CHAPTER I

PRELIMINARIES

This chapter deals with some concepts of graph theory needed in our study.

1.1 Graphs

A graph G is an ordered pair of disjoint sets (V, X), where V is a finite non-empty set and X is a set of 2-subsets of V. Elements of V and X will be referred to as points and lines of G, respectively. For any graph G we usually denote its set of points by $V(G)$ and its set of lines by $X(G)$. For brevity we shall denote a line $\{u, v\}$ by uv and will be referred to as the line that joins u and v. By the order of G we mean the number of points of G, i.e. $|V(G)|$.

The degree of any point v of G, denoted by $\deg_G(v)$ is the number of lines containing v. Observe that each line is joined by two points, thus it contributes 2 to the sum of the degrees of the points, giving

$$\sum_{v \in V(G)} \deg_G(v) = 2|X(G)|. \qquad (1.1)$$

For each graph G, its minimum degree will be denoted by $\delta(G)$.

For any two graphs G_1 and G_2, we say that G_1 and G_2 are disjoint graphs when $V(G_1) \cap V(G_1) = \varnothing$.

It is customary to represent a graph by means of a diagram. We represent points of a graph by geometrical points in a one-to-one fashion and any line uv is represented by a geometrical line joining the geometrical points that represent u and v. As an example, let $G = (V, X)$, where

$$V = \{a, b, c, d, e, f\},$$

$$X = \{ab, af, bc, bf, cd, ce, cf, df, ef\}.$$

G can be represented by the diagram in Figure 1.1.

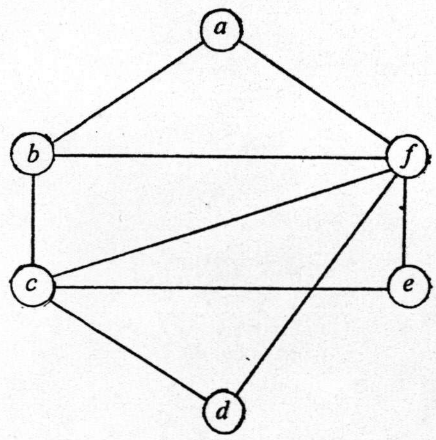


Figure 1.1

1.2 Isomorphism's

Two graphs G_1 and G_2 are said to be isomorphic, written $G_1 \cong G_2$ when there exists a one-to-one function θ from $V(G_1)$ onto $V(G_2)$ such that for any $u, v \in V(G_1)$, $uv \in X(G_1)$ if and only if $\theta(u)\theta(v) \in X(G_2)$. Such θ is said to be an isomorphism.

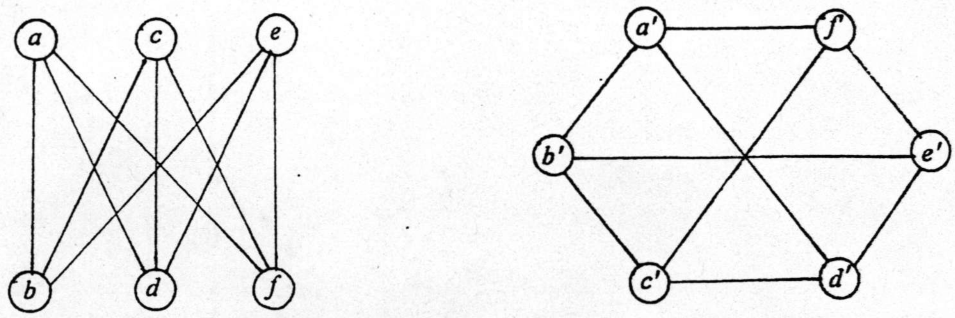


Figure 1.2

To illustrate the concept of isomorphism, consider the two graphs G_1 and G_2 shown in Figure 1.2. They look quite different, but they are isomorphic. An isomorphism θ from G_1 to G_2 is as follows:

$$\theta(a) = a', \ \ \theta(b) = b', \ \ \theta(c) = c',$$

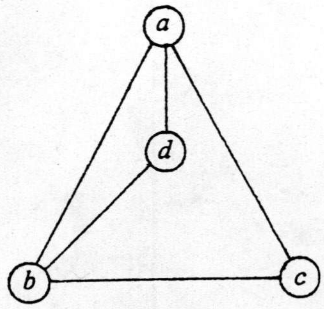
$$\theta(d) = d', \ \ \theta(e) = e' \text{ and } \theta(f) = f',$$

It is straightforward to verify that θ is an isomorphism.

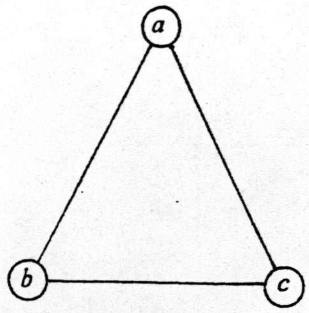
1.3 Subgraphs

By a subgraph of G we mean any graph $G' = (V', X')$ such that $V' \subseteq V$ and $X' \subseteq X$. When G' is a subgraph of G, we also say that G is a supergraph of G'.

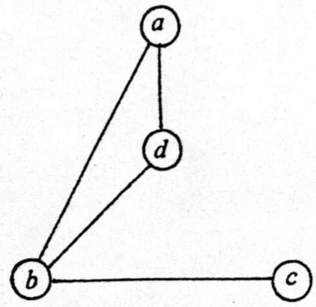
A subgraph G' of G is said to be a spanning subgraph of G if $V(G') = V(G)$. Let $S \subseteq V(G)$. Among the subgraphs of G that has S as the set of points there is one and only one, with the maximum number of lines. It includes all the lines of G that join points of S. This is called the subgraph of G induced by S and will be denoted by $<S>$. As an example let G, G_1 and G_2 be the graphs given in Figure 1.3 (i), (ii) and (iii) respectively. G_2 is a spanning subgraph of G but G_1 is not; G_1 is the subgraph of G induced by $\{ a, b, c \}$.



(i) G



(ii) G_1



(iii) G_2

Figure 1.3

1.4 Operations on graphs

Let G be a graph that contains at least two points. Let v be any point of G. We shall denote the graph obtained from G by deleting the point v and all lines containing v by $G-v$, i.e.

$$V(G-v) = V(G) - \{v\},$$

$$X(G-v) = X(G) \setminus \{ uv \mid u \in V(G), uv \in X(G) \}.$$

In general, if $T = \{ v_1, v_2,...,v_k\}$ is a subset of $V(G)$ we use the notation $G\setminus T$ to denote the graph obtained from G by deleting all the points $v_1, v_2,...,v_k$ and all the lines that contain them.

Let x be any line of a graph G. We shall denote the graph obtained from G by deleting the line x by $G-x$, i.e.

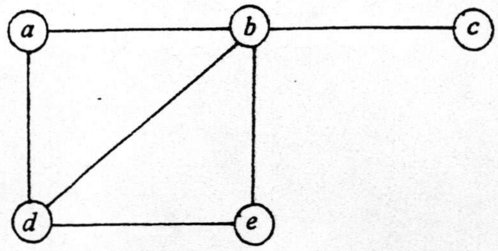
$$V(G-x) = V(G),$$

$$X(G-x) = X(G) \setminus \{x\}.$$

Let u, v be any two points of G such that $uv \notin X(G)$. We shall denote the graph obtained from G by adding the line joining u and v by $G+uv$, i.e.

$$V(G+uv) = V(G),$$

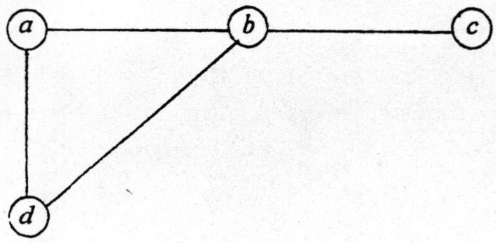
$$X(G+uv) = X(G) \cup \{uv\}.$$



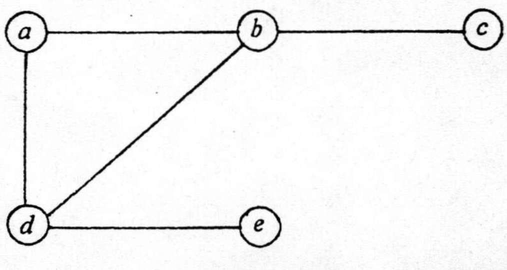
G

Figure 1.4

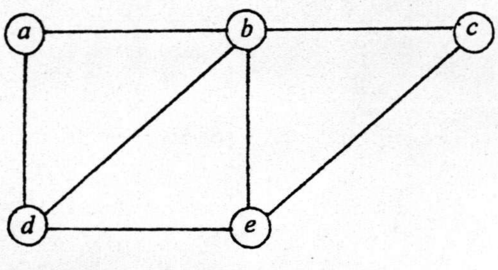
As an example, let G be the graph given in Figure 1.4. Then $G-e$, $G-be$ and $G+ce$ are given in Figure 1.5, (i), (ii), (iii) respectively.



(i) $G-e$



(ii) $G-be$



(iii) $G+ce$

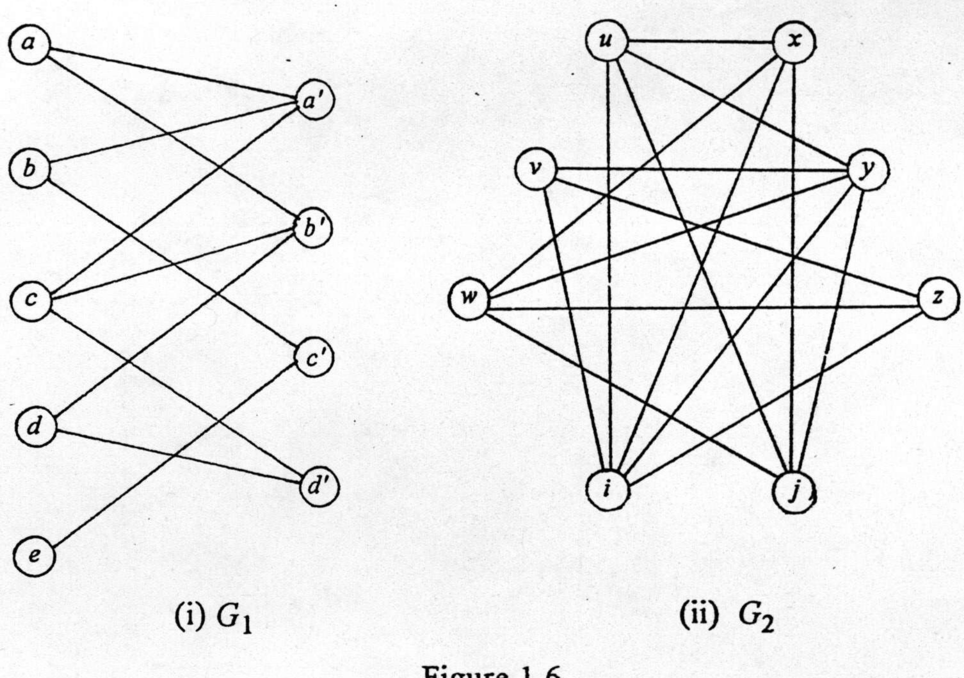
Figure 1.5

1.5 Special Classes of graphs

A graph G is said to be a complete graph if every pair of its points are joined by a line. Note that all complete graphs of the same order are isomorphic. So, they can be considered abstractly as the same graph. We shall denote any complete graph of order p by K_p. Any graph which is K_1 is said to be a trivial graph.

A graph G is said to be an empty graph if its line set $X(G)$ is empty. Note that all empty graphs of the same order are isomorphic. We shall denote any empty graph of order p by E_p

A graph G is said to be r-partite if its point set $V(G)$ can be partitioned into r disjoint non-empty subsets $V_1, V_2,...,V_r$ such that no line joins two points of the same subsets. The subsets $V_1, V_2,...,V_r$ are called the parts of G. 2-partite and 3-partite graphs are also referred to as a bipartite and a tripartite, respectively. The graph G_1 and G_2 of Figure 1.6 (i) and (ii) are examples of bipartite and tripartite graphs respectively.



(i) G_1 (ii) G_2

Figure 1.6

An r-partite graph with parts V_1, $V_2,\ldots,V_r$ is said to be a complete r-partite graph if every pair of points from distinct parts are joined by a line. A complete r-partite graph with parts V_1, $V_2,\ldots,V_r$ is said to have part sizes (p_1, $p_2,\ldots,p_r$) if $|V_i| = p_i$; $i = 1, 2,\ldots, r$. Two complete r-partite graphs G and G' with part sizes (p_1, $p_2,\ldots,p_r$) and (p'_1, $p'_2,\ldots,p'_r$) are said to have the same part sizes if there exist a permutation i_1, $i_2,\ldots$, i_r of 1, 2,..., r such that $p'_{i_1}=p_1$, $p'_{i_2}=p_2,\ldots,p'_{i_r}=p_r$. Note that all complete r-partite graphs that have the same part sizes are isomorphic. We shall denote a complete r-partite graph with parts sizes (p_1, $p_2,\ldots,p_r$) by $K_{p_1, p_2,\ldots,p_r}$.