CHAPTER II

THEORY OF SUPERCONDUCTIVITY

Superconductivity was first discovered in 1911 by H. Kamerlingh Onnes in Leiden just three years after he had first liquefied helium. In this chapter we will review their early phenomenological description. Then we will briefly sketch the evolution of the concepts of Bardeen, Cooper and Schrieffer. Some of these problems are collaborating of our studies that we will cite in the further chapter.

2.1 THE BASIC PHENOMENA

What Kamerlingh Onnes observed was that the electric resistance vanished completely at a critical temperature T_c. Thus perfect conductivity is the first property of superconductivity. The next property to be discovered was perfect diamagnetism by Meissner and Ochsenfeld. They found that not only that a magnetic field is excluded from a superconductor that may be explained by the theory of perfect conductivity but is also expelled from a normal metal when it is cooled down through T_c. For perfect conductivity prevents a field from penetrating a superconductor by setting up an eddy currents that just cancell the applied field. But Meissner effect cannot be explained by perfect conductivity which tends to trap flux in.

2.2 THE LONDON EQUATIONS

The two basic phenomena of electrodynamic properties were described in 1935 by F. and H. London. These two equations governing the macroscopic electric and magnetic fields are

$$\vec{E} = \frac{\partial}{\partial t}(\Lambda \vec{J}_s), \qquad (2.1)$$

and

$$\vec{H} = -c \nabla \times (\Lambda \vec{J}_s), \qquad (2.2)$$

where

$$\Lambda = 4\pi\lambda_L^2/c^2 = m/n_s e^2 \qquad (2.3)$$

is a phenomenological parameter, and n_s is the number density of superconducting electrons that varies continuously from zero at T_c to a limiting value of the order of n, the density of conduction electrons at $T<T_c$

Equation (2.1) describes perfect conductivity since the electric field will accelerate the superconducting electrons. The Eq. (2.2) combined with the Maxwell equation $\nabla \times \vec{H} = \frac{4\pi}{C}\vec{J}$ gives

$$\nabla^2\vec{H} = \frac{1}{\lambda_L^2}\vec{H}, \qquad (2.4)$$

this equation demonstates that a magnetic field is exponentially screened from the interior of a sample with a penetration depth λ_L that is the Meissner effect. Thus the parameter λ_L in Eq (2.3) is defined as a penetration depth.

A derivation of the London equations comes from noting that the canonical momentum $\vec{P} = m\vec{v} + e\vec{A}/c$, when there is no applied field we expect that the net momentum is zero at the ground state. The average velocity, in the presence of a field, is

$$\langle \vec{V}_s \rangle = -e\vec{A}/mc \qquad (2.5)$$

Let $\langle n_s \rangle$ be the number density of electrons in the ground state, Eq. (2.5) leads to

$$\vec{J}_s = n_s e \langle \vec{v}_s \rangle = -n_s e^2 \vec{A}/mc = -\vec{A}/\Lambda c \qquad (2.6)$$

Taking the time derivative of Eq. (2.6) it leads to (2.1), and taking the curl of Eq.(2.6) we obtain the Eq.(2.2)

Equation (2.6) is not gauge invariance. That is, when $\vec{A}' \longrightarrow \vec{A} + \nabla\chi$, this leaves the magnetic field $\vec{H} = \nabla \times \vec{A}$ unchanged. But it changes the current in Eq. (2.6), thus we must choose a unique gauge that Eq. (2.6) manifests the gauge invariance. The choice, known as London gauge, requires that $\nabla \cdot \vec{A} = 0$ so that $\nabla \cdot \vec{J}_s = 0$ in avoiding any build up of charge density. For changing $\vec{A}$ to $\vec{A}'$, scalar field χ satisfies Laplace equation $\nabla^2\chi = 0$. Thus the London gauge leaves both $\nabla \cdot \vec{A}$ and $\nabla \times \vec{A}$ unchanged.

2.3 THE GINZBURG-LANDAU THEORY

In 1950, Ginzburg and Landau had introduced a complex pseudowave function ψ as an order parameter for the superconducting electrons in such a way that the local density of superconducting electrons was given by

$$n_s = |\psi(\vec{x})|^2 \qquad (2.7)$$

By using the variational principle with an assumed expansion of the free energy in power of ψ and $\nabla\psi$, they obtained the conditions

$$\frac{1}{2m^*}\left(\frac{\hbar}{i}\nabla - \frac{e^*}{c}\vec{A}\right)^2\psi + |\psi|^2\psi = -\alpha\psi \qquad (2.8)$$

and
$$\vec{J}_s = \frac{e^*\hbar}{2m^*i}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{e^{*2}}{m^*c}|\psi|^2\vec{A}, \qquad (2.9)$$

which are called the Ginzburg-Landau equations. Equation (2.8) is analogous to Schrödinger equation for a free particle but without a nonlinear term and Eq. (2.9) is the equation of the supercurrent as usual as in quantum mechanics for particles of charge e^* and mass m^*. And $e^* = 2e$ is the charge of a pair of electrons and m^*, an arbitrary value, it can be conveniently chosen twice the mass of the free electron, i.e. $m^* = 2m$.

With this formalism, it provides an explanation beyond the London theory. These are nonlinear effects in the fields strong enough to change n_s and spatial variation of n_s. This theory can explain the intermediate state of superconductors, that is the interface between superconduction and normal domains when an applied magnetic field is approximated to be a critical magnetic field.

The Ginzburg-Landau theory is a limiting form of the microscopic theory of Bardeen, Cooper and Schrieffer (BCS), valid near T_c, in which ψ is directly proportional to the gap parameter Δ. More physically, ψ can be thought of as the wave function in the center of mass coordinate system of the Cooper pairs.

2.4 THE BCS THEORY (2)

The BCS theory evolved from the observation of Cooper(3) that if a pair of electrons were slightly excited from the Fermi sea, they could form a real bound state provided that there was a weak attractive potential. This state had the lowest energy if the net momentum is zero. But if the system has a bound state by putting two electrons to form a Cooper pair, the Fermi sea ground state is then unstable against this formation. The BCS theory provides a formalism for handling such a state.

In terms of second quantization, using $c^{*}_{\vec{k},\sigma}$ and $c_{\vec{k},\sigma}$ which are creation and annihilation operators with momentum $\vec{k}$ and spin index σ. These operators satisfy the anticommutation relations

$$\left\{ c_{\vec{k}\sigma}, c^{*}_{\vec{k}'\sigma'} \right\} = \delta_{\vec{k}\vec{k}'}\,\delta_{\sigma\sigma'}$$

$$\left\{ c_{\vec{k}\sigma}, c_{\vec{k}'\sigma'} \right\} = \left\{ c^{*}_{\vec{k}\sigma}, c^{*}_{\vec{k}'\sigma'} \right\} = 0 \qquad (2.10)$$

The single particle number operator is defined by

$$n_{\vec{k}} = c^{*}_{\vec{k}\sigma} c_{\vec{k}\sigma} \qquad (2.11)$$

In term of these operators, the interaction potential acting to scatter the electrons of each other can be written as

$$V(\vec{r}_1, \vec{r}_2) = \sum_{\vec{k},\vec{k}',\varkappa} V(\varkappa)\, c^{*}_{\vec{k}+\vec{\varkappa},\sigma}\, c^{*}_{\vec{k}'-\vec{\varkappa},\sigma'}\, c_{k',\sigma'}\, c_{k,\sigma} \qquad (2.12)$$

where $\vec{\varkappa}$ is the momentum of the exchanging phonon, that is, the electrons in the states with momentum $\vec{k}$ and $\vec{k}'$ exchange by a phonon of momentum $\vec{\varkappa}$ to new states $\vec{k} + \vec{\varkappa}$ and $\vec{k}' - \vec{\varkappa}$.

Next assuming that the ground state can be only occupied by paired electrons, that is if the state $\vec{k}\uparrow$ is occupied, so is $-\vec{k}\downarrow$; and if $\vec{k}\uparrow$ is unoccupied, neither is $-\vec{k}\downarrow$. With this assumption the effective potential is that connecting paired states, then Eq. (2.12) becomes

$$V = \sum_{\vec{k},\vec{\varkappa}} V(\vec{\varkappa})\, c^{*}_{\vec{k}+\vec{\varkappa},\uparrow}\, c^{*}_{-\vec{k}-\vec{\varkappa},\downarrow}\, c_{-k\downarrow}\, c_{k\uparrow} \qquad (2.13)$$

Introducing pair creation and annihilation operators $b^{*}_{\vec{k}}$ and $b_{\vec{k}}$ by

$$b^{*}_{\vec{k}} = c^{*}_{\vec{k}\uparrow}\, c^{*}_{-\vec{k}\downarrow} \quad b_{\vec{k}} = c_{-\vec{k}\downarrow}\, c_{\vec{k}\uparrow} \qquad (2.14)$$

Equation (2.13) in terms of these operatore is

$$V = \sum_{\vec{k},\vec{l}} V_{\vec{k}\vec{l}}\, b^{*}_{\vec{l}}\, b_{\vec{k}} \qquad (2.15)$$

The Hamiltonian treated by BCS is

$$H = 2\sum_{\vec{k}<\vec{k}_F} |\epsilon|\, b_{\vec{k}}\, b^{*}_{\vec{k}} + 2\sum_{\vec{k}>\vec{k}_F} \epsilon_{\vec{k}}\, b^{*}_{\vec{k}} b_{\vec{k}} + \sum_{\vec{k},\vec{l}} V_{\vec{k}\vec{l}}\, b^{*}_{\vec{l}}\, b_{\vec{k}} \qquad (2.16)$$

Where $\epsilon_{\vec{k}}$ are energies measured above the Fermi level and the factor of 2 comes from the two particles in the pair. The problem is then to solve for eigenfunctions and eigenvalues of this Hamiltonian.

Many-particle systems, such as the electron gas in metals at low temperatures, formed by particles obeying Fermi-Dirac statistics are called Fermi superfluids. In fact the superfluidity of the electron gas we call superconductivity. Therefore superconductivity is the systems of superfluid systems. Since Fermi systems can exhibit off-diagonal long-range order in two-body reduced density matrix and examples of off-diagonal long-range order in Fermi systems are superconductivity in metals. We will study reduced density matrices and off-diagonal long-range order in the next chapter.