it later.

Proposition 4.1. *Let Y be a spherical harmonic of degree k. Then*

$$\Delta_{S^d} Y = -k(k+d-1)Y.$$

Proof.

$$\begin{aligned}
-\Delta_{S^d} &= -\sum_{i<j}\Big(x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i}\Big)^2 \\
&= -\frac{1}{2}\sum_{i\neq j}\Big(x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i}\Big)^2 \\
&= -\frac{1}{2}\sum_{i\neq j}\Big(x_i^2\frac{\partial^2}{\partial x_j^2} + x_j^2\frac{\partial^2}{\partial x_i^2} - x_j\Big(\frac{\partial}{\partial x_j} + x_i\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\Big) - x_i\Big(\frac{\partial}{\partial x_i} + x_j\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\Big)\Big) \\
&= \frac{1}{2}\sum_{i\neq j}\Big(-x_i^2\frac{\partial^2}{\partial x_j^2} - x_j^2\frac{\partial^2}{\partial x_i^2} + x_j\frac{\partial}{\partial x_j} + x_i\frac{\partial}{\partial x_i} + 2x_ix_j\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\Big) \\
&= \sum_{i\neq j} -x_i^2\frac{\partial^2}{\partial x_j^2} + \sum_{i\neq j} x_ix_j\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + \frac{1}{2}\sum_{i\neq j}\Big(x_j\frac{\partial}{\partial x_j} + x_i\frac{\partial}{\partial x_i}\Big).
\end{aligned}$$

For fixed j,

$$\sum_{\substack{i=1\\(i\neq j)}}^{d+1} -x_i^2\frac{\partial^2}{\partial x_j^2} = -(x_1^2 + x_2^2 + \cdots + x_{d+1}^2)\frac{\partial^2}{\partial x_j^2} + x_j^2\frac{\partial^2}{\partial x_j^2}$$

$$\sum_{j=1}^{d+1}\sum_{\substack{i=1\\(i\neq j)}}^{d+1} -x_i^2\frac{\partial^2}{\partial x_j^2} = -|x|^2\Delta + \sum_{j=1}^{d+1} x_j^2\frac{\partial^2}{\partial x_j^2}, \quad \text{where} \quad \Delta = \sum_{j=1}^{d+1}\frac{\partial^2}{\partial x_j^2}.$$

Therefore

$$-\Delta_{S^d} = -|x|^2\Delta + \sum_{j=1}^{d+1} x_j^2\frac{\partial^2}{\partial x_j^2} + \sum_{i\neq j} x_ix_j\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + \frac{1}{2}\sum_{i\neq j}\Big(x_j\frac{\partial}{\partial x_j} + x_i\frac{\partial}{\partial x_i}\Big)$$

Since Y is homogeneous, it is enough to prove the statement for only a harmonic monomial $P = x_1^{n_1} \dots x_{d+1}^{n_{d+1}}$ where $n_1 + \dots + n_{d+1} = k$.

$$\begin{aligned}
-\Delta_{S^d} P &= \Big(\sum_{j=1}^{d+1} n_j(n_j - 1) + 2 \sum_{i<j} n_i n_j + \frac{1}{2} \sum_{i \neq j} (n_i + n_j) \Big) P \\
&= \Big(\sum_{j=1}^{d+1} n_j^2 + 2 \sum_{i<j} n_i n_j - \sum_{j=1}^{d+1} n_j + d \sum_{i=1}^{d+1} n_i \Big) P \\
&= \Big(\Big(\sum_{j=1}^{d+1} n_j \Big)^2 - k + kd \Big) P \\
&= (k^2 - k + kd) P \\
&= k(k + d - 1)P.
\end{aligned}$$

□

To prove essential self-adjointness of Δ_{S^d}, we recall the following theorem. For the definition of self-adjointness, a reader can refer to [6], p 256.

Theorem 4.2 ([6], p 257). *Let T be a densely defined symmetric operator on a Hilbert space H. Then the following statements are equivalent:*

a *T is essentially self-adjoint;*

b $\mathrm{Ker}(T^* \pm i) = 0$;

c $\mathrm{Range}(T \pm i)$ *are dense in H.*

Now we are ready to verify the following proposition.

Proposition 4.3. *The Laplacian Δ_{S^d} is essentially self-adjoint.*

Proof. Firstly we verify that Δ_{S^d} is symmetric. Let

$$X_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}.$$

Let $so(d)$ be defined by $\{A \in M_n(\mathbb{R}) \mid A^t = -A\}$. From the definition of Lie derivative we can write

$$X_{ij}f(x) = \frac{d}{dt}f(e^{-tA_{ij}}x)|_{t=0}, \text{ for some } A_{ij} \in so(d+1).$$

If f, g are smooth functions on S^d, then

$$\begin{aligned}
\langle X_{ij}f, g\rangle &= \int_{S^d} \left[\frac{d}{dt}f(e^{-tA_{ij}}y)\right]|_{t=0}g(y)\,dy \\
&= \frac{d}{dt}\Big[\int_{S^d} f(e^{-tA_{ij}}y)g(y)\,dy\Big]|_{t=0} \quad \text{(by compactness of } S^d) \\
&= \frac{d}{dt}\Big[\int_{S^d} f(y)g(e^{tA_{ij}}y)\,dy\Big]|_{t=0} \quad \text{(since } dy \text{ is rotationally invariant)} \\
&= -\langle f, X_{ij}g\rangle.
\end{aligned}$$

In particular, $\langle \Delta_{S^d} f, g\rangle = \langle f, \Delta_{S^d} g\rangle$, for any spherical harmonics f, g. Thus Δ_{S^d} is symmetric on H. Now to show that Δ_{S^d} is essentially self-adjoint, it is enough to show that

$$\overline{\text{Range}(\Delta_{S^d} + i)} = L^2(S^d, dx).$$

Recall that if Y_k is a spherical harmonic of degree k, then

$$(\Delta_{S^d} + i)Y_k = (-k(k+d-1) + i)Y_k.$$

Let $\{Y_{1,k}, \ldots, Y_{N_k,k}\}$ be an orthonormal basis for $\mathcal{H}_k$. and $f \in L^2(S^d)$. We write

$$\begin{aligned}
f &= \sum_{k=0}^{\infty}\sum_{n=1}^{N_k} \langle f, Y_{n,k}\rangle Y_{n,k} \\
&= \sum_{k=0}^{\infty}\sum_{n=1}^{N_k} \langle f, Y_{n,k}\rangle \frac{(\Delta_{S^d} + i)Y_{n,k}}{(-k(k+d-1) + i)}.
\end{aligned}$$

Hence $f \in \overline{\text{Range}(\Delta_{S^d} + i)}$. This shows that Δ_{S^d} is essentially self-adjoint. □

Now we turn to the Laplacian on a complex sphere. Let J_a^2 and $J_{\bar{a}}^2$ denote the differential operators on $S_{\mathbb{C}}^d$ given by

$$J_a^2 = \sum_{k<l} \left(a_l \frac{\partial}{\partial a_k} - a_k \frac{\partial}{\partial a_l} \right)^2$$

$$J_{\bar{a}}^2 = \sum_{k<l} \left(\bar{a}_l \frac{\partial}{\partial \bar{a}_k} - \bar{a}_k \frac{\partial}{\partial \bar{a}_l} \right)^2.$$

In the same way as Δ_{S^d}, J_a^2 and $J_{\bar{a}}^2$ can be interpreted as operators on $S_{\mathbb{C}}^d$.

Chapter 5

Segal-Bargmann transform

For each point $\mathbf{x} \in S^d$ and $t > 0$, there is the *heat kernel based at* $\mathbf{x}$ denoted by $\rho_t(\mathbf{x}, \cdot)$ with the property that

$$\frac{d}{dt}\rho_t(\mathbf{x}, \mathbf{y}) = \frac{1}{2}\Delta_{S^d}\rho_t(\mathbf{x}, \mathbf{y}), \quad \text{and}$$

$$\lim_{t \to 0^+} \int_{S^d} \rho_t(\mathbf{x}, \mathbf{y}) f(\mathbf{y})\, d\mathbf{y} = f(\mathbf{x}) \text{ for any } f \in C(S^d).$$

Given any function f in $L^2(S^d)$ we define the Segal-Bargmann transform $C_t f$ of f by

$$C_t f(\mathbf{a}) = \int_{S^d} \rho_t(\mathbf{a}, \mathbf{x}) f(\mathbf{x})\, d\mathbf{x}, \quad \mathbf{a} \in S^d_{\mathbb{C}}. \tag{5.1}$$

where ρ_t is the heat kernel on S^d, with $\rho_t(\cdot, \mathbf{x})$ extended by analytic continuation from S^d to $S^d_{\mathbb{C}}$.

It is easy to see that $C_t f$ is a holomorphic function on $S^d_{\mathbb{C}}$. Analogous to the Segal-Bargmann transform on $\mathbb{R}^d$, we expect that C_t maps onto a space of holomorphic functions which are square-integrable with respect to a certain measure on $S^d_{\mathbb{C}}$. We can describe this measure explicitly using the identification $S^d_{\mathbb{C}} \cong T^*(S^d)$ via the map

$$\mathbf{a}(\mathbf{x},\mathbf{p}) = (\cosh p)\mathbf{x} + i\frac{\sinh p}{p}\mathbf{p} \quad \text{for any } (\mathbf{x},\mathbf{p}) \in T^*(S^d).$$

Let ν_t be the function satisfying the following differential equation:

$$\frac{\partial}{\partial t}\nu_t(R) = \frac{1}{2}\left[\frac{\partial^2}{\partial R^2} + (d-1)\frac{\cosh R}{\sinh R}\frac{\partial}{\partial R}\right]\nu_t(R) \tag{5.2}$$

with the initial condition

$$\lim_{t\to 0^+} c_d \int_0^\infty f(R)\nu_t(R)(\sinh R)^{d-1}dR = f(0)$$

for all continuous functions f on $[0,\infty]$. Here c_d is the volume of the unit sphere in $\mathbb{R}^d$. The existence of ν_t is guaranteeed by [2], Section 5.7. Then the desired measure can be written as

$$\nu_{2t}(2p)\left(\frac{\sinh 2p}{2p}\right)^{d-1} 2^d\, d\mathbf{p}\, d\mathbf{x}$$

where $d\mathbf{p}$ is Lebesgue measure on a d-dimensional real vecter space, $p = |\mathbf{p}|$ and $d\mathbf{x}$ is the surface measure on S^d.

In fact,

$$\left(\frac{\sinh 2p}{2p}\right)^{d-1} 2^d\, d\mathbf{p}\, d\mathbf{x}$$

is the (complex) rotationally invariant measure on $S^d_{\mathbb{C}} \cong T^*(S^d)$ and $\nu_{2t}(2p)$ is the density with respect to this measure. We state it in the following proposition.

Proposition 5.1. *The measure*

$$\left(\frac{\sinh\ 2p}{2p}\right)^{d-1} 2^d\, d\mathbf{p}\, d\mathbf{x}$$

is invariant under the action of $SO(d+1,\mathbb{C})$ *on* $S^d_{\mathbb{C}} \cong T^*(S^d)$.

To prove this proposition, we need to use the next lemma. Let $R = 2p$ and let $\alpha = |\mathbf{a}|^2 = \sum |a_k|^2$. Then

$$\alpha := |\mathbf{a}|^2 = \cosh^2 p + \sinh^2 p = \cosh 2p.$$

Since $p \geq 0$, we have $R = 2p = \cosh^{-1}\alpha$. Hence p can be considered as a function of $\mathbf{a}$.

Lemma 5.2. *Let ϕ be a smooth, even, real-valued function on $\mathbb{R}$ and consider the function on $S^d_{\mathbb{C}}$ given by $p \mapsto \phi(2p)$ where p is regarded as a function of $\mathbf{a}$. Then*

$$J^2_{\mathbf{a}}\,\phi(2p) = J^2_{\overline{\mathbf{a}}}\,\phi(2p) = -\Big[\frac{\partial^2\phi}{\partial R^2} + (d-1)\,\frac{\cosh R}{\sinh R}\,\frac{\partial\phi}{\partial R}\Big]_{R=2p}$$

Proof. By Chain Rule, we have

$$\begin{aligned}
\frac{\partial\phi}{\partial a_k}(R) &= \frac{d\phi}{dR}\frac{dR}{d\alpha}\frac{\partial\alpha}{\partial a_k}\\
\frac{dR}{d\alpha} &= -\frac{1}{(1-\alpha^2)^{\frac{1}{2}}} = -\frac{1}{(1-|\mathbf{a}|^4)^{\frac{1}{2}}}\\
\frac{\partial\alpha}{\partial a_k} &= \overline{a}_k.
\end{aligned}$$

Then

$$\begin{aligned}
\left(a_l\frac{\partial}{\partial a_k} - a_k\frac{\partial}{\partial a_l}\right)\phi(R) &= a_l\frac{d\phi}{dR}\Big(-\frac{1}{(1-|\mathbf{a}|^4)^{\frac{1}{2}}}\Big)\overline{a}_k - a_k\frac{d\phi}{dR}\Big(-\frac{1}{(1-|\mathbf{a}|^4)^{\frac{1}{2}}}\Big)\overline{a}_l\\
&= (a_k\overline{a}_l - a_l\overline{a}_k)\frac{d\phi}{dR}\Big(\frac{1}{(1-|\mathbf{a}|^4)^{\frac{1}{2}}}\Big).
\end{aligned}$$

Consider

$$\begin{aligned}
&\frac{\partial}{\partial a_k}(a_k\overline{a_l} - a_l\overline{a}_k)\frac{d\phi}{dR}\Big(\frac{1}{(1-|\mathbf{a}|^4)^{\frac{1}{2}}}\Big)\\
&= \frac{d\phi}{dR}\Big(\frac{-1}{(1-|\mathbf{a}|^4)^{\frac{1}{2}}}\Big)\overline{a_l} + (a_k\overline{a}_l - a_l\overline{a}_k)\frac{d\phi}{dR}\frac{\partial}{\partial a_k}\Big(\frac{1}{(1-|\mathbf{a}|^4)^{\frac{1}{2}}}\Big) + \Big(\frac{1}{(1-|\mathbf{a}|^4)^{\frac{1}{2}}}\Big)\frac{\partial}{\partial a_k}\frac{d\phi}{dR}\\
&= \frac{d\phi}{dR}\Big(\frac{1}{(1-|\mathbf{a}|^4)^{\frac{1}{2}}}\overline{a_l}\Big) + (a_k\overline{a_l} - a_l\overline{a_k})(|\mathbf{a}|^2\frac{d\phi}{dR}\Big(\frac{\overline{a_k}}{(1-|\mathbf{a}|^4)^{\frac{3}{2}}} - \frac{\overline{a_k}}{(1-|\mathbf{a}|^4)^{\frac{3}{2}}}\frac{d^2\phi}{dR^2}\Big).
\end{aligned}$$

Then

$$\Big(a_l\frac{\partial}{\partial a_k}-a_k\frac{\partial}{\partial a_l}\Big)^2\phi(R)$$
$$=a_l\Big(\frac{d\phi}{dR}\Big(\frac{1}{(1-|\mathbf{a}|^4)^{\frac{1}{2}}}\overline{a_l}\Big)+(a_k\overline{a_l}-a_l\overline{a_k})\Big(|\mathbf{a}|^2\frac{d\phi}{dR}\Big(\frac{\overline{a_k}}{(1-|\mathbf{a}|^4)^{\frac{3}{2}}}-\frac{\overline{a_k}}{(1-|\mathbf{a}|^4)^{\frac{3}{2}}}\frac{d^2\phi}{dR^2}\Big)\Big)$$
$$+a_k\Big(\frac{d\phi}{dR}\Big(\frac{1}{(1-|\mathbf{a}|^4)^{\frac{1}{2}}}\overline{a_l}\Big)+(a_k\overline{a_l}-a_l\overline{a_k})\Big(|\mathbf{a}|^2\frac{d\phi}{dR}(\frac{\overline{a_k}}{(1-|\mathbf{a}|^4)^{\frac{3}{2}}}-\frac{\overline{a_k}}{(1-|\mathbf{a}|^4)^{\frac{3}{2}}}\frac{d^2\phi}{dR^2}\Big)\Big)$$
$$=\left(\frac{a_k\overline{a_l}-a_l\overline{a_k}}{|\mathbf{a}|^4-1}\right)\frac{\partial^2\phi}{\partial R^2}-\frac{(|a_k|^2+|a_l|^2)(|\mathbf{a}|^4-1)+|\mathbf{a}|^2(a_k\overline{a_l}-a_l\overline{a_k})^2}{(|\mathbf{a}|^4-1)^{\frac{3}{2}}}\frac{\partial\phi}{\partial R}.$$

Note that

$$\begin{aligned}\sum_{k<l}(|a_k|^2+|a_l|^2)&=\frac{1}{2}\sum_{k\neq l}(|a_k|^2+|a_l|^2)\\&=\frac{1}{2}\sum_{k,l}(1-\delta_{kl})(|a_k|^2+|a_l|^2)\\&=\frac{1}{2}[2(d+1)|\mathbf{a}|^2-|\mathbf{a}|^2]\\&=d|\mathbf{a}|^2.\end{aligned}$$

$$\begin{aligned}\sum_{k<l}(a_k\overline{a_l}-a_l\overline{a_k})^2&=\frac{1}{2}\sum_{k,l}(1-\delta_{kl})(a_k^2\overline{a_l}^2+a_l^2\overline{a_k}^2-2|a_l|^2|a_k|^2)\\&=\frac{1}{2}(|a^2|^2+|a^2|^2-2|\mathbf{a}|^4)+(\sum_k|a_k|^4+|a_k|^4-2|a_k|^4)\\&=-(|\mathbf{a}|^4-|a^2|^2)\\&=-(|\mathbf{a}|^4-1).\end{aligned}$$

Then

$$\sum_{k<l}\Big(a_l\frac{\partial}{\partial a_k}-a_k\frac{\partial}{\partial a_l}\Big)^2\phi(R)=-\frac{\partial^2\phi}{\partial R^2}-(d-1)\frac{|\mathbf{a}|^2}{\sqrt{|\mathbf{a}|^4-1}}\frac{\partial\phi}{\partial R}.$$

Since $|\mathbf{a}|^2=\cosh R$, $\sqrt{|\mathbf{a}|^4-1}=\sinh R$, so we obtain the claimed formula. □

Let us recall the following theorem about Haar measure on a Lie group.

Theorem 5.3 ([5], Theorem 8.36). *Let G be a Lie group, let H be a closed subgroup, and let $\triangle_G$ and $\triangle_H$ be the respective modular functions. Then a necessary and sufficient condition for G/H to have a nonzero G-invariant Borel measure is that the restriction to H of $\triangle_G$ is equal to $\triangle_H$. In this case such a measure $d\mu(gH)$ is unique up to a scalar multiplication.*

Proof of Proposition 5.1. We consider $S^d_{\mathbb{C}}$ as the quotient $SO(d+1,\mathbb{C})/SO(d,\mathbb{C})$. Since $SO(d+1,\mathbb{C})$ and $SO(d,\mathbb{C})$ are unimodular, it follows that the modular functions $\Delta_{SO(d+1,\mathbb{C})}$ and $\Delta_{SO(d,\mathbb{C})}$ are equal to 1. Hence by Theorem 5.3, there is a smooth $SO(d+1,\mathbb{C})$-invariant measure on $S^d_{\mathbb{C}}$ and it is unique up to a constant. Especially, this measure must be $SO(d+1)$-invariant. Since $d\mathbf{p}\,d\mathbf{x}$ is also $SO(d+1)$-invariant, this measure must be of the form $\gamma(p)d\mathbf{p}\,d\mathbf{x}$ for some smooth function γ. Let

$$\beta(p) = 2^d\Big(\frac{\sinh 2p}{2p}\Big)^{d-1}$$

and $g(p) = \dfrac{\alpha(p)}{\beta(p)}$. Therefore g is an $SO(d+1)$-invariant function. We will consider $J^2_{\mathbf{a}}$ only on the space of $SO(d+1)$-invariant functions. We already know that $J^2_{\mathbf{a}}$ must be self-adjoint with respect to the $SO(d+1,\mathbb{C})$-invariant measure. In particular, $J^2_{\mathbf{a}}$ must be self-adjoint when restricted to the space of $SO(d+1)$-invariant functions, which we can write in the form $f(a) = \phi(2p)$ for some smooth function ϕ on $\mathbb{R}$. By Lemma 5.2, on the space of $SO(d+1)$-invariant functions, $J^2_{\mathbf{a}}$ is equal to the hyperbolic Laplacian ([2], page 177). Since the measure $\beta(p)d\mathbf{p}\,d\mathbf{x}$ is just the hyperbolic volume measure, $J^2_{\mathbf{a}}$ is self-adjoint with respect to the measure $\beta(p)d\mathbf{p}\,d\mathbf{x}$. We can conclude that on $SO(d+1)$-invariant functions, $J^2_{\mathbf{a}}$ is self-adjoint with respect to both the measure $\gamma(p)\,d\mathbf{p}\,d\mathbf{x}$ and $\beta(p)\,d\mathbf{p}\,d\mathbf{x}$. Then

$$\begin{aligned}
\langle J_a^2 g(p), J_a^2 g(p)\rangle_\beta &= \int_{x\in S^d}\int_{p\cdot x=0} J_a^2(g(p))J_a^2(g(p))\beta(p)\,d\mathbf{p}\,d\mathbf{x} \\
&= \int_{x\in S^d}\int_{p\cdot x=0} J_a^2(J_a^2(g(p))\frac{\alpha(p)}{\beta(p)}\beta(p)\,d\mathbf{p}\,d\mathbf{x} \\
&= \int_{x\in S^d}\int_{p\cdot x=0} J_a^2(J_a^2(g(p))\alpha(p)\,d\mathbf{p}\,d\mathbf{x} \\
&= \int_{x\in S^d}\int_{p\cdot x=0} J_a^2(g(p))J_a^2(1)\alpha(p)\,d\mathbf{p}\,d\mathbf{x} \\
&= 0.
\end{aligned}$$

Thus $J_a^2 g(p) = 0$. By Lemma 5.2,

$$\Big[\frac{\partial^2 g}{\partial R^2} + (d-1)\frac{\cosh R}{\sinh R}\frac{\partial g}{\partial R}\Big]|_{R=2p} = 0.$$

Since g is a smooth $SO(d+1)$-invariant function on $S^d_{\mathbb{C}}$, g is an even function. Then $\dfrac{\partial g}{\partial R}|_{R=0} = 0$. Solving the equation gives

$$\frac{\partial g}{\partial R} = ce^{-(d-1)\int_R^1 \coth S dS}.$$

Then

$$\begin{aligned}
\frac{\partial g}{\partial R}|_{R=0} &= c\lim_{\epsilon\to 0^+} e^{-(d-1)\int_\epsilon^1 \coth S\,dS} \\
&= 0.
\end{aligned}$$

Therefore $c = 0$, and so is $\dfrac{\partial g}{\partial R}$. Hence g is constant, which implies that γ is a constant multiple of β. □

Let $\mathcal{H}L^2(S^d_{\mathbb{C}}, \nu_t)$ be the space of holomorphic functions F on $S^d_{\mathbb{C}} \cong T^*(S^d)$ for which

$$\int_{x\in S^d}\int_{p\cdot x=0} |F(\mathbf{a}(\mathbf{x},\mathbf{p}))|^2 \nu_{2t}(2p)\left(\frac{\sinh 2p}{2p}\right)^{d-1} 2^d\,d\mathbf{p}\,d\mathbf{x} < \infty.$$

Here is the main theorem of this work.

Theorem 5.4. *The Segal-Bargmann transform C_t defined by (5.1) is a unitary map from $L^2(S^d, d\mathbf{x})$ onto $\mathcal{H}L^2(S^d_{\mathbb{C}}, \nu_t)$.*

We divide the proof into two parts : isometry and surjectivity.

Proposition 5.5. *C_t is an isometry.*

Proof. Since the map $\mathbf{a} \mapsto \rho_t(\mathbf{a}, \mathbf{x})$ is holomorphic for each $\mathbf{x} \in S^d$, we have

$$J^2_{\bar{\mathbf{a}}}\, \rho_t(\mathbf{a}, \mathbf{x}) = 0 \text{ for each } \mathbf{x} \in S^d.$$

By definition of the heat kernel, we have

$$\frac{1}{2} J^2_{\mathbf{a}}\, \rho_t(\mathbf{a}, \mathbf{x}) = \frac{\partial}{\partial t} \rho_t(\mathbf{a}, \mathbf{x}).$$

Let $f, g \in L^2(S^d)$. Then

$$J^2_{\bar{\mathbf{a}}}\, C_t f(\mathbf{a}) = \int_{S^d} (J^2_{\bar{\mathbf{a}}}\, \rho_t(\mathbf{a}, \mathbf{x})) f(\mathbf{x})\, d\mathbf{x} = 0;$$

$$\frac{1}{2} J^2_{\mathbf{a}}\, C_t f(\mathbf{a}) = \int_{S^d} (\frac{1}{2} J^2_{\mathbf{a}}\, \rho_t(\mathbf{a}, \mathbf{x})) f(\mathbf{x})\, d\mathbf{x} = \frac{\partial}{\partial t} C_t f(\mathbf{a}).$$

Therefore

$$\begin{aligned}
\frac{1}{2}(J^2_{\mathbf{a}} + J^2_{\bar{\mathbf{a}}})\, C_t f(\mathbf{a}) \overline{C_t g(\mathbf{a})} &= \frac{1}{2}\{(J^2_{\mathbf{a}}\, C_t f(\mathbf{a})) \overline{C_t g(\mathbf{a})} + C_t f(\mathbf{a})(J^2_{\mathbf{a}}\, \overline{C_t g(\mathbf{a})}) + \\
&\qquad (J^2_{\bar{\mathbf{a}}}\, C_t f(\mathbf{a})) \overline{C_t g(\mathbf{a})} + C_t f(\mathbf{a})(J^2_{\bar{\mathbf{a}}}\, \overline{C_t g(\mathbf{a})})\} \\
&= \frac{\partial}{\partial t} C_t f(\mathbf{a}) \overline{C_t g(\mathbf{a})} + C_t f(\mathbf{a}) \frac{\partial}{\partial t} \overline{C_t g(\mathbf{a})} \\
&= \frac{\partial}{\partial t} (C_t f(\mathbf{a}) \overline{C_t g(\mathbf{a})}).
\end{aligned}$$

We know that $d\mathbf{a} = \beta(p)\, d\mathbf{p}\, d\mathbf{x}$ is $SO(d+1,\mathbb{C})$-invariant. Hence

$$
\begin{aligned}
&\frac{\partial}{\partial t}\int_{\mathbf{a}\in S^d_{\mathbb{C}}} C_tf(\mathbf{a})\overline{C_tg(\mathbf{a})}\nu_t(\mathbf{a})\, d\mathbf{a}\\
&= \int_{\mathbf{a}\in S^d_{\mathbb{C}}} \frac{\partial}{\partial t}(C_tf(\mathbf{a})\overline{C_tg(\mathbf{a})}\nu_t(\mathbf{a}))\, d\mathbf{a}\\
&= \int_{\mathbf{a}\in S^d_{\mathbb{C}}} \frac{1}{2}(J^2_{\mathbf{a}} + J^2_{\bar{\mathbf{a}}})(C_tf(\mathbf{a})\overline{C_tg(\mathbf{a})})\nu_t(\mathbf{a}) + C_tf(\mathbf{a})\overline{Cg(\mathbf{a})}\frac{\partial \nu_t(\mathbf{a})}{\partial t}\, d\mathbf{a}\\
&= \int_{\mathbf{a}\in S^d_{\mathbb{C}}} C_tf(\mathbf{a})\overline{C_tg(\mathbf{a})}\frac{1}{2}(J^2_{\mathbf{a}} + J^2_{\bar{\mathbf{a}}})\nu_t(\mathbf{a}) + C_tf(\mathbf{a})\overline{C_tg(\mathbf{a})}\frac{\partial \nu_t(\mathbf{a})}{\partial t}\, d\mathbf{a}\\
&= \int_{\mathbf{a}\in S^d_{\mathbb{C}}} C_tf(\mathbf{a})\overline{C_tg(\mathbf{a})}(\frac{1}{2}(J^2_{\mathbf{a}} + J^2_{\bar{\mathbf{a}}}) + \frac{\partial}{\partial t})\nu_t(\mathbf{a})\, d\mathbf{a}\\
&= 0.
\end{aligned}
$$

From Lemma 5.2 and the differential equation (5.2) satisfied by ν_t we see that the last integral is zero. Then $\int_{\mathbf{a}\in S^d_{\mathbb{C}}} C_tf(\mathbf{a})\overline{C_tg(\mathbf{a})}\nu_t(\mathbf{a})\, d\mathbf{a}$ is independent of t. By the initial condition for ν and since $\rho_t(\mathbf{a},\cdot) = \rho_t(\mathbf{x},\cdot)$ when $p = 0$ and $\lim_{t\to 0}\rho_t(\mathbf{x},\cdot) = \delta_{\mathbf{x}}$,

$$\lim_{t\to 0}\int_{\mathbf{a}\in S^d_{\mathbb{C}}} C_tf(\mathbf{a})\overline{C_tg(\mathbf{a})}\nu_t(\mathbf{a})\, d\mathbf{a} = \int_{S^d} f(\mathbf{x})\overline{g(\mathbf{x})}\, d\mathbf{x}.$$

Since the value of the first integral is independent of t, this shows that C_t is isometric.

□

Next, we we turn to the proof of surjectivity of C_t. As we already know that if Y_k is spherical harmonic of degree k,

$$\Delta_{S^d} Y_k = -k(k+d-1)Y_k.$$

Since $C_tf(\mathbf{a}(\mathbf{x},\mathbf{p})) = (\int_{S^d}\rho_t(\mathbf{x},\mathbf{y})f(\mathbf{y})\, d\mathbf{y})_{\mathbb{C}}$,

$$
\begin{aligned}
C_tY_k &= (e^{\frac{t\Delta_{S^d}}{2}}Y_k)_{\mathbb{C}}\\
&= (e^{\frac{t\lambda_k}{2}}Y_k)_{\mathbb{C}}\\
&= e^{\frac{t\lambda_k}{2}}(Y_k)_{\mathbb{C}},\ \text{where } \lambda_k = -k(k+d-1).
\end{aligned}
$$

Thus the image of C_t contains analytic continuation of all spherical harmonics. Since C_t is an isometry, its image is a closed subspace of $L^2(S^d_{\mathbb{C}}, \nu_t)$. Thus it suffices to show that every holomorphic L^2 function on $S^d_{\mathbb{C}}$ can be approximated by spherical harmonics of holomorphic representations.

Let F be any holomorphic function on $S^d_{\mathbb{C}}$. Then $F|_{S^d}$ is a smooth function. $F|_{S^d}$ can be written as a spherical harmonic expansion as follows:

$$F(\mathbf{x}) = \sum_{k=0}^{\infty} \sum_{n=1}^{N_k} \langle F, Y_{n,k} \rangle Y_{n,k}(\mathbf{x}) \quad (\mathbf{x} \in S^d).$$

From Chapter 3, we know that this series converges in $L^2(S^d, d\mathbf{x})$. We will verify that this series converges uniformly.

Proposition 5.6. *If F is a smooth function on S^d, then the spherical harmonic expansion converges uniformly to F.*

Proof.

$$\begin{aligned} \langle \Delta_{S^d} F, Y_{n,k} \rangle &= \langle F, \Delta_{S^d} Y_{n,k} \rangle \\ &= \langle F, -k(k+d-1) Y_{n,k} \rangle \\ &= -k(k+d-1) \langle F, Y_{n,k} \rangle. \end{aligned}$$

This implies that

$$\langle \Delta^m_{S^d} F, Y_{n,k} \rangle = (-1)^m (k(k+d-1))^m \langle F, Y_{n,k} \rangle.$$

Then we can estimates the series by

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{n=1}^{N_k} |\langle F, Y_{n,k}\rangle Y_{n,k}(\mathbf{x})| \\
&\leq \sum_{k=0}^{\infty}\sum_{n=1}^{N_k} |\langle F, Y_{n,k}\rangle| \sqrt{\frac{N_k}{C_d}}\, \|Y_{n,k}\|_2 \\
&= \frac{1}{\sqrt{C_d}} \sum_{k=0}^{\infty}\sum_{n=1}^{N_k} \sqrt{N_k} |\langle F, Y_{n,k}\rangle| \\
&= \frac{1}{\sqrt{C_d}} \sum_{k=0}^{\infty}\sum_{n=1}^{N_k} \Big(\frac{\sqrt{N_k}}{k^d (k+d-1)^d} |\langle \Delta_{S^d}^{d} F, Y_{n,k}\rangle| \Big) \\
&\leq \frac{1}{\sqrt{C_d}} \Big(\sum_{k=0}^{\infty} \frac{N_k^2}{k^{2m}(k+d-1)^{2m}} \Big)^{\frac{1}{2}} \Big(\sum_{k=0}^{\infty}\sum_{n=1}^{N_k} |\langle \Delta_{S^d}^{m} F, Y_{n,k}\rangle|^2 \Big)^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{C_d}} \Big(\sum_{k=0}^{\infty} \frac{N_k^2}{k^{2m}(k+d-1)^{2m}} \Big)^{\frac{1}{2}} \big\| \Delta_{S^d}^{2m} F \big\|_2 .
\end{aligned}
$$

By (3.1), we know that

$$
N_k = \frac{(d+2k-1)}{k} \binom{d+k-2}{k-1}.
$$

Hence,

$$
\begin{aligned}
&\sum_{k=0}^{\infty} \frac{N_k^2}{k^{2m}(k+d-1)^{2m}} \\
&= \frac{1}{((d-1)!)^2} \sum_{k=0}^{\infty} \frac{1}{k^{2m}} \Big(\frac{(k+d-2)!}{(k-1)!(k+d-1)^{m-2}} \Big)^2 \Big(\frac{(d+2k-1)}{k(d+k-1)^2} \Big)^2 .
\end{aligned}
$$

If $m \geq d$, then

$$
\frac{(k+d-2)!}{(k-1)!(k+d-1)^{m-2}} = \frac{(d+k-2)\dots(k)}{(k+d-1)^{m-2}} < 1.
$$

Since

$$
\frac{(d+2k-1)}{k(d+k-1)^2} \to 0 \text{ when } k \to \infty,
$$

the k^{th} term in the series is bounded by $\frac{\text{const}}{k^{2m}}$. By Comparison test, this series converges uniformly on S^d. □

Since F is assumed holomorphic, and each term in the Fourier series above has an analytic continuation to $S^d_{\mathbb{C}}$, it is natural to suggest the following series expansion for $F(a)$

$$F(a) = \sum_{k=0}^{\infty}\sum_{n=1}^{N_k}\langle F, Y_{n,k}\rangle Y_{n,k}(a).$$

We will refer to this series as the holomorphic Fourier series of F, abbreviated HFS.

Proposition 5.7. *For any holomorphic function F on $S^d_{\mathbb{C}}$ the holomorphic Fourier series of F converges to F uniformly on compact sets.*

Proof. For each $a \in S^d_{\mathbb{C}}$, define the function $F_a(\mathbf{x}) = F(A\mathbf{x})$, where $a = Ae_{d+1} \cong A\,SO(d,\mathbb{C})$. F_a is a smooth function on S^d. We will consider the Fourier series of F_a at point $\mathbf{x}$ and HFS of F at point $A\mathbf{x}$. Since Laplacian on $\mathbb{R}^d$ is invariant under the action of orthogonal matrix, $Y_{n,k}(A\mathbf{x})$ is a spherical harmonic of the same degree. Then

$$Y_{n,k}(A\mathbf{x}) = \sum_{l=1}^{N_k} a^k_{n,l} Y_{l,k}(\mathbf{x}),$$

and

$$\begin{aligned}\sum_{l=1}^{N_k} a^k_{n,l}a^k_{m,l} &= \int_{S^d} Y_{n,k}(A\mathbf{x})Y_{m,k}(A\mathbf{x})\,d\mathbf{x}\\ &= \int_{S^d} Y_{n,k}(\mathbf{x})Y_{m,k}(\mathbf{x})\,d\mathbf{x}\\ &= \delta_{mn}.\end{aligned}$$

Thus $(a^k_{n,l})_{1\le n,l\le N_k}$ is an orthogonal matrix. We can write $Y_{n,k}(\mathbf{x})$ in the form

$$Y_{n,k}(\mathbf{x}) = \sum_{l=1}^{N_k} a^k_{n,l} Y_{l,k}(A^{-1}\mathbf{x}).$$

Since $(a^k_{n,l})^{-1} = (a^k_{n,l})^t$, $Y_{n,k}(A^{-1}\mathbf{x}) = \sum_{l=1}^{N_k} a^k_{l,n} Y_{l,k}(\mathbf{x})$. Consider the coefficients of the Fourier series of F_a at point $\mathbf{x}$. We have

$$\begin{aligned}\langle F_a, Y_{n,k}\rangle &= \int_{S^d} F(A\mathbf{x})Y_{n,k}(\mathbf{x})\,d\mathbf{x}\\ &= \int_{S^d} F(\mathbf{x})Y_{n,k}(A^{-1}\mathbf{x})\,d\mathbf{x}\\ &= \sum_{l=1}^{N_k} a_{l,n}^k \int_{S^d} F(\mathbf{x})Y_{l,k}(\mathbf{x})\,d\mathbf{x}.\end{aligned}$$

This implies that

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{n=1}^{N_k}\langle F_a, Y_{n,k}\rangle Y_{n,k}(\mathbf{x}) &= \sum_{k=0}^{\infty}\sum_{n=1}^{N_k}(\sum_{l=1}^{N_k}\langle F, Y_{l,k}\rangle Y_{n,k}(\mathbf{x}))\\ &= \sum_{k=0}^{\infty}\sum_{l=1}^{N_k}\langle F, Y_{l,k}\rangle Y_{l,k}(A\mathbf{x}).\end{aligned}$$

We can see that the holomorphic Fourier series of F at the point $A\mathbf{x}$ is the same as the ordinary Fourier series of F_a at the point $\mathbf{x}$.

Let $\mathbf{x} = e_{d+1}$. Then

$$F(a) = F_a(e_{d+1}) = \text{HFS of } F \text{ at } Ae_{d+1} = \text{HFS of } F \text{ at } a.$$

This shows that the HFS converges pointwise. We will verify that HFS converges uniformly on compact sets in $S^d_{\mathbb{C}}$. Let K be any compact set in $S^d_{\mathbb{C}}$. Since K is compact, there exists $M > 0$ such that $|\Delta^m F(a)| \leq M$ for all $a \in K$. Because HFS of F at a is equal to the Fourier series of F_a at e_{d+1}, it suffices to show that the series $\sum_{k=0}^{\infty}\sum_{m=1}^{N_k}\langle F_a, Y_{n,k}\rangle Y_{n,k}(e_{d+1})$ as a function of a, converges uniformly on

K.

$$\sum_{k=0}^{\infty}\sum_{n=1}^{N_k}|\langle F_a, Y_{n,k}\rangle\, Y_{n,k}(e_{d+1})|$$
$$\leq \sum_{k=0}^{\infty}\sum_{n=1}^{N_k}|\langle F_a, Y_{n,k}\rangle|\sqrt{\frac{N_k}{C_d}}\,\|Y_{n,k}\|_2$$
$$= \frac{1}{\sqrt{C_d}}\sum_{k=0}^{\infty}\sum_{n=1}^{N_k}\frac{\sqrt{N_k}}{k^m(k+d-1)^m}|\langle \Delta^m F_a, Y_{n,k}\rangle|$$
$$\leq \frac{1}{\sqrt{C_d}}\Big(\sum_{k=0}^{\infty}\sum_{n=1}^{N_k}(\frac{N_k}{k^{2m}(k+d-1)^{2m}})\Big)^{\frac{1}{2}}(\sum_{k=0}^{\infty}\sum_{n=1}^{N_k}|\langle \Delta^m F_a, Y_{n,k}\rangle|^2)^{\frac{1}{2}}$$
$$= \frac{1}{\sqrt{C_d}}\Big(\sum_{k=0}^{\infty}\frac{N_k^2}{k^{2m}(k+d-1)^{2m}}\Big)^{\frac{1}{2}}\|\Delta^m F_a\|_2$$
$$\leq \frac{1}{\sqrt{C_d}}\Big(\sum_{k=0}^{\infty}\frac{N_k^2}{k^{2m}(k+d-1)^{2m}}\Big)^{\frac{1}{2}}\|\Delta^m F_a\|_\infty$$
$$\leq \frac{1}{\sqrt{C_d}}\Big(\sum_{k=0}^{\infty}\frac{N_k^2}{k^{2m}(k+d-1)^{2m}}\Big)^{\frac{1}{2}}M.$$

By proving in a similar way, we have holomorphic Fourier series of F converges uniformly on any compact subset K of $S^d_{\mathbb{C}}$. □

Proposition 5.8. *If $F \in \mathcal{H}L^2(S^d_{\mathbb{C}}, \nu_t)$, then the holomorphic Fourier series of F converges to F in $L^2(S^d_{\mathbb{C}}, \nu_t)$.*

Proof. At first, we claim that this series is an orthogonal series. Let Y_k and Y_l be spherical harmonic polynomials of degree k and l respectively, $k \neq l$. Since $v_t(p)\,da$ is $SO(d+1)$-invariant,

$$\begin{aligned}\int_{S^d_{\mathbb{C}}} Y_k(a)\overline{\Delta_{S^d}Y_l(a)}v_t(a)\,da &= \langle Y_k, \Delta_{S^d}Y_l\rangle \\ &= \langle \Delta_{S^d}Y_k, Y_l\rangle \\ &= \int_{S^d_{\mathbb{C}}} \Delta_{S^d}Y_k(a)\overline{Y_l(a)}v_t(a)\,da.\end{aligned}$$

Since Δ_{S^d} commutes with analytic continuation,

$$\Delta_{S^d}(Y_{n,k})_{\mathbb{C}} = (\Delta_{S^d} Y_{n,k})_{\mathbb{C}} = -k(k+d-1)(Y_{n,k})_{\mathbb{C}}.$$

Then

$$-l(l+d-1)\int_{S^d_{\mathbb{C}}} Y_k(a)\overline{Y_l(a)}v_t(a)\,da = -k(k+d-1)\int_{S^d_{\mathbb{C}}} Y_k(a)\overline{Y_l(a)}v_t(a)\,da.$$

Since $l \neq k$, $\int_{S^d_{\mathbb{C}}} Y_k(a)\overline{Y_l(a)}v_t(a)\,da = 0$. Since the series is orthogonal, it will converge provided that the sum of the squares of the norms is finite.

Let $E_n = \left\{\mathbf{a}(\mathbf{x},\mathbf{p}) \in S^d_{\mathbb{C}} \mid |\mathbf{p}| \leq n\right\}$. Claim that E_n is $SO(d+1)$-invariant. Let $A \in SO(d+1)$. Then

$$\begin{aligned} A(\mathbf{a}(\mathbf{x},\mathbf{p})) &= A\cosh p\mathbf{x} + iA\frac{\sinh p}{p}\mathbf{p} \\ &= \cosh p(A\mathbf{x}) + i\frac{\sinh p}{p}(A\mathbf{p}). \end{aligned}$$

Since $A \in SO(d+1)$, it follows that $|A\mathbf{p}| = |\mathbf{p}| \leq n$, $|A\mathbf{x}| = |\mathbf{x}| = 1$ and $A\mathbf{x}\cdot A\mathbf{p} = \mathbf{x}\cdot\mathbf{p} = 0$. Thus $A(\mathbf{a}(\mathbf{x},\mathbf{p})) = \mathbf{a}(A\mathbf{x}, A\mathbf{p}) \in E_n$.

Hence E_n is an increasing sequence of compact $SO(d+1)$-invariant sets, with $\cup_n E_n = S^d_{\mathbb{C}}$. Since $F = \sum_{k=0}^{\infty} Y_k$ on E_n, where $Y_k = \sum_{n=1}^{N_k}\langle F, Y_{n,k}\rangle Y_{n,k}$,

$$\|F|_{E_n}\|^2 = \sum_{k=0}^{\infty}\|Y_k|_{E_n}\|^2.$$

Note that $\|Y_k|_{E_n}\|^2$ increases with n.

We may apply Monotone Convergence Theorem so that

$$\sum_{k=0}^{\infty}\|Y_k\|^2 = \|F\|^2 < \infty.$$

Hence this series converges in $L^2(S^d_{\mathbb{C}}, \nu_t)$. □

This completes the proof of surjectivity.

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