CHAPTER III

EXTREMAL GRAPHS RELATED TO COMPLETE SUBGRAPHS

In this chapter, we determine the maximum number of lines of graphs with p points that contain exactly n complete subgraphs of order r and all of these n complete subgraphs are pairwise disjoint. The result is given in Theorem 3.10.

For each $G = (V, X)$ we associate a function η_G: $V \rightarrow P(V)$, the power set of V, by defining

$$\eta_G(u) = \{ v \mid v \in V, uv \in X \}.$$

The function η will be called the neighborhood function of G. Note that

$$\deg_G(v) = |\eta_G(v)|.$$

Proposition 3.1

For any graph G, η_G has the following properties:

(i). For each $v \in V(G)$, $v \notin \eta_G(v)$.

(ii). For any u, $v \in V(G)$, if $u \in \eta_G(v)$, then $v \in \eta_G(u)$.

Proof By definition of graphs, we see that $vv \notin X(G)$ for any $v \in V(G)$. Hence we have (i). Let u, v be any points of $V(G)$ such that $u \in \eta_G(v)$. Then $uv \in X$. But $vu = uv$. Hence $vu \in X$. Thus $v \in \eta_G(u)$. Therefore η_G has properties (ii). □

Proposition 3.2

Let V be a non-empty set. Let $\eta: V \to P(V)$. If η satisfies both of the following properties:

(i). For each $v \in V$, $v \notin \eta(v)$.

(ii). For any u, $v \in V$, if $u \in \eta(v)$, then $v \in \eta(u)$.

Then there exists a unique graph G such that $V(G) = V$ and $\eta_G = \eta$.

Proof Define

$$X = \{uv \mid u,v \in V,\ u \in \eta(v)\}.$$

By property (i), it can be seen that members of X, if exist, must be 2-subsets of V. By property (ii), we see that for any u, v in V we have

$$uv \in X \text{ if and only if } vu \in X.$$

Hence X is a well-defined set of 2-subsets of V. Therefore (V, X) is a graph.

Let $G = (V, X)$. Let v be any point in V. For any $u \in \eta_G(v)$, $uv \in X$. Thus $u \in \eta(v)$. Hence $\eta_G(v) \subseteq \eta(v)$. Similarly we can show that $\eta(v) \subseteq \eta_G(v)$. Hence $\eta_G(v) = \eta(v)$ for all $v \in V$. Therefore $\eta_G = \eta$.

Let G' be any graph such that $\eta_{G'} = \eta$. Thus $\eta_{G'} = \eta_G$. It is clear that

$$V(G') = V = V(G).$$

Observe that

$$\begin{aligned} uv \in X(G') &\leftrightarrow u \in \eta_{G'}(v) \\ &\leftrightarrow u \in \eta_G(v) \\ &\leftrightarrow uv \in X. \end{aligned}$$

Therefore $\qquad G' = G$ □

In the sequel, we shall denote the graph with neighborhood function η by $G[\eta]$.

Let V, V' be any two non-empty sets such that $V \subseteq V'$. Let η, η' be any two neighborhood functions on V and V' respectively. η is said to be a sub-neighborhood function of η' when $\eta(v)$ is a subset of $\eta'(v)$ for each v in V. We note that η being a sub-neighborhood function of η' does not mean that η is a subset of η'. For example, let

$$V = \{a, b, c\};$$

$$V' = \{a, b, c, d\};$$

$$\eta = \{ (a, \{b\}), (b, \{a, c\}), (c, \{b\}) \};$$

and

$$\eta' = \{ (a, \{b, c\}), (b, \{a, c\}), (c, \{a, b\}) \}.$$

It can be seen that η is a sub-neighborhood function according to the above definition, but η is not a subset of η'.

Proposition 3.3

If η is a sub-neighborhood function of η'. Then $G[\eta]$ is a subgraph of $G[\eta']$.

Proof Observe that

$$X(G[\eta']) = \{uv \mid u,v \in V(G[\eta']),\ u \in \eta'(v)\}$$

and $$X(G[\eta]) = \{uv \mid u,v \in V(G[\eta]),\ u \in \eta(v)\}.$$

Since η is a sub-neighborhood function of η', hence the domain of η must be a subset of the domain of η'. Therefore

$$V(G[\eta]) \subseteq V(G[\eta']).$$

That is each point of $G[\eta]$ is a point of $G[\eta']$. Next we show that every line of $G[\eta]$ is a line of $G[\eta']$. To do this, let $uv \in X(G[\eta])$. Then $u \in \eta(v)$. Since $\eta(v) \subseteq \eta'(v)$. Thus $u \in \eta'(v)$. Thus $uv \in X(G[\eta'])$. Hence

$$X(G[\eta]) \subseteq X(G[\eta']).$$

Therefore $G[\eta]$ is a subgraph of $G[\eta']$. □

A graph G is said to have the property $P(r)$ if every two distinct complete subgraphs of order r are disjoint. A graph G is said to have the property $P(r, n)$ if it contains exactly n complete subgraphs of order r and has property $P(r)$.

Proposition 3.4.

Let n, r be any positive integers such that $r \geq 2$. Let H be a graph of order rn with property $P(r, n)$. Let G be any graph with property $P(r)$ such that G is a supergraph of H. Then for any point $v \in V(G)\backslash V(H)$, there exist at most $(r-2)n$ lines from v to points of H.

Proof Let (V_i, X_i) $i = 1,\ldots,n$ be the n disjoint complete subgraphs of order r of H. Let

$$v \in V(G)\backslash V(H).$$

Thus

$$v \notin V_i,$$

$i = 1,\ldots,n$. Suppose there are more than $(r-2)$ lines from v to points of V_i for some i. Let $v_1, v_2,\ldots, v_{r-1}$ be $r-1$ points of V_i which are joined by lines to v. Thus $v, v_1, v_2,\ldots, v_{r-1}$ form a complete subgraphs of order r which is not disjoint from (V_i, X_i). This is contrary to the assumption that G has the property $P(r)$. Thus for each i, there are at most $(r-2)$ lines from v to points of V_i. Therefore there are at most $(r-2)n$ lines from v to points of H. □

In what follows, we shall use the symbol $\bar{n}$ to denote the set $\{1, 2, \ldots, n\}$.

Proposition 3.5

Let n, r be any positive integers such that $r \geq 2$. Let G be any graph of order rn that has the property $P(r, n)$. Then G can have atmost

$$\frac{rn(n(r\text{-}2)+1)}{2}$$

lines.

Proof Let (V_i, X_i) $i = 1, \ldots, n$ be the n disjoint complete subgraphs of order r of G. Let $i \in \bar{n}$. Since G has the property $P(r, n)$, hence $G \backslash V_i$ has property $P(r, n-1)$. Note that $G \backslash V_i$ has order $r(n-1)$. Since G is a supergraph of $G \backslash V_i$ which has property $P(r, n-1)$. Therefore, by Proposition 3.4, there exist at most $(r-2)(n-1)$ lines from any $v \in V_i$ to points of $G \backslash V_i$. Thus for any $v \in V_i$,

$$\begin{aligned} \deg_G(v) &= |\{ u \mid u \in G \backslash V_i, uv \in X(G) \}| + |\{ u \mid u \in V_i, uv \in X(G) \}| \\ &\leq (r-2)(n-1) + r - 1 \\ &= n(r-2) + 1. \end{aligned}$$

Therefore

$$|X(G)| = \frac{\sum_{v \in V(G)} \deg_G(v)}{2} \leq \frac{rn(n(r-2)+1}{2}. \quad \square$$

Note that Proposition 3.5 gives an upper bound of the number of lines among graphs of order rn that have the property $P(r, n)$. In the next proposition (Proposition 3.6), we shall show that this upper bound is attained. This will be done by constructing a graph of order rn that has the property $P(r, n)$. Our construction calls for arithemetic of the subscripts used in labelling parts of certain complete r-partite graphs, and this is best done by using the elements of Z_r, the set of residue classes modulo r, as subscripts.

Proposition 3.6

Let n, r be any positive integers such that $r \geq 2$. Let v_{ij}, $i \in Z_r$, $j \in \bar{n}$, be any rn distinct elements. Let

$$T = \{ v_{ij} \mid i \in Z_r, j \in \bar{n} \}.$$

For each $i_0 \in Z_r$ and $j_0 \in \bar{n}$, let

$$A_1(i_0, j_0) = \begin{cases} \phi & \text{if } j_0 = 1, \\ \{ v_{ij_0-1} \mid i \in Z_r, i \neq i_0 \text{ and } i \neq i_0 + 1 \} & \text{if } 1 < j_0 \leq n; \end{cases}$$

$$A_2(i_0, j_0) = \{ v_{ij_0} \mid v_{ij_0} \in T, i \neq i_0 \};$$

$$A_3(i_0, j_0) = \begin{cases} \{ v_{ij_0+1} \mid i \in Z_r, i \neq i_0 \text{ and } i \neq i_0 - 1 \} & \text{if } 1 \leq j_0 < n; \\ \phi & \text{if } j_0 = n. \end{cases}$$

Let η: $T \rightarrow P(T)$ be difined by

$$\eta(v_{i_0j_0}) = \bigcup_{j=1}^{j_0} A_1(i_0, j) \cup A_2(i_0, j_0) \cup \bigcup_{j=j_0}^{n} A_3(i_0, j).$$

Then η is a neighborhood function of a graph G of order rn with property $P(r, n)$ that has the maximum number of lines and

$$|X(G)| = \frac{rn(n(r-2)+1)}{2}.$$

Proof Let $v_{i_0j_0}$ be any element of T. For any $i \in Z_r$ and $j \in \bar{n}$, if $v_{ij_0} \in A_1(i_0, j)$, then, by the definition of $A_1(i_0, j)$, we have $i \neq i_0$. Hence

(3.1) $$v_{i_0j_0} \notin A_1(i_0, j)$$

for any $j \in \bar{n}$. By similar argument we see that

$$v_{i_0j_0} \notin A_2(i_0, j) \text{ and } v_{i_0j_0} \notin A_3(i_0, j)$$

for any $j \in \bar{n}$. Therefore

$$v_{i_0j_0} \notin \bigcup_{j'=1}^{j} A_1(i_0, j') \cup A_2(i_0, j) \cup \bigcup_{j'=j}^{n} A_3(i_0, j').$$

Since for any $j \in \bar{n}$

$$\eta(v_{i_0j}) = \bigcup_{j'=1}^{j} A_1(i_0, j') \cup A_2(i_0, j) \cup \bigcup_{j'=j}^{n} A_3(i_0, j'),$$

thus

(3.2) $$v_{i_0j_0} \notin \eta(v_{i_0j}).$$

In particular, we have

$$v_{i_0j_0} \notin \eta(v_{i_0j_0}).$$

Since $v_{i_0j_0}$ is arbitrary, so we have

(3.3) (i) $v_{ij} \notin \eta(v_{ij})$ for any $v_{ij} \in T$.

Let $v_{i_1j_1}, v_{i_2j_2} \in T$ be any elements such that $v_{i_2j_2} \in \eta(v_{i_1j_1})$. Since

$$\eta(v_{i_1 j_1}) = \bigcup_{j=1}^{j_1} A_1(i_1, j) \cup A_2(i_1, j_1) \cup \bigcup_{j=j_1}^{n} A_3(i_1, j),$$

thus $v_{i_2 j_2} \in \bigcup_{j=1}^{j_1} A_1(i_1, j)$ or $v_{i_2 j_2} \in A_2(i_1, j_1)$ or $v_{i_2 j_2} \in \bigcup_{j=j_1}^{n} A_3(i_1, j)$.

Case 1. Suppose that $v_{i_2 j_2} \in \bigcup_{j=1}^{j_1} A_1(i_1, j)$. Then there exists $j \in \bar{n}$ such that

(3.4) $$j \leq j_1,$$

(3.5) $$v_{i_2 j_2} \in A_1(i_1, j).$$

From (3.5), we see that

$$v_{i_2 j_2} = v_{ij\text{-}1}$$

for some $i \in Z_r$ such that $i \neq i_1$ and $i \neq i_1+1$. Therefore

(3.6) $$j_2 = j - 1,$$

(3.7) $$i_2 \neq i_1 \text{ and } i_2 \neq i_1+1.$$

From (3.4) and (3.6) we see that

$$j_1 > j_2.$$

Let $j' = j_1 - 1$. Then $j_1 = j'+1$ and $j'+1 > j_2$ and hence $j' \geq j_2$. Therefore by definition of $A_3(i_2, j')$, we see that

$$v_{i_1 j_1} \in A_3(i_2, j').$$

Since

$$A_3(i_2, j') \subseteq \bigcup_{j=j_2}^{n} A_3(i_2, j) \subseteq \eta(v_{i_2 j_2}),$$

so, we have

$$v_{i_1 j_1} \in \eta(v_{i_2 j_2}).$$

Case 2. Suppose that $v_{i_2 j_2} \in A_2(i_1, j_1)$. By the definition of $A_2(i_1, j_1)$,

$$i_2 = i_1 \text{ and } j_2 \neq j_1.$$

Thus

$$v_{i_1 j_1} \in A_2(i_2, j_2).$$

Since

$$A_2(i_2, j_2) \subseteq \eta(v_{i_2 j_2}).$$

Therefore

$$v_{i_1 j_1} \in \eta(v_{i_2 j_2}).$$

Case 3. Suppose that $v_{i_2 j_2} \in \bigcup_{j=j_1}^{n} A_3(i_1, j)$. Then there exists $j \in \bar{n}$ such that

(3.8) $$j \geq j_1,$$

(3.9) $$v_{i_2 j_2} \in A_3(i_1, j).$$

From (3.9), we see that

$$v_{i_2 j_2} = v_{ij+1}$$

for some $i \in Z_r$ such that $i \neq i_1$ and $i \neq i_1 - 1$. Therefore

(3.10) $$j_2 = j+1,$$

(3.11) $$i_2 \neq i_1 \text{ and } i_2 \neq i_1 - 1.$$

From (3.8) and (3.10) we see that

$$j_1 < j_2.$$

Let $j' = j_1 + 1$. Then $j_1 = j' - 1$ and $j' - 1 < j_2$ and hence $j' \leq j_2$. Therefore by definition of $A_1(i_2, j')$,

$$v_{i_1 j_1} \in A_1(i_2, j').$$

Since

$$A_1(i_2, j') \subseteq \bigcup_{j=1}^{j_2} A_1(i_2, j) \subseteq \eta(v_{i_2 j_2}),$$

so, we have

$$v_{i_1j_1} \in \eta(v_{i_2j_2}).$$

From the three cases considered, we see that η satisfies property

(3.12) (ii) for any $v_{i_1j_1}, v_{i_2j_2} \in T$, if $v_{i_2j_2} \in \eta(v_{i_1j_1})$, then $v_{i_1j_1} \in \eta(v_{i_2j_2})$.

Now we have (3.3) and (3.12). So, by Proposition 3.2, there is a graph $G[\eta]$ such that $V(G[\eta]) = T$ and η is the neighborhood function of $G[\eta]$.

For each $j \in \bar{n}$, let

$$V_j = \{ v_{0j}, v_{1j}, \ldots, v_{r-1j} \}.$$

Observe that for any V_j, any pair of distinct points of V_j must be of the form $v_{ij}, v_{i'j}$, where $i \neq i'$. Hence, by definitions of $A_2(i', j)$ and η, we have

$$v_{ij} \in A_2(i', j) \subseteq \eta(v_{i'j}).$$

Hence $v_{ij}v_{i'j} \in X(G[\eta])$. Therefore every pair of points of V_j forms a line. Hence

$$<V_j> \cong K_r.$$

Thus $<V_1>, <V_2>, \ldots, <V_n>$ are n complete subgraphs of order r of $G[\eta]$.

Suppose $G[\eta]$ has more than n complete subgraphs of order r. Let W be any subset of T consisting of r points such that $<W> \cong K_r$. Suppose that

(3.13) $$W = \{v_{i_0j_0}, v_{i_1j_1}, \ldots, v_{i_{r-1}j_{r-1}}\}$$

and

$$W \neq V_j$$

for any $j \in \bar{n}$. By (3.2) we know that for any $v_{ij}, v_{i'j'} \in T$ if $i = i'$ then $v_{ij} \notin \eta(v_{i'j'})$ i.e. $v_{ij}v_{i'j'} \notin X(G[\eta])$. Thus for any distinct points $v_{ij}, v_{i'j'} \in W$ such that $v_{ij}v_{i'j'} \in X(G[\eta])$, we must have i $\neq i'$. Hence $i_0, i_1, \ldots, i_{r-1}$ are distinct. Without loss of generality, we may assume that

$$i_0 = 0, i_1 = 1, \ldots, i_{r-1} = r-1.$$

Thus (3.13) may be rewritten as

(3.14) $$W = \{v_{0j_0}, v_{1j_1}, \ldots, v_{r\text{-}1j_{r\text{-}1}}\}.$$

Let $i \in Z_r$. Since $\langle W \rangle \cong K_r$, $v_{i\,j_i}v_{i+1j_{i+1}} \in X$ i.e.

(3.15) $$v_{ij_i} \in \eta(v_{i+1j_{i+1}}) = \bigcup_{j=1}^{j_{i+1}} A_1(i+1, j) \cup A_2(i+1, j_{i+1}) \cup \bigcup_{j=j_{i+1}}^{n} A_3(i+1, j).$$

For any $j \in \bar{n}$ and $v_{i'j'} \in T$, by definition of $A_3(i, j)$, if

$$i' = i \text{ or } i' = i - 1 \text{ or } j' \neq j+1,$$

then

$$v_{i'j'} \notin A_3(i, j).$$

Hence

$$v_{ij_i} \notin A_3(i+1, j)$$

for any $j \in \bar{n}$. Thus

$$v_{ij_i} \notin \bigcup_{j=j_{i+1}}^{n} A_3(i+1, j).$$

Therefore, by (3.15), we have

$$v_{ij_i} \in \bigcup_{j=1}^{j_{i+1}} A_1(i+1, j) \cup A_2(i+1, j_{i+1}).$$

Note that in case $v_{ij_i} \in \bigcup_{j=1}^{j_{i+1}} A_1(i+1, j)$, we have $v_{ij_i} \in A_1(i+1, j)$ for some $j \leq j_{i+1}$.

Thus

(3.16) $$j_i = j-1 < j_{i+1}$$

On the other hand, if $v_{ij_i} \in A_2(i+1, j_{i+1})$, then

(3.17) $$j_i = j_{i+1}.$$

In any case we have,

$$j_i \leq j_{i+1}.$$

Therefore

$$j_0 \le j_1 \le \ldots \le j_{r-1} \le j_0.$$

Hence there exists $j \in \bar{n}$ such that

$$j_0 = j_1 = \ldots = j_{r-2} = j_{r-1} = j.$$

Thus, by (3.14), we have

$$W = \{v_{0j}, v_{1j}, \ldots, v_{r-1j}\} = V_j,$$

which is a contradiction. Thus $G[\eta]$ has exactly n complete subgraphs of order r. Since $V_j \cap V_{j'} = \varnothing$, for distinct $j, j' \in \bar{n}$. Hence the n complete subgraphs of order r are disjoint. Therefore $G[\eta]$ has the property $P(r, n)$. For each $v_{i_0 j_0} \in T$,

$$\begin{aligned}
\deg_{G[\eta]}(v_{i_0 j_0}) &= |\eta(v_{i_0 j_0})| \\
&= |\bigcup_{j=1}^{j_0} A_1(i_0, j) \cup A_2(i_0, j_0) \cup \bigcup_{j=j_0}^{n} A_3(i_0, j)| \\
&= |\bigcup_{j=1}^{j_0} A_1(i_0, j)| + |A_2(i_0, j_0)| + |\bigcup_{j=j_0}^{n} A_3(i_0, j)| \\
&= (i-1)(r-2) + (r-1) + (n-i)(r-2) \\
&= n(r-2) + 1.
\end{aligned}$$

By (1.1), therefore

$$|X(G[\eta])| = \frac{\sum_{v \in T} \deg_G(v)}{2} = \frac{rn(n(r-2)+1)}{2}.$$

By the Proposition 3.5, $G[\eta]$ is a graph of order rn with property $P(r, n)$ that has the maximum number of lines. □

An illustration of the construction given in the above proof for the case $n = 4$, $r = 3$ can be found in Appendix A.

Proposition 3.7 Let n be a non-negative integer. Let r be any positive integer such that $r \geq 2$. If G is a graph of order $1+rn$ with property $P(r, n)$, then

$$|X(G)| \leq (r-2)n + \frac{rn(n(r-2)+1)}{2}.$$

Furthermore, the maximum number of lines of graphs of order $1+rn$ with property $P(r, n)$ is

$$(r-2)n + \frac{rn(n(r-2)+1)}{2}.$$

Proof Let v be the point of $V(G)$ such that v is not a point of any complete subgraphs of order r. Note that $G-v$ has order rn. Since G is a supergraph of $G-v$ which has property $P(r, n)$. Thus, by Proposition 3.5,

$$|X(G-v)| \leq \frac{rn(n(r-2)+1)}{2},$$

and by Proposition 3.4, there exist at most $(r-2)n$ lines from v to points of $G-v$. Therefore

$$|X(G)| = |X(G-v)| + \text{The number of lines from } v \text{ to points of } G-v$$
$$\leq \frac{rn(n(r-2)+1)}{2} + (r-2)n.$$

In order to prove that the maximum number of lines of graphs of order $1+rn$ with property $P(r, n)$ is $(r-2)n + \frac{rn(n(r-2)+1)}{2}$, it suffices to establish a graph of order $1+rn$ with property $P(r, n)$ that has $(r-2)n + \frac{rn(n(r-2)+1)}{2}$ lines.

Let T and η be as defined in Proposition 3.6. Thus $G[\eta]$ is a graph of order rn with property $P(r, n)$ such that

$$|X(G[\eta])| = \frac{rn(n(r-2)+1)}{2}.$$

Let v be an element that is distinct from all elements of T. For each $i \in Z_r$, let

$$C_i = \{ v_{ij} \mid v_{ij} \in T \text{ for } j \in \bar{n} \}.$$

Let G be the graph defined as follows.

$$V(G) = V(G[\eta]) \cup \{v\}.$$

$$X(G) = X(G[\eta]) \cup \{ vu \mid u \in T \setminus (C_{r-2} \cup C_{r-1}) \}.$$

Note that v is joined to all the points of $\bigcup_{i=0}^{r-3} C_i$.

Next, we shall show G has the property $P(r, n)$.

Consider any $r-1$ points that are joined to v. Since $r-1$ points are from the $r-2$ sets $C_0,\ldots,C_{r-3}$. Hence a pair of the points must be from the same C_i for some i. Assume that this pair of points are $v_{ij'}$, $v_{ij''}$. Since η is as given in Proposition 3.6. Hence it satisfies the relation (3.2) in the proof of the proposition. It follows from this condition that

$$v_{ij'}v_{ij''} \notin X(G[\eta]).$$

Thus

$$v_{ij'}v_{ij''} \notin X(G[\eta]) \cup \{ vu \mid u \in T \setminus (C_{r-2} \cup C_{r-1}) \}$$

i.e.

$$v_{ij'}v_{ij''} \notin X(G).$$

Thus for any $r-1$ points joined to v there are at least two points not joined by a line. Hence v is not an point in any complete subgraphs of order r of G. Hence the complete subgraphs of order r of G are those and only those of $G[\eta]$. Hence G contains exactly n disjoint complete subgraphs of order r. Therefore G has the property $P(r, n)$. Observe that

$$|X(G)| = |X(G[\eta])| + |\{ vu \mid u \in T \setminus (C_{r-2} \cup C_{r-1} \}|$$

$$= \frac{rn(n(r-2)+1)}{2} + (r-2)n.$$

Therefore, G is a graph of order $1+rn$ with property $P(r, n)$ that has the maximum number of lines. □

Proposition 3.8

Let n be a non-negative integer. Let p, r be any positive integers such that $p \geq r \geq 2$ and $p > rn$. Let $m = p - rn$, $k = [\frac{m}{r-1}]$ and $s = m - k(r-1)$. Then there exists a graph G of order p such that

(i). G has the property $P(r, n)$;

(ii). $\max\{\deg_G(v) \mid v \in V(G)\setminus T\} = m - k + (r-2)n$, where T is the set of all the points of the n disjoint complete subgraphs of order r of G;

(iii). $|X(G)| = \dfrac{m^2 - km - s(k+1) + 2mn(r-2) + rn(n(r-2)+1)}{2}$.

Proof Let

$$m_0 = m_1 = \ldots = m_{s-1} = k+1$$

and

$$m_s = m_{s+1} = \ldots = m_{r-2} = k.$$

Let

$v_{01}, v_{02}, \ldots\ldots\ldots\ldots\ldots, v_{0n},$

$v_{11}, v_{12}, \ldots\ldots\ldots\ldots\ldots, v_{1n},$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$

$v_{r-11}, v_{r-12}, \ldots\ldots\ldots, v_{r-1n},$

$u_{01}, u_{02}, \ldots\ldots\ldots\ldots\ldots\ldots, u_{0k+1},$

$u_{11}, u_{12}, \ldots\ldots\ldots\ldots\ldots\ldots, u_{1k+1},$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$

$u_{s-11}, u_{s-12}, \ldots\ldots\ldots\ldots\ldots, u_{s-1k+1},$

$u_{s1}, u_{s2}, \ldots\ldots\ldots\ldots\ldots, u_{sk},$

$u_{s+11}, u_{s+12}, \ldots\ldots\ldots, u_{s+1k},$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$

$u_{r-21}, u_{r-22}, \ldots\ldots\ldots\ldots, u_{r-2k}$

be distinct elements. For each $i \in Z_r$, let

$$C_i = \{ v_{i1}, v_{i2}, \ldots, v_{in} \}.$$

Define

$$T = \bigcup_{i \in Z_r} C_i .$$

For each $i \in Z_{r-1}$, let

$$U_i = \{ u_{i1}, u_{i2}, \ldots, u_{im_i} \}.$$

Define

$$U = \bigcup_{i \in Z_{r-1}} U_i .$$

Note that $|C_i| = n$; $i \in Z_r$, $|T| = rn$, $|U_i| = m_i$; $i \in Z_{r-1}$, $|U| = m$.

For each $i_0 \in Z_r$ and $j_0 \in \bar{n}$, define $A_1(i_0, j_0)$, $A_2(i_0, j_0)$, $A_3(i_0, j_0)$ and $\eta(v_{i_0 j_0})$ as in Proposition 3.6. Let $\eta' : U \cup T \to P(U \cup T)$ be defined by

$$\eta'(v) = \begin{cases} U \cup T - U_i \cup C_i \cup C_{r-1} & \text{when } v \in U_i \quad \text{for some } i \in Z_{r-1}. \\ \eta(v) \cup U - U_i & \text{when } v \in C_i \quad \text{for some } i \in Z_r \backslash \{r-1\}. \\ \eta(v) & \text{when } v \in C_{r-1}. \end{cases}$$

We shall show that η' is a neighborhood function on $U \cup T$. First we must show that

(3.18) $\quad v \notin \eta'(v)$

for all $v \in U \cup T$. This will be done according to the cases used in the definition of η'.

Case 1. $v \in U_i$ for some $i \in Z_{r-1}$. In this case $\eta'(v) = U \cup T - U_i \cup C_i \cup C_{r-1}$. Since $v \notin U \cup T - U_i$, thus $\eta'(v) \cap U_i = \varnothing$, and hence $v \notin \eta'(v)$. Therefore (3.18) holds.

Case 2. $v \in C_i$ for some $i \in Z_r \backslash \{r-1\}$. In this case $\eta'(v) = \eta(v) \cup U - U_i$. Since $v \notin \eta(v)$ and $v \notin U$, $v \notin \eta(v) \cup U - U_i$. Thus $v \notin \eta'(v)$. Therefore (3.18) holds.

Case 3. $v \in C_{r-1}$. Then $\eta'(v) = \eta(v)$. Since $v \notin \eta(v)$ for all $v \in T$, $v \notin \eta'(v)$. Therefore (3.18) holds.

Since v is arbitrary, so we have

(3.19) $\quad v \notin \eta'(v)$,

for all $v \in U \cup T$.

Next we shall show that

(3.20) $\quad v_2 \in \eta'(v_1)$

for all $v_1, v_2 \in U \cup T$ such that $v_1 \in \eta'(v_2)$. Let $v_1, v_2 \in U \cup T$ be such that $v_1 \in \eta'(v_2)$. By definition of η', we have three cases.

Case 1. $v_2 \in U_{i_0}$ for some $i_0 \in Z_{r-1}$. In this case

$$\eta'(v_2) = U \cup T - U_{i_0} \cup C_{i_0} \cup C_{r-1}$$

$$= \bigcup_{i \in Z_{r-1} \setminus \{i_0\}} U_i \cup \bigcup_{i \in Z_r \setminus \{i_0, r-1\}} C_i .$$

Thus

$$v_1 \in \bigcup_{i \in Z_{r-1} \setminus \{i_0\}} U_i \text{ and } v_1 \in \bigcup_{i \in Z_r \setminus \{i_0, r-1\}} C_i .$$

Subcase 1.1. $v_1 \in \bigcup_{i \in Z_{r-1} \setminus \{i_0\}} U_i$. Therefore there exists an $i \in Z_{r-1} \setminus \{i_0\}$ such that

$$v_1 \in U_i.$$

Note that

$$i_0 \neq i \quad \text{and} \quad i_0 \neq r-1.$$

Since

$$v_2 \in U_{i_0},$$

hence

$$v_2 \notin U_i \cup C_i \cup C_{r-1}.$$

Therefore

$$v_2 \in U \cup T - U_i \cup C_i \cup C_{r-1}$$

i.e. we have

$$v_2 \in \eta'(v_1).$$

Subcase 1.2. $v_1 \in \bigcup_{i \in Z_r \setminus \{i_0, r-1\}} C_i$. Therefore $i \in Z_r \setminus \{i_0, r-1\}$ such that

$$v_1 \in C_i.$$

Note that

$$i_0 \neq i.$$

Since

$$v_2 \in U_{i_0},$$

hence

$$v_2 \notin U_i.$$

Therefore

$$v_2 \in \eta(v) \cup U - U_i$$

i.e. we have

$$v_2 \in \eta'(v_1).$$

Therefore (3.20) holds.

Case 2. $v_2 \in C_{i_0}$ for some $i_0 \in Z_r \backslash \{i_0, r-1\}$. In this case

$$\eta'(v_2) = \eta(v_2) \cup U - U_{i_0}.$$

Thus

$$v_1 \in \eta(v_2) \cup U - U_{i_0}.$$

Subcase 2.1. $v_1 \in \eta(v_2)$. Then $v_2 \in \eta(v_1)$.

Since $\eta(v) \subseteq \eta'(v)$ for all $v \in T$, $v_2 \in \eta'(v_1)$.

Subcase 2.2. $v_1 \in U_i$ for some $i \in Z_{r-1} \backslash \{i_0\}$. Then

$$\eta(v_1) = U \cup T - U_i \cup C_i \cup C_{r-1}.$$

Note that

$$i_0 \neq i \quad \text{and} \quad i_0 \neq r-1.$$

Since

$$v_2 \in U_{i_0},$$

hence

$$v_2 \notin U_i \cup C_i \cup C_{r-1}.$$

Therefore

$$v_2 \in U \cup T - U_i \cup C_i \cup C_{r-1}$$

i.e. we have

$$v_2 \in \eta'(v_1).$$

Therefore (3.20) holds.

Case 3. $v_2 \in C_{r-1}$. Then $\eta'(v_2) = \eta(v_2)$. Thus $v_1 \in \eta(v_2)$. So that $v_2 \in \eta(v_1)$. Since $\eta(v) \subseteq \eta'(v)$ for all $v \in T$, $v_2 \in \eta'(v_1)$. Therefore (3.20) holds.

From the three cases considered, we see that η' satisfies property

(3.21) for all $v_1, v_2 \in U \cup T$ if $v_1 \in \eta(v_2)$ then $v_2 \in \eta(v_1)$

From (3.19) and (3.21) we see that η' satisfies (i) and (ii) of Proposition 3.2. Therefore there exists a graph $G[\eta']$ such that $V(G[\eta']) = U \cup T$ and η' is its neighborhood function of $G[\eta']$.

Next, we shall show (i).

Observe that η is a sub-neighborhood function of η' and $\eta(v) = \eta'(v) \setminus U$ for all $v \in T$, hence $<T> \cong G[\eta']$. Therefore, by Proposition 3.3, $G[\eta]$ is a subgraph of $G[\eta']$. Since $G[\eta]$ has property $P(r, n)$, thus $G[\eta']$ contains all the complete subgraphs of order r of $G[\eta]$, i.e. it contains at least n disjoint complete subgraphs of order r.

Let $u_0 \in U$. Then there is $i_0 \in Z_{r-1}$ such that $u_0 \in U_{i_0}$. Thus

$$\eta(u_0) = U \cup T - U_{i_0} \cup C_{i_0} \cup C_{r-1}.$$

Let $v_2, v_3, \ldots, v_r$ be distinct points in $\eta'(u_0)$. Then

$$v_2, v_3, \ldots, v_r \in U \cup T - U_{i_0} \cup C_{i_0} \cup C_{r-1}.$$

Since

$$|\{ i \mid i \in Z_{r-1} \setminus \{i_0\}, U_i \subseteq \eta'(u) \text{ or } C_i \subseteq \eta'(u) \}| = |Z_r \setminus \{i_0, r-1\}| = r-2$$

and

$$|\{ v_2, v_3, \ldots, v_r \}| = r-1,$$

then there exist $v_{j'}, v_{j''} \in \{ v_1, \ldots, v_{r-1} \}$ and $i \in Z_{r-1} \setminus \{i_0\}$ such that

$$v_{j'}, v_{j''} \in U_i$$

or

$$v_{j'}, v_{j''} \in C_i$$

or

$$v_{j'} \in U_i \text{ and } v_{j''} \in C_i$$

or

$$v_{j'} \in C_i \text{ and } v_{j''} \in U_i.$$

Case 1. $v_{j'}, v_{j''} \in U_i$ Then

$$\eta'(v_{j'}) = U \cup T - U_i \cup C_i \cup C_{r-1},$$

so that $v_{j''} \notin \eta'(v_{j'})$. Thus $v_{j'}v_{j''} \notin X$.

Case 2. $v_{j'}, v_{j''} \in C_i$ Then

$$\eta'(v_{j'}) = \eta(v_{j'}) \cup U - U_i.$$

By Proposition 3.6, $v_{j''} \notin \eta(v_{j'})$, so that $v_{j''} \notin \eta'(v_{j'})$. Thus $v_{j'}v_{j''} \notin X$.

Case 3. $v_{j'} \in U_i$ and $v_{j''} \in C_i$. Then

$$\eta'(v_{j'}) = U \cup T - U_i \cup C_i \cup C_{r-1}.$$

Since

$$\eta'(v_{j'}) \cap C_i = \varnothing,$$

thus

$$v_{j''} \notin \eta'(v_{j'})$$

i.e. we have that

$$v_{j'}v_{j''} \notin X.$$

Case 4. $v_{j'} \in C_i$ and $v_{j''} \in U_i$. It can be shown in the same way as Case 3 that

$$v_{j'}v_{j''} \notin X.$$

Thus for any $r-1$ points in $\eta'(u_0)$ there are at least two points not joined by a line. Hence u_0 is not a point in any complete subgraphs of order r. Therefore any complete subgraphs of order r not contain any points in U. Thus for any complete subgraphs of order r consist of r points contained in T. Since $\langle T \rangle$ is isomorphic to the graph in Proposition 3.6 and $G[\eta]$ has n disjoint complete subgraphs of order r. Thus $G[\eta']$ has n disjoint complete subgraphs of order r. Hence $G[\eta']$ has the property $P(r, n)$. Therefore (i) holds.

Let $u \in U$ be arbitrary. Thus there exists i such that $u \in U_i$ for some $i \in Z_{r-1}$.

Hence

$$\begin{aligned} \deg_{G[\eta']}(u) &= |\eta'(u)| \\ &= |\, U \cup T - U_i \cup C_i \cup C_{r-1} \,| \\ &= m - m_i + rn - 2n \\ &= m - m_i + (r-2)n. \end{aligned}$$

Since $m_i = k$ or $m_i = k + 1$, hence $m_i \geq k$. Therefore we have

$$\deg_{G[\eta']}(u) \leq m - k + (r-2)n,$$

for all $u \in U$.

Note that the value $m - k + (r-2)n$ is attained by $\deg_{G[\eta']}(u)$ for $u \in U_s$, hence

$$\max\{\deg_G(v) \mid v \in V(G)\backslash T\} = m - k + (r-2)n.$$

Therefore (ii) holds.

To show (iii), observe that

$$(3.22) \qquad \sum_{v \in U \cup T} \deg_{G[\eta']}(v) = \sum_{v \in U} \deg_{G[\eta']}(v) + \sum_{v \in T} \deg_{G[\eta']}(v).$$

Each sum on the right hand side of (3.22) can be calculated as follows.

Let $i \in Z_{r-1}$. Note that

$$\begin{aligned} \sum_{v \in U_i} \deg_{G[\eta']}(v) &= \sum_{v \in U_i} |\eta'(v)| \\ &= \sum_{v \in U_i} (m - m_i + rn - 2n) \\ &= \sum_{v \in U_i} (m - m_i + n(r-2)) \\ &= m_i\,(m - m_i + n(r-2)) \end{aligned}$$

$$= mm_i - m_i^2 + m_i n(r-2).$$

Therefore, we have

$$\sum_{v\in U} \deg_{G[\eta']}(v) = \sum_{i\in Z_{r-1}} \Big(\sum_{v\in U_i} |\eta'(v)| \Big)$$

$$= \sum_{i\in Z_{r-1}} (mm_i - m_i^2 + m_i n(r-2))$$

$$= m(m_0+...+ m_{r-2}) - m_0^2 - m_1^2 - ... - m_{r-2}^2 + (m_0+...+m_{r-2})n(r-2)$$

$$= m^2 - m_0^2 - m_1^2 - ... - m_{r-2}^2 + mn(r-2)$$

$$= m^2 - s(k+1)^2 - (r-1-s)k^2. + mn(r-2)$$

$$= m^2 - km - s(k+1) + mn(r-2).$$

Hence

$$\sum_{v\in U} \deg_{G[\eta']}(v) = m^2 - km - s(k+1) + mn(r-2). \quad (3.23)$$

Let $i \in Z_{r-1}\backslash\{r-1\}$. Note that, by definition of η and η',

$$\sum_{v\in C_i} \deg_{G[\eta']}(v) = \sum_{v\in C_i} |\eta'(v)|$$

$$= \sum_{v\in C_i} ((n (r-2) + 1) + m - m_i)$$

$$= n ((n (r-2) + 1) + m - m_i)$$

$$= n (n (r-2) + 1) + mn - m_i n.$$

Therefore

$$\sum_{v\in T} \deg_{G[\eta']}(v) = \sum_{i\in Z_r\backslash\{r-1\}} \Big(\sum_{v\in C_i} \deg_{G[\eta']}(v) \Big) + \sum_{v\in C_{r-1}} \deg_{G[\eta']}(v)$$

$$= \sum_{i\in Z_r\backslash\{r-1\}} (n (n (r-2) + 1) + mn - m_i n) + \sum_{v\in C_{r-1}} |\eta'(v)|$$

$$= (r-1)n(n(r-2)+1) + mn(r-1) - n(m_0+\ldots+m_{r-2}) + n(n(r-2)+1)$$

$$= mn(r-2) + rn(n(r-2)+1).$$

Hence

$$\sum_{v\in T} \deg_{G[\eta']}(v) = mn(r-2) + rn(n(r-2)+1). \tag{3.24}$$

Then, by substituting values from (3.23) and (3.24) in (3.22), we have,

$$\sum_{v\in U\cup T} \deg_{G[\eta']}(v) = \sum_{v\in U} \deg_{G[\eta']}(v) + \sum_{v\in T} \deg_{G[\eta']}(v)$$

$$= m^2 - km - s(k+1) + mn(r-2) + mn(r-2) + rn(n(r-2)+1)$$

$$= m^2 - km - s(k+1) + 2mn(r-2) + rn(n(r-2)+1).$$

Thus
$$|X(G')| = \frac{\sum_{v\in U\cup T} \deg_{G[\eta']}(v)}{2}$$

$$= \frac{m^2 - km - s(k+1) + 2mn(r-2) + rn(n(r-2)+1)}{2}.$$

Therefore (iii) holds. □

Note that for any n, p, r, the graph $G[\eta']$ constructed according to our proof of Proposition 3.8 is unique up to isomorphism. We shall refer to such a graph as $G(p, r, n)$. Note that the graph constructed in Proposition 3.6 is a special case of $G(p, r, n)$. It is $G(rn, r, n)$. Therefore the graph constructed in Appendix A is $G(12, 3, 4)$. In Appendix B we give a construction of $G(17, 3, 4)$ as another example of $G(p, r, n)$ in the general case.

Lemma 3.9

Let n be a non-negative integer. Let m^*, r be any positive integers. Let G be any graph of order $m^* + rn$ with property $P(r, n)$. Let

$$k^* = [\frac{m^*}{r-1}], \quad s^* = m^* - k^*(r-1),$$

$$m' = m^* - 1, \quad k' = [\frac{m'}{r-1}] \text{ and } s' = m' - k'(r-1).$$

If T is the set of all the points of the n disjoint complete subgraphs of order r of G. Then

(i). $\min\{\deg_G(v) \mid v \in V(G)\backslash T\} \leq m' - k' + (r-2)n,$

and (ii). $k'm' + s'(k'+1) + 2k' + 1 = k^*m^* + s^*(k^*+1).$

Proof. Observe that T consists of exactly those points of G that are points of the n disjoint subgraphs of G that are isomorphic to K_r. Hence $|T| = rn$ and $<T>$ has property $P(r, n)$. Let

$$U = V(G)\backslash T.$$

Thus $|U| = m^*$. By definition of U, it can be seen that $<U>$ is a subgraph of G not containing any complete subgraphs of order r.

Pick a point $v \in U$ for which $\deg_{<U>}(v)$ is minimum. We claim that

(3.25) $\qquad \deg_{<U>}(v) \leq m' - k'.$

To show this, we consider two cases:

Case 1. $s^* = 0$. In this case we have

$$m' = m^* - 1 = k^*(r-1) - 1 = (k^*-1)(r-1) + (r-2).$$

But we also have

$$m' = k'(r-1) + s',$$

where $0 \leq s' \leq r$. Hence, by the division algorithm, we have

(3.26) $k' = k^*-1$ and $s' = r-2$.

Thus

$$m^* - k^* = (m' + 1) - (k' + 1) = m' - k'.$$

Hence, by Proposition 2.5, we have

$$\deg_{<U>}(v) \le m' - k'.$$

Case 2. $s^* > 0$. In this case we have

$$k'(r-1) + s' = m' = m^* - 1 = k^*(r-1)+s^* - 1.$$

Again, hence, by the division algorithm, we have

(3.27) $k' = k^*$ and $s' = s^*-1$.

Thus

$$m^* - k^* - 1 = (m' + 1) - k' - 1 = m' - k'.$$

Hence, by Proposition 2.5, we have

$$\deg_{<U>}(v) \le m' - k'.$$

Since G and $<T>$ have the property $P(r, n)$ and G is a supergraph of $<T>$, hence, by Proposition 3.4, there exist at most $(r-2)n$ lines from v to points of $<T>$. Thus

(3.28) $|\{ u \mid u \in T, uv \in X(G) \}| \le (r-2)n.$

Observe that

$$\begin{aligned}\deg_G(v) &= |\{ u \mid u \in U, uv \in X(G) \}| + |\{ u \mid u \in T, uv \in X(G) \}| \\ &= \deg_{<U>}(v) + |\{ u \mid u \in T, uv \in X(G) \}|.\end{aligned}$$

By applying (3.25) and (3.28) to the right hand side of the above equation, we have

$$\deg_G(v) \le m' - k' + (r-2)n,$$

i.e. (i) holds.

To prove (ii) we consider 2 cases according to the values of s^*.

If $s^* = 0$, then, by (3.26),

$$\begin{aligned} k'm' + s'(k'+1) + 2k' + 1 &= k'm' + (r-2)(k'+1) + 2k' + 1 \\ &= k'(m'+1) + (r-1)(k'+1) \\ &= k'm^* + (r-1)k^* \\ &= k'm^* + m^* \\ &= m^*(k'+1) \\ &= m^*k^* \\ &= k^*m^* - s^*(k^*+1). \end{aligned}$$

If $s^* > 0$, then, by (3.27), we have

$$\begin{aligned} k'm' + s'(k'+1) + 2k' + 1 &= k'm' + k' + (s^*-1)(k'+1) + (k'+1) \\ &= k'(m'+1) + s(k'+1) \\ &= k^*m^* + s^*(k^*+1). \end{aligned}$$

Therefore (ii) holds. □

Theorem 3.10

Let n be a non-negative integer. Let p, r be any positive integers such that $p \geq r \geq 2$ and $p \geq rn$. If G is a graph of order p with property $P(r, n)$ that has the maximum number of lines then

$$|X(G)| = \frac{m^2 - km - s(k+1) + 2mn(r-2) + rn(n(r-2)+1)}{2},$$

where $m = p - rn$, $k = [\frac{m}{r-1}]$ and $s = m - k(r-1)$.

Proof Let n, r be fixed. Our proof will be by induction on p. By Proposition 3.6 and Proposition 3.7, the statement of the theorem holds for the case for $p = rn$ and $p = 1+rn$ respectively. Let p^* be any positive integer such that $p^* > 1+rn$.

Assume that the statement of the theorem holds for all p such that $rn \le p < p^*$. Let G^* be a graph of order p^*, with property $P(r, n)$ that has the maximum number of lines. Let T be the set of all the points of the n disjoint complete subgraphs of order r of G^*. Let

$$U = V(G^*)\backslash T.$$

Pick a point $v \in U$ for which $\deg_{<U>}(v)$ is minimum. Let

$$V' = V(G^*)\backslash\{v\}.$$

Let $p' = |V'|$. Thus $p' = p^* - 1$. Let $m' = p' - rn$, $k' = [\frac{m'}{r-1}]$ and $s' = m' - k'(r-1)$.

By Proposition 3.8, there exists a graph G' with point set V' such that

(i). G' has property $P(r, n)$;

(ii). $\max\{\deg_{G'}(w) \mid w \in V'\backslash T'\} = m' - k' + (r-2)n$ where T' is the set of all the points of the n disjoint complete subgraphs of order r of G';

(iii). $|X(G')| = \dfrac{m'^2 - k'm' - s'(k'+1) + 2m'n(r-2) + rn(n(r-2)+1)}{2}$.

By the induction hypothesis, any graphs with p' points that has the property $P(r, n)$ and has the maximum number of lines has

$$\frac{m'^2 - k'm' - s'(k'+1) + 2m'n(r-2) + rn(n(r-2)+1)}{2}$$

lines. Hence G' is a graph of order p' which has the maximum number of lines.

Note that the set $V'\backslash T' \neq \varnothing$, hence

$$\{\deg_{G'}(w) \mid w \in V'\backslash T'\} \neq \varnothing.$$

So it has a maximum element. Let $u \in V'\backslash T'$ be such that

$$\deg_{G'}(u) = \max\{\deg_{G'}(w) \mid w \in V'\backslash T'\}.$$

Thus, by (ii), we have

(3.29) $$\deg_{G'}(u) = m' - k' + (r-2)n.$$

Let $G^{\#}$ be the graph defined as follows.

$$V(G^{\#}) = V(G^{*}).$$

$$X(G^{\#}) = X(G') \cup \{ vw \mid w \in G', uw \in X(G') \}.$$

Note that v is joined to those and only those points of G' that are joined to u. Hence

(3.30) $$\deg_{G^{\#}}(v) = \deg_{G'}(u), \\ = m' - k' + (r-2)n.$$

The last equality follows from (3.29).

Note that the n complete subgraphs of order r of G' are also complete subgraphs of order r of $G^{\#}$ which are disjoint. We claim that $G^{\#}$ has no other complete subgraphs of order r. Suppose the contrary. Let H be a complete subgraph of order r of $G^{\#}$ which is distinct from the n disjoint complete subgraphs of order r of G'. By construction of $G^{\#}$ we see that v must be a point of H. Let $v_2,\ldots,v_r$ be the other points of H. Since a point is joined to v if and only if it is joined to u. Hence $v_2,\ldots,v_r$ are joined to u. So u together $v_2,\ldots,v_r$ form complete subgraph of order r of G'. This is contrary to the fact that G' has exactly n disjoint complete subgraphs of order r. Hence our supposition must be wrong. Hence $G^{\#}$ has the property $P(r, n)$.

Observe that

$$|X(G^{\#})| = |X(G^{\#}-v)| + |\{ vw \mid w \in G', uw \in X(G') \}|$$

$$= |X(G')| + \deg_{G^{\#}}(v)$$

$$= \frac{m'^2 - k'm' - (k'+1) + 2m'n(r-2) + rn(n(r-2)+1)}{2} + m' - k' + (r-2)n$$

$$= \frac{m'^2 - k'm' - s'(k'+1) + 2m'n(r-2) + rn(n(r-2)+1) + 2m' - 2k' + 2(r-2)n}{2}$$

$$= \frac{(m'+1)^2 - k'm' - s'(k'+1) - 2k' - 1 + 2(m'+1)n(r-2) + rn(n(r-2)+1)}{2}.$$

Let $m^* = p^* - rn$, $k^* = [\frac{m^*}{r-1}]$ and $s^* = m^* - k^*(r-1)$.

By Lemma 3.9(ii), we have

$$k'm' + s'(k'+1) + 2k' + 1 \quad = k^*m^* + s^*(k^*+1).$$

Therefore

$$|X(G^{\#})| = \frac{m^{*2} - m^*k^* - s^*(k^*+1) + 2m^*n(r-2) + rn(n(r-2)+1)}{2}.$$

Observe that

(3.31) $\qquad |X(G^*)| = |X(G^*-v)| + \deg_{G^*}(v).$

Since G^*-v and G' are graphs of order p' that have the property $P(r, n)$ and G' is one with the maximum number of lines. Hence

(3.32) $\qquad |X(G^*-v)| \leq |X(G')|.$

By Lemma 3.9(i) and (3.30), we have

(3.33) $\qquad \deg_{G^*}(v) \leq m' - k' + (r-2)n = \deg_{G^{\#}}(v).$

Thus, by (3.31), (3.32) and (3.33), we have

$$|X(G^*)| \leq |X(G')| + \deg_{G^{\#}}(v).$$

By definition of $X(G^{\#})$, we have

$$|X(G^{\#})| = |X(G')| + \deg_{G^{\#}}(v).$$

Thus

$$|X(G^*)| \leq |X(G^{\#})|.$$

Note that both G^* and $G^{\#}$ are graphs of order p^* that have property $P(r, n)$ and G^* is such a graph with the maximum number of lines. Thus

$$|X(G^*)| \geq |X(G^{\#})|.$$

Therefore

$$
\begin{aligned}
|X(G^*)| &= |X(G^\#)| \\
&= \frac{m^{*2} - m^*k^* - s^*(k^*+1) + 2m^*n(r-2) + rn(n(r-2)+1)}{2}. \qquad \square
\end{aligned}
$$