CHAPTER V

REDUCED DENSITY MATRICES FOR FERMI SUPERFLUID SYSTEMS

From earlier chapter we examine the concepts of reduced density matrices and ODLRO, also with the two-fluid model. In this chapter we will give some applications of these concepts toasystem of Fermi superfluid. First, we will find the equation of motion of the condensate macroscopic wave function, then to show that it remains invariant under the gauge transformations. Next we find that this wave equation corresponds to the conventional microscopic theory of BCS. Finally we will give some of the thermo-hydrodynamic equations of Fermi superfluid.

5.1 EQUATION OF MOTION. (10)

From the theory of reduced density matices, the n^{th} order reduced density matrix is defined by

$$\Omega_n(x_1',\ldots x_n' , x_1'',\ldots x_n'') = Tr\rho(t)\psi^{\dagger}(x_1'')\ldots\psi^{\dagger}(x_n'')\psi(x_n')\ldots\psi(x_1') \quad (5.1)$$

where $\rho(t)$ is the statistical density matrix at time t and ψ^+, ψ are fermion creation and annihilation operators respectively. Each coordinate $\vec{x}$ includes the space coordinate and spin coordinate.

For Fermi system, the one-body reduced density matrix, Ω_1, can not be factorized because of the exclusion principle. But in the limit that $\vec{x}'_1$ is very far from $\vec{x}''_1$, Ω_1 becomes to zero. That is

$$\Omega_1(\vec{x}'_1,\vec{x}''_1) \dashrightarrow 0 \quad \text{as} \quad |\vec{x}'_1-\vec{x}''_1| \dashrightarrow \infty \tag{5.2}$$

The **two-particle reduced density matrix**, Ω_2, can be factorized as

$$\Omega_2(\vec{x}'_1,\vec{x}'_2,\vec{x}''_1,\vec{x}''_2)$$
$$= \phi^*(\vec{x}''_1,\vec{x}''_2)\,\phi(\vec{x}'_1,\vec{x}'_2) + \tilde{\Omega}_2(\vec{x}'_1,\vec{x}'_2,\vec{x}''_1,\vec{x}''_2) \tag{5.3}$$

where ϕ is the condensate macroscopic wave function of the system. $\tilde{\Omega}_2$ is the two-particle correlation function. Equation (5.3) is refered to as define the condensate macroscopic wave function.

When we consider ODLRO, the two-particle correlation function vanishes when $\vec{x}'_1,\vec{x}'_2$ is very far from $\vec{x}''_1,\vec{x}''_2$. That is

$$\Omega_2(\vec{x}'_1,\vec{x}'_2,\vec{x}''_1,\vec{x}''_2) \dashrightarrow \phi^*(\vec{x}''_1,\vec{x}''_2)\,\phi(\vec{x}'_1,\vec{x}'_2)$$
$$\tilde{\Omega}_2(\vec{x}'_1,\vec{x}'_2,\vec{x}''_1,\vec{x}''_2) \dashrightarrow 0 \text{ as } \vec{x}'_1 \approx \vec{x}''_2, \vec{x}''_1 \approx \vec{x}''_2$$
$$\text{and } |\vec{x}'_1 - \vec{x}''_1| \dashrightarrow \infty \tag{5.4}$$

The **three-particle reduced density matrix can be factorized** (11), (12), as

$$\Omega_3(\vec{x}'_1,\vec{x}'_2,\vec{x}'_3,\vec{x}''_1,\vec{x}''_2,\vec{x}''_3) =$$
$$A\Omega_1(\vec{x}'_1,\vec{x}''_1)\,\Omega_2(\vec{x}'_2,\vec{x}'_3,\vec{x}''_2,\vec{x}''_3)$$
$$+\tilde{\Omega}_3(\vec{x}'_1,\vec{x}'_2,\vec{x}'_3,\vec{x}''_1,\vec{x}''_2,\vec{x}''_3 \tag{5.5}$$

where A is the usual antisymmetrizer, so that the three-particle correlation function, $\tilde{\Omega}_3$, has the Fermi symmetry. $\tilde{\Omega}_3$ vanishes when any two coordinates are sufficiently far from the others.

The Hamiltonian of the system is

$$H = \frac{\hbar^2}{2m}\int\left\{\left(\nabla - \frac{ie}{\hbar c}\vec{A}(\vec{x})\right)\psi^\dagger(\vec{x})\cdot\left(\nabla - \frac{ie}{\hbar c}\vec{A}(\vec{x})\right)\psi(\vec{x})\right\} d\vec{x}$$
$$+\frac{1}{2}\iint V(\vec{x}-\vec{y})\,\psi^\dagger(\vec{x})\,\psi^\dagger(\vec{y})\,\psi(\vec{y})\psi(\vec{x})\; d\vec{y}d\vec{x} \qquad (5.6)$$

where A is a vector potential and V is the interparticle potential with finite spatial range.

The Heisenberg equation of motion of Ω_2 is

$$i\hbar\frac{\partial\Omega_2}{\partial t}(\vec{x}_1',\vec{x}_2',\vec{x}_1'',\vec{x}_2'')$$
$$= \left[\Omega_2(\vec{x}_1',\vec{x}_2',\vec{x}_1'',\vec{x}_2''),H\right]$$
$$= \mathrm{Tr}\,\rho(t)\left[\psi^\dagger(\vec{x}_1'')\,\psi^\dagger(\vec{x}_2'')\,\psi(\vec{x}_2')\,\psi(\vec{x}_1')\;,\;H\right] \qquad (5.7)$$

Substituting the Hamiltonian from Eq. (5.6) and using a usual anticommutation relations, Eq. (5.7) yields

$$i\hbar\frac{\partial}{\partial t}\Omega_2(\vec{x}_1',\vec{x}_2',\vec{x}_1'',\vec{x}_2'')$$
$$= \mathrm{Tr}\,\rho(t)\Bigg[-\frac{\hbar^2}{2m}\sum_{i=1}^{2}\left\{\left(\nabla_{\vec{x}_i'}-\frac{ie}{\hbar c}\vec{A}(\vec{x}_i')\right)^2-\left(\nabla_{\vec{x}_i''}-\frac{ie}{\hbar c}\vec{A}(\vec{x}_i'')\right)^2\right\}\psi^\dagger(\vec{x}_1'')\psi^\dagger(\vec{x}_2'')\psi(\vec{x}_2')\psi(\vec{x}_1')$$
$$+\left\{V(\vec{x}_1'-\vec{x}_2')-V(\vec{x}_1''-\vec{x}_2'')\right\}\psi^\dagger(\vec{x}_1'')\psi^\dagger(\vec{x}_2'')\psi(\vec{x}_2')\psi(\vec{x}_1')+\int\{V(\vec{x}_1'-\vec{y})$$
$$+V(\vec{x}_2'-\vec{y})-V(\vec{x}_1''-\vec{y})-V(\vec{x}_2''-\vec{y})\}\psi^\dagger(\vec{x}_1'')\psi^\dagger(\vec{x}_2'')\psi^\dagger(\vec{y})\psi(\vec{y})\psi(\vec{x}_2')\psi(\vec{x}_1')\,d\vec{y}\Bigg]$$

Or
$$i\hbar\frac{\partial}{\partial t}\Omega_2(\vec{x}_1',\vec{x}_2',\vec{x}_1'',\vec{x}_2'')$$
$$=\Bigg[-\frac{\hbar^2}{2m}\sum_{i=1}^{2}\left\{\left(\nabla_{\vec{x}_i'}-\frac{ie}{\hbar c}\vec{A}(\vec{x}_i)\right)^2-\left(\nabla_{\vec{x}_i''}-\frac{ie}{\hbar c}\vec{A}(\vec{x}_i'')\right)^2\right\}$$
$$+V(\vec{x}_1'-\vec{x}_2')-V(\vec{x}_1''-\vec{x}_2'')\Bigg]\Omega_2(\vec{x}_1',\vec{x}_2',\vec{x}_1'',\vec{x}_2'')+\int\left\{V(\vec{x}_1'-\vec{y})+V(\vec{x}_2'-\vec{y})-V(\vec{x}_1''-\vec{y})\right.$$
$$\left.-V(\vec{x}_2''-\vec{y})\right\}\;\Omega_3(\vec{x}_1',\vec{x}_2',\vec{y},\vec{x}_1'',\vec{x}_2'',\vec{y})\,d\vec{y} \qquad (5.8)$$

Substituting Eq. (5.3) (5.5) into (5.8)

$$i\hbar\frac{\partial}{\partial t}\left\{\phi^*(\vec{x}''_1,\vec{x}''_2)\phi(\vec{x}'_1,\vec{x}'_2)+\tilde{\Omega}_2(\vec{x}'_1,\vec{x}'_2,\vec{x}''_1,\vec{x}''_2)\right\}$$
$$=\Big[-\frac{\hbar^2}{2m}\sum_{i=1}^{2}\Big\{\Big(\nabla_{\vec{x}'_i}-\frac{ie}{\hbar c}\vec{A}(\vec{x}_i)\Big)^2-\Big(\nabla_{\vec{x}''_i}-\frac{ie}{\hbar c}\vec{A}(\vec{x}''_i)\Big)^2\Big\}$$
$$+\Big\{V(\vec{x}'_1-\vec{x}'_2)-V(\vec{x}''_1-\vec{x}''_2)\Big\}\Big]\Big[\phi^*(\vec{x}''_1,\vec{x}''_2)\phi(\vec{x}'_1,\vec{x}'_2)+\tilde{\Omega}_2(\vec{x}'_1,\vec{x}'_2,\vec{x}''_1,\vec{x}''_2)\Big]$$
$$+\int\Big\{V(\vec{x}'_1-\vec{y})+V(\vec{x}'_2-\vec{y})-V(\vec{x}''_1-\vec{y})-V(\vec{x}''_2-\vec{y})\Big\}\Big\{A\Omega_1(\vec{x}'_1,\vec{x}''_1)\Omega_2(\vec{x}'_2,\vec{y},\vec{x}''_2,\vec{y})$$
$$+\tilde{\Omega}_3(\vec{x}'_1,\vec{x}'_2,\vec{y},\vec{x}''_1,\vec{x}''_2,\vec{y})\Big\}\,d\vec{y} \qquad (5.9)$$

Expanding the term $A\Omega_1(\vec{x}'_1\vec{x}''_1)\Omega_2(\vec{x}'_2,\vec{x}'_3,\vec{x}''_2,\vec{x}''_3)$ in all possible permutation, and using the symmetry relation for fermions (13),

$$\Omega_n(\vec{x}'_1,\ldots,\vec{x}'_{r-1},\vec{x}'_r,\ldots\vec{x}'_n,\vec{x}''_1,\ldots,\vec{x}''_{r-1},\vec{x}''_r,\ldots\vec{x}''_n)$$
$$=-\Omega_n(\vec{x}'_1,\ldots,\vec{x}'_r,\vec{x}'_{r-1},\ldots\vec{x}'_n,\vec{x}''_1,\ldots\vec{x}''_{r-1},\vec{x}''_r,\ldots\vec{x}''_n) \qquad (5.10)$$

Then

$$A\Omega_1(\vec{x}'_1,\vec{x}''_1)\Omega_2(\vec{x}'_2,\vec{x}'_3,\vec{x}''_2,\vec{x}''_3)$$
$$=\Omega_1(\vec{x}'_1,\vec{x}''_1)\Omega_2(\vec{x}'_2,\vec{x}'_3,\vec{x}''_2,\vec{x}''_3)-\Omega_1(\vec{x}'_1,\vec{x}''_2)\Omega_2(\vec{x}'_2,\vec{x}'_3,\vec{x}''_1,\vec{x}''_3)$$
$$+\Omega_1(\vec{x}'_1,\vec{x}''_3)\Omega_2(\vec{x}'_2,\vec{x}'_3,\vec{x}''_1,\vec{x}''_2)-\Omega_1(\vec{x}'_2,\vec{x}''_1)\Omega_2(\vec{x}'_1,\vec{x}'_3,\vec{x}''_2,\vec{x}''_3)$$
$$+\Omega_1(\vec{x}'_2,\vec{x}''_2)\Omega_2(\vec{x}'_1,\vec{x}'_3,\vec{x}''_1,\vec{x}''_3)-\Omega_1(\vec{x}'_2,\vec{x}''_3)\Omega_2(\vec{x}'_1,\vec{x}'_3,\vec{x}''_1,\vec{x}''_2)$$
$$+\Omega_1(\vec{x}'_3,\vec{x}''_1)\Omega_2(\vec{x}'_1,\vec{x}'_2,\vec{x}''_2,\vec{x}''_3)-\Omega_1(\vec{x}'_3,\vec{x}''_2)\Omega_2(\vec{x}'_1,\vec{x}'_2,\vec{x}''_1,\vec{x}''_3)$$
$$+\Omega_1(\vec{x}'_3,\vec{x}''_3)\Omega_2(\vec{x}'_1,\vec{x}'_2,\vec{x}''_1,\vec{x}''_2)$$

Equation (5.9) becomes

$$
\begin{aligned}
& i\hbar\phi^*(\vec{x}''_1,\vec{x}''_2)\frac{\partial}{\partial t}\phi(\vec{x}'_1,\vec{x}'_2)+i\hbar\phi(\vec{x}'_1,\vec{x}'_2)\frac{\partial}{\partial t}\phi^*(\vec{x}''_1,\vec{x}''_2)+i\hbar\frac{\partial}{\partial t}\tilde{\Omega}_2(\vec{x}'_1,\vec{x}'_2,\vec{x}''_1,\vec{x}''_2) \\
&= -\frac{\hbar^2}{2m}\Big[\phi^*(\vec{x}''_1,\vec{x}''_2)\sum_{i=1}^{2}\Big(\nabla_{\vec{x}'_i}-\frac{ie}{\hbar c}\vec{A}(\vec{x}'_i)\Big)^2\phi(\vec{x}'_1,\vec{x}'_2)-\phi(\vec{x}'_1,\vec{x}'_2)\sum_{i=1}^{2}\Big(\nabla_{\vec{x}''_i}-\frac{ie}{\hbar c}\vec{A}(\vec{x}''_i)\Big)^2\phi^*(\vec{x}''_1,\vec{x}''_2) \\
&\quad +\sum_{i=1}^{2}\Big\{\Big(\nabla_{\vec{x}'_i}-\frac{ie}{\hbar c}\vec{A}(\vec{x}'_i)\Big)^2-\Big(\nabla_{\vec{x}''_i}-\frac{ie}{\hbar c}\vec{A}(\vec{x}''_i)\Big)^2\Big\}\tilde{\Omega}_2(\vec{x}'_1,\vec{x}'_2,\vec{x}''_1,\vec{x}''_2)\Big]+\Big\{V(\vec{x}'_1-\vec{x}'_2)-V(\vec{x}''_1-\vec{x}''_2)\Big\} \\
&\quad \Big\{\phi^*(\vec{x}''_1,\vec{x}''_2)\phi(\vec{x}'_1,\vec{x}'_2)+\tilde{\Omega}_2(\vec{x}'_1,\vec{x}'_2,\vec{x}''_1,\vec{x}''_2)\Big\}+\int\Big\{V(\vec{x}'_1-\vec{y})+V(\vec{x}'_2-\vec{y})-V(\vec{x}''_1-\vec{y})-V(\vec{x}''_2-\vec{y})\Big\} \\
&\quad \Big\{\Omega_1(\vec{x}'_1,\vec{x}''_1)\Omega_2(\vec{x}'_2,\vec{y},\vec{x}''_2,\vec{y})-\Omega_1(\vec{x}'_1,\vec{x}''_2)\Omega_2(\vec{x}'_2,\vec{y},\vec{x}''_1,\vec{y})+\Omega_1(\vec{x}'_1,\vec{y})\Omega_2(\vec{x}'_2,\vec{y},\vec{x}''_1,\vec{x}''_2) \\
&\quad -\Omega_1(\vec{x}'_2,\vec{x}''_1)\Omega_2(\vec{x}'_1,\vec{y},\vec{x}''_2,\vec{y})+\Omega_1(\vec{x}'_2,\vec{x}''_2)\Omega_2(\vec{x}'_1,\vec{y},\vec{x}''_1,\vec{y})-\Omega_1(\vec{x}'_2,\vec{y})\Omega_2(\vec{x}'_1,\vec{y},\vec{x}''_1,\vec{x}''_2) \\
&\quad +\Omega_1(\vec{y},\vec{x}''_1)\Omega_2(\vec{x}'_1,\vec{x}'_2,\vec{x}''_2,\vec{y})-\Omega_1(\vec{y},\vec{x}''_2)\Omega_2(\vec{x}'_1,\vec{x}'_2,\vec{x}''_1,\vec{y})+\Omega_1(\vec{y},\vec{y})\Omega_2(\vec{x}'_1,\vec{x}'_2,\vec{x}''_1,\vec{x}''_2)\Big\}\,d\vec{y}
\end{aligned}
$$

Dividing both sides by $\phi^*(\vec{x}''_1,\vec{x}''_2)\phi(\vec{x}'_1,\vec{x}'_2)$, then take the limit $\vec{x}'_1,\vec{x}'_2 \longrightarrow \vec{x}''_1,\vec{x}''_2$ and using Eqs. (5.2), (5.4). The equation of motion becomes

$$
\begin{aligned}
& \frac{i\hbar}{\phi(\vec{x}'_1,\vec{x}'_2)}\frac{\partial}{\partial t}\phi(\vec{x}'_1,\vec{x}'_2)+\frac{i\hbar}{\phi^*(\vec{x}''_1,\vec{x}''_2)}\frac{\partial}{\partial t}\phi^*(\vec{x}''_1,\vec{x}''_2) \\
&= -\frac{\hbar^2}{2m}\Big[\frac{1}{\phi(\vec{x}'_1,\vec{x}'_2)}\sum_{i=1}^{2}\Big(\nabla_{\vec{x}'_i}-\frac{ie}{\hbar c}\vec{A}(\vec{x}'_i)\Big)^2\phi(\vec{x}'_1,\vec{x}'_2) \\
&\quad -\frac{1}{\phi^*(\vec{x}'',\vec{x}'')}\sum_{i=1}^{2}\Big(\nabla_{\vec{x}''_i}-\frac{ie}{\hbar c}\vec{A}(\vec{x}''_i)\Big)^2\phi^*(\vec{x}''_1,\vec{x}''_2)\Big] \\
&\quad +\Big\{V(\vec{x}'_1-\vec{x}'_2)-V(\vec{x}''_1-\vec{x}''_2)\Big\}+\frac{1}{\phi^*(\vec{x}''_1,\vec{x}''_2)\phi(\vec{x}'_1,\vec{x}'_2)}\int \\
&\quad \Big\{V(\vec{x}'_1-\vec{y})+V(\vec{x}'_2-\vec{y})-V(\vec{x}''_1-\vec{y})-V(\vec{x}''_2-\vec{y})\Big\}\Big\{\Omega_1(\vec{x}'_1,\vec{y})\Omega_2(\vec{x}'_2,\vec{y},\vec{x}''_1,\vec{x}''_2) \\
&\quad -\Omega_1(\vec{x}'_2,\vec{y})\Omega_2(\vec{x}'_1,\vec{y},\vec{x}''_1,\vec{x}''_2)+\Omega_1(\vec{y},\vec{y})\Omega_2(\vec{x}'_1,\vec{x}'_2,\vec{x}''_1,\vec{x}''_2)\Big\}\,d\vec{y}
\end{aligned}
$$

Separation of variables $\vec{x}'_1$, $\vec{x}'_2$ and $\vec{x}''_1$, $\vec{x}''_2$

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\phi(\vec{x}_1,\vec{x}_2)
&= -\frac{\hbar^2}{2m}\sum_{i=1}^{2}\left(\nabla_{\vec{x}_i}-\frac{ie}{\hbar c}\vec{A}(\vec{x}_i)\right)^2\phi(\vec{x}_1,\vec{x}_2)+V(\vec{x}_1-\vec{x}_2)\phi(\vec{x}_1,\vec{x}_2)\\
&+\int\left\{V(\vec{x}_1-\vec{y})+V(\vec{x}_2-\vec{y})\right\}\left\{\Omega_1(\vec{y},\vec{y})\left(\phi(\vec{x}_1,\vec{x}_2)+\frac{\tilde{\Omega}_2(\vec{x}_1,\vec{x}_2,\vec{x}''_1,\vec{x}''_2)}{\phi^*(\vec{x}''_1,\vec{x}''_2)}\right)\right.\\
&+\Omega_1(\vec{x}_1,\vec{y})\left(\phi(\vec{x}_2,\vec{y})+\frac{\tilde{\Omega}_2(\vec{x}_2,\vec{y},\vec{x}''_1,\vec{x}''_2)}{\phi^*(\vec{x}''_1,\vec{x}''_2)}\right)\\
&\left.-\Omega_1(\vec{x}_2,\vec{y})\left(\phi(\vec{x}_1,\vec{y})+\frac{\tilde{\Omega}_2(\vec{x}_1,\vec{y},\vec{x}''_1,\vec{x}''_2)}{\phi^*(\vec{x}''_1,\vec{x}''_2)}\right)\right\}d\vec{y} \qquad (5.11)
\end{aligned}
$$

Equation (5.11) is the wave equation of the condensate macroscopic wave function ϕ

5.2 GAUGE INVARIANCE (14)

In this section we will consider the gauge transformations of the wave equation and its gauge invariance under this transformations. When the magnetic field is coupled with the system the total potential includes the term $\sum_{i=1}^{2} e\varphi(\vec{x}_i)$ where φ is a scalar potential. The gauge transformations have the form

$$
\begin{aligned}
\vec{A}(\vec{x}) &\longrightarrow \vec{A}(\vec{x}) + \nabla_{\vec{x}}\chi(\vec{x},t)\\
\varphi(\vec{x}) &\longrightarrow \varphi(\vec{x}) - \frac{1}{c}\frac{\partial}{\partial t}\chi(\vec{x},t) \qquad (5.12)
\end{aligned}
$$

where χ is some scalar function of space and time. We assume the macroscopic wave function has the form

$$\phi(\vec{x}_1,\vec{x}_2) \longrightarrow a(\vec{x}_1,\vec{x}_2)\,\phi(\vec{x}_1,\vec{x}_2) \qquad (5.13)$$

Substituting Eqs.(5.12) and (5.13) into (5.11),

$$i\hbar\frac{\partial}{\partial t}\left(a(\vec{x}_1,\vec{x}_2)\phi(\vec{x}_1,\vec{x}_2)\right)$$

$$= -\frac{\hbar^2}{2m}\sum_{i=1}^{2}\left(\nabla_{\vec{x}_i} - \frac{ie}{\hbar c}\vec{A}(\vec{x}_i) - \frac{ie}{\hbar c}\nabla_{\vec{x}_i}\chi(\vec{x})\right)^2 a(\vec{x}_1,\vec{x}_2)\,\phi(\vec{x}_1,\vec{x}_2)$$

$$+V(\vec{x}_1-\vec{x}_2)a(\vec{x}_1,\vec{x}_2)\phi(\vec{x}_1,\vec{x}_2)+\sum_{i=1}^{2} e\left(\varphi(\vec{x}_i)-\frac{1}{c}\frac{\partial}{\partial t}\chi(\vec{x}_i)\right)$$

$$a(\vec{x}_1,\vec{x}_2)\phi(\vec{x}_1,\vec{x}_2) \quad +\int\Big\{V(\vec{x}_1-\vec{y})+v(\vec{x}_2-\vec{y})$$

$$+\sum_{i=1}^{2} e\left(\varphi(\vec{x}_i)-\frac{1}{c}\frac{\partial}{\partial t}\chi(\vec{x}_i)\right)+2e\left(\varphi(\vec{y})-\frac{1}{c}\frac{\partial}{\partial t}\chi(\vec{y})\right)\Big\}$$

$$\Big\{\Omega_1(\vec{y},\vec{y})\left(a(\vec{x}_1,\vec{x}_2)\phi(\vec{x}_1,\vec{x}_2)+\frac{\tilde{\Omega}_2(\vec{x}_1,\vec{x}_2,\vec{x}_1'',\vec{x}_2'')}{a^*(\vec{x}_1'',\vec{x}_2'')\phi^*(\vec{x}_1'',\vec{x}_2'')}\right.$$

$$+\Omega_1(\vec{x}_1,\vec{y})\left(a(\vec{x}_2,\vec{y})\phi(\vec{x}_2,\vec{y})+\frac{\tilde{\Omega}_2(\vec{x}_2,\vec{y},\vec{x}_1'',\vec{x}_2'')}{a^*(\vec{x}_1'',\vec{x}_2'')\phi^*(\vec{x}_1'',\vec{x}_2'')}\right.$$

$$\left.-\Omega_1(\vec{x}_2,\vec{y})\left(a(\vec{x}_1,\vec{y})\phi(\vec{x}_1,\vec{y})+\frac{\tilde{\Omega}_2(\vec{x}_1,\vec{y},\vec{x}_1'',\vec{x}_2'')}{a^*(\vec{x}_1'',\vec{x}_2'')\phi^*(\vec{x}_1'',\vec{x}_2'')}\right)\right\}d\vec{y} \qquad (5.14)$$

Since

$$\sum_{i=1}^{2}\left(\nabla_{\vec{x}_i} - \frac{ie}{\hbar c}\vec{A}(\vec{x}_i) - \frac{ie}{\hbar c}\nabla_{\vec{x}_i}\chi(\vec{x})\right)^2 a(\vec{x}_1,\vec{x}_2)\phi(\vec{x}_1,\vec{x}_2)$$

$$= a(\vec{x}_1,\vec{x}_2)\sum_{i=1}^{2}\left(\nabla_{\vec{x}_i}+\nabla_{\vec{x}_i}\ln a\,(\vec{x}_1,\vec{x}_2)-\frac{ie}{\hbar c}A(\vec{x}_i)-\frac{ie}{\hbar c}\nabla_{\vec{x}_i}\chi(\vec{x}_i)\right)^2\phi(\vec{x}_1,\vec{x}_2)$$

Dividing both sides of Eq. (5.14) by $a\,(\vec{x}_1,\vec{x}_2)\;\phi\;(\vec{x}_1,\vec{x}_2)$

$$\frac{i\hbar}{a(\vec{x}_1,\vec{x}_2)}\frac{\partial}{\partial t}a(\vec{x}_1,\vec{x}_2)+\frac{i\hbar}{\phi(\vec{x}_1,\vec{x}_2)}\frac{\partial}{\partial t}\phi(\vec{x}_1,\vec{x}_2)$$

$$= -\frac{\hbar^2}{2m\phi(\vec{x}_1,\vec{x}_2)}\sum_{i=1}^{2}\Big(\nabla_{\vec{x}_i}+\nabla_{\vec{x}_i}\ln a(\vec{x}_1,\vec{x}_2)$$

$$-\frac{ie}{\hbar c}\vec{A}(\vec{x}_i)-\frac{ie}{\hbar c}\nabla_{\vec{x}_i}\chi(\vec{x}_i)\Big)^2\phi(\vec{x}_1,\vec{x}_2)+V(\vec{x}_1-\vec{x}_2)+\sum_{i=1}^{2} e\left(\varphi(\vec{x}_i)-\frac{1}{c}\frac{\partial}{\partial t}\chi(\vec{x}_i)\right)$$

$$+\int\Big\{V(\vec{x}_1-\vec{y})+V(\vec{x}_2-\vec{y})+\sum_{i=1}^{2} e\left(\varphi(\vec{x}_i)-\frac{1}{c}\frac{\partial}{\partial t}\chi(\vec{x}_i)\right)+2e\left(\varphi(\vec{y})-\frac{1}{c}\frac{\partial}{\partial t}\chi(\vec{y})\right)\Big\}$$

$$\Big\{\Omega_1(\vec{y},\vec{y})\Big(1+\frac{\tilde{\Omega}_2(\vec{x}_1,\vec{x}_2,\vec{x}_1'',\vec{x}_2'')}{a^*(\vec{x}_1'',\vec{x}_2'')a(\vec{x}_1,\vec{x}_2)\phi^*(\vec{x}_1'',\vec{x}_2'')\phi(\vec{x}_1,\vec{x}_2)}\Big)$$

$$\Omega_1(\vec{x}_1,\vec{y})\Big(\frac{a(\vec{x}_2,\vec{y})\phi(\vec{x}_2,\vec{y})}{a(\vec{x}_1,\vec{x}_2)\phi(\vec{x}_1,\vec{x}_2)}+\frac{\tilde{\Omega}_2(\vec{x}_2,\vec{y},\vec{x}_1'',\vec{x}_2'')}{a^*(\vec{x}_1'',\vec{x}_2'')a(\vec{x}_1,\vec{x}_2)\phi^*(\vec{x}_1'',\vec{x}_2'')\phi(\vec{x}_1,\vec{x}_2)}\Big)$$

$$-\Omega_1(\vec{x}_2,\vec{y})\Big(\frac{a(\vec{x}_1,\vec{y})\phi(\vec{x}_1,\vec{y})}{a(\vec{x}_1,\vec{x}_2)\phi(\vec{x}_1,\vec{x}_2)}+\frac{\tilde{\Omega}_2(\vec{x}_1,\vec{y},\vec{x}_1'',\vec{x}_2'')}{a^*(\vec{x}_1'',\vec{x}_2'')a(\vec{x}_1,\vec{x}_2)\phi^*(\vec{x}_1'',\vec{x}_2'')\phi(\vec{x}_1,\vec{x}_2)}\Big)\Big\}d\vec{y}$$

Since the equation of motion must preserve gauge invariance, then $\sum_{i=1}^{2}\Big\{\nabla_{\vec{x}_i}\ln a(\vec{x}_1,\vec{x}_2)-\frac{ie}{\hbar c}\nabla_{\vec{x}_i}\chi(\vec{x}_i)\Big\} = 0.$ From this we find that $a(\vec{x}_1,\vec{x}_2)$ has the form $a(\vec{x}_1,\vec{x}_2) = \exp\frac{ie}{\hbar c}\Big\{\chi(\vec{x}_1)+\chi(\vec{x}_2)\Big\}$.

Substituting $a(\vec{x}_1,\vec{x}_2)$ into the above equation, it yields

$$i\hbar\frac{\partial}{\partial t}\phi(\vec{x}_1,\vec{x}_2)$$

$$= -\frac{\hbar^2}{2m}\sum_{i=1}^{2}\Big(\nabla_{\vec{x}_i}-\frac{ie}{\hbar c}\vec{A}(\vec{x}_i)\Big)^2\phi(\vec{x}_1,\vec{x}_2)+V(\vec{x}_1-\vec{x}_2)\phi(\vec{x}_1,\vec{x}_2)$$
$$+\int\Big[V(\vec{x}_1-\vec{y})+V(\vec{x}_2-\vec{y})-\frac{e}{c}\frac{\partial}{\partial t}\Big\{\chi(\vec{x}_1)+\chi(\vec{x}_2)+2\chi(\vec{y})\Big\}\Big]$$
$$\Big[\Omega_1(\vec{y},\vec{y})\Big\{\phi(\vec{x}_1,\vec{x}_2)+\frac{\tilde{\Omega}_2(\vec{x}_1,\vec{x}_2,\vec{x}_1'',\vec{x}_2'')}{\exp\frac{ie}{\hbar c}\big(\chi(\vec{x}_1)+\chi(\vec{x}_2)-\chi(\vec{x}_1'')-\chi(\vec{x}_2'')\big)\phi^*(\vec{x}_1'',\vec{x}_2'')}\Big\}$$
$$+\Omega_1(\vec{x}_1,\vec{y})\Big\{\frac{\phi(\vec{x}_2,\vec{y})}{\exp\frac{ie}{\hbar c}\big(\chi(\vec{x}_1)-\chi(\vec{y})\big)}+\frac{\tilde{\Omega}_2(\vec{x}_2,\vec{y},\vec{x}_1'',\vec{x}_2'')}{\exp\frac{ie}{\hbar c}\big(\chi(\vec{x}_1)+\chi(\vec{x}_2)-\chi(\vec{x}_1'')-\chi(\vec{x}_2'')\big)\phi^*(\vec{x}_1'',\vec{x}_2'')}\Big\}$$
$$-\Omega_1(\vec{x}_2,\vec{y})\Big\{\frac{\phi(\vec{x}_1,\vec{y})}{\exp\frac{ie}{\hbar c}\big(\chi(\vec{x}_2)-\chi(\vec{y})\big)}+\frac{\tilde{\Omega}_2(\vec{x}_1,\vec{y},\vec{x}_1'',\vec{x}_2'')}{\exp\frac{ie}{\hbar c}\big(\chi(\vec{x}_1)+\chi(\vec{x}_2)-\chi(\vec{x}_1'')-\chi(\vec{x}_2'')\big)\phi^*(\vec{x}_1'',\vec{x}_2'')}\Big\}\Big]d\vec{y}$$

Since the macroscopic wave function must be single-valued we choose the gauge choice $\chi = \frac{2\pi\hbar c}{e}n$, where $n = 0,\pm1,\pm2,\ldots$ When we substitute this quantization condition, the orignal equation of motion is preserved.

5.3 FOURIER TRANSFORMATION OF THE WAVE EQUATION

The wave equation (5.11), if it is expected to be right, it must be satisfied by the conventional miroscopic theories of superconductivity. In this section we want to show that the wave equation (5.11) corresponds to the theory of BCS. First we assume antiparallel spin pairing only and define the Fourier components (12)

$$\phi(\vec{x}_1,\vec{x}_2) = \delta_{\sigma_1,\sigma_2} \frac{1}{(2\pi)^{3/2}} \int \phi_k \exp\left\{ i\vec{k}\cdot(\vec{x}_1-\vec{x}_2) - 2i\mu t/\hbar \right\} d\vec{k} \tag{5.15}$$

$$\Omega_1(\vec{x}_1,\vec{x}_2) = \frac{1}{2}\delta_{\sigma_1,-\sigma_2} \frac{1}{(2\pi)^3} \int n_k \exp\left\{ i\vec{k}\cdot(\vec{x}_1-\vec{x}_2) \right\} d\vec{k} \tag{5.16}$$

$$V(\vec{x}_1-\vec{x}_2) = \frac{1}{(2\pi)^3} \int v_k \exp\left\{ i\vec{k}\cdot(\vec{x}_1-\vec{x}_2) \right\} d\vec{k} \tag{5.17}$$

where μ is some constant. We rewrite the wave equation with the exception that now an applied magnetic field is excluded and we neglect the term $\tilde{\Omega}_2$ in the integrand.

$$\begin{aligned} i\hbar \frac{\partial}{\partial t}\phi(\vec{x}_1,\vec{x}_2) &= -\frac{\hbar^2}{2m}\left(\nabla^2_{\vec{x}_1} + \nabla^2_{\vec{x}_2}\right)\phi(\vec{x}_1,\vec{x}_2) + V(\vec{x}_1,\vec{x}_2)\phi(\vec{x}_1,\vec{x}_2) + \int \left\{ V(\vec{x}_1-\vec{y}) + V(\vec{x}_2-\vec{y}) \right\} \\ &\quad \left\{ \Omega_1(\vec{y},\vec{y})\phi(\vec{x}_1,\vec{x}_2) + \Omega_1(\vec{x}_1,\vec{y})\phi(\vec{x}_2,\vec{y}) - \Omega_1(\vec{x}_2,\vec{y})\phi(\vec{x}_1,\vec{y}) \right\} d\vec{y} \end{aligned} \tag{5.18}$$

The Fourier transform of the left hand side is

$$\frac{1}{(2\pi)^{3/2}}\int \exp\left\{-i\vec{k}\cdot(\vec{x}_1,\vec{x}_2)+2i\mu t/\hbar\right\}\left[\, i\hbar\frac{\partial}{\partial t}\,\frac{1}{(2\pi)^{3/2}}\int\phi_{k'}\exp\left\{i\vec{k}'\cdot(\vec{x}_1-\vec{x}_2)\right.\right.$$
$$\left.\left.-2i\mu t/\hbar\right\}d\vec{k}'\right]d(\vec{x}_1-\vec{x}_2)$$
$$=\frac{1}{(2\pi)^3}\iint\exp\left\{-i\vec{k}\cdot(\vec{x}_1,\vec{x}_2)+2i\mu t/\hbar+i\vec{k}'\cdot(\vec{x}_1-\vec{x}_2)\right.$$
$$\left.-2i\mu t/\hbar\right\}d(\vec{x}_1-\vec{x}_2)\,2\mu\phi_{k'}d\vec{k}'$$
$$=\int\delta(\vec{k}-\vec{k}')2\mu\phi_{k'}d\vec{k}'$$
$$=2\mu\phi_k$$

The Fourier transform of the first term on the right hand side is

$$\frac{1}{(2\pi)^{3/2}}\int\exp\left\{-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)+2i\mu t/\hbar\right\}\left[-\frac{\hbar^2}{2m}\left(\nabla^2_{x_1}+\nabla^2_{x_2}\right)\frac{1}{(2\pi)^{3/2}}\int\phi_{k'}\right.$$
$$\left.\exp\left\{i\vec{k}'\cdot(\vec{x}_1-\vec{x}_2)-2i\mu t/\hbar\right\}d\vec{k}'\right]d(\vec{x}_1-\vec{x}_2)$$
$$=\frac{1}{(2\pi)^3}\iint\exp\left\{-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)+2i\mu t/\hbar+i\vec{k}'\cdot(\vec{x}_1-\vec{x}_2)\right.$$
$$\left.-2i\mu t/\hbar\right\}d(\vec{x}_1-\vec{x}_2)\frac{h^2k'^2}{m}\phi_{k'}d\vec{k}'$$
$$=\int\delta(\vec{k}-\vec{k}')\frac{h^2k'^2}{m}\phi_{k'}d\vec{k}'$$
$$=\frac{h^2k^2}{m}\phi_k$$

The Fourier transform of the second term on the right hand side is

$$\frac{1}{(2\pi)^{3/2}}\int \exp\left\{-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)+2i\mu t/\hbar\right\}\left[\left\{\frac{1}{(2\pi)^3}\int v_{k'}\exp\left\{i\vec{k}'\cdot(\vec{x}_1-\vec{x}_2)\right\}\ d\vec{k}'\right\}\right.$$

$$\left.\left\{\frac{1}{(2\pi)^{3/2}}\int \phi_{k''}\exp\left\{i\vec{k}''\cdot(\vec{x}_1-\vec{x}_2)-2i\mu t/\hbar\right\}\ d\vec{k}''\right\}\right]d(\vec{x}_1-\vec{x}_2)$$

$$= \frac{1}{(2\pi)^6}\iiint \exp\left\{-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)+2i\mu t/\hbar+i\vec{k}'\cdot(\vec{x}_1-\vec{x}_2)\right.$$

$$\left.+i\vec{k}''\cdot(\vec{x}_1-\vec{x}_2)-2i\mu t/\hbar\right\}d(\vec{x}_1-\vec{x}_2)\ v_{k'}\phi_{k''}d\vec{k}''\,d\vec{k}'$$

$$= \frac{1}{(2\pi)^3}\iint\delta(\vec{k}''-(\vec{k}-\vec{k}'))v_{k'}\phi_{k''}d\vec{k}''d\vec{k}'$$

$$= \frac{1}{(2\pi)^3}\int v_{k'}\phi_{k-k'}d\vec{k}'$$

$$= \sum_{k'} v_{k'}\phi_{k-k'}$$

where the integral is replaced by the sum according to the rule $\frac{1}{(2\pi)^3}\int d\vec{k}\ldots \;\text{------>}\; \frac{1}{V}\sum_{\vec{k}}\ldots$

Now for the terms in the integrand, the first term is zero since from definition $\delta_{\vec{y},\vec{y}}$ is zero. The second term has its Fourier transform

$$\frac{1}{(2\pi)^{3/2}}\int \exp\{-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)+2i\mu t/\hbar\}\Big[\iint\Big(\frac{1}{(2\pi)^3}\int v_{k'}\exp\{i\vec{k}'\cdot(\vec{x}_1-\vec{y})\}$$

$$+\frac{1}{(2\pi)^3}\int v_{k'}\exp\{i\vec{k}'(\vec{y}-\vec{x}_2)\}\Big)\, d\vec{k}'\, d\vec{y}\ \frac{1}{2(2\pi)^3}\int n_{k''}\exp\{i\vec{k}''\cdot(\vec{x}_1-\vec{y})\}\,d\vec{k}''$$

$$\frac{(-1)}{(2\pi)^{3/2}}\int \phi_{k'''}\exp\{i\vec{k}'''\cdot(\vec{y}-\vec{x}_2)-2i\mu t/\hbar\}\,d\vec{k}'''\Big]\, d(\vec{x}_1-\vec{x}_2)$$

$$= -\frac{1}{2(2\pi)^9}\int\!\!\int\!\!\int\!\!\int\!\!\int\Big[\exp\{-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)+2i\mu t/\hbar+i(\vec{k}'+\vec{k}'')\cdot\vec{x}_1-i\vec{k}'''\cdot\vec{x}_2-i(\vec{k}'+\vec{k}''-\vec{k}''')\cdot\vec{y}$$

$$-2i\mu t/\hbar\}+\exp\{-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)+2i\mu t/\hbar+i\vec{k}''\cdot\vec{x}_1-i(\vec{k}'-\vec{k}''')\cdot\vec{x}_2-i(\vec{k}''-\vec{k}'-\vec{k}''')\cdot\vec{y}$$

$$-2i\mu t/\hbar\}\Big]\ v_{k'}n_{k''}\phi_{k'''}\,dy\,d\vec{k}'''\, d\vec{k}''\, d\vec{k}'\ d(\vec{x}-\vec{x})$$

$$= -\frac{1}{2(2\pi)^6}\iiiint\Big[\exp\{-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)+i(\vec{k}'+\vec{k}'')\cdot\vec{x}_1-i\vec{k}'''\cdot\vec{x}_2\}\,\delta(\vec{k}'+\vec{k}''-\vec{k}''')\,v_{k'}n_{k''}\phi_{k'''}$$

$$d\vec{k}'''\ d\vec{k}''\ d\vec{k}'\ d(\vec{x}_1-\vec{x}_2)+\exp\{-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)+i\vec{k}''\cdot\vec{x}_1-i(\vec{k}'+\vec{k}''')\cdot\vec{x}_2\}$$

$$\delta(\vec{k}''-\vec{k}'-\vec{k}''')\,v_{k'}n_{k''}\phi_{k'''}\,d\vec{k}'''\ d\vec{k}''\ d\vec{k}'\,d(\vec{x}_1-\vec{x}_2)$$

$$= -\frac{1}{2(2\pi)^6}\iiint\Big[\exp -i(\vec{k}-\vec{k}'-\vec{k}'')\cdot(\vec{x}_1-\vec{x}_2)\ v_{k'}n_{k''}\phi_{k'+k''}+\exp\{-i(\vec{k}-\vec{k}'')\cdot(\vec{x}_1-\vec{x}_2)\}$$

$$v_{k'}n_{k''}\phi_{k''-k'}\Big]\,d\vec{k}''\ d\vec{k}'\ d(\vec{x}_1-\vec{x}_2)$$

$$= -\frac{1}{2(2\pi)^3}\iint\Big[\delta(\vec{k}-\vec{k}'-\vec{k}'')\ v_{k'}n_{k''}\phi_{k'+k''}+\delta(\vec{k}-\vec{k}'')\ v_{k'}n_{k''}\phi_{k''-k'}\Big]\,d\vec{k}''\ d\vec{k}'$$

$$= -\frac{1}{2(2\pi)^3}\int v_{k-k''}n_{k''}\phi_k\ d\vec{k}''-\frac{1}{2(2\pi)^3}\int v_{k'}n_k\phi_{k-k'}\ d\vec{k}'$$

$$= -\frac{1}{2}\sum_{k'} v_{k-k'}\,n_{k'}\phi_k-\frac{1}{2}\sum_{k'} v_{k'}n_k\phi_{k-k'}$$

where we use the symmetry relations of macroscopic wave function, $\phi(\vec{x}_1,\vec{x}_2) = -\phi(\vec{x}_2,\vec{x}_1)$. The last term in the integrand has its Fourier transform

$$\frac{1}{(2\pi)^{3/2}}\int \exp\left\{-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)+2i\mu t/\hbar\right\}\left[\iint\left(\frac{1}{(2\pi)^3}\int v_{k'}\exp\left\{i\vec{k}'\cdot(\vec{x}_1-\vec{y})\right\}\right.\right.$$

$$\left.+\frac{1}{(2\pi)^3}\int v_{k'}\exp\left\{i\vec{k}'\cdot(\vec{y}-\vec{x}_2)\right\}\right)d\vec{k}'d\vec{y}\;\frac{1}{2(2\pi)^3}\int n_{k''}\exp\left\{i\vec{k}''\cdot(\vec{y}-\vec{x}_2)\right\}d\vec{k}''$$

$$\left.\frac{1}{(2\pi)^{3/2}}\int \phi_{k'''}\exp\left\{i\vec{k}'''\cdot(\vec{x}_1-\vec{y})-2i\mu t/\hbar\right\}d\vec{k}'''\right]d(\vec{x}_1-\vec{x}_2)$$

$$=\frac{1}{2(2\pi)^9}\iiiint\!\!\int\Big[\exp\left\{-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)+2i\mu t/\hbar+i(\vec{k}'+\vec{k}''')\cdot\vec{x}_1-i\vec{k}''\cdot\vec{x}_2-i(\vec{k}'-\vec{k}''+\vec{k}''')\cdot\vec{y}\right.$$

$$\left.-2i\mu t/\hbar\right\}+\exp\left\{-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)+2i\mu t/\hbar+i\vec{k}'''\cdot\vec{x}_1-i(\vec{k}'+\vec{k}'')\cdot\vec{x}_2-i(-\vec{k}'-\vec{k}''+\vec{k}''')\cdot\vec{y}\right.$$

$$\left.-2i\mu t/\hbar\right\}\Big]\,v_{k'}n_{k''}\phi_{k'''}\,d\vec{y}\,d\vec{k}'''\,d\vec{k}''\,d\vec{k}'\,d(\vec{x}_1-\vec{x}_2)$$

$$=\frac{1}{2(2\pi)^6}\iiiint\Big[\exp\left\{-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)+i(\vec{k}'+\vec{k}''')\cdot\vec{x}_1-i\vec{k}''\cdot\vec{x}_2\right\}\delta(k'''-k''+k')\,v_{k'}n_{k''}\phi_{k'''}$$

$$d\vec{k}'''\,d\vec{k}''\,d\vec{k}'\,d(\vec{x}_1-\vec{x}_2)+\exp\left\{-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)+i\vec{k}'''\cdot\vec{x}_1-i(\vec{k}'+\vec{k}'')\cdot\vec{x}_2\right\}$$

$$\delta(\vec{k}'''-\vec{k}''-\vec{k}')\,v_{k'}n_{k''}\phi_{k'''}\,d\vec{k}'''\,d\vec{k}''\,d\vec{k}'\,d(\vec{x}_1-\vec{x}_2)$$

$$=\frac{1}{2(2\pi)^6}\iiint\Big[\exp\left\{-i(\vec{k}-\vec{k}'')\cdot(\vec{x}_1-\vec{x}_2)\right\}\,v_{k'}n_{k''}\phi_{k''-k'}\,d\vec{k}''\,d\vec{k}'\,d(\vec{x}_1-\vec{x}_2)$$

$$+\exp\left\{-i(\vec{k}-\vec{k}'-\vec{k}'')\cdot(\vec{x}_1-\vec{x}_2)\right\}\,v_{k'}n_{k''}\phi_{k'+k''}\,d\vec{k}''\,d\vec{k}'\,d(\vec{x}_1-\vec{x}_2)$$

$$=\frac{1}{2(2\pi)^3}\iint\Big[\delta(\vec{k}-\vec{k}'')\,v_{k'}n_{k''}\phi_{k''-k'}\,d\vec{k}''\,d\vec{k}'+\delta(\vec{k}-\vec{k}'-\vec{k}'')$$

$$v_{k'}n_{k''}\phi_{k'+k''}\,d\vec{k}''\,d\vec{k}'\Big]$$

$$=\frac{1}{2(2\pi)^3}\int v_{k'}n_k\phi_{k-k'}\,d\vec{k}'+\frac{1}{2(2\pi)^3}\int v_{k-k''}n_{k''}\phi_k\,d\vec{k}''$$

$$=\frac{1}{2}\sum_{k'}v_{k'}n_k\phi_{k-k'}+\frac{1}{2}\sum_{k'}v_{k-k'}n_{k'}\phi_k$$

The Fourier transform of the wave equation is

$$2\mu\phi_k = \frac{\hbar^2k^2}{m}\phi_k + \sum_{k'} v_{k'}\phi_{k-k'} - \sum_{k'} v_{k-k'}n_{k'}\phi_k - \sum_{k'} v_{k'}n_k\phi_{k-k'}$$

$$2\left(\mu - \frac{\hbar^2k^2}{2m} + \frac{1}{2}\sum_{k'} v_{k-k'}n_{k'}\right)\phi_k = \sum_{k'} v_{k'}\phi_{k-k'}(1-n_k)$$

If we introduce the notation $\xi_k = \frac{\hbar^2k^2}{2m} - \frac{1}{2} v_{k-k'} n_{k'}$, then the wave equation (5.18) becomes

$$2(\mu - \xi_k)\phi_k = \sum_{k'} v_{k'}\phi_{k-k'}(1-n_k) \tag{5.19}$$

Now in the usual notation of the microscopic theory, μ, the chemical potential and

$$n_k = 2\left\{u_k^2 f_k + v_k^2 (1-f_k)\right\} \tag{5.20}$$

$$\phi_k = u_k v_k (1-2f_k) \tag{5.21}$$

where u_k, v_k are the parameters of the Bogolubov transformation and f_k is the quasiparticle occupation number. By setting

$u_k = \sin\theta_k$, $v_k = \cos\theta_k$, then $n_k = 2\left\{\sin^2\theta_k f_k + \cos^2\theta_k - \cos^2\theta_k f_k\right\} = 2\left\{\cos^2\theta_k - \cos 2\theta_k f_k\right\}$ and $\phi_k = \frac{1}{2}\sin 2\theta_k (1-2f_k)$

when we substitute n_k and ϕ_k into Eq. (5.19), the result is

$$(\mu - \xi_k)(1-2f_k)\sin 2\theta_k$$

$$= \frac{1}{2}\sum_{k'} v_{k'}\sin 2\theta_{k-k'}(1-2f_{k-k'})(1-2\cos^2\theta_k + 2\cos 2\theta_k f_k)$$

$$= \frac{1}{2}\sum_{k'} v_{k'}\sin 2\theta_{k-k'}(1-2f_{k-k'})(2\cos 2\theta_k f_k - \cos 2\theta_k)$$

$$= -\frac{1}{2}\sum_{k'} v_{k'}\cos 2\theta_k \sin 2\theta_{k-k'}(1-2f_k)(1-2f_{k-k'})$$

$$\text{or}\quad 2(\xi_k-\mu)(1-2f_k)\sin 2\theta_k = \sum_{k'} v_{k'}\cos 2\theta_k \sin 2\theta_{k-k'}(1-2f_k)(1-2f_{k-k'}) \quad (5.22)$$

Equation (5.22) is one of the standard equations of the microscopic theory of superconductivity (1) (15). Thus the time dependent wave equation (5.18) is satisfied in equilibrium by the conventional microscopic theories.

In the microscopic theory of superconductivity Eq. (5.19) is obtained by minimizing the free energy with respect to u_k and v_k. With minimizing the free energy with respect to f_k will complete the theory. Therefore the wave equation (5.19) is only partially of the microscopic theory.

5.4 TWO-FLUID HYDRODYNAMICS.

This section is devoted to find the thermo-hydrodynamic equations of Fermi superfluids. In analogy to the case of Bose superfluids (16), we define the condensate macroscopic wave function by

$$\phi(\vec{x}_1,\vec{x}_2) = (2\rho_s(\vec{x}_1,\vec{x}_2))^{\frac{1}{2}} \exp(i\theta(\vec{x}_1,\vec{x}_2)) \qquad (5.23)$$

where $\rho_s(\vec{x}_1,\vec{x}_2)$ is the condensate density and $\theta(\vec{x}_1,\vec{x}_2)$ the phase of the condensate macroscopic wave function. By substituting Eq. (5.23) into the wave equation (5.11),

$$i\hbar \frac{\partial}{\partial t}\left\{ (2\rho_s)^{\frac{1}{2}} e^{i\theta}\right\}$$

$$= -\frac{\hbar^2}{2m}\left\{\left(\nabla_{\vec{x}_i} - \frac{ie}{\hbar c}\vec{A}(\vec{x}_1)\right)^2 + \left(\nabla_{\vec{x}_2} - \frac{ie}{\hbar c}\vec{A}(\vec{x}_2)\right)^2\right\}$$

$$\left\{(2\rho_s)^{\frac{1}{2}} e^{i\theta}\right\} + v(\vec{x}_1,\vec{x}_2)\left\{ (2\rho_s)^{\frac{1}{2}} c^{i\theta}\right\} + (2\rho_s)^{\frac{1}{2}} c^{i\theta}\; F \qquad (5.24)$$

where

$$F = \int\left\{V(\vec{x}_1-\vec{y})+V(\vec{x}_2-\vec{y})\right\}\left\{\Omega_1(\vec{y},\vec{y})\left(1+\frac{\tilde{\Omega}_2(\vec{x}_1,\vec{x}_2,\vec{x}_1'',\vec{x}_2'')}{\phi(\vec{x}_1,\vec{x}_2)\phi^*(\vec{x}_1'',\vec{x}_2'')}\right)\right.$$

$$+\Omega_1(\vec{x}_1,\vec{y})\left(\frac{\phi(\vec{x}_2,\vec{y})}{\phi(\vec{x}_1,\vec{x}_2)} + \frac{\tilde{\Omega}_2(\vec{x}_2,\vec{y},\vec{x}_1'',\vec{x}_2'')}{\phi(\vec{x}_1,\vec{x}_2)\phi^*(\vec{x}_1'',\vec{x}_2'')}\right)$$

$$\left.-\Omega_1(\vec{x}_2,\vec{y})\left(\frac{\phi(\vec{x}_1,\vec{y})}{\phi(\vec{x}_1,\vec{x}_2)} + \frac{\tilde{\Omega}_2(\vec{x}_1,\vec{y},\vec{x}_1'',\vec{x}_2'')}{\phi(\vec{x}_1,\vec{x}_2)\phi^*(\vec{x}_1'',\vec{x}_2'')}\right)\right\} d\vec{y}$$

Now

$$i\hbar\frac{\partial}{\partial t}\left\{(2\rho_s)^{\frac{1}{2}} c^{i\theta}\right\}$$

$$= \left\{\frac{i\hbar}{2\rho_s}\frac{\partial}{\partial t}\rho_s - \hbar\frac{\partial}{\partial t}\theta\right\}(2\rho_s)^{\frac{1}{2}} e^{i\theta} \; -\frac{\hbar^2}{2m}\left\{\left(\nabla_{\vec{x}_i} - \frac{ie}{\hbar c}\vec{A}(\vec{x}_1)\right)^2\right.$$

$$\left. + \left(\nabla_{x_2} - \frac{ie}{\hbar c}\vec{A}(\vec{x}_2)\right)^2\right\}\left\{(2\rho_s)^{\frac{1}{2}} c^{i\theta}\right\}$$

$$= -\frac{\hbar^2}{2m}\left[\frac{1}{2\rho_s}\left(\nabla^2_{\vec{x}_1}+\nabla^2_{\vec{x}_2}\right)\rho_s - \frac{1}{4\rho_s^2}\left\{(\nabla_{\vec{x}_1}\rho_s)^2+(\nabla_{\vec{x}_2}\rho_s)^2\right\}\right.$$

$$-\left\{(\nabla_{\vec{x}_1}\theta)^2+(\nabla_{\vec{x}_2}\theta)^2\right\} + i\left\{\frac{1}{\rho_s}\left(\nabla_{\vec{x}_1}\rho_s\cdot\nabla_{\vec{x}_1}\theta+\nabla_{\vec{x}_2}\rho_s\cdot\nabla_{\vec{x}_2}\theta\right)+\left(\nabla^2_{\vec{x}_1}+\nabla^2_{\vec{x}_2}\right)\theta\right\}$$

$$-\frac{ie}{\hbar c}\left\{\frac{1}{\rho_s}\left(\vec{A}(\vec{x}_1)\cdot\nabla_{\vec{x}_1}\rho_s+\vec{A}(\vec{x}_2)\cdot\nabla_{\vec{x}_2}\rho_s\right)+\left(\nabla_{\vec{x}_1}\cdot\vec{A}(\vec{x}_1)+\nabla_{\vec{x}_2}\cdot\vec{A}(\vec{x}_2)\right)\right.$$

$$\left.\left.+2i\left(\vec{A}(\vec{x}_1)\cdot\nabla_{\vec{x}_1}\theta+\vec{A}(\vec{x}_2)\cdot\nabla_{\vec{x}_2}\theta\right)\right\} - \frac{e^2}{\hbar^2c^2}\left\{\vec{A}^2(\vec{x}_1)+\vec{A}^2(\vec{x}_2)\right\}\right](2\rho_s)^{\frac{1}{2}} e^{i\theta}$$

Equation (5.24) becomes

$$\frac{i\hbar}{2\rho_s}\frac{\partial\rho_s}{\partial t}-\hbar\frac{\partial\theta}{\partial t}$$

$$= -\frac{\hbar^2}{2m}\Big[\frac{1}{2\rho_s}(\nabla^2_{\vec{x}_1}+\nabla^2_{\vec{x}_2})\rho_s-\frac{1}{4\rho_s^2}\Big\{(\nabla_{\vec{x}_1}\rho_s)^2+(\nabla_{\vec{x}_2}\rho_s)^2\Big\}-\Big\{(\nabla_{\vec{x}_1}\theta)^2+(\nabla_{\vec{x}_2}\theta)^2\Big\}$$

$$+\frac{2e}{\hbar c}\Big(\vec{A}(\vec{x}_1)\cdot\nabla_{\vec{x}_1}\theta+\vec{A}(\vec{x}_2)\cdot\nabla_{\vec{x}_2}\theta\Big)-\frac{e^2}{\hbar^2c^2}\Big\{\vec{A}^2(\vec{x}_1)+\vec{A}^2(\vec{x}_2)\Big\}\Big]-i\frac{\hbar^2}{2m}\Big[\frac{1}{\rho_s}$$

$$\Big\{\nabla_{\vec{x}_1}\rho_s\cdot\nabla_{\vec{x}_1}\theta+\nabla_{\vec{x}_2}\rho_s\cdot\nabla_{\vec{x}_2}\theta\Big\}+(\nabla^2_{\vec{x}_1}+\nabla^2_{\vec{x}_2})\theta-\frac{e}{\hbar c}\Big\{\frac{1}{\rho_s}\Big(\vec{A}(\vec{x}_1)\cdot\nabla_{\vec{x}_1}\rho_s+\vec{A}(\vec{x}_2)\cdot\nabla_{\vec{x}_2}\rho_s\Big)$$

$$+\Big(\nabla_{\vec{x}_1}\cdot\vec{A}(\vec{x}_1)+\nabla_{\vec{x}_2}\cdot\vec{A}(\vec{x}_2)\Big)\Big\}\Big]+V(\vec{x}_1-\vec{x}_2)+F$$

Take the real part of the above equation,

$$-\hbar\frac{\partial}{\partial t}\theta$$

$$= -\frac{\hbar^2}{2m}\Big[\frac{1}{2\rho_s}(\nabla^2_{\vec{x}_1}+\nabla^2_{\vec{x}_2})\rho_s-\frac{1}{4\rho_s^2}\Big\{(\nabla_{\vec{x}_1}\rho_s)^2+(\nabla_{\vec{x}_2}\rho_s)^2\Big\}$$

$$-\Big\{(\nabla_{\vec{x}_1}\theta)^2+(\nabla_{\vec{x}_2}\theta)^2\Big\}+\frac{2e}{\hbar c}\Big\{\vec{A}(\vec{x}_1)\cdot\nabla_{\vec{x}_1}\theta+\vec{A}(\vec{x}_2)\cdot\nabla_{\vec{x}_2}\theta\Big\}$$

$$-\frac{e^2}{\hbar^2c^2}\Big\{\vec{A}^2(\vec{x}_1)+\vec{A}^2(\vec{x}_2)\Big\}\Big]+V(\vec{x}_1-\vec{x}_2)+\mathrm{Re}\,F \qquad (5.25)$$

and the imginary part of it is

$$\frac{\partial}{\partial t}\rho_s = -\frac{\hbar}{m}\rho_s\Big[\frac{1}{\rho_s}\Big\{\nabla_{\vec{x}_1}\rho_s\cdot\nabla_{\vec{x}_1}\theta+\nabla_{\vec{x}_2}\rho_s\cdot\nabla_{\vec{x}_2}\theta\Big\}$$

$$+(\nabla^2_{\vec{x}_1}+\nabla^2_{\vec{x}_2})\theta-\frac{e}{\hbar c}\Big\{\frac{1}{\rho_s}\Big(\vec{A}(\vec{x}_1)\cdot\nabla_{\vec{x}_1}\rho_s+\vec{A}(\vec{x}_2)\cdot\nabla_{\vec{x}_2}\rho_s\Big)$$

$$+\Big(\nabla_{\vec{x}_1}\cdot\vec{A}(\vec{x}_1)+\nabla_{\vec{x}_2}\cdot\vec{A}(\vec{x}_2)\Big)\Big\}\Big]+\frac{2\rho_s}{\hbar}\,\mathrm{Im}\,F \tag{5.26}$$

We then define

$$2\rho_s\vec{v}_s = \frac{\hbar}{4mi}\lim_{\substack{\vec{x}_1\to\vec{x}''_1\\ \vec{x}_2\to\vec{x}''_2}}\Big\{\nabla_{\vec{x}_1}+\nabla_{\vec{x}_2}-\nabla_{\vec{x}''_1}-\nabla_{\vec{x}''_2}$$

$$-\frac{ie}{\hbar c}\Big(\vec{A}(\vec{x}_1)+\vec{A}(\vec{x}_2)+\vec{A}(\vec{x}''_1)+\vec{A}(\vec{x}''_2)\Big)\Big\}\phi^*(\vec{x}''_1,\vec{x}''_2)\phi(\vec{x}_1,\vec{x}_2) \tag{5.27}$$

Where $\vec{v}_s$ is the condensate velocity. By substituting Eq. (5.23) into (5.27)

$$2\rho_s\vec{v}_s = \frac{\hbar}{4mi}\lim_{\substack{\vec{x}_1\to\vec{x}''_1\\ \vec{x}_2\to\vec{x}''_2}}\Big[\nabla_{\vec{x}_1}+\nabla_{\vec{x}_2}-\nabla_{\vec{x}''_1}-\nabla_{\vec{x}''_2}-\frac{ie}{\hbar c}\Big(\vec{A}(\vec{x}_1)+\vec{A}(\vec{x}_2)+\vec{A}(\vec{x}''_1)+\vec{A}(\vec{x}''_2)\Big)\Big]$$

$$\Big[\Big\{2\rho_s(\vec{x}''_1,\vec{x}''_2)\Big\}^{\frac{1}{2}}\exp\Big\{-i\theta(\vec{x}''_1,\vec{x}''_2)\Big\}\Big\{2\rho_s(\vec{x}_1,\vec{x}_2)\Big\}^{\frac{1}{2}}\exp\Big\{i\theta(\vec{x}_1,\vec{x}_2)\Big\}\Big]$$

$$=\frac{\hbar}{4mi}\lim_{\substack{\vec{x}_1\to\vec{x}''_1\\ \vec{x}_2\to\vec{x}''_2}}\Big[\phi^*(\vec{x}''_1,\vec{x}''_2)\Big\{\nabla_{\vec{x}_1}+\nabla_{\vec{x}_2}\Big\}\Big\{2\rho_s(\vec{x}_1,\vec{x}_2)\Big\}^{\frac{1}{2}}\exp\Big\{i\theta(\vec{x}_1,\vec{x}_2)\Big\}$$

$$-\phi(\vec{x}_1,\vec{x}_2)\Big\{\nabla_{\vec{x}''_1}+\nabla_{\vec{x}''_2}\Big\}\Big\{2\rho_s(\vec{x}''_1,\vec{x}''_2)\Big\}^{\frac{1}{2}}\exp\Big\{-i\theta(\vec{x}''_1,\vec{x}''_2)\Big\}$$

$$-\frac{ie}{\hbar c}\Big\{\vec{A}(\vec{x}_1)+\vec{A}(\vec{x}_2)+\vec{A}(\vec{x}''_1)+\vec{A}(\vec{x}''_2)\Big\}\phi^*(\vec{x}''_1,\vec{x}''_2,)\phi(\vec{x}_1,\vec{x}_2)\Big]$$

$$=\frac{\hbar}{4mi}\lim_{\substack{\vec{x}_1\to\vec{x}''_1\\ \vec{x}_2\to\vec{x}''_2}}\Big[\frac{1}{2\rho_s(\vec{x}_1,\vec{x}_2)}(\nabla_{\vec{x}_1}+\nabla_{\vec{x}_2})\rho_s(\vec{x}_1,\vec{x}_2)+i(\nabla_{\vec{x}_1}+\nabla_{\vec{x}_2})\theta(\vec{x}_1,\vec{x}_2)$$

$$-\frac{1}{2\rho_s(\vec{x}''_1,\vec{x}''_2)}(\nabla_{\vec{x}''_1}+\nabla_{\vec{x}''_2})\rho_s(\vec{x}''_1,\vec{x}''_2)+i(\nabla_{\vec{x}''_1}+\nabla_{\vec{x}''_2})\theta(\vec{x}''_1,\vec{x}''_2)$$

$$-\frac{ie}{\hbar c}\Big(\vec{A}(\vec{x}_1)+\vec{A}(\vec{x}_2)+\vec{A}(\vec{x}''_1)+\vec{A}(\vec{x}''_2)\Big)\Big]\phi^*(\vec{x}''_1,\vec{x}''_2)\phi(\vec{x}_1,\vec{x}_2)$$

For a bulk system $(\nabla_{\vec{x}_1}+\nabla_{\vec{x}_2})\,\rho_s(\vec{x}_1,\vec{x}_2)$ is equal to zero, then

$$2\rho_s\vec{v}_s = \frac{\hbar}{4mi}\left[2i(\nabla_{\vec{x}_1}+\nabla_{\vec{x}_2})\theta(\vec{x}_1,\vec{x}_2)-2\frac{ie}{\hbar c}\left(\vec{A}(\vec{x}_1)+\vec{A}(\vec{x}_2)\right)\right]2\rho_s(\vec{x}_1,\vec{x}_2)$$

$$\text{or } \vec{v}_s = \frac{\hbar}{2m}(\nabla_{\vec{x}_1}+\nabla_{\vec{x}_2})\theta(\vec{x}_1,\vec{x}_2) - \frac{e}{2mc}\left(\vec{A}(\vec{x}_1)+\vec{A}(\vec{x}_2)\right)$$

let $\vec{R} = \frac{1}{2}(\vec{x}_1+\vec{x}_2)$ and $\vec{r} = \vec{x}_1-\vec{x}_2$, we rewrite $\vec{v}_s$ in terms of a new variables

$$\vec{v}_s = \frac{\hbar}{2m}\nabla_{\vec{R}}\theta(\vec{R},\vec{r}) - \frac{e}{mc}\vec{A}(\vec{R},\vec{r}) \qquad (5.28)$$

Equation (5.28) is refered to as define the condensate velocity. Also we define the relative velocity $\vec{v}_R$ which plays the role of the relative velocity of the particles of the pair.

$$\vec{v}_R = \frac{2\hbar}{m}\nabla_{\vec{r}}\theta(\vec{R},\vec{r}) - \frac{4e}{mc}\vec{A}(\vec{R},\vec{r}) \qquad (5.29)$$

We rewrite Eq. (5.25) in terms of new variables

$$-\hbar\frac{\partial}{\partial t}\theta = -\frac{\hbar^2}{2m}\left[\frac{1}{4\rho_s}(\nabla^2_{\vec{R}}+4\nabla^2_{\vec{r}})\rho_s - \frac{1}{8\rho_s^2}\left\{(\nabla_{\vec{R}}\rho_s)^2+4(\nabla_{\vec{r}}\rho_s)^2\right\}\right.$$
$$\left. -\frac{1}{2}\left\{(\nabla_{\vec{R}}\theta)^2+4(\nabla_{\vec{r}}\theta)^2\right\} + \frac{2e}{\hbar c}\vec{A}(\vec{R},\vec{r})\cdot\nabla_{\vec{R}}\theta - \frac{2e^2}{\hbar^2c^2}\vec{A}^2(\vec{R},\vec{r})\right]$$
$$+ V(\vec{r}) + \mathrm{Re}\,F.$$

Taking $-\frac{1}{2m}\nabla_{\vec{R}}$ on both sides of the above equation and using Eqs. (5.28) and (5.29)

$$\frac{\partial}{\partial t}\left[\vec{v}_s+\frac{e}{mc}\vec{A}(\vec{R},\vec{r})\right] = \frac{\hbar^2}{4m^2}\nabla_{\vec{R}}\left[\frac{1}{4_s}(\nabla^2_{\vec{R}}+4\nabla^2_{\vec{r}})\rho_s - \frac{1}{8\rho_s^2}\left\{(\nabla_{\vec{R}}\rho_s)^2+4(\nabla_{\vec{r}}\rho_s)^2\right\}\right.$$
$$-\frac{m^2}{\hbar^2}\left\{2\vec{v}_s^2+\frac{1}{2}v_{\vec{R}}^2+\frac{4e}{mc}(\vec{v}_s+\vec{v}_R)\cdot\vec{A}(\vec{R},\vec{r})+\frac{10e^2}{m^2c^2}\vec{A}^2(\vec{R},\vec{r})\right\}$$
$$\left.+\frac{4mc}{h^2c}\left\{\vec{v}_s\cdot\vec{A}(\vec{R},\vec{r}) + \frac{e}{mc}\vec{A}^2(\vec{R},\vec{r})\right\} - \frac{2e^2}{\hbar^2c^2}\vec{A}^2(\vec{R},\vec{r})\right]$$
$$-\frac{1}{2m}\nabla_{\vec{R}}V(\vec{r}) - \frac{1}{2m}\nabla_{\vec{R}}\,\mathrm{Re}\,F.$$

Finally

$$\frac{\partial}{\partial t}\vec{v}_s+(\vec{v}_s\cdot\nabla_{\vec{R}})\vec{v}_s+\frac{e}{mc}\frac{\partial}{\partial t}\vec{A}(\vec{R},\vec{r})$$

$$= -\nabla_{\vec{R}}\Big[-\frac{\hbar^2}{16m^2}\frac{1}{\rho_s}(\nabla^2_{\vec{R}}+4\nabla^2_{\vec{r}})\rho_s+\frac{\hbar^2}{32m^2}\frac{1}{\rho_s^2}\left\{(\nabla_{\vec{R}}\rho_s)^2+4(\nabla_{\vec{r}}\rho_s)^2\right\}$$

$$+\frac{1}{8}\vec{v}_R^{\,2}+\frac{e}{mc}\vec{v}_R\cdot\vec{A}(\vec{R},\vec{r})+\frac{2e^2}{m^2c^2}\vec{A}^2(\vec{R},\vec{r})+\frac{1}{2m}V(\vec{r})+\frac{1}{2m}\,\mathrm{Re}\,F\Big] \quad (5.30)$$

Equation (5.30) is the equation of motion of the condensate velocity $\vec{v}_S$. Next we will find the equation of motion of the condensate density ρ_s by rewriting Eq.(5.26) in terms of new variables

$$\frac{\partial}{\partial t}\rho_s = -\frac{\hbar}{m}\rho_s\Big[\frac{1}{2\rho_s}\left\{\nabla_{\vec{R}}\rho_s\cdot\nabla_{\vec{R}}\theta+4\nabla_{\vec{r}}\rho_s\cdot\nabla_{\vec{r}}\theta\right\}+\frac{1}{2}\left\{\nabla^2_{\vec{R}}+4\nabla^2_{\vec{r}}\right\}\theta$$

$$-\frac{e}{\hbar c}\left\{\frac{1}{\rho_s}\left(\vec{A}(\vec{R},\vec{r})\cdot\nabla_{\vec{R}}\rho_s\right)+\nabla_{\vec{R}}\cdot\vec{A}(\vec{R},\vec{r})\right\}\Big]+\frac{2\rho_s}{\hbar}\,\mathrm{Im}\,F.$$

Using a definition of $\vec{v}_s$ and $\vec{v}_r$

$$\frac{\partial}{\partial t}\rho_s = -\frac{\hbar}{m}\rho_s\Big[\frac{1}{2\rho_s}\left\{\frac{2m}{\hbar}\nabla_{\vec{R}}\rho_s\cdot\vec{v}_s+\frac{2m}{\hbar}\nabla_{\vec{r}}\rho_s\cdot\vec{v}_R+\frac{2e}{\hbar c}(\nabla_{\vec{R}}\rho_s+4\nabla_{\vec{r}}\rho_s)\cdot\vec{A}(\vec{R},\vec{r})\right\}$$

$$+\frac{1}{2}\left\{\frac{2m}{\hbar}\nabla_{\vec{R}}\cdot\vec{v}_s+\frac{2m}{\hbar}\nabla_{\vec{r}}\cdot\vec{v}_R+\frac{2e}{\hbar c}\nabla_{\vec{R}}\cdot\vec{A}(\vec{R},\vec{r})+\frac{8e}{\hbar c}\nabla_{\vec{r}}\cdot\vec{A}(\vec{R},\vec{r})\right\}$$

$$-\frac{e}{\hbar c}\frac{1}{\rho_s}\left(\nabla_{\vec{R}}\rho_s\cdot\vec{A}(\vec{R},\vec{r})\right)-\frac{e}{\hbar c}\nabla_{\vec{R}}\cdot\vec{A}(\vec{R},\vec{r})\Big]+\frac{2\rho_s}{h}\,\mathrm{Im}\,F.$$

Or

$$\frac{\partial}{\partial t}\rho_s = -\nabla_{\vec{R}}\rho_s\cdot\vec{v}_s-\nabla_{\vec{r}}\rho_s\cdot\vec{v}_R-\frac{e}{mc}(\nabla_{\vec{R}}\rho_s+4\nabla_{\vec{r}}\rho_s)\cdot\vec{A}(\vec{R}\cdot\vec{r})-\rho_s\nabla_{\vec{R}}\cdot\vec{v}_s$$

$$-\rho_s\nabla_{\vec{r}}\cdot\vec{v}_R-\frac{e}{mc}\rho_s\nabla_{\vec{R}}\cdot\vec{A}(\vec{R},\vec{r})-\frac{4e}{mc}\rho_s\nabla_{\vec{r}}\cdot\vec{A}(\vec{R},\vec{r})+\frac{e}{mc}\nabla_{\vec{R}}\rho_s\cdot\vec{A}(\vec{R},\vec{r})$$

$$+\frac{e}{mc}\rho_s\nabla_{\vec{R}}\cdot\vec{A}(\vec{R},\vec{r})+\frac{2\rho_s}{\hbar}\,\mathrm{Im}\,F.$$

Finally

$$\frac{\partial}{\partial t}\rho_s + \nabla_{\vec{R}}\cdot(\rho_s \vec{v}_s) = -\nabla_{\vec{r}}\cdot\left[\rho_s \vec{v}_R + \frac{4e}{mc}\rho_s \cdot \vec{A}(\vec{R},\vec{r})\right] + \frac{2\rho_s}{\hbar}\,\mathrm{Im}\,F \qquad (5.31)$$

Equation (5.31) is the equation of motion of the condensate density ρ_s. We can see that the two equations of motion are incomplete because of the last term F. For simpliflying F, let us neglect the term $\tilde{\Omega}_2$.

We can simplifly F by assuming that the function multiplying $V(\vec{x}_1-\vec{y}) + V(\vec{x}_2-\vec{y})$ vary slowly compared with $V(\vec{x}_1-\vec{y}) + V(\vec{x}_2-\vec{y})$ so that they can be developed into $\vec{y}-\vec{x}_1$ or $\vec{y}-\vec{x}_2$ near $\vec{x}_1$ or $\vec{x}_2$. By introduction $\vec{R}_i = 1/2\,(\vec{x}_i + \vec{y})$ and $\vec{r}_i = \vec{x}_i-\vec{y}$, $i = 1, 2$, then equate $\vec{r}_1=\vec{r}_2=\vec{r}$. We then find

$$\phi(\vec{x}_1,\vec{y}) = \phi(\vec{x}_1,\vec{x}_2-\vec{r})$$

$$= \phi(\vec{x}_1,\vec{x}_2) - r_k\,\partial_{x_{2_k}}\phi(\vec{x}_1,\vec{x}_2) + \frac{1}{2} r_k r_l\,\partial_{x_{2_k}}\partial_{x_{2_l}}\phi(\vec{x}_1,\vec{x}_2) + \ldots$$

$$\phi(\vec{x}_2,\vec{y}) = \phi(\vec{x}_2,\vec{x}_1) - r_k\,\partial_{x_{1_k}}\phi(\vec{x}_2,\vec{x}_1) + \frac{1}{2} r_k r_l\,\partial_{x_{1_k}}\partial_{x_{2_l}}\phi(\vec{x}_2,\vec{x}_1) + \ldots$$

$$\Omega_1(\vec{x}_1,\vec{y}) = \Omega_1(\vec{x}_1,\vec{x}_2) - r_k\,\partial_{x_{2_k}}\Omega_1(\vec{x}_1,\vec{x}_2) + \frac{1}{2} r_k r_l\,\partial_{x_{2_k}}\partial_{x_{2_l}}\Omega_1(\vec{x}_1,\vec{x}_2) + \ldots$$

$$\Omega_1(\vec{x}_2,\vec{y}) = \Omega_1(\vec{x}_2,\vec{x}_1) - r_k\,\partial_{x_{1_k}}\Omega_1(\vec{x}_2,\vec{x}_1) + \frac{1}{2} r_k r_l\,\partial_{x_{1_k}}\partial_{x_{1_l}}\Omega_1(\vec{x}_2,\vec{x}_1) + \ldots$$

$$\Omega_1(\vec{y},\vec{y}) = \Omega_1(\vec{x}_1,\vec{x}_2) - r_k(\partial_{x_{1_k}} + \partial_{x_{2_k}})\Omega_1(\vec{x}_1,\vec{x}_2) + \frac{1}{2} r_k r_l(\partial_{x_{1_k}}\partial_{x_{1_l}} + \partial_{x_{2_k}}\partial_{x_{2_l}})$$

$$\Omega_1(\vec{x}_1,\vec{x}_2) + \ldots$$

And $\Omega_1(\vec{x}_1,\vec{y})\,\dfrac{\phi(\vec{x}_2,\vec{y})}{\phi(\vec{x}_1,\vec{x}_2)}$

$$=\left\{\Omega_1(\vec{x}_1,\vec{x}_2)-r_k\partial_{x_{2_k}}\Omega_1(\vec{x}_1,\vec{x}_2)+\frac{1}{2}r_kr_l\partial_{x_{2_k}}\partial_{x_{2_l}}\Omega_1(\vec{x}_1,\vec{x}_2)\right\}$$

$$\left\{-1+\frac{r_k\partial_{x_{1_k}}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)}-\frac{\frac{1}{2}r_kr_l\partial_{x_{1_k}}\partial_{x_{1_l}}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)}\right\}$$

$$=-\Omega_1(\vec{x}_1,\vec{x}_2)+r_k\partial_{x_{2_k}}\Omega_1(\vec{x}_1,\vec{x}_2)-\frac{1}{2}r_kr_l\partial_{x_{2_k}}\partial_{x_{2_l}}\Omega_1(\vec{x}_1,\vec{x}_2)+\Omega_1(\vec{x}_1,\vec{x}_2)$$

$$\frac{r_k\partial_{x_{1_k}}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)}-r_kr_l\partial_{x_{2_k}}\Omega_1(\vec{x}_1,\vec{x}_2)\frac{\partial_{x_{1_l}}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)}-\frac{1}{2}\Omega_1(\vec{x}_1,\vec{x}_2)r_kr_l$$

$$\frac{\partial_{x_{1_k}}\partial_{x_{1_l}}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)}$$

By using the relation ($f(\vec{r})$ is a function of $|\vec{r}|$)

$$\int r_k\; f(\vec{r})\; d\vec{r} = 0$$

$$\int r_k\, r_l\; f(\vec{r})\; d\vec{r} = \delta_{kl}\,\frac{1}{3}\int \vec{r}^{\,2} f(\vec{r})\; d\vec{r} \qquad (5.32)$$

$$\int\left\{V(\vec{x}_1-\vec{y})+V(\vec{x}_2-\vec{y})\right\}\Omega_1(\vec{x}_1,\vec{y})\frac{\phi(\vec{x}_2,\vec{y})}{\phi(\vec{x}_1,\vec{x}_2)}\,d\vec{y}$$

$$=-2\int V(\vec{r})\left[-\Omega_1(\vec{x}_1,\vec{x}_2)+r_k\left\{\partial_{x_{2_k}}\Omega_1(\vec{x}_1,\vec{x}_2)+\Omega_1(\vec{x}_1,\vec{x}_2)\frac{\partial_{x_{1_k}}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)}\right\}\right.$$

$$-r_kr_l\left\{\frac{1}{2}\partial_{x_{2_k}}\partial_{x_{2_l}}\Omega_1(\vec{x}_1,\vec{x}_2)+\partial_{x_{2_k}}\Omega_1(\vec{x}_1,\vec{x}_2)\frac{\partial_{x_{1_l}}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)}\right.$$

$$\left.\left.+\frac{1}{2}\Omega_1(\vec{x}_1,\vec{x}_2)\frac{\partial_{x_{1_k}}\partial_{x_{1_l}}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)}\right\}\right]d\vec{r}$$

$$=2\Omega_1(\vec{x}_1,\vec{x}_2)\int V(\vec{r})d\vec{r}+\left[\nabla^2_{\vec{x}_2}\Omega_1(\vec{x}_1,\vec{x}_2)+2\nabla_{\vec{x}_2}\Omega_1(\vec{x}_1,\vec{x}_2)\frac{\nabla_{\vec{x}_1}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)}+\Omega_1(\vec{x}_1,\vec{x}_2)\right.$$

$$\left.\frac{\nabla^2_{\vec{x}_1}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)}\right]\frac{1}{3}\int \vec{r}^{\,2}V(\vec{r})\,d\vec{r}$$

$$= 2\Omega_1(\vec{x}_1,\vec{x}_2)\omega_1 + \Big[\nabla^2_{\vec{x}_2}\Omega_1(\vec{x}_1,\vec{x}_2) + 2\nabla_{\vec{x}_2}\Omega_1(\vec{x}_1,\vec{x}_2)\frac{\nabla_{\vec{x}_1}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)}$$

$$+ \Omega_1(\vec{x}_1,\vec{x}_2)\frac{\nabla^2_{\vec{x}_1}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)}\Big]\omega_2 \qquad (5.33)$$

$$\omega_1 = \int V(\vec{r})\, d\vec{r}$$

$$\omega_2 = \frac{1}{3}\int \vec{r}^{\,2} V(\vec{r})\, d\vec{r} \qquad (5.34)$$

Furthermore $\Omega_1(\vec{x}_2,\vec{y})\dfrac{\phi(\vec{x}_1,\vec{y})}{\phi(\vec{x}_1,\vec{x}_2)}$

$$= \Big\{\Omega_1(\vec{x}_1,\vec{x}_2) - r_k\partial_{x_{1_k}}\Omega_1(\vec{x}_1,\vec{x}_2) + \frac{1}{2}r_k r_l\partial_{x_{1_k}}\partial_{x_{1_l}}\Omega_1(\vec{x}_1,\vec{x}_2)\Big\}$$

$$\Big\{1 - r_k\frac{\partial_{x_{2_k}}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)} + \frac{1}{2}r_k r_l\frac{\partial_{x_{2_k}}\partial_{x_{2_l}}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)}\Big\}$$

$$= \Omega_1(\vec{x}_1,\vec{x}_2) - r_k\partial_{x_{1_k}}\Omega_1(\vec{x}_1,\vec{x}_2) + \frac{1}{2}r_k r_l\partial_{x_{1_k}}\partial_{x_{1_l}}\Omega_1(\vec{x}_1,\vec{x}_2)$$

$$-\Omega_1(\vec{x}_1,\vec{x}_2)r_k\frac{\partial_{x_{2_k}}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)} + r_k r_l\partial_{x_{1_k}}\Omega_1(\vec{x}_1,\vec{x}_2)\frac{\partial_{x_{2_l}}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)}$$

$$+\frac{1}{2}\Omega_1(\vec{x}_1,\vec{x}_2)r_k r_l\frac{\partial_{x_{2_k}}\partial_{x_{2_l}}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)}$$

$$-\int\Big\{V(\vec{x}_1-\vec{y}) + V(\vec{x}_2-\vec{y})\Big\}\Omega_1(\vec{x}_2,\vec{y})\frac{\phi(\vec{x}_1,\vec{y})}{\phi(\vec{x}_1,\vec{x}_2)}\, d\vec{y}$$

$$= 2\int V(\vec{r})\Big[\Omega_1(\vec{x}_1,\vec{x}_2) - r_k\Big\{\partial_{x_{1_k}}\Omega_1(\vec{x}_1,\vec{x}_2) + \Omega_1(\vec{x}_1,\vec{x}_2)\frac{\partial_{x_{2_k}}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)}\Big\}$$

$$+ r_k r_l\Big\{\frac{1}{2}\partial_{x_{1_k}}\partial_{x_{1_l}}\Omega_1(\vec{x}_1,\vec{x}_2) + \partial_{x_{1_k}}\Omega_1(\vec{x}_1,\vec{x}_2)\frac{\partial_{x_{2_l}}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)}$$

$$+\frac{1}{2}\Omega_1(\vec{x}_1,\vec{x}_2)\frac{\partial_{x_{2_k}}\partial_{x_{2_l}}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)}\Big\}\Big]\, d\vec{r}$$

$$= 2\Omega_1(\vec{x}_1,\vec{x}_2)\omega_1 + \Big[\nabla^2_{\vec{x}_1}\Omega_1(\vec{x}_1,\vec{x}_2) + 2\nabla_{\vec{x}_1}\Omega_1(\vec{x}_1,\vec{x}_2)\frac{\nabla_{\vec{x}_2}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)}$$

$$+\Omega_1(\vec{x}_1,\vec{x}_2)\frac{\nabla^2_{\vec{x}_2}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)}\Big]\omega_2 \qquad (5.35)$$

Finally

$$\int\left\{V(\vec{x}_1-\vec{y})+V(\vec{x}_2-\vec{y})\right\}\Omega_1(\vec{y},\vec{y})\,d\vec{y}$$

$$= -2\int V(\vec{r})\left\{\Omega_1(\vec{x}_1,\vec{x}_2)-r_k(\partial_{x_{1_k}}+\partial_{x_{2_k}})\Omega_1(\vec{x}_1,\vec{x}_2)\right.$$

$$\left.+\ \frac{1}{2}r_k r_l(\partial_{x_{1_k}}\partial_{x_{1_l}}+\partial_{x_{2_k}}\partial_{x_{2_l}})\Omega_1(\vec{x}_1,\vec{x}_2)\right\}d\vec{r}$$

$$= -2\Omega_1(\vec{x}_1,\vec{x}_2)\omega_1-(\nabla^2_{\vec{x}_1}+\nabla^2_{\vec{x}_2})\Omega_1(\vec{x}_1,\vec{x}_2)\,\omega_2 \tag{5.36}$$

From Eqs.(5.33), (5.35) and (5.36) we get

$$\int\left\{V(\vec{x}_1-\vec{y})+V(\vec{x}_2-\vec{y})\right\}\left\{\Omega_1(\vec{y},\vec{y})+\Omega_1(\vec{x}_1,\vec{y})\frac{\phi(\vec{x}_2,\vec{y})}{\phi(\vec{x}_1,\vec{x}_2)}-\Omega_1(\vec{x}_2,\vec{y})\frac{\phi(\vec{x}_1,\vec{y})}{\phi(\vec{x}_1,\vec{x}_2)}\right\}d\vec{y}$$

$$= 2\Omega_1(\vec{x}_1,\vec{x}_2)\omega_1+\left[2\nabla_{\vec{x}_1}\Omega_1(\vec{x}_1,\vec{x}_2)\frac{\nabla_{\vec{x}_2}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)}+2\nabla_{\vec{x}_2}\Omega_1(\vec{x}_1,\vec{x}_2)\frac{\nabla_{\vec{x}_1}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)}\right.$$

$$\left.+\Omega_1(\vec{x}_1,\vec{x}_2)\frac{\nabla^2_{\vec{x}_1}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)}+\Omega_1(\vec{x}_1,\vec{x}_2)\frac{\nabla^2_{\vec{x}_2}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)}\right]\omega_2$$

Now

$$\nabla_{\vec{x}}\phi(\vec{x}_1,\vec{x}_2) = \left\{\frac{1}{2\rho_s}\nabla_{\vec{x}}\rho_s+i\nabla_{\vec{x}}\theta\right\}\phi(\vec{x}_1,\vec{x}_2)$$

$$\nabla^2_{\vec{x}}\phi(\vec{x}_1,\vec{x}_2) = \left\{\frac{1}{2\rho_s}(\nabla^2_{\vec{x}}\rho_s)-\frac{1}{4\rho_s^2}(\nabla_{\vec{x}}\rho_s)^2-(\nabla_{\vec{x}}\theta)^2\right.$$

$$\left.+\ i\,\frac{1}{\rho_s}(\nabla_{\vec{x}}\rho_s\cdot\nabla_{\vec{x}}\theta)+i\nabla^2_{\vec{x}}\theta\right\}\phi(\vec{x}_1,\vec{x}_2)$$

Then

$$2\nabla_{\vec{x}_1}\Omega_1(\vec{x}_1,\vec{x}_2)\cdot\frac{\nabla_{\vec{x}_2}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)}+2\nabla_{\vec{x}_2}\Omega_1(\vec{x}_1,\vec{x}_2)\frac{\nabla_{\vec{x}_1}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)}$$

$$= \frac{1}{\rho_s}(\nabla_{\vec{x}_1}\Omega_1\cdot\nabla_{\vec{x}_2}\rho_s+\nabla_{\vec{x}_2}\Omega_1\cdot\nabla_{\vec{x}_1}\rho_s)+2i(\nabla_{\vec{x}_1}\Omega_1\cdot\nabla_{\vec{x}_2}\theta+\nabla_{\vec{x}_2}\Omega_1\cdot\nabla_{\vec{x}_1}\theta)$$

$$= \frac{1}{2\rho_s}(\nabla_{\vec{R}}\Omega_1\cdot\nabla_{\vec{R}}\rho_s-4\nabla_{\vec{r}}\Omega_1\cdot\nabla_{\vec{r}}\rho_s)+i(\nabla_{\vec{R}}\Omega_1\cdot\nabla_{\vec{R}}\theta-4\nabla_{\vec{r}}\Omega_1\cdot\nabla_{\vec{r}}\theta)$$

$$= \frac{1}{2\rho_s}\left(\nabla_{\vec{R}}\Omega_1\cdot\nabla_{\vec{R}}\rho_s - 4\nabla_{\vec{r}}\Omega_1\cdot\nabla_{\vec{r}}\rho_s\right) + \frac{2mi}{\hbar}\Big\{\nabla_{\vec{R}}\Omega_1\cdot\vec{V}_s - \nabla_{\vec{r}}\Omega_1\cdot\vec{V}_R$$

$$+ \frac{e}{mc}\left(\nabla_{\vec{R}}\Omega_1 - 4\nabla_{\vec{r}}\Omega_1\right)\cdot\vec{A}(\vec{R},\vec{r})\Big\}$$

And $$\Omega_1(\vec{x}_1,\vec{x}_2)\left\{\frac{\nabla^2_{\vec{x}_1}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)} + \frac{\nabla^2_{\vec{x}_2}\phi(\vec{x}_1,\vec{x}_2)}{\phi(\vec{x}_1,\vec{x}_2)}\right\}$$

$$= \Omega_1\Big[\frac{1}{2\rho_s}\left(\nabla^2_{\vec{x}_1} + \nabla^2_{\vec{x}_2}\right)\rho_s - \frac{1}{4\rho_s^2}\left((\nabla_{\vec{x}_1}\rho_s)^2 + (\nabla_{\vec{x}_2}\rho_s)^2\right) - (\nabla_{\vec{x}_1}\theta)^2 - (\nabla_{\vec{x}_2}\theta)^2$$

$$+ i\left\{\frac{1}{\rho_s}\left(\nabla_{\vec{x}_1}\rho_s\cdot\nabla_{\vec{x}_1}\theta + \nabla_{\vec{x}_2}\rho_s\cdot\nabla_{\vec{x}_2}\theta\right) + \left(\nabla^2_{\vec{x}_1} + \nabla^2_{\vec{x}_2}\right)\theta\right\}\Big]$$

$$= \Omega_1\Big[\frac{1}{4\rho_s}\left(\nabla^2_{\vec{R}} + 4\nabla^2_{\vec{r}}\right)\rho_s - \frac{1}{8\rho_s^2}\left((\nabla_{\vec{R}}\rho_s)^2 + 4(\nabla_{\vec{r}}\rho_s)^2\right) - \frac{1}{2}\left((\nabla_{\vec{R}}\theta)^2 + 4(\nabla_{\vec{r}}\theta)^2\right)$$

$$+ i\left\{\frac{1}{2\rho_s}\left(\nabla_{\vec{R}}\rho_s\cdot\nabla_{\vec{R}}\theta + 4\nabla_{\vec{r}}\rho_s\cdot\nabla_{\vec{r}}\theta\right) + \frac{1}{2}\left(\nabla^2_{\vec{R}} + 4\nabla^2_{\vec{r}}\right)\theta\right\}\Big]$$

$$= \Omega_1\Big[\frac{1}{4\rho_s}\left(\nabla^2_{\vec{R}} + 4\nabla^2_{\vec{r}}\right)\rho_s - \frac{1}{8\rho_s^2}\left((\nabla_{\vec{R}}\rho_s)^2 + 4(\nabla_{\vec{r}}\rho_s)^2\right) - \frac{2m^2}{\hbar^2}\left(\vec{v}_s^{\,2} + \frac{1}{4}\vec{v}_R^{\,2}\right)$$

$$- \frac{4me}{\hbar^2 c}\left(\vec{v}_s + \vec{v}_R\right)\cdot\vec{A}(\vec{R},\vec{r}) - \frac{10e^2}{\hbar^2 c^2}\vec{A}^2(\vec{R},\vec{r}) + i\Big\{\frac{m}{\hbar}\frac{1}{\rho_s}\left(\nabla_{\vec{R}}\rho_s\cdot\vec{v}_s + \nabla_{\vec{r}}\rho_s\cdot\vec{v}_R\right)$$

$$+ \frac{e}{\hbar c}\frac{1}{\rho_s}\left(\nabla_{\vec{R}}\rho_s + 4\nabla_{\vec{r}}\rho_s\right)\cdot\vec{A}(\vec{R},\vec{r}) + \frac{m}{\hbar}\left(\nabla_{\vec{R}}\cdot\vec{v}_s + \nabla_{\vec{r}}\cdot\vec{v}_r\right) + \frac{e}{\hbar c}\left(\nabla_{\vec{R}} + 4\nabla_{\vec{r}}\right)\cdot\vec{A}(\vec{R},\vec{r})\Big\}\Big]$$

At last we get the expressions for F, that is

$$F = 2\Omega_1\omega_1 + \frac{1}{2\rho_s}\left(\nabla_{\vec{R}}\Omega_1\cdot\nabla_{\vec{R}}\rho_s - 4\nabla_{\vec{r}}\Omega_1\cdot\nabla_{\vec{r}}\rho_s\right)\omega_2 + \Omega_1\Big[\frac{1}{4\rho_s}\left(\nabla^2_{\vec{R}} + 4\nabla^2_{\vec{r}}\right)\rho_s$$

$$- \frac{1}{8\rho_s^2}\left((\nabla_{\vec{R}}\rho_s)^2 + 4(\nabla_{\vec{r}}\rho_s)^2\right) - \frac{2m^2}{\hbar^2}\left(\vec{v}_s^{\,2} + \frac{1}{4}\vec{v}_R^{\,2}\right) - \frac{4me}{\hbar^2 c}\left(\vec{v}_s + \vec{v}_R\right)\cdot\vec{A}(\vec{R},\vec{r})$$

$$- \frac{10e^2}{\hbar^2 c^2}\vec{A}^2(\vec{R},\vec{r})\Big]\omega_2 + i\Big[\frac{2m}{\hbar}\Big\{\nabla_{\vec{R}}\Omega_1\cdot\vec{v}_s - \nabla_{\vec{r}}\Omega_1\cdot\vec{v}_R + \frac{e}{mc}\left(\nabla_{\vec{R}}\Omega_1 - 4\nabla_{\vec{r}}\Omega_1\right)\cdot\vec{A}(\vec{R},\vec{r})\Big\}$$

$$+ \Omega_1\Big\{\frac{m}{\hbar}\frac{1}{\rho_s}\left(\nabla_{\vec{R}}\rho_s\cdot\vec{v}_s + \nabla_{\vec{r}}\rho_s\cdot\vec{v}_R\right) + \frac{e}{\hbar c}\frac{1}{\rho_s}\left(\nabla_{\vec{R}}\rho_s + 4\nabla_{\vec{r}}\rho_s\right)\cdot\vec{A}(\vec{R},\vec{r})$$

$$+ \frac{m}{\hbar}\left(\nabla_{\vec{R}}\cdot\vec{v}_s + \nabla_{\vec{r}}\cdot\vec{v}_R\right) + \frac{e}{\hbar c}\left(\nabla_{\vec{R}} + 4\nabla_{\vec{r}}\right)\cdot\vec{A}(\vec{R},\vec{r})\Big\}\Big]\omega_2 \qquad (5.37)$$

From Eq. (5.37) we can find

$$
\begin{aligned}
\frac{1}{2m}\,\mathrm{Re}\,F &= \frac{\omega_1}{m}\Omega_1 + \frac{\omega_2}{4m}\frac{1}{\rho_s}\left(\nabla_{\vec R}\Omega_1\cdot\nabla_{\vec R}\rho_s - 4\nabla_{\vec r}\Omega_1\cdot\nabla_{\vec r}\rho_s\right) + \frac{\omega_2}{2m}\Omega_1\Bigg[\frac{1}{4\rho_s}\left(\nabla^2_{\vec R}+4\nabla^2_{\vec r}\right)\rho_s \\
&= -\frac{1}{8\rho_s^2}\Big((\nabla_{\vec R}\rho_s)^2+4(\nabla_{\vec r}\rho_s)^2\Big) - \frac{2m^2}{\hbar^2}\left(\vec v_s^{\,2}+\frac{1}{4}\vec v_R^{\,2}\right) \\
&\quad - \frac{4me}{\hbar^2 c}(\vec v_s+\vec v_R)\cdot\vec A(\vec R,\vec r) - \frac{10e^2}{\hbar^2c^2}\vec A^2(\vec R,\vec r)\Bigg] \qquad (5.38)
\end{aligned}
$$

$$
\begin{aligned}
\frac{2\rho_s}{\hbar}\,\mathrm{Im}\,F &= \omega_2\Bigg[\frac{4m}{\hbar^2}\rho_s\left(\nabla_{\vec R}\Omega_1\cdot\vec v_s - \nabla_{\vec r}\Omega_1\cdot\vec v_R\right) + \frac{4e}{\hbar^2 c}\rho_s\left(\nabla_{\vec R}\Omega_1 - 4\nabla_{\vec r}\Omega_1\right)\cdot\vec A(\vec R,\vec r) \\
&\quad + \frac{2m}{\hbar^2}\Omega_1\left(\nabla_{\vec R}\rho_s\cdot\vec v_s + \nabla_{\vec r}\rho_s\cdot\vec v_R\right) + \frac{2e}{\hbar^2 c}\Omega_1\left(\nabla_{\vec R}\rho_s + 4\nabla_{\vec r}\rho_s\right)\cdot\vec A(\vec R,\vec r) \\
&\quad + \frac{2m}{\hbar^2}\Omega_1\rho_s\left(\nabla_{\vec R}\cdot\vec v_s + \nabla_{\vec r}\cdot\vec v_R\right) + \frac{2e}{\hbar^2 c}\Omega_1\rho_s\left(\nabla_{\vec R}+4\nabla_{\vec r}\right)\cdot\vec A(\vec R,\vec r)\Bigg] \\
&= \omega_2\Bigg[\frac{2m}{\hbar^2}\Omega_1\left\{\nabla_{\vec R}\cdot(\rho_s\vec v_s) + \nabla_{\vec r}\cdot(\rho_s\vec v_R)\right\} \\
&\quad + \frac{2e}{\hbar^2 c}\Omega_1\left\{\nabla_{\vec R}\cdot(\rho_s\vec A(\vec R,\vec r)) + 4\nabla_{\vec r}\cdot(\rho_s\vec A(\vec R,\vec r))\right\} \\
&\quad + \frac{4m}{\hbar^2}\rho_s\left(\nabla_{\vec R}\Omega_1\cdot\vec v_s - \nabla_{\vec r}\Omega_1\cdot\vec v_R\right) + \frac{4e}{\hbar^2 c}\rho_s\left(\nabla_{\vec R}\Omega_1 - 4\nabla_{\vec r}\Omega_1\right)\cdot\vec A(\vec R,\vec r)\Bigg] \qquad (5.39)
\end{aligned}
$$

The last two equations, when we substitute into Eqs. (5.30) and (5.31), complete the equations of motion of the condensate velocity and density.