ทฤษฎีบทลิมิตเข้าสู่ส่วนกลางของผลบวกของกำลังของฟังก์ชันของตัวแปรสุ่มที่เป็นอิสระต่อกัน

นายกิตติพงษ์ ไหลภาภรณ์

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THE CENTRAL LIMIT THEOREMS OF SUMS OF POWERS OF FUNCTION OF INDEPENDENT RANDOM VARIABLES

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กำหนดให้ (X_{nk}), $k = 1,\ldots,k_n$; $n = 1,2,\ldots$ เป็นลำดับสองชั้นของตัวแปรสุ่มกณิกนันต์ซึ่งเป็นอิสระเชิงแถว ภายใต้เงื่อนไขที่ว่าลำดับของฟังก์ชันการแจกแจงของตัวแปรสุ่ม

$$S_n = X_{n1} + \ldots + X_{nk_n} - A_n$$

ลู่เข้าอย่างอ่อนโดยที่ A_n คือค่าคงตัวที่เหมาะสม ในวิทยานิพนธ์นี้ จะให้เงื่อนไขจำเป็นและเพียงพอของการลู่เข้าอย่างอ่อนของลำดับของฟังก์ชันการแจกแจงของตัวแปรสุ่ม

$$S_n^r = (g(X_{n1}))^r + \ldots + (g(X_{nk_n}))^r - B_n(r)$$

สู่ฟังก์ชันการแจกแจง F^r นอกจากนั้นจะให้เงื่อนไขจำเป็นและเพียงพอของการลู่เข้าอย่างอ่อนของลำดับของฟังก์ชันการแจกแจง F^r อีกด้วย โดยที่ $B_n(r)$ เป็นค่าคงตัวที่เหมาะสมและ $g : \mathbf{R} \to \mathbf{R}$ เป็นฟังก์ชันซึ่งมีสมบัติที่กำหนดให้บางประการ

ภาควิชา ......คณิตศาสตร์......... ลายมือชื่อนิสิต ....

สาขาวิชา ......คณิตศาสตร์......... ลายมือชื่ออาจารย์ที่ปรึกษา ....

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Let $(X_{nk}), k=1,\dots,k_n \ ; n=1,2,\dots$ be a double sequence of infinitesimal random variables which are rowwise independent, under a condition of convergence of distribution function of the sums

$$S_n = X_{n1} + \dots + X_{nk_n} - A_n$$

where A_n are suitably chosen constants, we give necessary and sufficient conditions for the sequence of distribution functions of

$$S_n^r = \left(g\left(X_{n1}\right)\right)^r + \dots + \left(g\left(X_{nk_n}\right)\right)^r - B_n(r)$$

to weakly converge to a limiting distribution function F^r for each natural number r, and also for convergence of the sequence of distribution function (F^r) ,where $g : \mathbf{R} \to \mathbf{R}$ has certain properties and $B_n(r)$ are suitably chosen constants.

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สาขาวิชา.....คณิตศาสตร์........ ลายมือชื่ออาจารย์ที่ปรึกษา..........................
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CONTENTS

INTRODUCTION

This thesis deals with probability limit theorem, mainly related to positive integer power of function of independent random variables . The problem was first investigated by J.M.Shapiro [5], and can be precisely stated as follows:

Let (X_{nk})., $k = 1,\dots,k_n$; $n = 1,2,\dots$ be a double sequence of infinitesimal random variables which are rowwise independent. The distribution functions of the sums

$$S_n^r \quad = \quad |X_{n1}|^r + \dots + |X_{nk_n}|^r - B_n(r) \qquad\qquad (*)$$

where $B_n(r)$ are suitably chosen constants and $r \in N$ is considered. Necessary and sufficient conditions are found for the sequence of distribution functions of the sum (*) to weakly converge to a limit distribution function F^r for each natural number r ,and also for convergence of the sequence of distribution function (F^r) .

In fact, the sum (*) is of the form

$$S_n^r = (g(X_{n1}))^r + \dots + (g(X_{nk_n}))^r - B_n(r)$$

where $g : \mathbf{R} \to \mathbf{R}$ is defined by $g(x) = |x|$.

In this present work, we extend the results of [5] to the case where the function $g : \mathbf{R} \to \mathbf{R}$ has the following properties:

(g-1) $g(0) = 0$.

(g-2) g is continuous , strictly decreasing on $(-\infty, 0]$ and strictly increasing on $[0, \infty)$.

(g-3) for some positive δ there exist a positive constants c such that,

$$\left|\frac{g(x)}{x}\right| < c \text{ for all } x \in (-\delta, \delta).$$

(g-4) $g(-\infty) = g(\infty) = \infty$.

In Chapter I, some important preliminary results and notations, which are necessary for the work, are presented.

Chapter II gives the main results.

CHAPTER I

PRELIMINARIES

In this chapter, we present some basic concepts and facts of probability theory that are needed in this thesis. The proofs of the statements are omitted as they can be found in [1]-[4].

1.1 Random Variables, Distribution Functions and Characteristic Functions

A **probability space** is a measure space $(\Omega, \mathcal{E}, P)$ for which $P(\Omega) = 1$. The measure P is called a **probability measure**. The set Ω will be referred to as a **sample space** and its elements are called **points** or **elementary events** . The elements of $\mathcal{E}$ are called **events**. For any event A, the value $P(A)$ is called the **probability of** A .

Let $(\Omega, \mathcal{E}, P)$ be a probability space. A function $X : \Omega \to \mathbf{R}$ is called a **random variable** if for every Borel set B in $\mathbf{R}$, $X^{-1}(B)$ belongs to $\mathcal{E}$. We shall use the notation $P(X \in B)$ inplace of $P(\{\omega \in \Omega | X(\omega) \in B\})$.
In the case where $B = (-\infty, a]$ and $[a, b]$, $P(X \in B)$ is denoted by $P(X \leq a)$ and $P(a \leq X \leq b)$, respectively.

Random variables $X_1, X_2, \ldots, X_n$ are said to be **independent** if for all real numbers $x_1, x_2, \ldots, x_n$,

$$P\left(\bigcap_{i=1}^{n} \{\omega | X_i(\omega) \leq x_i\}\right) = \prod_{i=1}^{n} P(X_i \leq x_i)$$

A sequence (X_n) is called a **sequence of independent random variables** if for each natural number n , the random variables $X_1, X_2, \ldots, X_n$ are independent.

A double sequence (X_{nk}) , $k = 1,2,\ldots, k_n$; $n = 1,2,\ldots$ of random variables is said to be **rowwise independent** if for each natural number n , $X_{n1}, X_{n2}, \ldots, X_{nk_n}$ are independent.

If for each positive real number ε, $\lim_{n\to\infty} \max_{1\le k\le k_n} P(|X_{nk}| > \varepsilon) = 0$ then (X_{nk}) has an **infinitesimal** property.

Proposition 1.1.1 Let $X_1, X_2, \ldots, X_n$ be random variables and g a Borel function. If $X_1, X_2, \ldots, X_n$ are independent then $g(X_1), g(X_2), \ldots, g(X_n)$ are also independent.

Let X be a random variable. A function $F : \mathbf{R} \to [0,1]$ defined by

$$F(x) = P(X \le x), \text{ for each real number } x,$$

is called **the distribution function of the random variable** X .

A function $F : \mathbf{R} \to [0,1]$ is the distribution function of some random variable if and only if

1. F is nondecreasing,
2. F is right continuous,
3. $F(-\infty) = 0$ and $F(+\infty) = 1$.

Let X be a random variable on a probability space $(\Omega, \mathcal{E}, P)$. X is said to be a **discrete random variable** if the image of X is countable and X is called a **continuous random variable** if F can be written in the form

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

for some nonnegative integrable function f on $\mathbf{R}$ and in this case ,we say that f is the **probability function of** X.

Now we will give some examples of random variables .

Example 1.1.1 X is said to be a **Poisson** random variable with parameter λ, written as X ~Poi(λ), if its image is $\{0,1,2,...\}$ and

$$P(X=k)=\frac{1}{k!}\lambda^k e^{-\lambda}.$$

Example 1.1.2 We say that X is a **normal** random variable with parameter μ and σ^2, written as X ~N(μ,σ^2), if its probability function is

$$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{1}{2\sigma^2}(x-\mu)^2).$$

Example 1.1.3 We say that X is a **degenerate** random variable with parameter a, if the distribution function of X is

$$F(x)=\begin{cases}0 & \text{if } x<a;\\ 1 & \text{if } x\geq a.\end{cases}$$

A sequence (F_n) of bounded nondecreasing functions **converges weakly** to F if for every continuity point x of F, $F_n(x)\to F(x)$ which will be written as $F_n \xrightarrow{w} F$.

For a distribution function F of a random variable X. The function $\varphi:\mathbf{R}\to\mathbf{C}$ defined by

$$\varphi(t) = \int_{\mathbf{R}} e^{itx}\,dF(x)$$

is called the **characteristic function of the distribution function** F and it is also called the **characteristic function of the random variable** X.

Proposition 1.1.2 If the random variables $X_1, X_2, ..., X_n$ are independent with characteristic functions $\varphi_1, \varphi_2, ..., \varphi_n$ respectively, then the characteristic function φ of the sum $X_1 + X_2 + ... + X_n$ is given by

$$\varphi(t) \quad = \quad \varphi_1(t)\varphi_2(t)...\varphi_n(t).$$

The correspondence between the distribution function and the characteristic function is one-to-one. Moreover, if a sequence of distribution functions (F_n) converges to a distribution function F at every continuity point of F, then the corresponding sequence of characteristic functions (φ_n) converges to the characteristic function φ of F and the converse is also true.

1.2 Infinitely Divisible Distribution Functions

A random variable X is said to be **infinitely divisible** if, for every natural number n, it can be represented as the sum

$$X = X_{n1} + X_{n2} + ... + X_{nn}$$

of n independent identically distributed random variables $X_{n1}, X_{n2}, ..., X_{nn}$.

The distribution function and the corresponding characteristic function of an infinitely divisible random variable are also said to be **infinitely divisible distribution function** and **infinitely divisible characteristic function**, respectively.

Theorem 1.2.1 The characteristic function φ is infinitely divisible if and only if for every natural number n, there exists a characteristic function φ_n such that

$$\varphi = (\varphi_n)^n.$$

Theorem 1.2.2 The distribution function of finite sum of independent infinitely divisible random variables is also infinitely divisible.

Theorem 1.2.3 The weak limit of a sequence of infinitely divisible distribution function is itself infinitely divisible.

It is well known that the characteristic function φ is infinitely divisible if and only if its logarithm can be represented in the form

$$\log\varphi(t) = i\gamma t + \int_{-\infty}^{\infty} f(t,u)\,dG(u)$$

where γ is a real constant , the function f is given by

$$f(t,u) = \begin{cases} (e^{itu} - 1 - \dfrac{itu}{1+u^2})(\dfrac{1+u^2}{u^2}) & \text{if } u \neq 0; \\ -\dfrac{t^2}{2} & \text{if } u = 0, \end{cases} \qquad \ldots(1.2.1)$$

and G is a bounded non-decreasing function which is right continuous with $G(-\infty) = 0$.

The representation of φ given by (1.2.1) is unique , and it is known as the formula of **Le′vy and Khintchine** .

There is another representation of the logarithm of an infinitely divisible characteristic function φ , known as **Le′vy's formula** :

$$\begin{aligned} \log\varphi(t) \;=\; & i\gamma t - \frac{\sigma^2 t^2}{2} + \int_{-\infty}^{0^-} (e^{itx} - 1 - \frac{itx}{1+x^2})\,dM(x) \\ & + \int_{0^+}^{+\infty} (e^{itx} - 1 - \frac{itx}{1+x^2})\,dN(x) \end{aligned} \qquad \ldots(1.2.2)$$

where $\sigma^2 \geq 0$ and γ are real constants, M and N are non-decreasing functions defined on $(-\infty,0)$ and $(0,+\infty)$ respectively with $M(-\infty) = N(\infty) = 0$ and

$$\int_{-\varepsilon}^{0^-} x^2 dM(x) + \int_{0^+}^{\varepsilon} x^2 dN(x) < +\infty \qquad \ldots(1.2.3)$$

for every positive real number ε and we denoted $\lim_{x\to\infty} N(x)$ and $\lim_{x\to-\infty} M(x)$ by $N(\infty)$ and $M(-\infty)$, respectively.

The characteristic function φ is infinitely divisible if and only if its logarithm can be represented by Le'vy's formula (1.2.2).

For an infinitely divisible characteristic function φ, representations (1.2.1) and (1.2.2) are related by

$$M(x) = \int_{-\infty}^{x} \frac{1+u^2}{u} dG(u) \qquad \text{for } x < 0$$

$$N(x) = -\int_{x}^{+\infty} \frac{1+u^2}{u} dG(u) \qquad \text{for } x > 0 \qquad \ldots(1.2.4)$$

and $$\sigma^2 = G(0^+) - G(0^-).$$

We can give (1.2.2) in the following form :

$$\log\varphi(t) = i\gamma(\tau)t - \frac{\sigma^2 t^2}{2} + \int_{-\infty}^{-\tau}(e^{itx}-1)dM(x) + \int_{\tau}^{+\infty}(e^{itx}-1)dN(x)$$
$$+ \int_{-\tau}^{0}(e^{itx}-1-itx)dM(x) + \int_{0^+}^{\tau}(e^{itx}-1-itx)dN(x) \qquad \ldots(1.2.5)$$

where M and N have the same meanings as above, τ and $-\tau$ are arbitrary continuity points of the functions M and N, respectively, and

$$\gamma(\tau) = \gamma + \int_{|t|<\tau} t\,dG(t) + \int_{|t|\geq\tau} \frac{1}{t} dG(t). \qquad \ldots(1.2.6)$$

Theorem 1.2.4 For the convergence of infinitely divisible distribution functions F_n to the limit distribution function F it is necessary and sufficient that

1. $M_n(u) \to M(u)$, $N_n(u) \to N(u)$ at continuity points of the functions M and N
2. $\gamma_n(\tau) \to \gamma(\tau)$
3. $\lim_{\varepsilon \to 0^+} \overline{\lim_{n\to\infty}} \left\{ \int_{-\varepsilon}^{0} u^2 dM_n(u) + \sigma_n^2 + \int_0^{\varepsilon} u^2 dN_n(u) \right\}$

$= \lim_{\varepsilon \to 0^+} \underline{\lim_{n\to\infty}} \left\{ \int_{-\varepsilon}^{0} u^2 dM_n(u) + \sigma_n^2 + \int_0^{\varepsilon} u^2 dN_n(u) \right\} = \sigma^2$

where the functions M_n, N_n and M, N and the constants σ_n, $\gamma_n(\tau)$ and $\sigma, \gamma(\tau)$ are defined by (1.2.4) and (1.2.6) for the distribution functions F_n and F respectively.

Theorem 1.2.5 There exists a sequence of constants A_n such that the distribution functions of the sum

$$X_{n1} + X_{n2} + \ldots + X_{nk_n} - A_n \qquad \ldots(1.2.7)$$

of independent infinitesimal random variables converge to a limit if and only if there exist non-decreasing functions M and N, defined on the intervals $(-\infty, 0)$ and $(0, +\infty)$ respectively with $M(-\infty) = 0$ and $N(+\infty) = 0$, and a constant $\sigma \geq 0$ such that

1. $\lim_{n\to\infty} \sum_{k=1}^{k_n} F_{nk}(x) = M(x)$
2. $\lim_{n\to\infty} \sum_{k=1}^{k_n} [F_{nk}(x) - 1] = N(x)$

at every continuity point of M and N, and

3. $$\lim_{\varepsilon \to 0^+} \underline{\lim}_{n\to\infty} \sum_{k=1}^{k_n} \{ \int_{|x|<\varepsilon} x^2 dF_{nk}(x) - (\int_{|x|<\varepsilon} x dF_{nk}(x))^2 \}$$

$$= \lim_{\varepsilon \to 0^+} \overline{\lim}_{n\to\infty} \sum_{k=1}^{k_n} \{ \int_{|x|<\varepsilon} x^2 dF_{nk}(x) - (\int_{|x|<\varepsilon} x dF_{nk}(x))^2 \} = \sigma^2 ,$$

where F_{nk} denotes the distribution function of X_{nk} .

The constants A_n may be chosen according to the formula

$$A_n = \sum_{k=1}^{k_n} \int_{|x|<\tau} x dF_{nk}(x) - \gamma(\tau)$$

where γ is the function in (1.2.6) and $-\tau$ and $+\tau$ are continuity points of M and N , respectively. The logarithm of the characteristic function of the limit of the distribution function of the sum (1.2.7) is defined by the formula in (1.2.2) with the functions M, N and constants $\gamma(\tau), \sigma$.

It is well- known that normal, degenerate and Poisson random variables are infinitely divisible with Le$'$vy's formula as follow:

Table of canonical representation of infinitely divisible characteristic function

distribution function	characteristic function	Le$'$vy's formula			
		γ	σ	$M(x)$	$N(x)$
Normal	$\exp\,(iat - \frac{1}{2}\sigma^2 t^2)$	a	σ	0	0
Degenerate	$\exp(iat)$	a	0	0	0
Poisson	$\exp(\lambda(e^{it} - 1))$	$\frac{\lambda}{2}$	0	0	$\begin{cases} 0 & \text{if } x > 1; \\ -\lambda & \text{if } x < 1. \end{cases}$

Table 1.2.1

CHAPTER II

THE CENTRAL LIMIT THEOREMS OF SUMS OF POWERS OF FUNCTION OF INDEPENDENT RANDOM VARIABLES

In this chapter, we let $(X_{nk}), k = 1,2,\ldots,k_n$; $n = 1,2,\ldots$ be a double sequence of infinitesimal random variables which are rowwise independent.

Let

$$S_n = X_{n1} + \ldots + X_{nk_n} - A_n ,$$

where A_n are constants and let F_n, F_{nk} be the distribution functions of S_n and X_{nk}, respectively. Necessary and sufficient conditions for (F_n) to converge to a distribution function F are known, and in particular it is well known that F is infinitely divisible.

In 1957, Shapiro considered the distribution functions of the sums

$$T_n^r = |X_{n1}|^r + \ldots + |X_{nk_n}|^r - B_n(r) \qquad (*)$$

where $B_n(r)$ are suitably chosen constants and $r \in N$. For each natural number r, let G_n^r, G_{nk}^r be the distribution functions of $T_n^r, |X_{nk}|^r$, respectively. The results are as follow.

THEOREM A

Assume that for each $r \in N$, $G_n^r \xrightarrow{w} G^r$ as $n \to \infty$ and $G^r \xrightarrow{w} H$ as $r \to \infty$. Then H is a degenerate or normal or Poisson distribution function or the distribution function of the sum of two independent random variables, one a normal and the other a Poisson distribution function.

THEOREM B

If $F_n \xrightarrow{w} F$ as $n \to \infty$, then for suitably chosen constants $B_n(r)$, $G_n^r \xrightarrow{w} G^r$ as $n \to \infty$ if and only if

$$\lim_{\varepsilon \to 0^+} \overline{\lim_{n \to \infty}} \sum_{k=1}^{k_n} \left\{ \int_0^{\varepsilon} x^{2r} d\left[F_{nk}(x) - F_{nk}(-x^-)\right] - \left(\int_0^{\varepsilon} x^r d\left[F_{nk}(x) - F_{nk}(-x^-)\right] \right)^2 \right\}$$

$$= \lim_{\varepsilon \to 0^+} \underline{\lim_{n \to \infty}} \sum_{k=1}^{k_n} \left\{ \int_0^{\varepsilon} x^{2r} d\left[F_{nk}(x) - F_{nk}(-x^-)\right] - \left(\int_0^{\varepsilon} x^r d\left[F_{nk}(x) - F_{nk}(-x^-)\right] \right)^2 \right\}$$

$$= \sigma_r^2 < \infty ,$$

where $\lim_{t \to x^-} f(t)$ is denoted by $f(x^-)$.

THEOREM C

Under the condition $F_n \xrightarrow{w} F$ as $n \to \infty$ and for suitably chosen constants $B_n(r)$, a necessary and sufficient condition for $G^r \xrightarrow{w} H$ as $r \to \infty$, is that

1. $M(x) = 0$ for $x < -1$,
2. $N(x) = 0$ for $x > 1$, and
3. $\lim_{r \to \infty} \sigma_r^2 = (\sigma^*)^2$,

where M, N are functions satisfying the Le'vy's formula with respect to F and σ_r, σ^* are constants satisfying Le'vy's formula with respect to G^r and H, respectively.

In fact the sum (*) is of the form

$$S_n^r = (g(X_{n1}))^r + \ldots + (g(X_{nk_n}))^r - B_n(r)$$

where $g: \mathbf{R} \to \mathbf{R}$ is defined by $g(x) = |x|$. In this work, we generalize THEOREM A , B and C to the case of

$$S_n^r = (g(X_{n1}))^r + \ldots + (g(X_{nk_n}))^r - B_n(r)$$

where $g:\mathbf{R}\rightarrow\mathbf{R}$ is with the following properties:

(g-1) $g(0)=0$.

(g-2) g is continuous, strictly decreasing on $(-\infty,0]$ and strictly increasing on $[0,\infty)$.

(g-3) for some positive δ, there exist a constants c such that $\left|\frac{g(x)}{x}\right|<c$ for all $x\in(-\delta,\delta)$.

(g-4) $g(-\infty)=g(+\infty)=\infty$.

Since g satisfies (g-1) and (g-2), we can write

$$g(x)=\begin{cases} g_1(x) & \text{if} \quad x\geq 0; \\ g_2(x) & \text{if} \quad x<0. \end{cases}$$

where $g_1:\mathbf{R}_0^+\rightarrow\mathbf{R}_0^+$ is defined by $g_1(x)=g(x)$ and $g_2:\mathbf{R}_0^-\rightarrow\mathbf{R}_0^+$ is defined by $g_2(x)=g(x)$. Since g is continuous at 0, we can assume the δ in (g-3) to be such that $g_1(\delta)<1$ and $g_2(-\delta)<1$.

Now, we give some examples of g.

1. $g(x) \;=\; c|\,x\,|^n,\ c\in\mathbf{R}_0^+$ and $n\in N$.

2. $g(x) \;=\; \begin{cases} x+\sin x & \text{if} \quad x\geq 0; \\ -x+\sin x & \text{if} \quad x<0. \end{cases}$

From now on, we let F_n^r, F_{nk}^r be the distribution functions of $S_n^r, (g(X_{nk}))^r$ respectively and for infinitely divisible distribution function F^r, we let $M^r, N^r, \gamma^r, \sigma^r$ be M, N, γ, σ in Le′vy's formula of F^r.

Lemma2.1

$$1.\quad F_{nk}^r(x)=\begin{cases} 0 & \text{if } x<0; \\ P(X_{nk}=0) & \text{if } x=0; \\ F_{nk}(g_1^{-1}(x^{\frac{1}{r}}))-F_{nk}(g_2^{-1}(x^{\frac{1}{r}})^-) & \text{if } x>0. \end{cases}$$

2. $F_{nk}^{r}(x)=\begin{cases}0 & \text{if } x<0;\\ F_{nk}^{1}(x^{\frac{1}{r}}) & \text{if } x\geq 0.\end{cases}$

Proof.

1. Let $x \in \mathbf{R}$.

Case 1 $x<0$.

Since g is nonnegative and $x<0$,

$$\begin{aligned} F_{nk}^{r}(x) &= P((g(X_{nk}))^{r} \leq x) \\ &= P(\phi) \\ &= 0. \end{aligned}$$

Case 2 $x=0$.

$$\begin{aligned} F_{nk}^{r}(0) &= P((g(X_{nk}))^{r} \leq 0) \\ &= P(g(X_{nk})=0) \\ &= P(X_{nk}=0). \end{aligned}$$

Case 3 $x>0$.

$$\begin{aligned} F_{nk}^{r}(x) &= P((g(X_{nk}))^{r} \leq x) \\ &= P(0 \leq g(X_{nk}) \leq x^{\frac{1}{r}}) \\ &= P(g_{2}^{-1}(x^{\frac{1}{r}}) \leq X_{nk} \leq g_{1}^{-1}(x^{\frac{1}{r}})) \\ &= F_{nk}(g_{1}^{-1}(x^{\frac{1}{r}})) - F_{nk}(g_{2}^{-1}(x^{\frac{1}{r}})^{-}). \end{aligned}$$

2. Let $x \in \mathrm{R}$.

Case 1 $x<0$.

Since g is nonnegative and $x<0$,

$$\begin{aligned} F_{nk}^{r}(x) &= P((g(X_{nk}))^{r} \leq x) \\ &= 0. \end{aligned}$$

Case 2 $x=0$.

$$\begin{aligned} F_{nk}^{r}(0) &= P(g(X_{nk})^{r} \leq 0) \\ &= P(g(X_{nk}) \leq 0) \\ &= F_{nk}^{1}(0). \end{aligned}$$

Case 3 $x > 0$.

$$\begin{aligned} F_{nk}^{r}(x) &= P\left((g(X_{nk}))^{r} \le x\right) \\ &= P(0 \le g(X_{nk}) \le x^{\frac{1}{r}}) \\ &= P(g(X_{nk}) \le x^{\frac{1}{r}}) \\ &= F_{nk}^{1}(x^{\frac{1}{r}}) . \end{aligned}$$

#

Lemma 2.2 If (X_{nk}) is a sequence of infinitesimal random variables then so is $(g(X_{nk}))^{r}$ for all $r > 0$.

Proof.

Let ε be any positive real number. Then

$$\begin{aligned} P\left((g(X_{nk}))^{r} > \varepsilon\right) &= P\left(g(X_{nk}) > \varepsilon^{\frac{1}{r}}\right) \\ &= P\left(X_{nk} > g_{1}^{-1}(\varepsilon^{\frac{1}{r}})\right) + P\left(X_{nk} < g_{2}^{-1}(\varepsilon^{\frac{1}{r}})\right) \\ &\le P\left(|X_{nk}| > g_{1}^{-1}(\varepsilon^{\frac{1}{r}})\right) + P\left(|X_{nk}| > -g_{2}^{-1}(\varepsilon^{\frac{1}{r}})\right). \end{aligned}$$

Since (X_{nk}) is infinitesimal, $\lim\limits_{n\to\infty} \max\limits_{1\le k\le k_n} P\left(|X_{nk}| > g_{1}^{-1}(\varepsilon^{\frac{1}{r}})\right) = 0$ and $\lim\limits_{n\to\infty} \max\limits_{1\le k\le k_n} P\left(|X_{nk}| > g_{2}^{-1}(\varepsilon^{\frac{1}{r}})\right) = 0$. Then $\lim\limits_{n\to\infty} \max\limits_{1\le k\le k_n} P\left(g(X_{nk})^{r} > \varepsilon\right) = 0$.

#

Lemma 2.3 Let $X \sim \mathrm{N}(a, \sigma^{2})$ and $Y \sim \mathrm{Poi}(\lambda)$. If X and Y are independent, then Le′vy's formula of the characteristic function of $X + Y$ is

$$\log \varphi_{X+Y}(t) = i(a+\frac{\lambda}{2})t - \frac{1}{2}\sigma^2 t^2 + \int_0^\infty (e^{itx} - 1 - \frac{itx}{1+x^2}) dK(x),$$

where $K : \mathbf{R}^+ \to \mathbf{R}$ is defined by $K(x) = \begin{cases} -\lambda & \text{if } 0 < x \le 1; \\ 0 & \text{if } x > 1. \end{cases}$

Proof.

By Table 1.2.1, we have $\log\varphi_X(t) = iat - \frac{1}{2}\sigma^2 t^2$ and

$$\log\varphi_Y(t) = i\frac{\lambda}{2}t + \int_0^\infty (e^{itx} - 1 - \frac{itx}{1+x^2}) dK(x),$$

where φ_X and φ_Y are the characteristic functions of X and Y, respectively. Since X and Y are independent, $\varphi_{X+Y} = \varphi_X \varphi_Y$. Then

$$\begin{aligned} \log\varphi_{X+Y}(t) &= \log\varphi_X(t)\varphi_Y(t) \\ &= \log\varphi_X(t) + \log\varphi_Y(t) \\ &= iat - \frac{1}{2}\sigma^2 t^2 + i\frac{\lambda}{2}t + \int_0^\infty (e^{itx} - 1 - \frac{itx}{1+x^2}) dK(x) \\ &= i(a+\frac{\lambda}{2})t - \frac{1}{2}\sigma^2 t^2 + \int_0^\infty (e^{itx} - 1 - \frac{itx}{1+x^2}) dK(x) \end{aligned}$$

#

Theorem 2.1 Assume that $F_n^r \xrightarrow{w} F^r$ for every $r \in N$. Then for every $r \in N$,

1. $M^r(x) = 0$ on $(-\infty, 0)$ and
2. $N^r(x) = N^1(x^{\frac{1}{r}})$ a.e. on $(0, \infty)$.

Proof.

Let $r \in N$. By Lemma 2.2 and Proposition 1.1.1, $(g(X_{nk})^r)$ is infinitesimal and rowwise independent. Since $F_n^r \xrightarrow{w} F^r$, by Theorem 1.2.5 we know that F^r is infinitely divisible with M^r and N^r are M and N in Le$'$vy's formula such that

$$\lim_{n\to\infty}\sum_{k=1}^{k_n} F_{nk}^r(x) = M^r(x) \qquad \ldots(2.1.1)$$

and
$$\lim_{n\to\infty}\sum_{k=1}^{k_n}\left[F_{nk}^r(x) - 1\right] = N^r(x) \qquad \ldots(2.1.2)$$

for all continuity points of M^r and N^r, respectively.

By Lemma 2.1(2), $F_{nk}^r(x) = \begin{cases} 0 & \text{if } x < 0; \\ F_{nk}^1(x^{\frac{1}{r}}) & \text{if } x \geq 0. \end{cases}$

Then $\lim_{n\to\infty}\sum_{k=1}^{k_n} F_{nk}^r(x) = 0$ for all $x < 0$. By (2.1.1), $M^r(x) = 0$ a.e. on $(-\infty,0)$.

But M^r is nondecreasing. Hence $M^r(x) = 0$ on $(-\infty,0)$.

Since $\lim_{n\to\infty}\sum_{k=1}^{k_n}[\ F_{nk}^r(x) - 1\ ] = \lim_{n\to\infty}\sum_{k=1}^{k_n}[\ F_{nk}^1(x^{\frac{1}{r}}) - 1\ ] = N^1(x^{\frac{1}{r}})$ for all continuity points $x^{\frac{1}{r}}$ of N^1, by (2.1.2) we have $N^r(x) = N^1(x^{\frac{1}{r}})$ a.e. on $(0,\infty)$.

#

Theorem 2.2 If $F_n \xrightarrow{w} F$ as $n \to \infty$ then for every $r \in N$

1. $\lim_{n\to\infty}\sum_{k=1}^{k_n} F_{nk}^r(x) = 0$ for all $x < 0$ and
2. $\lim_{n\to\infty}\sum_{k=1}^{k_n}(F_{nk}^r(x) - 1) = N(g_1^{-1}(x^{\frac{1}{r}})) - M(g_2^{-1}(x^{\frac{1}{r}}))$ a.e. on $(0,\infty)$.

Furthermore, if $F_n^r \xrightarrow{w} F^r$ for every $r \in N$ then for each $r \in N$, we have

3. $M^r \equiv 0$ on $(-\infty,0)$ and
4. $N^r(x) = N(g_1^{-1}(x^{\frac{1}{r}})) - M(g_2^{-1}(x^{\frac{1}{r}}))$ a.e. on $(0,\infty)$.

Proof.

1. follows from Lemma 2.1(1).

To prove 2, let $r \in N$. Since $F_n \xrightarrow{w} F$, by Theorem 1.2.5 we know that

$$\lim_{n\to\infty}\sum_{k=1}^{k_n} F_{nk}(x) = M(x) \qquad \ldots(2.2.1)$$

$$\lim_{n\to\infty}\sum_{k=1}^{k_n} (F_{nk}(x)-1) = N(x) \qquad \ldots(2.2.2)$$

for all continuity points of M and N, where M and N are functions in Le$'$vy's formula of F. By Lemma 2.1(1), we have

$F_{nk}^r(x) = F_{nk}(g_1^{-1}(x^{\frac{1}{r}})) - F_{nk}(g_2^{-1}(x^{\frac{1}{r}})^-)$ for $x > 0$. Hence

$$\begin{aligned}
\lim_{n\to\infty}\sum_{k=1}^{k_n}(F_{nk}^r(x)-1) &= \lim_{n\to\infty}\sum_{k=1}^{k_n}\left[F_{nk}(g_1^{-1}(x^{\frac{1}{r}}))-1-F_{nk}(g_2^{-1}(x^{\frac{1}{r}})^-)\right] \\
&= \lim_{n\to\infty}\sum_{k=1}^{k_n}\left[F_{nk}(g_1^{-1}(x^{\frac{1}{r}}))-1\right] - \lim_{n\to\infty}\sum_{k=1}^{k_n}\left[F_{nk}(g_2^{-1}(x^{\frac{1}{r}})^-)\right] \\
&= N(g_1^{-1}(x^{\frac{1}{r}})) - M(g_2^{-1}(x^{\frac{1}{r}})^-) \text{ a.e. on } (0,\infty) \\
&= N(g_1^{-1}(x^{\frac{1}{r}})) - M(g_2^{-1}(x^{\frac{1}{r}})) \quad \text{a.e. on } (0,\infty).
\end{aligned}$$

Now, we suppose that $F_n^r \xrightarrow{w} F^r$ for every $r \in N$.

By Lemma 2.2 and Proposition 1.1.1, $(g(X_{nk})^r)$ is infinitesimal and is rowwise independent. Since $F_n^r \xrightarrow{w} F^r$, by Theorem 1.2.5 we know that F^r is infinitely divisible with M^r and N^r as the functions M and N in Le$'$vy's formula and are such that

$$\lim_{n\to\infty}\sum_{k=1}^{k_n} F_{nk}^r(x) = M^r(x) \qquad \ldots(2.2.3)$$

$$\lim_{n\to\infty}\sum_{k=1}^{k_n} (F_{nk}^r(x)-1) = N^r(x) \qquad \ldots(2.2.4)$$

for all continuity points of M^r and N^r, respectively.

Hence, by (1) and (2.2.3) we have $M^r(x) = 0$ a.e. on $(-\infty,0)$. But M^r is nondecreasing. Hence $M^r(x) = 0$ on $(-\infty,0)$. By (2) and (2.2.4) we have

$N^r(x) = N(g_1^{-1}(x^{\frac{1}{r}})) - M(g_2^{-1}(x^{\frac{1}{r}}))$ a.e. on $(0,\infty)$.

#

Theorem 2.3 Assume that

1. for every $r \in N$, $F_n^r \xrightarrow{w} F^r$ as $n \to \infty$ and

2. $F^r \xrightarrow{w} H$ as $r \to \infty$.

Then H is one of the followings

1. a degenerate distribution function
2. a Poisson distribution function
3. a normal distribution function
4. the distribution function of the sum of two independent random variables one of which is normal and the other is Poisson.

Proof.

Let r be any natural number. Then, by Theorem 2.1 we have $M^r \equiv 0$ on $(-\infty,0)$ and $N^r(x) = N^1(x^{\frac{1}{r}})$ a.e. on $(0,\infty)$. Since $F^r \xrightarrow{w} H$ as $r \to \infty$, by Theorem 1.2.4, we have

$$\lim_{r\to\infty} M^r(x) = M^*(x) \text{ for all continuity points } x \text{ of } M^* \qquad \ldots(2.3.1)$$

$$\lim_{r\to\infty} N^r(x) = N^*(x) \quad \text{for all continuity points } x \text{ of } N^* \qquad \ldots(2.3.2)$$

$$\lim_{r\to\infty} \gamma^r(\tau) = \gamma^*(\tau) \qquad \ldots(2.3.3)$$

$$\lim_{\varepsilon\to 0^+} \overline{\lim_{r\to\infty}} \left\{ \int_{-\varepsilon}^{0} u^2 dM^r(u) + (\sigma^r)^2 + \int_{0}^{\varepsilon} u^2 dN^r(u) \right\}$$

$$= \lim_{\varepsilon\to 0+} \underline{\lim_{r\to\infty}} \left\{ \int_{-\varepsilon}^{0} u^2 dM^r(u) + (\sigma^r)^2 + \int_{0}^{\varepsilon} u^2 dN^r(u) \right\} = (\sigma^*)^2 \qquad \ldots(2.3.4)$$

where M^*, N^*, γ^* and σ^* are associated with H in Le$'$vy's formula.

This shows that $M^* \equiv 0$ a.e. on $(-\infty,0)$. Since M^* is nondecreasing, $M^* \equiv 0$ on $(-\infty,0)$. And by appendix,

$$N^*(x) = \lim_{r\to\infty} N^1(x^{\frac{1}{r}}) = \begin{cases} N^1(1^+) & \text{if } x > 1 \quad ; \\ N^1(1^-) & \text{if } 0 < x < 1. \end{cases}$$

a.e. on $(0,\infty)$.

Since N^* is nondecreasing, $N^*(x) = \begin{cases} N^1(1^+) & \text{if } x > 1 \ ; \\ N^1(1^-) & \text{if } 0 < x < 1, \end{cases}$

on $(0,\infty)$.

But $N^*(\infty) = 0$, so $N^1(1^+) = 0$.

Thus

$$N^*(x) = \begin{cases} 0 & \text{if } x > 1 \ ; \\ N^1(1^-) & \text{if } 0 < x < 1, \end{cases}$$

and $M^* \equiv 0$.

Case 1. $\sigma^* = 0$ and $N^* \equiv 0$.

It follows from Table 1.2.1 that H is degenerate.

Case 2. $\sigma^* \neq 0$ and $N^* \equiv 0$.

It follows from Table 1.2.1 that H is a normal distribution function.

Case 3. $\sigma^* = 0$ and N^* takes one jump .

Subcase 3.1 $\gamma^* = -\dfrac{N^1(1^-)}{2}$.

From Table 1.2.1 we see that H is a Poisson distribution function.

Subcase 3.2 $\gamma^* \neq -\dfrac{N^1(1^-)}{2}$.

For any constant m ,we note that the characteristic $\varphi_m^*(t)$ of $H(x-m)$ is $e^{imt}\varphi^*(t)$, where φ^* is the characteristic function of H . Hence for $m = -\dfrac{2\gamma^* + N^1(1^-)}{2}$ we see that

$$\begin{aligned} \log\varphi_m^*(t) &= \log e^{imt}\varphi^*(t) \\ &= imt + \log\varphi^*(t) \\ &= imt + i\gamma^* t + 0 + 0 + \int_{-\infty}^{0^-} (e^{itx} - 1 - \frac{itx}{1+x^2}) dN^*(x) \\ &= i(-\frac{N^1(1^-)}{2})t + 0 + 0 + \int_{-\infty}^{0^-} (e^{itx} - 1 - \frac{itx}{1+x^2}) dN^*(x). \end{aligned}$$

By Table 1.2.1 we see that $H(x-m)$ is a Poisson distribution function.

Case 4. $\sigma^* \neq 0$ and N^* takes one jump.

By Lemma 2.3, H is the distribution function of the sum of two independent random variables one of which is a Poisson and the other is a normal .

#

Theorem 2.4 Assume that $F_n^r \xrightarrow{w} F^r$ as $n \to \infty$ for every $r \in N$ and $F_n \xrightarrow{w} F$ as $n \to \infty$. If $F^r \xrightarrow{w} H$ as $r \to \infty$ then

1. $M^*(x) = 0$ on $(-\infty, 0)$,
2. $N^*(x) = \begin{cases} 0 & \text{if } \quad x > 1 \ ; \\ N(g_1^{-1}(1^-)) - M(g_2^{-1}(1^-)) & \text{if } 0 < x < 1, \end{cases}$

on $(0, \infty)$,

3. $M(g_2^{-1}(1^+)) = N(g_1^{-1}(1^+)) = 0$,

where M and N are in Le$'$vy 's formula of F and M^* and N^* are in Le$'$vy 's formula of H.

Proof.

Let r be any natural number. Then by Theorem 2.2 we have

$$M^r \equiv 0 \text{ on } (-\infty, 0) \qquad \ldots(2.4.1)$$

and $N^r(x) = N(g_1^{-1}(x^{\frac{1}{r}})) - M(g_2^{-1}(x^{\frac{1}{r}}))$ a.e. on $(0, \infty)$ $\qquad \ldots(2.4.2)$

Since $F^r \xrightarrow{w} H$ as $r \to \infty$, it follows from Theorem 1.2.4 that there exist nondecreasing functions $M^* : \mathbf{R}^- \to \mathbf{R}$ and $N^* : \mathbf{R}^+ \to \mathbf{R}$ where $M^*(-\infty) = N^*(\infty) = 0$ satisfy (2.3.1) - (2.3.4).

By (2.3.1) and (2.4.1), we see that $M^* \equiv 0$ a.e. on $(-\infty, 0)$ .Since M^* is nondecreasing, $M^* \equiv 0$ on $(-\infty, 0)$. Then (1) holds. By appendix, (2.3.2) and (2.4.2) we have

$$N^*(x) = \lim_{r\to\infty} N(g_1^{-1}(x^{\frac{1}{r}})) - M(g_2^{-1}(x^{\frac{1}{r}}))$$
$$= \begin{cases} N(g_1^{-1}(1^+)) - M(g_2^{-1}(1^+)) & \text{if } x > 1\,; \\ N(g_1^{-1}(1^-)) - M(g_2^{-1}(1^-)) & \text{if } 0 < x < 1. \end{cases}$$

a.e. on $(0,\infty)$.

Since N^* is nondecreasing,

$$N^*(x) = \begin{cases} N(g_1^{-1}(1^+)) - M(g_2^{-1}(1^+)) & \text{if } x > 1\,; \\ N(g_1^{-1}(1^-)) - M(g_2^{-1}(1^-)) & \text{if } 0 < x < 1. \end{cases} \quad \ldots(2.4.3)$$

on $(0,\infty)$.

But $N^*(+\infty) = 0$, so

$$N^*(x) = \begin{cases} 0 & \text{if } x > 1\,; \\ N(g_1^{-1}(1^-)) - M(g_2^{-1}(1^-)) & \text{if } 0 < x < 1. \end{cases} \quad \ldots(2.4.4)$$

on $(0,\infty)$, i.e. (2) holds. From (2.4.3) and (2.4.4) we have $M(g_2^{-1}(1^+)) = N(g_1^{-1}(1^+))$.

Since $0 = M(-\infty) \le M(g_2^{-1}(1^+)) = N(g_1^{-1}(1^+)) \le N(+\infty) = 0$, (3) holds.

#

Theorem 2.5 Assume that $F_n \xrightarrow{w} F$ as $n \to \infty$. Then for each $r \in N$ and suitably chosen constants $B_n(r)$, $F_n^r \xrightarrow{w} F^r$ as $n \to \infty$ if and only if

1. $$\lim_{\varepsilon\to 0^+} \overline{\lim_{n\to\infty}} \sum_{k=1}^{k_n} \{ \int_0^{g_1^{-1}(\varepsilon^{\frac{1}{r}})} g(t)^{2r}\, dF_{nk}(t) + \int_{g_2^{-1}(\varepsilon^{\frac{1}{r}})}^{0} g(t)^{2r}\, dF_{nk}(t^-) - (\int_0^{g_1^{-1}(\varepsilon^{\frac{1}{r}})} g(t)^{r}\, dF_{nk}(t) + \int_{g_2^{-1}(\varepsilon^{\frac{1}{r}})}^{0} g(t)^{r}\, dF_{nk}(t^-))^2 \} = \sigma_r^2 < \infty$$ and

2. $$\lim_{\varepsilon\to 0^+} \underline{\lim_{n\to\infty}} \sum_{k=1}^{k_n} \{ \int_0^{g_1^{-1}(\varepsilon^{\frac{1}{r}})} g(t)^{2r}\, dF_{nk}(t) + \int_{g_2^{-1}(\varepsilon^{\frac{1}{r}})}^{0} g(t)^{2r}\, dF_{nk}(t^-) - (\int_0^{g_1^{-1}(\varepsilon^{\frac{1}{r}})} g(t)^{r}\, dF_{nk}(t) + \int_{g_2^{-1}(\varepsilon^{\frac{1}{r}})}^{0} g(t)^{r}\, dF_{nk}(t^-))^2 \} = \sigma_r^2 < \infty.$$

Proof.

Note that, for each $\varepsilon > 0$ we have

$$\int\limits_{|x|<\varepsilon} x^2 dF_{nk}^r(x) - \Big(\int\limits_{|x|<\varepsilon} x dF_{nk}^r(x)\Big)^2$$

$$= \int_0^{\varepsilon} x^2 d\Big[\, F_{nk}(g_1^{-1}(x^{\frac{1}{r}})) - F_{nk}(g_2^{-1}(x^{\frac{1}{r}})^-)\,\Big]$$

$$-\Big(\int_0^{\varepsilon} x\, d\Big[\, F_{nk}(g_1^{-1}(x^{\frac{1}{r}})) - F_{nk}(g_2^{-1}(x^{\frac{1}{r}})^-)\,\Big]\Big)^2$$

$$= \int_0^{g_1^{-1}(\varepsilon^{\frac{1}{r}})} g(t_1)^{2r} dF_{nk}(t_1) + \int_{g_2^{-1}(\varepsilon^{\frac{1}{r}})}^{0} g(t_2)^{2r} dF_{nk}(t_2^-)$$

$$-\Big(\int_0^{g_1^{-1}(\varepsilon^{\frac{1}{r}})} g(t_1)^{r} dF_{nk}(t_1) + \int_{g_2^{-1}(\varepsilon^{\frac{1}{r}})}^{0} g(t_2)^{r} dF_{nk}(t_2^-)\Big)^2 \quad [\; t_1 = g_1^{-1}(x^{\frac{1}{r}}) \text{ and } t_2 = g_2^{-1}(x^{\frac{1}{r}})]$$

$$= \int_0^{g_1^{-1}(\varepsilon^{\frac{1}{r}})} g(t)^{2r} dF_{nk}(t) + \int_{g_2^{-1}(\varepsilon^{\frac{1}{r}})}^{0} g(t)^{2r} dF_{nk}(t^-)$$

$$-\Big(\int_0^{g_1^{-1}(\varepsilon^{\frac{1}{r}})} g(t)^{r} dF_{nk}(t) + \int_{g_2^{-1}(\varepsilon^{\frac{1}{r}})}^{0} g(t)^{r} dF_{nk}(t^-)\Big)^2. \qquad \ldots(2.5.1)$$

To prove necessity, we suppose that $F_n^r \xrightarrow{w} F^r$ as $n \to \infty$.

Then 1. and 2. follow from Theorem 1.2.5 and (2.5.1).

For sufficiency, since $F_n \xrightarrow{w} F$ as $n \to \infty$, by Theorem 2.2 we have

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} F_{nk}^r(x) = 0 \text{ for } x < 0 \qquad \ldots(2.5.2)$$

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \Big[F_{nk}^r(x) - 1\Big] = N(g_1^{-1}(x^{\frac{1}{r}})) - M(g_2^{-1}(x^{\frac{1}{r}})) \text{ a.e. on } (0,\infty). \qquad \ldots(2.5.3)$$

Define $M^r : (-\infty, 0) \to \mathbf{R}$ and $N^r : (0,\infty) \to \mathbf{R}$ by

$$M^r(x) = 0 \text{ and } N^r(x) = N(g_1^{-1}(x^{\frac{1}{r}})) - M(g_2^{-1}(x^{\frac{1}{r}})).$$

Clearly, M^r is nondecreasing and $M^r(-\infty) = 0$.

Now, we show that N^r is nondecreasing and $N^r(\infty) = 0$.

Let $x, y \in \mathbf{R}^+$ be such that $x \le y$. Since M , N are nondecreasing, g_1^{-1} is increasing and g_2^{-1} is decreasing, $N(g_1^{-1}(x^{\frac{1}{r}})) \le N(g_1^{-1}(y^{\frac{1}{r}}))$ and $M(g_2^{-1}(y^{\frac{1}{r}})) \le M(g_2^{-1}(x^{\frac{1}{r}}))$. Then

$$\begin{aligned} N^r(x) &= N(g_1^{-1}(x^{\frac{1}{r}})) - M(g_2^{-1}(x^{\frac{1}{r}})) \\ &\le N(g_1^{-1}(y^{\frac{1}{r}})) - M(g_2^{-1}(y^{\frac{1}{r}})) \\ &= N^r(y). \end{aligned}$$

Thus N^r is nondecreasing. By property (g-2) and (g-4), we have

$\lim\limits_{x\to\infty} g_1^{-1}(x^{\frac{1}{r}}) = \infty$ and $\lim\limits_{x\to\infty} g_2^{-1}(x^{\frac{1}{r}}) = -\infty$. Hence $N^r(+\infty) = 0$.

By assumptions 1, 2 and (2.5.1) we have

$$\lim_{\varepsilon\to 0^+} \overline{\lim_{n\to\infty}} \sum_{k=1}^{k_n} \{ \int_{|x|<\varepsilon} x^2 dF_{nk}^r(x) - (\int_{|x|<\varepsilon} x dF_{nk}^r(x))^2 \}$$

$$= \lim_{\varepsilon\to 0^+} \underline{\lim_{n\to\infty}} \sum_{k=1}^{k_n} \{ \int_{|x|<\varepsilon} x^2 dF_{nk}^r(x) - (\int_{|x|<\varepsilon} x dF_{nk}^r(x))^2 \} = \sigma_r^2 < \infty. \qquad \ldots(2.5.4)$$

By Lemma 2.2 ,(2.5.2)-(2.5.4) and Theorem 1.2.5, $F_n^r \xrightarrow{w} F^r$, where F^r is a distribution with respect to M^r, N^r and σ_r^2 .

#

Theorem 2.6 Let $F_n \xrightarrow{w} F$ and $F_n^r \xrightarrow{w} F^r$ as $n \to \infty$ for all $r \in N$.

Then $F^r \xrightarrow{w} H$ as $r \to \infty$ if and only if

1. $M(x) = 0$ for all $x < g_2^{-1}(1)$
2. $N(x) = 0$ for all $x > g_1^{-1}(1)$
3. $\lim\limits_{r\to\infty} \sigma_r^2 = (\sigma^*)^2$

where the functions M, N are associated with F, the constant σ_r^2 is associated with F^r and σ^* is associated with H, through the Le'vy's formula.

Moreover, we know that

4. if $\sigma^* = 0$, M is continuous at $g_2^{-1}(1)$ and N is continuous at $g_1^{-1}(1)$ then H is a degenerate distribution function.
5. if $\sigma^* \neq 0$, M is continuous at $g_2^{-1}(1)$ and N is continuous at $g_1^{-1}(1)$ then H is a normal distribution function.
6. if $\sigma^* = 0$, M is discontinuous at $g_2^{-1}(1)$ or N is discontinuous at $g_1^{-1}(1)$ then $H(x-m)$ is a Poisson distribution function, for some constant m.
7. if $\sigma^* \neq 0$, M is discontinuous at $g_2^{-1}(1)$ or N is discontinuous at $g_1^{-1}(1)$ then H is the distribution function of the sum of two independent random variables one of which is a normal and the other is a Poisson.

Proof.

For $r \geq 2$ and $0 < \varepsilon < \min((g_1(\delta))^r, (g_2(-\delta))^r)$, we have

$\max(g_1^{-1}(\varepsilon^{\frac{1}{r}}), |\, g_2^{-1}(\varepsilon^{\frac{1}{r}})\,|) \leq \delta$ and

$$0 \leq \int_{-\varepsilon}^{0^-} u^2 dM^r(u) + \int_{0^-}^{\varepsilon} u^2 dN^r(u)$$

$$= 0 + \int_{0^+}^{\varepsilon} u^2 d[\; N(g_1^{-1}(u^{\frac{1}{r}})) - M(g_2^{-1}(u^{\frac{1}{r}}))\;] \qquad \text{(by Theorem 2.2 (3) and(4))}$$

$$= \int_{0^+}^{g_1^{-1}(\varepsilon^{\frac{1}{r}})} (g_1(t_1))^{2r} dN(t_1) - \int_{0^-}^{g_2^{-1}(\varepsilon^{\frac{1}{r}})} (g_2(t_2))^{2r} dM(t_2) \;\; [t_1 = g_1^{-1}(u^{\frac{1}{r}}) \text{ and } t_2 = g_2^{-1}(u^{\frac{1}{r}})]$$

$$= \int_{0^+}^{g_1^{-1}(\varepsilon^{\frac{1}{r}})} (g_1(t))^{2r} dN(t) + \int_{g_2^{-1}(\varepsilon^{\frac{1}{r}})}^{0^-} (g_2(t))^{2r} dM(t)$$

$$\leq \varepsilon\Big(\int_{0^+}^{g_1^{-1}(\varepsilon^{\frac{1}{r}})} (g_1(t))^r dN(t)\Big) + \varepsilon\Big(\int_{g_2^{-1}(\varepsilon^{\frac{1}{r}})}^{0^-} (g_2(t))^r dM(t)\Big)$$

$$\leq \varepsilon\left[\int_{0^+}^{\delta}(g_1(t))^r\,dN(t) + \int_{-\delta}^{0^-}(g_2(t))^r\,dM(t)\right]$$

$$\leq \varepsilon\left[\int_{0^+}^{\delta}(g_1(t))^2 dN(t) + \int_{-\delta}^{0^-}(g_2(t))^2 dM(t)\right]$$

$$= \varepsilon\left[\int_{0^+}^{\delta}\left(\frac{g_1(t)}{t}\right)^2 t^2 dN(t) + \int_{-\delta}^{0^-}\left(\frac{g_2(t)}{t}\right)^2 t^2 dM(t)\right]$$

$$\leq c^2\varepsilon\left[\int_{0^+}^{\delta} t^2 dN(t) + \int_{-\delta}^{0^-} t^2 dM(t)\right].\ (\text{by property (g-3)}) \qquad \ldots(2.6.1)$$

Then $0 \leq \lim\limits_{\varepsilon\to 0^+}\overline{\lim\limits_{r\to\infty}}\left\{ \int_{-\varepsilon}^{0^-} u^2 dM^r(u) + \int_{0^+}^{\varepsilon} u^2 dN^r(u) \right\}$

$$\leq \lim_{\varepsilon\to 0^+}\overline{\lim_{r\to\infty}}\ c^2\varepsilon\left[\int_{0^+}^{\delta} t^2 dN(t) + \int_{-\delta}^{0^-} t^2 dM(t)\right]$$

$$= 0.$$

Hence $\lim\limits_{\varepsilon\to 0^+}\overline{\lim\limits_{r\to\infty}}\left\{ \int_{-\varepsilon}^{0^-} u^2 dM^r(u) + \int_{0^+}^{\varepsilon} u^2 dN^r(u) \right\} = 0.$ ...(2.6.2)

Similarly, we have

$$\lim_{\varepsilon\to 0^+}\underline{\lim_{r\to\infty}}\left\{ \int_{-\varepsilon}^{0^-} u^2 dM^r(u) + \int_{0^+}^{\varepsilon} u^2 dN^r(u) \right\} = 0. \qquad \ldots(2.6.3)$$

To prove necessity, we suppose that $F^r \xrightarrow{w} H$ as $r \to \infty$.

Since $F_n \xrightarrow{w} F$, by Theorem1.2.5 we have $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{k_n} F_{nk}(x) = M(x)$ and $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{k_n}\left[F_{nk}(x) - 1\right] = N(x)$ for all continuity points of M and N. Then (1) and (2) follow from Theorem 2.4(3) and the fact that M and N are nondecreasing and $M(-\infty) = N(\infty) = 0$.

Now, we will show (3).

Since $F^r \xrightarrow{w} H$, by Theorem 1.2.4 we have

$$\lim_{\varepsilon\to 0^+} \overline{\lim_{r\to\infty}}\{ \int_{-\varepsilon}^{0} u^2 dM^r(u) + \sigma_r^2 + \int_{0}^{\varepsilon} u^2 dN^r(u) \}$$

$$= \lim_{\varepsilon\to 0^+} \underline{\lim_{r\to\infty}}\{ \int_{-\varepsilon}^{0} u^2 dM^r(u) + \sigma_r^2 + \int_{0}^{\varepsilon} u^2 dN^r(u) \} = (\sigma^*)^2 \qquad \ldots(2.6.4)$$

By (2.6.2) - (2.6.3), we see that $\overline{\lim_{r\to\infty}}\sigma_r^2 = (\sigma^*)^2$ and $\underline{\lim_{r\to\infty}}\sigma_r^2 = (\sigma^*)^2$.

So $\lim_{r\to\infty} \sigma_r^2 = (\sigma^*)^2$.

To prove sufficiency, we assume that (1), (2) and (3) hold.

Since $F_n \xrightarrow{w} F$ and $F_n^r \xrightarrow{w} F^r$ as $n \to \infty$, by Theorem 2.2 , $M^r \equiv 0$ and

$N^r(x) = N(g_1^{-1}(x^{\frac{1}{r}})) - M(g_2^{-1}(x^{\frac{1}{r}}))$ a.e. on $(0,\infty)$.

Let $N^* : \mathbf{R}^+ \to \mathbf{R}$ be defined by $N^*(x) = \lim_{r\to\infty} N^r(x)$ and $M^* : \mathbf{R}^- \to \mathbf{R}$ be defined by $M^*(x) = \lim_{r\to\infty} M^r(x)$.

Then $M^* \equiv 0$ on $(-\infty,0)$ and by assumptions (1) , (2) and appendix

$$N^*(x) = \begin{cases} 0 & \text{if } x > 1 ; \\ N(g_1^{-1}(1^-)) - M(g_2^{-1}(1^-)) & \text{if } 0 < x < 1. \end{cases}$$

a.e. on $(0,\infty)$.

Since N^* is nondecreasing,

$$N^*(x) = \begin{cases} 0 & \text{if } x > 1 ; \\ N(g_1^{-1}(1^-)) - M(g_2^{-1}(1^-)) & \text{if } 0 < x < 1. \end{cases}$$

on $(0,\infty)$. $\qquad\ldots(2.6.5)$

That is $M^*(-\infty) = N^*(+\infty) = 0$.

From assumptions 3 and (2.6.2) we have

$$\lim_{\varepsilon\to 0^+}\overline{\lim_{r\to\infty}}\{\int_{-\varepsilon}^{0} u^2 dM^r(u)+\sigma_r^2+\int_0^{\varepsilon} u^2 dN^r(u)\}$$
$$= \lim_{r\to\infty}\sigma_r^2$$
$$= (\sigma^*)^2.$$

Similarly by (2.6.3), $\lim_{\varepsilon\to 0^+}\underline{\lim_{r\to\infty}}\{\int_{-\varepsilon}^{0} u^2 dM^r(u)+\sigma_r^2+\int_0^{\varepsilon} u^2 dN^r(u)\} = (\sigma^*)^2$.

By Theorem 1.2.4, we have $\lim_{r\to\infty} F^r(x) = H(x)$, where H is the infinitely divisible distribution determined by M^*, N^*, γ^* and $(\sigma^*)^2$.

We note that, by (2.6.5) $N^*(x)=0$ if M is continuous at $g_2^{-1}(1)$ and N is continuous at $g_1^{-1}(1)$ and N^* takes one jump if M is discontinuous at $g_2^{-1}(1)$ or N is discontinuous at $g_1^{-1}(1)$.

Hence Le′vy's formula of H can be represented by $M^*, N^*, (\sigma^*)^2$ and γ^*.

Case 1. $\sigma^*=0, M$ is continuous at $g_2^{-1}(1)$ and N is continuous at $g_1^{-1}(1)$.
It follows from Table 1.2.1 that H is degenerate.

Case 2. $\sigma^*\neq 0, M$ is continuous at $g_2^{-1}(1)$ and N is continuous at $g_1^{-1}(1)$.
It follows from Table 1.2.1 that H is a normal distribution function.

Case 3. $\sigma^*=0$ and M is discontinuous at $g_2^{-1}(1)$ or N is discontinuous at $g_1^{-1}(1)$.

Subcase 3.1 $\gamma^* = -\dfrac{N(g_1^{-1}(1))-M(g_2^{-1}(1))}{2}$.

From Table 1.2.1 we see that H is a Poisson distribution function.

Subcase 3.2 $\gamma^* \neq -\dfrac{N(g_1^{-1}(1))-M(g_2^{-1}(1))}{2}$.

We note that, for any constant m the characteristic $\varphi_m^*(t)$ of $H(x-m)$ is $e^{imt}\varphi^*(t)$ where φ^* is the characteristic function of H. Hence for

$m = -\dfrac{2\gamma^* + N(g_1^{-1}(1))-M(g_2^{-1}(1))}{2}$ we see that

$$\begin{aligned}
\log \varphi_m^*(t) &= \log e^{imt} \varphi^*(t) \\
&= imt + \log \varphi^*(t) \\
&= imt + i\gamma^* t + 0 + 0 + \int_{-\infty}^{0^-} (e^{itx} - 1 - \frac{itx}{1+x^2}) dN^*(x) \\
&= i(-\frac{N(g_1^{-1}(1)) - M(g_2^{-1}(1))}{2})t + \int_{-\infty}^{0^-} (e^{itx} - 1 - \frac{itx}{1+x^2}) dN^*(x) .
\end{aligned}$$

By table1.2.1 we see that $H(x-m)$ is a Poisson distribution function.

Case 4. if $\sigma^* \neq 0$ and M is discontinuous at $g_2^{-1}(1)$ or N is discontinuous at $g_1^{-1}(1)$ then, by Lemma 2.3, H is the sum of two independent random variables of Poisson and normal .

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REFERENCES

[1] Eugene Lukacs. **Characteristic Functions** . Newyork: Hafner, 1960.

[2] Geoffrey Grimmett and Dominic Welsh. **Probability and Introduction**. Oxford:Clarendon Press, 1991.

[3] Gnedenko, B.V., and Kolmogorov, A.N. **Limit Distributions for Sums of Independent Random Variables** . Cambridge:Addison-Wesley, 1954.

[4] Petrov, V.V. **Sums of Independent Random Variables** . Berlin: Springer-Verlag, 1975.

[5] Shapiro, J.M. Sums of Powers of Independent Random Variables. **the American Mathematical Society** 29 (August 1957): 515-522.

Appendix

Let A and B be any neighborhoods of 1.

Let $N: A \to \mathbf{R}$ and $h: B \to \mathbf{R}$.

Then 1. For each $x > 1$, if $\lim_{r\to\infty} N(h(x^{\frac{1}{r}})) = l_1$ and $\lim_{t\to 1^+} N(h(t)) = l_2$

then $l_1 = l_2$,

and 2. For each $0 < x < 1$, if $\lim_{r\to\infty} N(h(x^{\frac{1}{r}})) = l_1$ and $\lim_{t\to 1^-} N(h(t)) = l_2$

then $l_1 = l_2$.

Proof.

1. Suppose that the assumption holds. Let $\varepsilon > 0$ be given.

Then there exists $K \in N$, with $\left|N(h(x^{\frac{1}{r}})) - l_1\right| < \frac{\varepsilon}{2}$, for all natural number $r \geq K$

and there exists $\delta > 0$ such that for all t, if $0 < t - 1 < \delta$, then $|N(h(t)) - l_2| < \frac{\varepsilon}{2}$.

For $N_1 > \max\{\ K, \log_{1+\delta} x\ \}$, we have $\left|N(h(x^{\frac{1}{N_1}})) - l_1\right| < \frac{\varepsilon}{2}$.

Since $N_1 > \log_{1+\delta} x$, $(1+\delta)^{N_1} > x$, $1 + \delta > x^{\frac{1}{N_1}} > 1$ which implies that

$0 < x^{\frac{1}{N_1}} - 1 < \delta$. Hence $\left|N(h(x^{\frac{1}{N_1}})) - l_2\right| < \frac{\varepsilon}{2}$.

Thus $|l_1 - l_2| \leq \left|l_1 - N(h(x^{\frac{1}{N_1}}))\right| + \left|N(h(x^{\frac{1}{N_1}})) - l_2\right| < \varepsilon$. So $l_1 = l_2$.

2. Similar to 1.

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