CHAPTER I

PRELIMINARIES

A triple $(S,+,\cdot)$ is called a semiring if

(i) $(S,+)$ and $(S,\cdot)$ are semigroups and

(ii) $x\cdot(y+z) = x\cdot y+x\cdot z$ and $(y+z)\cdot x = y\cdot x+z\cdot x$ for all $x,y,z \in S$, and the operations $+$ and $\cdot$ are called the addition and multiplication of the semiring, respectively.

A semiring $(S,+,\cdot)$ is said to be additively commutative [multiplicatively commutative] if $(S,+)$ [$(S,\cdot)$] is a commutative semigroup. A commutative semiring is a semiring which is both additively and multiplicatively commutative.

Let $S = (S,+,\cdot)$ be a semiring. An element 0 of S is called a zero of S if $x+0 = 0+x = x$ and $x\cdot 0 = 0\cdot x = 0$ for every $x \in S$. An element 1 of S is called a multiplicative identity or an identity of S if $x\cdot 1 = 1\cdot x = x$ for every $x \in S$. By a semiring with 0 [a semiring with 1], we shall mean a semiring which has 0 [1] as its zero [multiplicative identity]. If S has a zero 0 [a multiplicative identity 1] and $x \in S$ is such that $x+y = y+x = 0$ [$x\cdot y = y\cdot x = 1$] for some $y \in S$, then x is said to be additively invertible [multiplicatively invertible] in S.

In a semiring $S = (S,+,\cdot)$, for any element $x \in S$ and any positive integer n, let the notation nx denote the element $x+x+\ldots+x$ (n times) of S and the notation x^n denote the element $x\cdot x\cdot\ldots\cdot x$ (n times) of S.

An element x of a semiring $S = (S,+,\cdot)$ is called an additive idempotent [multiplicative idempotent] of S if $x+x = x$ [$x^2 = x$]. A semiring S is called an additively idempotent semiring if every element of S is an additive idempotent of S. A Boolean semiring or multiplicatively idempotent semiring is a semiring in which every element is a multiplicative idempotent, that is, a Boolean semiring is a semiring S with $x^2 = x$ for every $x \in S$. An idempotent semiring is a semiring which is both additively idempotent and multiplicatively idempotent.

A commutative semiring S with 0,1 is called a semifield if $S \setminus \{0\}$ is a group under the multiplication of S.

An element a of a semigroup S is said to be regular if $a = axa$ for some $x \in S$. A semigroup S is said to be regular if every element of S is regular. Then every idempotent of a semigroup S is a regular element of S. It is known that a semigroup S is a group if and only if S is a regular semigroup containing exactly one idempotent.

A semiring $(S,+,\cdot)$ is called a regular semiring if $(S,+)$ and $(S,\cdot)$ are regular semigroups. Therefore, a ring R is a regular ring if and only if for each $x \in R$, $x = xyx$ for some $y \in R$.

A Boolean algebra is a triple $(B,+,\cdot)$ such that

(i) $(B,+)$ and $(B,\cdot)$ are commutative semigroups,

(ii) $a\cdot(b+c) = a\cdot b+a\cdot c$ and $a+b\cdot c = (a+b)\cdot(a+c)$ for all $a,b,c \in B$,

(iii) there exist 2 elements 0 and 1 of B such that $0 \neq 1$, $0+a = a$ and $1\cdot a = a$ for every $a \in B$,

(iv) for every $a \in B$, there exists an element $a' \in B$ such that $a+a' = 1$ and $a\cdot a' = 0$.

Every Boolean algebra is a commutative idempotent semiring with 0,1. A commutative idempotent semiring $(S,+,\cdot)$ with 0,1 is a Boolean algebra if $a+b\cdot c = (a+b)\cdot(a+c)$ for all $a,b,c \in S$ and for every element $a \in S$, there exists an element $a' \in S$ such that $a+a' = 1$ and $a\cdot a' = 0$.

Let $S = (S,+,\cdot)$ and $T = (T,\oplus,\odot)$ be semirings. A map $\Phi : S \to T$ is called a <u>homomorphism</u> if

$$\Phi(x+y) = \Phi(x)\oplus\Phi(y),\ \Phi(x\cdot y) = \Phi(x)\odot\Phi(y)$$

for all $x,y \in S$. A homomorphism from S into T is called an <u>isomorphism</u> if it is one-to-one. The semirings S and T are said to be <u>isomorphic</u> and written as $S \cong T$, if there exists an isomorphism from S onto T.

Let $\{S_\alpha\}_{\alpha\in\Lambda}$ be a nonempty family of semirings. The <u>direct product</u> of the semirings S_α, $\alpha \in \Lambda$, is the semiring $S = \prod_{\alpha\in\Lambda} S_\alpha$ under the componentwise addition and multiplication, that is, for $(x_\alpha),(y_\alpha) \in S = \prod_{\alpha\in\Lambda} S_\alpha$,

$$(x_\alpha)+(y_\alpha) = (x_\alpha+y_\alpha),\ (x_\alpha)(y_\alpha) = (x_\alpha y_\alpha),$$

and if $(x_\alpha) \in \prod_{\alpha\in\Lambda} S_\alpha$, then for each $\beta \in \Lambda$, x_β is called the <u>βth component</u> of (x_α).

In particular, if $\{S_1,S_2,\ldots,S_n\}$ is a finite set of semirings, the direct product of the semirings $S_1,S_2,\ldots,S_n$ may be denoted by $S_1\times S_2\times\ldots\times S_n$ and its addition and multiplication are defined, respectively, by

$$(x_1,x_2,\ldots,x_n)+(y_1,y_2,\ldots,y_n) = (x_1+y_1,x_2+y_2,\ldots,x_n+y_n),$$

$$(x_1,x_2,\ldots,x_n)(y_1,y_2,\ldots,y_n) = (x_1y_1,x_2y_2,\ldots,x_ny_n)$$

for all $(x_1,x_2,\ldots,x_n),(y_1,y_2,\ldots,y_n) \in S_1\times S_2\times\ldots\times S_n$.

If S is an additively commutative semiring and n is a positive integer , let $M_n(S)$ be the set of all n×n matrices over S, so $M_n(S)$ is a semigroup under the matrix multiplication.

By a <u>matrix semigroup</u> over an additively commutative semiring S, we shall mean a semigroup of matrices over S under the matrix multiplication.

For an n×n matrix A over an additively commutative semiring S, the notation A_{ij} will denote the element of the matrix A in the <u>ith</u> row and <u>jth</u> column.

The following known results will be used in this thesis.

<u>Theorem 1.1.</u> For any ring R and for any positive integer n, the matrix semigroup $M_n(R)$ is regular if and only if R is a regular ring [1].

In particular, the matrix semigroup $M_n(F)$ is regular for any field F and for any positive integer n.

<u>Theorem 1.2.</u> Let S be an additively commutative semiring with 0, n a positive integer and $n \geqslant 2$. If the matrix semigroup $M_n(S)$ is regular, then S is a regular semiring [2].

<u>Theorem 1.3.</u> Let S be an additively commutative semiring with 0 and n a positive integer and $n \geqslant 3$. Then the matrix semigroup $M_n(S)$ is regular if and only if S is a regular ring [2].

<u>Theorem 1.4.</u> Let S be a commutative idempotent semiring with 0,1, n a positive integer and $n \geqslant 2$. Then the matrix semigroup $M_n(S)$ is regular if and only if n = 2 and S is a Boolean algebra [2].

Theorem 1.5. Let N be the set of all positive integers. Then there exists a binary operation $*$ on $N\cup\{0\}$ such that $(N\cup\{0\},*,\cdot)$ is a ring where $\cdot$ is the usual multiplication [3].

In this thesis, we let N, Q^+ and R^+ denote the set of all positive integers, the set of all positive rational numbers and the set of all positive real numbers, respectively.

Semirings that we are interested in the thesis are commutative semirings with 0,1. Then, for convenience, the word "Boolean semirings" are always assumed to be commutative and have 0,1.