CHAPTER II

PRELIMINARIES

In this chapter, we presents some basic concepts and facts of theory of probability that are needed in this work. The proofs of the statements are omitted; they can be found in [3] and [4].

Let Ω be a nonempty set of elements and $\mathfrak{U}$ be a set of subsets of Ω having the properties that

1) $\Omega \in \mathfrak{U}$,
2) if $A \in \mathfrak{U}$, then $\Omega \setminus A \in \mathfrak{U}$,
3) if $A_1, A_2, \ldots$ is any sequence of subsets of Ω belonging to $\mathfrak{U}$, then $\bigcup_n A_n \in \mathfrak{U}$.

Then $\mathfrak{U}$ is called a <u>σ-algebra</u>. A nonnegative countably additive function P defined on $\mathfrak{U}$ with $P(\Omega) = 1$, is called a <u>probability measure</u>. A triplet $(\Omega, \mathfrak{U}, P)$ is called a <u>probability space</u>. The elements of Ω are called <u>points</u> or <u>elementary events</u> and the set Ω is called a sample space. The elements of $\mathfrak{U}$ are called <u>events</u> and the value of P(A) is called the <u>probability of the event A</u>.

Let X be a real-valued function defined on Ω. If the set $X^{-1}(B) = \{w \mid X(w) \in B\}$ belongs to $\mathfrak{U}$ for any Borel subset B of $\mathbb{R}$, then the function X is called a <u>random variable</u>.

Let X be a random variable defined on a probability space and B be a Borel subset of $\mathbb{R}$. We shall usually use the notation $P(X \varepsilon B)$ instead of $P(\{w|X(w) \varepsilon B\})$, in case of $B = (-\infty,x]$, $[x_1,x_2]$ and $\{x\}$, we shall denote $P(X \varepsilon B)$ by $P(X \leq x)$, $P(x_1 \leq X \leq x_2)$ and $P_X(x)$ respeclively.

Let X be a random variable. The function F_X defined by

$$F_X(x) = P(X \leq x),$$

is called the <u>distribution function</u> of X.

We shall say that a random variable X has a <u>normal distribution</u> if its distribution function is given by

$$F_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-\frac{(t-a)^2}{2\sigma^2}} dt \qquad (-\infty < x < \infty)$$

where a and σ are real numbers such that $\sigma > 0$.

Let X be a random variable and $g: \mathbb{R} \to \mathbb{C}$ be a Borel function. We defined the <u>expectation</u> of $g \circ X$ by

$$\int_{-\infty}^{\infty} g(X)dF_X(x)$$

provided that the Lebesgue-integral $\int_{-\infty}^{\infty} g(X)dF_X(x)$ exists. We denote the expectation of $g \circ X$ by $E(g(X))$. For any nonnegative integer k, we define the k^{th} moment and the k^{th} absolutely moment of X by $E(X^k)$ and $E(|X|^k)$, respectively. It follows from integration theory that

i) $E(X^k)$ is finite if and only if $E(|X|^k)$ is finite,

ii) if $E(X^k)$ is finite, then $E(X^m)$ is finite for $0 < m \le k$.

By convention, $E(X^o) = E(|X^o|) = 1$.

The <u>variance</u> $\sigma^2(X)$ of a random variable X is defined as the expectation of $(X-E(X))^2$, provided that the expectation is finite.

For a random variable X, we defined the <u>characteristic function of X</u> by

$$\varphi_X(t) = E(e^{itX}), \qquad (-\infty < t < \infty).$$

We shall denote the argument of $\varphi_X(t)$ by $\theta_X(t)$.

For a random variable X, we write

$$\alpha(t) = \theta_X(t) - E(X)t$$

and

$$R(T) = \frac{1}{2\pi} \int_0^\pi \frac{|\varphi_X(t)|\sin(T\sqrt{\sigma^2(X)}t-\alpha(t))dt}{\sin \frac{t}{2}}.$$

For an integral-valued random variable X, we have

$$E(X^k) = \sum_{j=-\infty}^{\infty} j^k p_X(j)$$

and

$$\varphi_X(t) = \sum_{j=-\infty}^{\infty} e^{ijt} p_X(j).$$

In the following theorem, we shall show that for any integral-valued random variable X, if $\sigma^2(X) > 0$, then

$$P(k_1 \le X \le k_2) = R(T_2) - R(T_1)$$

where

$$T_1 = \frac{k_1 - E(X) - \frac{1}{2}}{\sqrt{\sigma^2(X)}} \quad \text{and} \quad T_2 = \frac{k_2 - E(X) + \frac{1}{2}}{\sqrt{\sigma^2(X)}} .$$

<u>Theorem 2.1</u>. Let X be any integral-valued random variable with finite variance. Assume that $\sigma^2(X) > 0$. Then

$$P(k_1 \le X \le k_2) = R(T_2) - R(T_1)$$

where

$$T_1 = \frac{k_1 - E(X) - \frac{1}{2}}{\sqrt{\sigma^2(X)}} \quad \text{and} \quad T_2 = \frac{k_2 - E(X) + \frac{1}{2}}{\sqrt{\sigma^2(X)}} .$$

<u>Proof</u>. Observe that for each integer k,

$$\int_{-\pi}^{\pi} e^{-ikt}\varphi_X(t)dt = \int_{-\pi}^{\pi} e^{-ikt}\left(\sum_{j=-\infty}^{\infty} p_X(j)e^{ijt}\right)dt ,$$

$$= \sum_{j=-\infty}^{\infty} p_X(j) \int_{-\pi}^{\pi} e^{i(j-k)t}\, dt ,$$

$$= 2\pi P_X(k)$$

where the second equality follows from the fact that $\sum_{j=-\infty}^{\infty} p_X(j)e^{ijt}$ converges uniformly on $[-\pi,\pi]$.

Therefore

$$P(k_1 \le X \le k_2) = \sum_{k=k_1}^{k_2} P_X(k) \ ,$$

$$= \frac{1}{2\pi} \sum_{k=k_1}^{k_2} \int_{-\pi}^{\pi} e^{-ikt} \varphi_X(t)dt \ ,$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=k_1}^{k_2} e^{-ikt} \varphi_X(t)dt \ ,$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin(\frac{(k_2 - k_1+1)}{2} t)}{\sin \frac{t}{2}} e^{-i(\frac{k_1 + k_2}{2})t} \varphi_X(t)dt \ ,$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin(\frac{(k_2 - k_1+1)}{2} t)}{\sin \frac{t}{2}} e^{\frac{-i(k_1 + k_2)t}{2}} |\varphi_X(t)| e^{i\theta_X(t)} dt,$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\varphi_X(t)| \frac{\sin(\frac{(k_2-k_1+1)}{2} t)}{\sin \frac{t}{2}} e^{i(\theta_X(t) - \frac{(k_1+k_2)t}{2})} dt,$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\varphi_X(t)| \frac{\sin(\frac{(k_2-k_1+1)}{2} t)}{\sin \frac{t}{2}} \cos(\theta_X(t) - \frac{(k_1+k_2)t}{2})dt$$

$$+ \frac{i}{2\pi} \int_{-\pi}^{\pi} |\varphi_X(t)| \frac{\sin(\frac{(k_2-k_1+1)}{2} t)}{\sin \frac{t}{2}} \sin(\theta_X(t) - \frac{(k_1+k_2)t}{2})dt$$

Since $P(k_1 \le X \le k_2)$ is real, hence

$$(2.1.1)\ldots \quad P(k_1 \le X \le k_2) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\varphi_X(t)| \frac{\sin(\frac{(k_2-k_1+1)}{2} t)}{\sin \frac{t}{2}} \cos(\theta_X(t) - \frac{(k_1+k_2)}{2} t)dt.$$

To obtain $|\varphi_X(t)|$ and $\theta_X(t)$, observe that

$$\varphi_X(t) = \sum_{k=-\infty}^{\infty} P_X(k)e^{ikt} ,$$

$$= \sum_{k=-\infty}^{\infty} P_X(k)\cos kt + i \sum_{k=-\infty}^{\infty} p_X(k)\sin kt.$$

Therefore

$$|\varphi_X(t)| = [(\sum_{k=-\infty}^{\infty} p(k)\cos kt)^2+(\sum_{k=-\infty}^{\infty} p_X(k)\sin kt)^2]^{\frac{1}{2}}$$

and

$$\theta_X(t) = \arctan \frac{\sum_{k=-\infty}^{\infty} p_X(k)\sin kt}{\sum_{k=-\infty}^{\infty} p_X(k)\cos kt} .$$

Note that $|\varphi_X(t)|$ is even and $\theta_X(t)$ is odd. It follows that

$$|\varphi_X(t)|\frac{\sin(\frac{(k_2-k_1+1)}{2}t)}{\sin\frac{t}{2}} \cos(\theta_X(t) - (\frac{k_1+k_2}{2})t) \quad \text{is even.}$$

Therefore, from (2.1.1), we have

$$P(k_1 \le X \le k_2) = \frac{1}{\pi}\int_0^{\pi} |\varphi_X(t)|\frac{\sin(\frac{(k_2-k_1+1)}{2}t)}{\sin\frac{t}{2}} \cos(\theta_X(t)-(\frac{k_1+k_2}{2})t)dt$$

$$= \frac{1}{2\pi}\int_0^{\pi} 2|\varphi_X(t)|\frac{\sin(\frac{(T_2-T_1)}{2}\sqrt{\sigma^2(X)}t)}{\sin\frac{t}{2}} \cos(\frac{(T_2+T_1)\sqrt{\sigma^2(X)}t-2\alpha(t)}{2})dt$$

$$= \frac{1}{2\pi}\int_0^{\pi} |\varphi_X(t)|\frac{[\sin(T_2\sqrt{\sigma^2(X)}t-\alpha(t))-\sin(T_1\sqrt{\sigma^2(X)}t-\alpha(t))]}{\sin\frac{t}{2}} dt,$$

$$= R(T_2) - R(T_1).$$

#

Let $X_1, X_2, \ldots, X_n$ be random variables with finite expectations.

Then

$$E(X_1 + X_2 + \ldots + X_n) = \sum_{j=1}^{n} E(X_j) .$$

Any random variables $X_1, X_2, \ldots, X_n$ are called <u>independent</u>

if

$$P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) = P(X_1 \leq x_1) P(X_2 \leq x_2) \ldots P(X_n \leq x_n)$$

holds for all real values of $x_1, x_2, \ldots, x_n$.

If the random variables $X_1, X_2, \ldots, X_n$ are independent, then

$$\varphi_{X_1 + X_2 + \ldots + X_n}(t) = \varphi_{X_1}(t) \varphi_{X_2}(t) \ldots \varphi_{X_n}(t)$$

and

$$\sigma^2(X_1 + X_2 + \ldots + X_n) = \sum_{j=1}^{n} \sigma^2(X_j).$$

Let $X_1, X_2, \ldots, X_n$ be independent integral-valued random variables. Throughout this thesis, we use the following notations:

$$S_n = X_1 + X_2 + \ldots + X_n,$$

$$B_n = \sum_{j=1}^{n} \sigma^2(X_j).$$