CHAPTER II

STRUCTURES OF CERTAIN EXTREMAL GRAPHS

In this chapter, we construct of extremal graphs that does not contain any subgraph isomorphic to K_r. Our knowledge on these structures will be needed in the next chapter.

Proposition 2.1

Let G be a complete r-partite graph of order p that has the maximum number of lines. Then the parts of G are as equal as possible, i.e. if $(p_1, p_2,\ldots, p_r)$ are the part sizes of G, then $|p_i - p_j| \leq 1$ for all i, j.

Proof Let $V_1, V_2,\ldots,V_r$ be the r parts of G and assume that $|V_i| = p_i$, $i = 1, 2,\ldots,r$.

Suppose the parts are not as equal as possible. Without loss of generality, we may assume that $p_2 \geq p_1 + 2$. Let $u \in V_2$. Let

$$V_1' = V_1 \cup \{u\},$$

$$V_2' = V_2 \setminus \{u\},$$

and

$$V_i' = V_i.$$

for all $i > 2$. Let G' be the complete r-partite graph with $V_1', V_2',\ldots,V_r'$ as its parts. Observe that

$$|X(G')| - |X(G)| = \frac{\sum_{v\in V(G')} deg_{G'}(v) - \sum_{v\in V(G)} deg_G(v)}{2}$$

$$= \frac{\sum_{v\in V_1'} deg_{G'}(v) + \ldots + \sum_{v\in V_r'} deg_{G'}(v) - \sum_{v\in V_1} deg_G(v) - \ldots - \sum_{v\in V_r} deg_G(v)}{2}$$

$$= \frac{\sum_{v\in V_1'} deg_{G'}(v) + \sum_{v\in V_2'} deg_{G'}(v) - \sum_{v\in V_1} deg_G(v) - \sum_{v\in V_2} deg_G(v)}{2}$$

$$= \frac{p_1'(p - p_1') + p_2'(p - p_2') - p_1(p - p_1) - p_2(p - p_2)}{2}$$

$$= \frac{(p_1+1)(p - p_1 - 1) + (p_2 - 1)(p - p_2 + 1) - p_1(p - p_1) - p_2(p - p_2)}{2}$$

$$= \frac{2p_2 - 2p_1 - 2}{2}$$

$$= p_2 - p_1 - 1$$

$$\geq 1.$$

Hence G' is a complete r-partite graph of order p such that $|X(G')| > |X(G)|$. Thus G is a complete r-partite graph of order p that does not have the maximum number of lines.

Therefore, for a complete r-partite graph of order p to have the maximum number of lines, its parts must be as equal as possible. □

Proposition 2.2

Let p, r be positive integers such that $p \geq r \geq 2$. Let $k = [\frac{p}{r}]$ and $s = p-kr$. Let G be a complete r-partite graph of order p that has the maximum number of lines. Then (i). G has s parts with k+1 points and r–s parts with k points;

(ii). $|X(G)| = \dfrac{p(p-k)-s(k+1)}{2}$.

Proof Let $V_1,\ldots,V_r$ be the r parts of G. Assume that V_i has p_i points where $i = 1, 2,\ldots,r$. Then each point u in any part V_i the lines containing u are of the form uv, where $v \in V \backslash V_i$. Hence we have

$$\begin{aligned} \deg_G(u) &= |V \backslash V_i| \\ &= |V| - |V_i| \\ &= p - p_i. \end{aligned}$$

Now we show that any p_i must be k or k+1. Suppose that $p_i > k+1$ for some i. So $p_i \geq k+2$. Then there must exist some j such that $p_j \leq k$. Otherwise we would have

$$\begin{aligned} p &= p_1 + p_2 + \ldots + p_r \\ &> (k+1)r \\ &= kr + r \\ &> kr + s \\ &= p, \end{aligned}$$

which is a contradiction. Thus

$$p_i - p_j \geq k+2-k = 2,$$

which is contrary to Proposition 2.1, since G has the maximum number of lines. So we have

$$p_i \leq k+1$$

for all $i = 1, 2,\ldots,r$.

Next we show that $p_i \geq k$ for $i = 1, 2, \ldots, r$. Suppose that this is not the case. So $p_i \leq k-1$ for some i. Then there must exist some j such that $p_j \geq k+1$. Otherwise we would have

$$\begin{aligned} p &= p_1 + p_2 + \ldots + p_r \\ &< kr \\ &\leq kr + s \\ &= p, \end{aligned}$$

which is a contradiction. Thus

$$p_j - p_i \geq (k+1)-(k-1) = 2,$$

which is contrary to Proposition 2.1, since G has the maximum number of lines. So we have

$$p_i \geq k$$

for all $i = 1, 2, \ldots, r$. Hence $p_i = k$ or $p_i = k+1$, for all $i = 1, 2, \ldots, r$. Let

$$\text{B} = \{\, i \mid p_i = k+1 \text{ where } i = 1, 2, \ldots, r\}.$$

So

$$\begin{aligned} p &= p_1 + p_2 \ldots + p_r \\ &= (k+1)|\text{B}| + k(r-|\text{B}|) \\ &= kr + |\text{B}| \end{aligned}$$

Thus $|\text{B}| = s$, i.e. the number of parts that contain $k+1$ points is s. According to what have been shown above, all the remaining parts contain k points. So the number of parts that contain k points is $r-s$. Hence (i) holds.

Without loss of generality, we may assume that $V_1, V_2, \ldots, V_s$ have $k+1$ points and $V_{s+1}, \ldots, V_r$ have k points. It follows that

$$\deg_G(u) = p-(k+1) \quad \text{for all } u \in V_1 \cup \ldots \cup V_s;$$

$$\text{and} \quad \deg_G(u) = p-k \quad \text{for all } u \in V_{s+1} \cup \ldots \cup V_r.$$

Thus, by (1.1), we have

$$
\begin{aligned}
2|X(G)| &= \sum_{v \in V(G)} \deg_G(v) \\
&= |V_1 \cup \ldots \cup V_s|(p-(k+1)) + |V_{s+1} \cup \ldots \cup V_r|(p-k) \\
&= s(k+1)(p-k-1) + (r-s)k(p-k) \\
&= (kr+s)(p-k) - s(k+1) \\
&= p(p-k) - s(k+1).
\end{aligned}
$$

Hence, we have

$$|X(G)| = \frac{p(p-k) - s(k+1)}{2}.$$

Therefore (ii) holds. □

By Proposition 2.2, it follows that complete r-partite graphs of order p with the maximum number of lines is unique up to isomorphism. It will be denoted by $T_r(p)$.

By an extremal graph with p points for a given forbidden graph H, we mean any graph of order p with the maximum number of lines that contains no subgraph isomorphic to H. The number of lines of such a graph will be denoted by $ex(p, H)$. The following theorem, due to Bóllobás, tells us when the value $ex(p, K_r)$ is attained.

Theorem 2.3 (Bóllobás[1])

Let p, r be any two positive integers such that $p \geq r \geq 2$. $\text{ex}(p, K_r) = |X(T_{r-1}(p))|$ and $T_{r-1}(p)$ is the unique graph of order p that does not contain a complete graph of order r.

A proof of this theorem can be found in [1]. □

Remark 2.4

According to Proposition 2.2 $|X(T_{r-1}(p))| = \dfrac{p(p-k)-s(k+1)}{2}$ where $k = [\dfrac{p}{r-1}]$ and $s = p-k(r-1)$. Therefore, it follows from Theorem 2.3 that

$$\mathrm{ex}(p, K_r) = \frac{p(p-k)-s(k+1)}{2}.$$

This can be rewritten as

$$\mathrm{ex}(p, K_r) = \frac{(r-2)(p^2-s^2)}{2(r-1)} + \frac{s(s-2)}{2}.$$

This is the formula given as Turán's result in [4].

Proposition 2.5

Let p, r be any two positive integers such that $p \geq r \geq 2$. Let G be any graph of order p not containing any subgraph isomorphic to K_r. Let $k = [\frac{p}{r}]$ and $s = p - kr$. Then the minimum degree of G, i.e. $\delta(G)$, satisfies following inequality.

$$\delta(G) \leq \begin{cases} p-k & \text{if } s=0; \\ p-k-1 & \text{if } s>0. \end{cases}$$

Proof Let u be a point of G with the minimum degree.

First we consider the case $s = 0$. Suppose $\deg_G(u) > p-k$. Then

$$\sum_{v \in V(G)} \deg_G(v) > p(p-k).$$

Thus $|X(G)| > \frac{p(p-k)}{2}$, which is contrary to Remark 2.4. Thus

$$\deg_G(u) \le p-k.$$

Next we consider the case $s > 0$. Suppose $\deg_G(u) > p-(k+1)$. Then

$$\deg_G(u) \ge p-(k+1)+1 = p-k.$$

So

$$\sum_{v \in V(G)} \deg_G(v) \ge p(p-k) > p(p-k)-s(k+1).$$

Thus $|X(G)| > \frac{p(p-k)-s(k+1)}{2}$, which is contrary to Remark 2.4. Thus

$$\deg_G(u) \le p-(k-1).$$

Hence

$$\delta(G) \le \begin{cases} p-k & \text{if } s=0; \\ p-k-1 & \text{if } s>0. \end{cases}$$

□

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