## CHAPTER II

## EVALUATION OF SOME WIENER INTEGRALS

In this chapter, we evaluate $\int_C \exp(\lambda \int_0^1 P(t)x^2(t)dt)dW(x)$ for several functions P which are positive (also nonnegative) on $[0,1]$.

<u>Theorem 2.1</u> Let P be a continuous and positive function on $[0,1]$ and let $\lambda_0$ be the least characteristic value of the differential equation

(1) $$f''(t) + \lambda P(t)\, f(t) = 0 ,$$

subject to the boundary conditions

(2) $$f(0) = f'(1) = 0.$$

Then, if $-\infty < \lambda < \lambda_0$ and $f_\lambda$ is any nontrivial solution of (1) satisfying

(3) $$f_\lambda'(1) = 0 ,$$

we have

(4) $$\int_C \exp\,(\lambda \int_0^1 P(t)x^2(t)dt)dW(x) = \left(\frac{f_\lambda(1)}{f_\lambda(0)}\right)^{\frac{1}{2}} .$$

Consequently, if $\lambda < \lambda_0$ and $f_1$, $f_2$ are any two linearly independent solutions of (1), we have

(5) $$\int_C \exp(\lambda \int_0^1 P(t)x^2(t)dt)dW(x) = \left(\frac{f_2'(a)f_1(a)-f_1'(a)f_2(a)}{f_2'(1)f_1(0)-f_1'(1)f_2(0)}\right)^{\frac{1}{2}} ,$$

where a is any convenient point in $0 \leq a \leq 1$.

Before proving this theorem, we shall prove Theorem 2.2 which is more general than Theorem 2.1.

Theorem 2.2 Let P be continuous and positive on $0 \leq t \leq 1$, and let $\lambda_0$ be the least characteristic value of (1) subject to the boundary conditions (2). Then if F is any Wiener measurable functional on C, $\lambda < \lambda_0$, and $f_\lambda(t)$ is any nontrivial solution of (1) satisfying (3), we have

$$\int_C F(x) \exp\left(\lambda \int_0^1 P(t)x^2(t)dt\right) dW(x)$$

$$(6) \qquad = \left(\frac{f_\lambda(1)}{f_\lambda(0)}\right)^{\frac{1}{2}} \int_C F\left[y(\cdot)+f_\lambda(\cdot) \int_0^{(\cdot)} \frac{f_\lambda'(s)}{\left[f_\lambda(s)\right]^2} y(s)ds\right] dW(y),$$

where the existence of one Wiener integral implies the existence of the other.

Proof : Consider the linear transformation

$$(7) \qquad y(t) = x(t) - \int_0^t \frac{f_\lambda'(s)}{f_\lambda(s)} x(s)\, ds ,$$

where $\lambda$ is a fixed value, $-\infty < \lambda < \lambda_0$, and $f_\lambda$ is any non-trivial solution of (1) satisfying (3), $f_\lambda(s) \neq 0$, $0 \leq s \leq 1$ (Theorem 1.16). This transformation takes the space C into a part of C. We shall show that it takes C into the whole of C in a 1-1 manner. First, multiplying (7) by $\dfrac{f_\lambda'(t)}{\left[f_\lambda(t)\right]^2}$, and get

$$\frac{f'_\lambda(t)}{[f_\lambda(t)]^2}\, y(t) = \frac{f'_\lambda(t)}{[f_\lambda(t)]^2}\, x(t) - \frac{f'_\lambda(t)}{[f_\lambda(t)]^2} \int_0^t \frac{f'_\lambda(s)}{f_\lambda(s)}\, x(s)ds$$

$$= \frac{d}{dt}\left[\frac{1}{f_\lambda(t)} \int_0^t \frac{f'_\lambda(s)}{f_\lambda(s)}\, x(s)ds\right],$$



so

$$(8) \qquad \int_0^t \frac{f'_\lambda(s)}{[f_\lambda(s)]^2}\, y(s)ds = \frac{1}{f_\lambda(t)} \int_0^t \frac{f'_\lambda(s)}{f_\lambda(s)}\, x(s)\, ds .$$

Multiplying (8) by $f_\lambda(t)$ and adding to (7), we obtain

$$(9) \qquad y(t) + f_\lambda(t) \int_0^t \frac{f'_\lambda(s)}{[f_\lambda(s)]^2}\, y(s)ds = x(t).$$

Thus to every function x of C there corresponds a function y of C defined by (7), and this y satisfies the relation (9). This implies that the transformation is 1-1. To show onto, let y be any function of C and define x by (9). Multiplying (9) by $\frac{f'_\lambda(t)}{f_\lambda(t)}$, we get

$$\frac{f'_\lambda(t)}{f_\lambda(t)}\, y(t) + f'_\lambda(t) \int_0^t \frac{f'_\lambda(s)}{[f_\lambda(s)]^2}\, y(s)\, ds = \frac{f'_\lambda(t)}{f_\lambda(t)}\, x(t),$$

$$\frac{d}{dt}\left[f_\lambda(t) \int_0^t \frac{f'_\lambda(s)}{[f_\lambda(s)]^2}\, y(s)\, ds\right] = \frac{f'_\lambda(t)}{f_\lambda(t)}\, x(t),$$

so that

$$(10) \qquad f_\lambda(t) \int_0^t \frac{f'_\lambda(s)}{[f_\lambda(s)]^2}\, y(s)\, ds = \int_0^t \frac{f'_\lambda(s)}{f_\lambda(s)}\, x(s)ds.$$

Subtracting (10) from (9), we obtain (7). Hence (7) is a 1-1 transformation of C onto itself.

Next, we want to show that the transformation (7) satisfies the hypotheses of the following theorem :

<u>Theorem 2.3</u> Let $K^1$ be continuous on $\{(t,s) \mid 0 \leq t \leq s,\ 0 \leq s \leq 1\}$ and let it vanish on $\{(0,s) \mid 0 \leq s \leq 1\}$. Let $K^2$ be continuous on $\{(t,s) \mid 0 \leq t \leq 1,\ 0 \leq s \leq t\}$ and let

$$K(t,s) = \begin{cases} K^1(t,s) & \text{when } 0 \leq t < s,\ 0 < s \leq 1, \\ K^2(t,s) & \text{when } s < t \leq 1,\ 0 \leq s < 1, \\ \dfrac{K^1(t,s)+K^2(t,s)}{2} & \text{when } t = s,\quad 0 \leq s \leq 1, \end{cases}$$

$$J(s) = K^2(s,s) - K^1(s,s) \quad , \quad 0 \leq s \leq 1,$$

$$D = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^1 \ldots \int_0^1 \begin{vmatrix} K(s_1,s_1) \ldots K(s_1, s_n) \\ \ldots\ldots\ldots\ldots \\ K(s_n,s_1) \ldots K(s_n,s_n) \end{vmatrix} ds_1 \ldots ds_n .$$

Assume furthermore that K satisfies the following conditions :

(A) For almost all s, K is absolutely continuous in t on $0 \leq t \leq 1$ after the jump at $t = s$ is removed by the addition of a step function.

(B) There exists a measurable function H which is of bounded variation in t for each s and which for almost all (t,s) in $[0,1] \times [0,1]$ is equal to $\frac{\partial K}{\partial t}(t,s)$.

(C) The function H mentioned in (B) can be chosen so that

$$\int_0^1 \sup_{0 \leq t \leq 1} |H(t,s)|\, ds < \infty \quad \text{and} \quad \int_0^1 \operatorname*{Var}_{0 \leq t \leq 1} [H(t,s)]\, ds < \infty .$$

(D) The function J is of bounded variation on $[0,1]$.

(E) The determinant $D \neq 0$.

Let S be a Wiener measurable subset of C, and let TS be the image of S under the transformation T defined by $y = T(x)$;

$$y(t) = x(t) + \int_0^1 K(t,s)\, x(s)\, ds.$$

Then we have $\text{meas}_W(TS) = |D| \int_S \exp(-\Phi[x])\, dW(x)$, where

$$\Phi[x] = \int_0^1 \left[\frac{d}{dt}\int_0^1 K(t,s)x(s)ds\right]^2 dt + 2\int_0^1 \left[\int_0^1 \frac{\partial}{\partial t}K(t,s)x(s)ds\right] dx(t)$$

$$+ \int_0^1 J(t)\, d\{[x(t)]^2\}.$$

Moreover, if $\mathscr{F}$ is any Wiener measurable function defined on TS, then

$$\int_{TS} \mathscr{F}[y]\, dW(y) = |D| \int_S \mathscr{F}\left[x + \int_0^1 K(\cdot,s)x(s)ds\right] \exp(-\Phi[x])\, dW(x),$$

in the sense that the existences of one side implies that of the other and the validity of the equality.

(For a proof of this theorem see [3])

To prove that the transformation (7) satisfies all the hypotheses of Theorem 2.3, we let $K^1(t,s) = 0$ $(0 \leq t \leq s,\ 0 \leq s \leq 1)$, $K^2(t,s) = -f_\lambda'(s)/f_\lambda(s)$ $(0 \leq s \leq t,\ 0 \leq s \leq 1)$. Clearly, $K^1$ is continuous. $K^2$ is continuous, since $f_\lambda'$ and $f_\lambda$ are continuous on $[0,1]$ and $f_\lambda(s) \neq 0$ for all s in $[0,1]$ (Theorem 1.16).

$$\text{Let} \quad K(t,s) = \begin{cases} 0 & , \ 0 \leq t < s, \ 0 < s \leq 1 \\ -f'_\lambda(s)/f_\lambda(s) & , \ s < t \leq 1, \ 0 \leq s < 1 \\ -f'_\lambda(s)/2f_\lambda(s) & , \ t = s, \quad 0 \leq s \leq 1 \end{cases}$$

$$\text{and} \quad J(s) = -f'_\lambda(s)/f_\lambda(s) \quad , \ 0 \leq s \leq 1.$$

To show that (A) holds, we define a step function $S_s$ as follow :

$$S_s(t) = \begin{cases} -f'_\lambda(s)/2f_\lambda(s) & , \ 0 \leq t < s \\ 0 & , \ t = s \\ f'_\lambda(s)/2f_\lambda(s) & , \ s < t \leq 1. \end{cases}$$

Therefore, for each s in $[0,1]$, $K(t,s)+S_s(t) = -f'_\lambda(s)/2f_\lambda(s)$ $(0 \leq t \leq 1)$, and it is also absolutely continuous in t on $0 \leq t \leq 1$, hence (A) holds. If we take $H(t,s) = 0$ on the square $[0,1]\times[0,1]$, then clearly (B) and (C) hold. To show that (D) holds, we consider that J is continuous on $[0,1]$ and

$$\begin{aligned} J'(s) &= (f'_\lambda(s)/f_\lambda(s))' \\ &= (f_\lambda(s)f''_\lambda(s)-f'^2_\lambda(s))/f^2_\lambda(s) \\ &= -\lambda P(s)-f'^2_\lambda(s)/f^2_\lambda(s). \end{aligned}$$

Since P, $f_\lambda$, $f'_\lambda$ are continuous on $[0,1]$ and $f_\lambda(s) \neq 0$ for any s in $[0,1]$ (Theorem 1.16), we have that J' is bounded on $[0,1]$, and hence J is of bounded variation. This shows that (D) holds. To show that K(t,s) satisfies (E), we consider the Fredholm determinant of the transformation,

$$D = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^1 \cdots \int_0^1 \begin{vmatrix} K(s_1,s_1) & \cdots & K(s_1,s_n) \\ \cdot & \cdots & \cdot \\ K(s_n,s_1) & \cdots & K(s_n,s_n) \end{vmatrix} ds_1 \ldots ds_n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ \int_0^1 K(s,s)ds \right\}^n$$

$$= \exp \left( \int_0^1 K(s,s)ds \right)$$

$$= \exp \left( \frac{1}{2} \int_0^1 (-f'_\lambda(s)/f_\lambda(s))\, ds \right)$$

$$= \exp \left( -\frac{1}{2} \left[ \log f_\lambda(1) - \log f_\lambda(0) \right] \right)$$

$$= \exp \left( \frac{1}{2} \log \frac{f_\lambda(0)}{f_\lambda(1)} \right)$$

$$= \left( \frac{f_\lambda(0)}{f_\lambda(1)} \right)^{\frac{1}{2}} \neq 0 .$$

Hence by Theorem 2.3 with $\mathcal{F}[y] = F[x] = F[T^{-1}(y)]$, we have

$$\int_C F\left[y(\cdot) + f_\lambda(\cdot) \int_0^{(\cdot)} \frac{f'_\lambda(s)}{f_\lambda(s)} y(s)\, ds\right] dW(y)$$

$$= \left( \frac{f_\lambda(0)}{f_\lambda(1)} \right)^{\frac{1}{2}} \int_C F[x] \exp\left\{ - \int_0^1 \left[\frac{d}{dt} \int_0^t \left(- \frac{f'_\lambda(s)}{f_\lambda(s)} x(s)\right)ds\right]^2 dt \right.$$

$$\left. - 2 \int_0^1 \left[ \int_0^t \frac{\partial}{\partial t}\left( \frac{-f'_\lambda(s)}{f_\lambda(s)}\right) x(s)ds\right] dx(t) - \int_0^1 \left(- \frac{f'_\lambda(t)}{f_\lambda(t)}\right) d[x(t)]^2 \right\} dW(x)$$

(12)

$$= \left( \frac{f_\lambda(0)}{f_\lambda(1)} \right)^{\frac{1}{2}} \int_C F[x] \exp \left\{ - \int_0^1 \left[ - \frac{f'_\lambda(t)}{f_\lambda(t)} x(t)\right]^2 dt + \int_0^1 \frac{f'_\lambda(t)}{f_\lambda(t)} d[x(t)]^2 \right\} d\, W(x)$$

where the existence of either side implies the existence of the other. On integrating by parts to the right side of (12) and using $f'_\lambda(1) = 0$, $x(0) = 0$ and (1), we obtain

$$\int_C F\left[y(\cdot) + f_\lambda(\cdot) \int_0^{(\cdot)} \frac{f'_\lambda(s)}{f_\lambda(s)} y(s)ds\right] dW(y)$$

$$= \left(\frac{f_\lambda(0)}{f_\lambda(1)}\right)^{\frac{1}{2}} \int_C F[x] \exp\{-\int_0^1 \left[\frac{f'_\lambda(t)}{f_\lambda(t)} x(t)\right]^2 dt + \left(\frac{f'_\lambda(1)}{f_\lambda(1)}[x(1)]^2 - \frac{f'_\lambda(0)}{f_\lambda(0)}[x(0)]^2\right)$$

$$- \int_0^1 x^2(t) d\left[\frac{f'_\lambda(t)}{f_\lambda(t)}\right]\} dW(x)$$

$$= \left(\frac{f_\lambda(0)}{f_\lambda(1)}\right)^{\frac{1}{2}} \int_C F[x] \exp\{-\int_0^1 \frac{[f'_\lambda(t)]^2}{f^2_\lambda(t)} x^2(t)dt$$

$$- \int_0^1 x^2(t)\left(\frac{f_\lambda(t)f''_\lambda(t) - [f'_\lambda(t)]^2}{f^2_\lambda(t)}\right)dt\} dW(x)$$

$$= \left(\frac{f_\lambda(0)}{f_\lambda(1)}\right)^{\frac{1}{2}} \int_C F[x] \exp\{-\int_0^1 \frac{f''_\lambda(t)}{f_\lambda(t)} x^2(t)dt\} dW(x)$$

$$= \left(\frac{f_\lambda(0)}{f_\lambda(1)}\right)^{\frac{1}{2}} \int_C F[x] \exp\left(\int_0^1 \lambda P(t)x^2(t)dt\right) dW(x).$$

Therefore

$$\int_C F[x] \exp\left(\lambda \int_0^1 P(t)x^2(t)dt\right) dW(x)$$

$$= \left(\frac{f_\lambda(1)}{f_\lambda(0)}\right)^{\frac{1}{2}} \int_C F\left[y(\cdot) + f_\lambda(\cdot) \int_0^{(\cdot)} \frac{f'_\lambda(s)}{f_\lambda(s)} y(s)ds\right] dW(y). \qquad \#$$

Proof of Theorem 2.1

Let the function F in Theorem 2.2 be identically unity, that is, $F(u) = 1$, for all u, we obtain (4).

To prove the second part, let $f_1$ and $f_2$ be any two linearly independent solutions of (1) and let $f_\lambda(t) = f_2'(1)f_1(t) - f_1'(1)f_2(t)$. Then $f_\lambda$ is a non-trivial solution of (1) since

$$\begin{aligned} f_\lambda''(t) + \lambda P(t)f_\lambda(t) &= f_2'(1)f_1''(t) - f_1'(1)f_2''(t) + \lambda P(t)f_2'(1)f_1(t) - \lambda P(t)f_1'(1)f_2(t) \\ &= f_2'(1)(f_1''(t)+\lambda P(t)f_1(t)) - f_1'(1)(f_2''(t)+\lambda P(t)f_2(t)) \\ &= 0 \end{aligned}$$

and $f_\lambda'(1) = f_2'(1)f_1'(1) - f_1'(1)f_2'(1) = 0$.

By relation (4), we have

$$\int_C \exp\left(\lambda \int_0^1 P(t)x^2(t)dt\right)dW(x) = \left(\frac{f_2'(1)f_1(1) - f_1'(1)f_2(1)}{f_2'(1)f_1(0) - f_1'(1)f_2(0)}\right)^{\frac{1}{2}}.$$

That is the relation (5) true for a = 1. Since the differential equation (1) has no term in f'(t), by Theorem 1.17 its wronskian is constant. Hence, we may use any convenient point a instead of a = 1. This yields Theorem 2.1. #

<u>Example 2.1.1</u> Let P(t) = 1.

Therefore the differential equation (1) becomes

(13) $$f''(t) + \lambda f(t) = 0$$

and the solution is $f(t) = A\cos(\sqrt{\lambda}t) + B\sin(\sqrt{\lambda}t)$ where A, B are constants. Since f(0) = 0, A = 0. Thus $f(t) = B\sin(\sqrt{\lambda}t)$ and $f'(t) = B\sqrt{\lambda}\cos(\sqrt{\lambda}t)$. And since f'(1) = 0, $B\sqrt{\lambda}\cos\sqrt{\lambda} = 0$. We may assume that $\lambda \neq 0$, if $\lambda = 0$ we have f(t) is a trivial solution because f''(t) = 0 has solution f(t) = Ct+D where C,D, are constants and $f(0) = f'(1) = 0$. Hence $\cos\sqrt{\lambda} = 0$. But $\cos (2n+1)\frac{\pi}{2} = 0$, $n = 0,1,2,\ldots$ . Thus $\lambda = (2n+1)^2 \frac{\pi^2}{4}$, $n = 0,1,2,\ldots$ . That is the least characteristic value is $\frac{\pi^2}{4}$ and the characteristic function which satisfies characteristic value $\lambda$ is $\sin \sqrt{\lambda}\, t$.

A nontrivial solution of (13) satisfying $f_\lambda'(1) = 0$ is $f_\lambda(t) = \cos(\sqrt{\lambda}(t-1))$. Therefore, by relation (4), for $-\infty < \lambda < \frac{\pi^2}{4}$,

$$\int_C \exp\left(\lambda \int_0^1 x^2(t)dt\right)dW(x) = \left(\frac{\cos 0}{\cos(-\sqrt{\lambda})}\right)^{\frac{1}{2}} = \left(\frac{1}{\cos\sqrt{\lambda}}\right)^{\frac{1}{2}} .$$

This means that

$$\int_C \exp\left(\lambda \int_0^1 x^2(t)dt\right)dW(x) = \begin{cases} (1/\cos\sqrt{\lambda})^{\frac{1}{2}} , & 0 \le \lambda < \frac{\pi^2}{4} \\ (1/\cosh\sqrt{|\lambda|})^{\frac{1}{2}} , & -\infty < \lambda < 0 . \end{cases}$$

<u>Example 2.1.2</u> Let $P(t) = (t+\alpha)^{-2}$, $0 < \alpha < \infty$.

Therefore the differential equation (1) becomes

(14) $$f''(t) + \frac{\lambda}{(t+\alpha)^2} f(t) = 0.$$

Take $t+\alpha = e^z$, then $\frac{dz}{dt} = \frac{1}{t+\alpha}$. This implies that

$$f'(t) = \frac{d}{dz} f(t) \frac{dz}{dt} = \frac{1}{t+\alpha} \frac{d}{dz} f(t),\quad f''(t) = \frac{1}{(t+\alpha)^2}\frac{d^2f(t)}{dz^2} - \frac{1}{(t+\alpha)^2}\frac{df(t)}{dz}.$$

Replacing $f''(t)$ in (14) and multiplying this equation by $(t+\alpha)^2$, we obtain

(15) $$\frac{d^2}{dz^2} f(t) - \frac{d}{dz} f(t) + \lambda f(t) = 0 .$$

If we write $\lambda = \frac{1}{4} - \mu^2$ , $0 < \mu < \infty$ , then we have the general solution of (15),

$$f(t) = Ae^{(\frac{1}{2}+\mu)z} + Be^{(\frac{1}{2}-\mu)z}$$

$$= A(t+\alpha)^{\frac{1}{2}+\mu} + B(t+\alpha)^{\frac{1}{2}-\mu} , \qquad (16)$$

where A and B are constants. That is the two linearly independent solutions of (14) are $(t+\alpha)^{\frac{1}{2}+\mu}$ and $(t+\alpha)^{\frac{1}{2}-\mu}$ .

Claim that there is no linear combination of $(t+\alpha)^{\frac{1}{2}+\mu}$ and $(t+\alpha)^{\frac{1}{2}-\mu}$ which satisfies the boundary conditions $f(0) = f'(1) = 0$ .

In fact,from (16), $f'(t) = A(\frac{1}{2}+\mu)(t+\alpha)^{-\frac{1}{2}+\mu} + B(\frac{1}{2}-\mu)(t+\alpha)^{-\frac{1}{2}-\mu}$.

If (16) satisfies the boundary conditions $f(0) = f'(1) = 0$, then we have

$$A\alpha^{\frac{1}{2}+\mu} + B\alpha^{\frac{1}{2}-\mu} = 0 ,$$

and $$A(\frac{1}{2}+\mu)(1+\alpha)^{-\frac{1}{2}+\mu} + B(\frac{1}{2}-\mu)(1+\alpha)^{-\frac{1}{2}-\mu} = 0 .$$

Consider

$$\begin{vmatrix} \alpha^{\frac{1}{2}+\mu} & \alpha^{\frac{1}{2}-\mu} \\ (\frac{1}{2}+\mu)(1+\alpha)^{-\frac{1}{2}+\mu} & (\frac{1}{2}-\mu)(1+\alpha)^{-\frac{1}{2}-\mu} \end{vmatrix} = (\frac{1}{2}-\mu)\alpha^{\frac{1}{2}+\mu}(1+\alpha)^{-\frac{1}{2}-\mu} - (\frac{1}{2}+\mu)\alpha^{\frac{1}{2}-\mu}(1+\alpha)^{-\frac{1}{2}+\mu}$$

$$= (\frac{\alpha}{1+\alpha})^{\frac{1}{2}+\mu}((\frac{1}{2}-\mu) - (\frac{1}{2}+\mu)(\frac{\alpha}{1+\alpha})^{-2\mu})$$

$$\neq 0 ,$$

since $0 < \alpha < \infty$ and $0 < \mu < \infty$ . Therefore $A = B = 0$, so we have the claim. It follows that $\lambda = \frac{1}{4} - \mu^2$ , $0 < \mu < \infty$ , is not a characteristic value and the least characteristic value of (14) is $\lambda_0 \geq \frac{1}{4}$. Hence relation (5) yields

$$\int_C \exp\left( \left(\frac{1}{4} - \mu^2\right) \int_0^1 \frac{x^2(t)}{(t+\alpha)^2} dt \right) d W(x)$$

$$= \left( \frac{(\frac{1}{2} - \mu)(a+\alpha)^{-\frac{1}{2}-\mu}(a+\alpha)^{\frac{1}{2}+\mu} - (\frac{1}{2} + \mu)(a+\alpha)^{-\frac{1}{2}+\mu}(a+\alpha)^{\frac{1}{2}-\mu}}{(\frac{1}{2} - \mu)(1+\alpha)^{-\frac{1}{2}-\mu}\alpha^{\frac{1}{2}+\mu} - (\frac{1}{2} + \mu)(1+\alpha)^{-\frac{1}{2}+\mu}\alpha^{\frac{1}{2}-\mu}} \right)^{\frac{1}{2}}$$

$$= \left( \frac{(\frac{1}{2} - \mu) - (\frac{1}{2} + \mu)}{-\alpha^{\frac{1}{2}-\mu}(1+\alpha)^{-\frac{1}{2}-\mu}\left[(\mu - \frac{1}{2})\alpha^{2\mu} + (\mu + \frac{1}{2})(1+\alpha)^{2\mu}\right]} \right)^{\frac{1}{2}}$$

$$= \left( \frac{2\mu\alpha^{\mu-\frac{1}{2}}(1+\alpha)^{\mu+\frac{1}{2}}}{(\mu - \frac{1}{2})\alpha^{2\mu} + (\mu + \frac{1}{2})(1+\alpha)^{2\mu}} \right)^{\frac{1}{2}} .$$

<u>Example 2.1.3</u> Let $P(t) = (t^2 + \alpha)^{-2}$, $0 < \alpha < \infty$.

Therefore the differential equation (1) becomes

$$f''(t) + \frac{\lambda}{(t^2 + \alpha)^2} f(t) = 0 \tag{17}$$

Take $t^2 + \alpha = e^u$, then $\frac{du}{dt} = \frac{2t}{t^2 + \alpha}$. This implies that

$$f'(t) = \frac{d}{du} f(t) \frac{du}{dt} = \frac{2t}{t^2 + \alpha} \frac{d}{du} f(t) ,$$

$$f''(t) = \frac{4t^2}{(t^2+\alpha)^2} \frac{d^2 f(t)}{du^2} + \frac{2(t^2+\alpha) - 4t^2}{(t^2 + \alpha)^2} \frac{df(t)}{du}$$

$$= \frac{4t^2}{(t^2+\alpha)^2} \frac{d^2 f(t)}{du^2} + \frac{2\alpha - 2t^2}{(t^2+\alpha)^2} \frac{df(t)}{du} .$$

Replacing $f''(t)$ in (17), we obtain

$$4t^2 \frac{d^2 f(t)}{du^2} + (2\alpha - 2t^2)\frac{d\,f(t)}{du} + \lambda f(t) = 0 ,$$

$$4(e^u - \alpha)\frac{d^2 f(t)}{du^2} + (2\alpha - 2(e^u - \alpha))\frac{d\,f(t)}{du} + \lambda f(t) = 0 ,$$

$$(18)\quad (4e^u - 4\alpha)\frac{d^2 f(t)}{du^2} + (4\alpha - 2e^u)\frac{d\,f(t)}{du} + \lambda f(t) = 0 .$$

If we write $\lambda = -\alpha$ , then $-\infty < \lambda < 0$ and $f_1(t) = e^{\frac{u}{2}}$ is a solution of (18). To show $f_1(t) = (t^2 + \alpha)^{\frac{1}{2}}$ is a solution of (17).

Since $f_1'(t) = \frac{t}{(t^2+\alpha)^{\frac{1}{2}}}$ and $f_1''(t) = \left[(t^2+\alpha)^{\frac{1}{2}} - \frac{t^2}{(t^2+\alpha)^{\frac{1}{2}}}\right] / (t^2+\alpha)$,

therefore (17) becomes

$$\frac{1}{t^2+\alpha}\left[(t^2+\alpha)^{\frac{1}{2}} - \frac{t^2+\alpha-\alpha}{(t^2+\alpha)^{\frac{1}{2}}} - \frac{\alpha(t^2+\alpha)^{\frac{1}{2}}}{t^2+\alpha}\right] = 0 .$$

By the method of finding another solution of differential equation, we have $f_2(t) = (t^2 + \alpha)^{\frac{1}{2}}\, v(t)$ where $v(t) = \int \frac{c e^{-\int \frac{0}{(t^2+\alpha)^2} dt}}{((t^2+\alpha)^{\frac{1}{2}})^2} dt$

$$= \int \frac{c}{t^2+\alpha}\, dt = \frac{c}{\sqrt{\alpha}} \tan^{-1}\frac{t}{\sqrt{\alpha}} ,\quad c \text{ is a non zero constant.}$$

Thus the second solution of (17) is $f_2(t) = \frac{c(t^2+\alpha)^{\frac{1}{2}}}{\sqrt{\alpha}} \tan^{-1}\frac{t}{\sqrt{\alpha}}$ .

And $f_1$, $f_2$ are linearly independent solutions since

$$\begin{vmatrix} f_1(t) & f_2(t) \\ f_1'(t) & f_2'(t) \end{vmatrix} = \begin{vmatrix} (t^2+\alpha)^{\frac{1}{2}} & \frac{c(t^2+\alpha)^{\frac{1}{2}}}{\sqrt{\alpha}} \tan^{-1} \frac{t}{\sqrt{\alpha}} \\ t(t^2+\alpha)^{-\frac{1}{2}} & \frac{c(t^2+\alpha)^{\frac{1}{2}}}{\sqrt{\alpha}} \frac{\sqrt{\alpha}}{t^2+\alpha} + \frac{ct(t^2+\alpha)^{-\frac{1}{2}}}{\sqrt{\alpha}} \tan^{-1} \frac{t}{\sqrt{\alpha}} \end{vmatrix}$$

$$= c + \frac{ct}{\sqrt{\alpha}} \tan^{-1} \frac{t}{\sqrt{\alpha}} - \frac{ct}{\sqrt{\alpha}} \tan^{-1} \frac{t}{\sqrt{\alpha}}$$

$$= c \neq 0 \ .$$

The least characteristic value of (17) is $\lambda_0 > 0$, because for $\lambda_0 > \lambda > -\infty$ there is no linear combination of $f_1$ and $f_2$ satisfies the boundary conditions $f(0) = f'(1) = 0$. Hence relation (5) yields

$$\int_C \exp\left(-\alpha \int_0^1 \frac{x^2(t)}{(t^2+\alpha)^2} dt\right) dW(x) = \left(\frac{c}{c\{(1+\alpha)^{-\frac{1}{2}} + \frac{(1+\alpha)^{-\frac{1}{2}}}{\sqrt{\alpha}} \tan^{-1} \frac{1}{\sqrt{\alpha}}\}\alpha^{\frac{1}{2}} - (1+\alpha)^{-\frac{1}{2}} \cdot 0}\right)^{\frac{1}{2}}$$

$$= \frac{(1+\alpha)^{\frac{1}{4}}}{(\sqrt{\alpha} + \tan^{-1} \frac{1}{\sqrt{\alpha}})^{\frac{1}{2}}} \ .$$

<u>Example 2.1.4</u> Let $P(t) = e^{\alpha t}$, $\alpha$ is a real number, $\alpha \neq 0$. Therefore the differential equation (1) becomes

$$(19) \qquad f''(t) + \lambda e^{\alpha t} f(t) = 0 \ .$$

To find the solution, we let $f(t) = \sum_{n=0}^{\infty} a_n e^{n\alpha t}$, then $f''(t) = \sum_{n=0}^{\infty} a_n n^2 \alpha^2 e^{n\alpha t}$

and (19) becomes

$$\sum_{n=0}^{\infty} a_n n^2 \alpha^2 e^{n\alpha t} = -\lambda \sum_{n=0}^{\infty} a_n e^{(n+1)\alpha t}$$

$$= -\lambda \sum_{n=1}^{\infty} a_{n-1} e^{n\alpha t} \ .$$

On equating coefficients we find

$$a_n n^2 \alpha^2 = -\lambda\, a_{n-1}\ , \quad n = 1,2,\dots\ .$$

This implies that $a_n = \dfrac{-\lambda\, a_{n-1}}{n^2\alpha^2} = \dfrac{(-\lambda)^n a_0}{(n!)^2 \alpha^{2n}}$ .

Choosing $a_0 = 1$ , we obtain

$$(20) \qquad f_1(t) = \sum_{n=0}^{\infty} \frac{(-1)^n \lambda^n e^{n\alpha t}}{\alpha^{2n}(n!)^2}$$

as a solution of (19) for every value of $\lambda \neq 0$, $-\infty < \lambda < \infty$.

To show this, we replace $f''(t)$ and $f(t)$ in (19) by

$$f_1''(t) = \sum_{n=1}^{\infty} \frac{(-1)^n \lambda^n e^{n\alpha t}}{\alpha^{2(n-1)}((n-1)!)^2}$$

and $$f_1(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\lambda^{n-1} e^{(n-1)\alpha t}}{\alpha^{2(n-1)}((n-1)!)^2}\ ,$$ respectively ,

then we get

$$\sum_{n=1}^{\infty} \frac{(-1)^n \lambda^n e^{n\alpha t}}{\alpha^{2(n-1)}((n-1)!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\lambda^n e^{n\alpha t}}{\alpha^{2(n-1)}((n-1)!)^2} = 0\ .$$

Also this $f_1$ is the classical of Bessel function of order zero,

$$J_0\left(\frac{2\lambda^{\frac{1}{2}} e^{\frac{\alpha t}{2}}}{\alpha}\right) \quad \text{where} \quad J_n(t) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{t}{2}\right)^{n+2k}}{k!\ \Gamma(k+n+1)}$$

A second linearly independent solution of (19) for $\lambda$ real, $\lambda \neq 0$ is given by $f_2(t) = Y_0\left(\dfrac{2\lambda^{\frac{1}{2}} e^{\frac{\alpha t}{2}}}{\alpha}\right)$ , the Bessel function of the second kind of order zero. The general Bessel function of the second kind is defined by the equation

$$Y_n(z) = \frac{J_n(z)\cos n\pi - J_{-n}(z)}{\sin n\pi}$$ , $n \neq$ integer, or by the limit

of this expression when n is an integer.

Since $f_1'(t) = \frac{d}{dt} J_0\left(\frac{2\lambda^{\frac{1}{2}} e^{\frac{\alpha t}{2}}}{\alpha}\right) = \lambda^{\frac{1}{2}} e^{\frac{\alpha t}{2}} J_0'\left(\frac{2\lambda^{\frac{1}{2}} e^{\frac{\alpha t}{2}}}{\alpha}\right)$ and

$f_2'(t) = \lambda^{\frac{1}{2}} e^{\frac{\alpha t}{2}} Y_0'\left(\frac{2\lambda^{\frac{1}{2}} e^{\frac{\alpha t}{2}}}{\alpha}\right)$, thus

$$f_2'(a)f_1(a) - f_1'(a)f_2(a) = \lambda^{\frac{1}{2}} e^{\frac{\alpha a}{2}} \left[J_0(z)Y_0'(z) - J_0'(z)Y_0(z)\right]_{z = \frac{2\lambda^{\frac{1}{2}} e^{\frac{\alpha a}{2}}}{\alpha}}$$

$$= \lambda^{\frac{1}{2}} e^{\frac{\alpha a}{2}} \text{ Wronskian } \left[J_0(z), Y_0(z)\right]_{z = \frac{2\lambda^{\frac{1}{2}} e^{\frac{\alpha a}{2}}}{\alpha}} .$$

But for n is not an integer, we have

$$\text{Wronskian } (J_n(z), Y_n(z)) = J_n(z)Y_n'(z) - J_n'(z)\,Y_n(z)$$

$$= \frac{1}{\sin n\pi}\left[J_n(z)J_n'(z)\cos n\pi - J_n(z)J'_{-n}(z)\right.$$

$$\left. - J_n'(z)J_n(z)\cos n\pi + J_n'(z)J_{-n}(z)\right]$$

(21) $$= \frac{1}{\sin n\pi}\left[J_n'(z)J_{-n}(z) - J_n(z)J'_{-n}(z)\right],$$

and we know that

(22) $$J_n'(z)J_{-n}(z) - J_n(z)J'_{-n}(z) = \frac{2\sin n\pi}{\pi z} ,$$

which we show as follows. Since

$$z\,J_n'(z) = z\sum_{k=0}^{\infty}\frac{(-1)^k\,(2k+n)\,(\frac{z}{2})^{2k+n-1}}{2\;k!\;\Gamma(k+n+1)}$$

$$= n\sum_{k=0}^{\infty}\frac{(-1)^k\,(\frac{z}{2})^{2k+n}}{k!\;\Gamma(k+n+1)} + z\sum_{k=1}^{\infty}\frac{(-1)^k\,(\frac{z}{2})^{2k+n-1}}{(k-1)!\;\Gamma(k+n+1)}$$

$$= n\,J_n(z) - z\sum_{k=0}^{\infty}\frac{(-1)^k\,(\frac{z}{2})^{2k+n+1}}{k!\;\Gamma(k+n+2)}$$

$$= n\,J_n(z) - z\,J_{n+1}(z)\;,$$

it follows that

$$\lim_{z\to 0} z(J_n'(z)J_{-n}(z)-J_n(z)J_{-n}'(z)) = \lim_{z\to 0}\{(nJ_n(z)-zJ_{n+1}(z))J_{-n}(z)-J_n(z)(-nJ_{-n}(z)-zJ_{-n+1}(z))\}$$

(23)

$$= \lim_{z\to 0}(2nJ_n(z)J_{-n}(z)-zJ_{n+1}(z)J_{-n}(z)+zJ_n(z)J_{-n+1}(z))\;.$$

And since $J_m(z)J_n(z) = \sum_{s=0}^{\infty}\frac{(-1)^s\,(\frac{z}{2})^{m+n+2s}\;\Gamma(m+n+2s+1)}{\Gamma(m+s+1)\Gamma(n+s+1)s!\Gamma(m+n+s+1)}$,

relation (23) becomes

$$\lim_{z\to 0} z(J_n'(z)J_{-n}(z)-J_n(z)J_{-n}'(z)) = \frac{2n}{\Gamma(n+1)\Gamma(-n+1)}$$

$$= \frac{2}{\Gamma(n)\;\Gamma(1-n)}$$

$$= \frac{2\sin n\pi}{\pi}\;,$$

this yields (22).

Hence (21) becomes

(24) $$\text{Wronskian } (J_n(z), Y_n(z)) = \frac{1}{\sin n\pi} \frac{2 \sin n\pi}{\pi z} = \frac{2}{\pi z} .$$

On taking limits, (24) is also true when n is an integer,

hence $$f_2'(a)\, f_1(a) - f_1'(a) f_2(a) = \lambda^{\frac{1}{2}} e^{\frac{\alpha a}{2}} \frac{2\alpha}{\pi 2\lambda^{\frac{1}{2}} e^{\frac{\alpha a}{2}}}$$

$$= \frac{\alpha}{\pi}$$

and $$f_2'(1) f_1(0) - f_1'(1) f_2(0) = \lambda^{\frac{1}{2}} e^{\frac{\alpha}{2}} \left[J_0\left(\frac{2\lambda^{\frac{1}{2}}}{\alpha}\right) Y_0'\left(\frac{2\lambda^{\frac{1}{2}} e^{\frac{\alpha}{2}}}{\alpha}\right) - J_0'\left(\frac{2\lambda^{\frac{1}{2}} e^{\frac{\alpha}{2}}}{\alpha}\right) Y_0\left(\frac{2\lambda^{\frac{1}{2}}}{\alpha}\right)\right].$$

On using these two equations in relation (5), we obtain

$$\int_C \exp\left(\lambda \int_0^1 x^2(t) e^{\alpha t} dt\right) dW(x) = \frac{\alpha^{\frac{1}{2}}}{\pi^{\frac{1}{2}} \lambda^{\frac{1}{4}} e^{\frac{\alpha}{4}}} \left[J_0\left(\frac{2\lambda^{\frac{1}{2}}}{\alpha}\right) Y_0'\left(\frac{2\lambda^{\frac{1}{2}} e^{\frac{\alpha}{2}}}{\alpha}\right) - J_0'\left(\frac{2\lambda^{\frac{1}{2}} e^{\frac{\alpha}{2}}}{\alpha}\right) Y_0\left(\frac{2\lambda^{\frac{1}{2}}}{\alpha}\right)\right]^{-\frac{1}{2}}$$

for all $\alpha \neq 0$ , $\alpha$ real and $-\infty < \lambda < 0$ or $0 < \lambda < \lambda_0$ where $\lambda_0$ is the least characteristic value of the differential equation (19) with the boundary conditions $f(0) = f'(1) = 0$.

<u>Example 2.1.5</u> Let $P(t) = (t+\alpha)^\beta$ , $0 < \alpha < \infty$ , $\beta \neq -2$, $\beta$ real. Therefore the differential equation (1) becomes

(25) $$f''(t) + \lambda (t+\alpha)^\beta f(t) = 0 .$$

By experiment, we take a solution of (25) as the series

$$f(t) = \sum_{n=0}^{\infty} a_n (t+\alpha)^{(\beta+2)n+1} , \text{ then}$$

(26) $$f''(t) = \sum_{n=0}^{\infty} a_n n(\beta+2) \left[n(\beta+2)+1\right] (t+\alpha)^{(\beta+2)n-1} .$$

And from (25), we have

$$f''(t) = -\lambda(t+\alpha)^{\beta} \sum_{n=0}^{\infty} a_n (t+\alpha)^{(\beta+2)n+1}$$

$$= -\lambda \sum_{n=0}^{\infty} a_n (t+\alpha)^{(\beta+2)n+\beta+1}$$

$$= -\lambda \sum_{n=1}^{\infty} a_{n-1} (t+\alpha)^{(\beta+2)(n-1)+(\beta+2)-1}$$

(27) $$= -\lambda \sum_{n=1}^{\infty} a_{n-1} (t+\alpha)^{(\beta+2)n-1} .$$



On equating coefficient from (26) and (27), we get

$$n(\beta+2) \left[n(\beta+2)+1\right] a_n = -\lambda a_{n-1} .$$

This implies that $a_n = -\lambda a_{n-1} / (\beta+2)^2 (n+\frac{1}{\beta+2})\, n$

$$= (-1)^n \lambda^n a_0 / (\beta+2)^{2n} (n+\frac{1}{\beta+2}) \ldots (1+\frac{1}{\beta+2}) n! .$$

If we let $a_0 = \dfrac{1}{\Gamma(1+\frac{1}{\beta+2})} \left(\dfrac{\lambda^{1/2}}{\beta+2}\right)^{\frac{1}{\beta+2}}$, then we have

$$f_1(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\frac{1}{\beta+2}+1)} \left(\frac{\lambda^{1/2}}{\beta+2}\right)^{2n+\frac{1}{\beta+2}} (t+\alpha)^{(\beta+2)n+1}$$

$$= (t+\alpha)^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\frac{1}{\beta+2}+1)} \left(\frac{\lambda^{1/2}}{\beta+2}\right)^{2n+\frac{1}{\beta+2}} (t+\alpha)^{\frac{\beta+2}{2}(2n+\frac{1}{\beta+2})}$$

(28) $$= (t+\alpha)^{1/2} J_{\frac{1}{\beta+2}} \left[\frac{2(t+\alpha)^{\frac{\beta}{2}+1} \lambda^{1/2}}{\beta+2}\right]$$

to show that $f_1$ satisfies the differential equation (25), let

(29) $$z = \frac{2(t+\alpha)^{\frac{\beta}{2}+1}\lambda^{\frac{1}{2}}}{\beta+2},$$

then $$\frac{dz}{dt} = (t+\alpha)^{\frac{\beta}{2}}\lambda^{\frac{1}{2}}.$$

With this notation we obtain from (28) that

$$f_1'(t) = \frac{1}{2}(t+\alpha)^{-\frac{1}{2}} J_{\frac{1}{\beta+2}}(z) + \lambda^{\frac{1}{2}}(t+\alpha)^{\frac{\beta}{2}+\frac{1}{2}} J'_{\frac{1}{\beta+2}}(z)$$

and

$$f_1''(t) = -\frac{1}{4}(t+\alpha)^{-\frac{3}{2}} J_{\frac{1}{\beta+2}}(z) + \frac{\lambda^{\frac{1}{2}}}{2}(t+\alpha)^{\frac{\beta}{2}-\frac{1}{2}} J'_{\frac{1}{\beta+2}}(z)$$

$$+ \lambda^{\frac{1}{2}}\left(\frac{\beta}{2}+\frac{1}{2}\right)(t+\alpha)^{\frac{\beta}{2}-\frac{1}{2}} J'_{\frac{1}{\beta+2}}(z) + \lambda(t+\alpha)^{\beta+\frac{1}{2}} J''_{\frac{1}{\beta+2}}(z)$$

$$= -\frac{1}{4}(t+\alpha)^{-\frac{3}{2}} J_{\frac{1}{\beta+2}}(z) + \lambda^{\frac{1}{2}}\left(\frac{\beta}{2}+1\right)(t+\alpha)^{\frac{\beta}{2}-\frac{1}{2}} J'_{\frac{1}{\beta+2}}(z)$$

$$+ \lambda(t+\alpha)^{\beta+\frac{1}{2}} J''_{\frac{1}{\beta+2}}(z).$$

Hence $f_1''(t) + \lambda(t+\alpha)^{\beta} f_1(t)$

$$= \lambda(t+\alpha)^{\beta+\frac{1}{2}} \left[J''_{\frac{1}{\beta+2}}(z) + \left(\frac{\frac{\beta}{2}+1}{\lambda^{\frac{1}{2}}(t+\alpha)^{\frac{\beta}{2}+1}}\right) J'_{\frac{1}{\beta+2}}(z) - \frac{1}{4\lambda(t+\alpha)^{\beta+2}} J_{\frac{1}{\beta+2}}(z) + J_{\frac{1}{\beta+2}}(z)\right]$$

$$= \lambda(t+\alpha)^{\beta+\frac{1}{2}} \left[J''_{\frac{1}{\beta+2}}(z) + \frac{1}{z} J'_{\frac{1}{\beta+2}}(z) - \frac{1}{z^2(\beta+2)^2} J_{\frac{1}{\beta+2}}(z) + J_{\frac{1}{\beta+2}}(z)\right]$$

$$= 0 ,$$

since $J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+k+1)}\left(\frac{z}{2}\right)^{n+2k} = -\sum_{k=1}^{\infty} \frac{(-1)^k 4k(n+k)}{4k!\,\Gamma(n+k+1)}\left(\frac{z}{2}\right)^{n+2k-2}$,

$$\frac{n^2}{z^2} J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k n^2}{4k!\,\Gamma(n+k+1)}\left(\frac{z}{2}\right)^{n+2k-2} ,$$

$$J_n'(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (n+2k)}{2k!\,\Gamma(n+k+1)}\left(\frac{z}{2}\right)^{n+2k-1} ,$$

$$\frac{1}{z} J_n'(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (n+2k)}{4k!\,\Gamma(n+k+1)}\left(\frac{z}{2}\right)^{n+2k-2} ,$$

$$J_n''(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (n+2k)(n+2k-1)}{4k!\,\Gamma(n+k+1)}\left(\frac{z}{2}\right)^{n+2k-2} ,$$

and it follows that

$$J_n''(z) + \frac{1}{z} J_n'(z) - \frac{n^2}{z^2} J_n(z) + J_n(z)$$

$$= \frac{n(n-1)+n-n^2}{4\Gamma(n+1)}\left(\frac{z}{2}\right)^{n-2} + \sum_{k=1}^{\infty} \frac{(-1)^k}{4k!\Gamma(n+k+1)}\{(n+2k)(n+2k-1)+(n+2k)-n^2-4k(n+k)\}\left(\frac{z}{2}\right)^{n+2k-2}$$

$$= 0 .$$

A second solution of (25) is given by

$$f_2(t) = (t+\alpha)^{\frac{1}{2}} Y_{\frac{1}{\beta+2}} \left[\frac{2(t+\alpha)^{\frac{\beta}{2}+1} \lambda^{\frac{1}{2}}}{\beta+2}\right]$$ which can be showns as in the case of $f_1$.

The functions $f_1$ and $f_2$ are linearly independent solutions of (25) for real number $\lambda$, $\lambda \neq 0$ because

$$\begin{vmatrix} f_1(t) & f_2(t) \\ f_1'(t) & f_2'(t) \end{vmatrix}$$

$$= \begin{vmatrix} (t+\alpha)^{\frac{1}{2}} J_{\frac{1}{\beta+2}}(z) & (t+\alpha)^{\frac{1}{2}} Y_{\frac{1}{\beta+2}}(z) \\ \frac{1}{2}(t+\alpha)^{-\frac{1}{2}} J_{\frac{1}{\beta+2}}(z) + \lambda^{\frac{1}{2}}(t+\alpha)^{\frac{\beta}{2}+\frac{1}{2}} J'_{\frac{1}{\beta+2}}(z) & \frac{1}{2}(t+\alpha)^{-\frac{1}{2}} Y_{\frac{1}{\beta+2}}(z) + \lambda^{\frac{1}{2}}(t+\alpha)^{\frac{\beta}{2}+\frac{1}{2}} Y'_{\frac{1}{\beta+2}}(z) \end{vmatrix}$$

$$= \lambda^{\frac{1}{2}}(t+\alpha)^{\frac{\beta}{2}+1}\left[J_{\frac{1}{\beta+2}}(z) Y'_{\frac{1}{\beta+2}}(z) - J'_{\frac{1}{\beta+2}}(z) Y_{\frac{1}{\beta+2}}(z)\right]$$

(24)

$$= \lambda^{\frac{1}{2}}(t+\alpha)^{\frac{\beta}{2}+1} \frac{2}{\pi z}$$

$$= \frac{\beta+2}{\pi} \neq 0.$$

Thus $f_2'(a)f_1(a) - f_1'(a)f_2(a) = \frac{\beta+2}{\pi}$. Also by (28) and (29), we obtain

$$f_1'(t) = \lambda^{\frac{1}{2}}(t+\alpha)^{\frac{\beta}{2}+\frac{1}{2}}\left[J'_{\frac{1}{\beta+2}}(z) + \frac{1}{z(\beta+2)} J_{\frac{1}{\beta+2}}(z)\right]$$

$$= \lambda^{\frac{1}{2}}(t+\alpha)^{\frac{\beta}{2}+\frac{1}{2}}\left[\sum_{k=0}^{\infty} \frac{(-1)^k \left(2k+\frac{1}{\beta+2}\right)\left(\frac{z}{2}\right)^{\frac{1}{\beta+2}+2k-1}}{2k!\,\Gamma\left(k+\frac{1}{\beta+2}+1\right)} + \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{\frac{1}{\beta+2}+2k}}{k!\,\Gamma\left(k+\frac{1}{\beta+2}+1\right) z(\beta+2)}\right]$$

$$= \lambda^{\frac{1}{2}}(t+\alpha)^{\frac{\beta}{2}+\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{\frac{1}{\beta+2}+2k-1}}{k!\,\Gamma\left(k+\frac{1}{\beta+2}\right)} \left( \frac{2k+\frac{1}{\beta+2}}{2\left(k+\frac{1}{\beta+2}\right)} + \frac{1}{2\left(k+\frac{1}{\beta+2}\right)(\beta+2)} \right)$$

$$= \lambda^{\frac{1}{2}}(t+\alpha)^{\frac{\beta}{2}+\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{\frac{1}{\beta+2}+2k-1}}{k!\;\Gamma\left(k+\frac{1}{\beta+2}\right)}$$

$$= \lambda^{\frac{1}{2}}(t+\alpha)^{\frac{\beta}{2}+\frac{1}{2}} J_{\frac{1}{\beta+2}-1}(z) .$$

Similarly $f_2'(t) = \lambda^{\frac{1}{2}}(t+\alpha)^{\frac{\beta}{2}+\frac{1}{2}} Y_{\frac{1}{\beta+2}-1}(z)$ .

Hence relation (5) yields

$$\int_C \exp\left(\lambda \int_0^1 (t+\alpha)^\beta x^2(t)dt\right) d\,W(x)$$

(30)

$$= \frac{(\beta+2)^{\frac{1}{2}}}{\pi^{\frac{1}{2}}\lambda^{\frac{1}{4}}(1+\alpha)^{\frac{\beta+1}{4}}\alpha^{\frac{1}{4}}} \left[ Y_{\frac{1}{\beta+2}-1}\left(\frac{2(1+\alpha)^{\frac{\beta}{2}+1}\lambda^{\frac{1}{2}}}{\beta+2}\right) J_{\frac{1}{\beta+2}}\left(\frac{2\alpha^{\frac{\beta}{2}+1}\lambda^{\frac{1}{2}}}{\beta+2}\right) - J_{\frac{1}{\beta+2}-1}\left(\frac{2(1+\alpha)^{\frac{\beta}{2}+1}\lambda^{\frac{1}{2}}}{\beta+2}\right) Y_{\frac{1}{\beta+2}}\left(\frac{2\alpha^{\frac{\beta}{2}+1}\lambda^{\frac{1}{2}}}{\beta+2}\right)\right]^{-\frac{1}{2}} .$$

This relation holds for $\beta \neq -2$, $0 < \alpha < \infty$ and $-\infty < \lambda < \lambda_0$ , $\lambda \neq 0$.

If we put $\beta = 0$, we obtain

(31) $$\int_C \exp\left(\lambda\int_0^1 x^2(t)dt\right)dW(x) = \frac{2^{\frac{1}{2}}}{\pi^{\frac{1}{2}}\lambda^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}\alpha^{\frac{1}{4}}}\left[Y_{-\frac{1}{2}}((1+\alpha)\lambda^{\frac{1}{2}})J_{\frac{1}{2}}(\alpha\lambda^{\frac{1}{2}})\right.$$

$$\left. - J_{-\frac{1}{2}}((1+\alpha)\lambda^{\frac{1}{2}})Y_{\frac{1}{2}}(\alpha\lambda^{\frac{1}{2}})\right]^{-\frac{1}{2}}$$

But $$J_{\frac{1}{2}}(u) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\frac{1}{2}+1)}\left(\frac{u}{2}\right)^{2k+\frac{1}{2}}$$

$$= \left(\frac{2}{u}\right)^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+\frac{1}{2})(k+\frac{1}{2}-1)\ldots\frac{3}{2}\cdot\frac{1}{2}\,\Gamma(\frac{1}{2})}\left(\frac{u}{2}\right)^{2k+1}$$

$$= \left(\frac{2}{u\pi}\right)^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k\ u^{2k+1}}{2^k k!(2k+1)(2k-1)\ldots 3.1}$$

$$= \left(\frac{2}{u\pi}\right)^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k u^{2k+1}}{(2k+1)!}$$

$$= \left(\frac{2}{u\pi}\right)^{\frac{1}{2}} \sin u \quad ,$$

$$J_{-\frac{1}{2}}(u) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k-\frac{1}{2}+1)}\left(\frac{u}{2}\right)^{2k-\frac{1}{2}}$$

$$= \left(\frac{2}{u}\right)^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k\ u^{2k}}{2^k\ k!(2k-1)(2k-3)\ldots 3.1}$$

$$= \left(\frac{2}{u\pi}\right)^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k\ u^{2k}}{(2k)!}$$

$$= \left(\frac{2}{u\pi}\right)^{\frac{1}{2}} \cos u \quad ,$$

$$Y_{\frac{1}{2}}(u) = \frac{J_{\frac{1}{2}}(u)\ \cos(\frac{\pi}{2}) - J_{-\frac{1}{2}}(u)}{\sin\frac{\pi}{2}} = -\left(\frac{2}{\pi u}\right)^{\frac{1}{2}} \cos u$$

and

$$Y_{-\frac{1}{2}}(u) = \frac{J_{-\frac{1}{2}}(u)\cos(-\frac{\pi}{2}) - J_{\frac{1}{2}}(u)}{\sin(-\frac{\pi}{2})} = \left(\frac{2}{\pi u}\right)^{\frac{1}{2}} \sin u.$$

Hence (31) becomes

$$\int_C \exp\left(\lambda \int_0^1 x^2(t)dt\right) d\,W(x)$$

$$= \frac{2^{\frac{1}{2}}}{\pi^{\frac{1}{2}}\lambda^{\frac{1}{4}}(1+\alpha)^{\frac{1}{4}}\alpha^{\frac{1}{4}}} \left[\frac{2}{\pi\lambda^{\frac{1}{2}}(1+\alpha)^{\frac{1}{2}}\alpha^{\frac{1}{2}}}\{\sin((1+\alpha)\lambda^{\frac{1}{2}})\sin(\alpha\lambda^{\frac{1}{2}}) - \cos((1+\alpha)\lambda^{\frac{1}{2}})\cos(\alpha\lambda^{\frac{1}{2}})\}\right]^{-\frac{1}{2}}$$

$$= \left[\cos\ ((1+\alpha)\lambda^{\frac{1}{2}} - \alpha\lambda^{\frac{1}{2}})\right]^{-\frac{1}{2}}$$

$$= \left[\cos\ \lambda^{\frac{1}{2}}\right]^{-\frac{1}{2}} ,$$

which is the same as in Example 2.1.1.

Next, we will find $\int_C \exp(\lambda \int_0^1 P(t)x^2(t)dt)dW(x)$ under the weaker hypothesis $P(t) \geq 0$, $0 \leq t \leq 1$, and this is our main purpose. To do this, we need

Theorem 2.4 Let P be a nonnegative continuous function on $[0,1]$, and $(P_n)$ a decreasing sequence of positive continuous functions on $[0,1]$ with $\lim_{n\to\infty} P_n = P$. Let $\lambda_{0,n}$ be the least characteristic value of (1) with $P = P_n$ subject to (2). Then if $f_{\lambda,n}$ is any nontrivial solution of (1) with $P = P_n$ satisfying $f'_{\lambda,n}(1) = 0$, we have, for $\lambda < \lim_{n\to\infty} \lambda_{0,n}$,

$$(32) \quad \int_C \exp\left(\lambda \int_0^1 P(t)x^2(t)dt\right)dW(x) = \lim_{n\to\infty} \left(\frac{f_{\lambda,n}(1)}{f_{\lambda,n}(0)}\right)^{\frac{1}{2}}.$$

Consequently, if $\lambda < \lim_{n\to\infty} \lambda_{0,n}$ and $f_{1,n}$ and $f_{2,n}$ are any two linearly independent solutions of (1) with $P = P_n$, we have

$$(33) \quad \int_C \exp\left(\lambda \int_0^1 P(t)x^2(t)dt\right)dW(x) = \lim_{n\to\infty} \left(\frac{f'_{2,n}(a)f_{1,n}(a)-f'_{1,n}(a)f_{2,n}(a)}{f'_{2,n}(1)f_{1,n}(0)-f'_{1,n}(1)f_{2,n}(0)}\right)^{\frac{1}{2}},$$

where $0 \leq a \leq 1$.

Remark : 1. We can construct a decreasing sequence $(P_n)$ as in Theorem 2.4 in several ways, for example define $P_n(t) = \max(P(t), \frac{1}{n})$ or $P_n(t) = P(t) + \frac{1}{n}$.

2. By Theorem 1.18, we know that the sequence $(\lambda_{0,n})$ of the first eigenvalue corresponds to a decreasing sequence $(P_n)$ is an increasing sequence. This guarantees the existence of $\lim_{n\to\infty} \lambda_{0,n}$ in Theorem 2.4.

Proof of Theorem 2.4

Let $\lambda < \lim_{n\to\infty} \lambda_{0,n}$, then $\lambda < \lambda_{0,n}$ for sufficiently large n where $\lambda_{0,n}$ is the least characteristic value of (1) with $P = P_n$ subject to (2), and $P_n$ is positive and continuous function on $[0,1]$. Thus by relation (4) and (5), we obtain

(34) $$\int_C \exp\left(\lambda \int_0^1 P_n(t)x^2(t)dt\right)dW(x) = \left(\frac{f_{\lambda,n}(1)}{f_{\lambda,n}(0)}\right)^{\frac{1}{2}}$$

and

(35) $$\int_C \exp\left(\lambda\int_0^1 P_n(t)x^2(t)dt\right)dW(x) = \left(\frac{f'_{2,n}(a)f_{1,n}(a)-f'_{1,n}(a)f_{2,n}(a)}{f'_{2,n}(1)f_{1,n}(0)-f'_{1,n}(1)f_{2,n}(0)}\right)^{\frac{1}{2}},$$

where $f_{\lambda,n}$ is any nontrivial solution of (32) satisfying $f'_{\lambda,n}(1) = 0$, and $f_{1,n}$, $f_{2,n}$ are two linearly independent solutions of (1) with $P = P_n$.

<u>Case 1</u> If $-\infty < \lambda < 0$, then $\lambda P_n(t)x^2(t)$ increases to $\lambda P(t)x^2(t)$, hence by Monotone convergence Theorem,

$$\int_C \exp\left(\lambda \int_0^1 P(t)x^2(t)dt\right)dW(x) = \lim_{n\to\infty} \int_C \exp\left(\lambda \int_0^1 P_n(t)\, x^2(t)dt\right)dW(x)$$

By (34) and (35), we have (32) and (33), respectively.

<u>Case 2</u> If $0 < \lambda < \lim_{n\to\infty} \lambda_{0,n}$, then $\lambda P_n(t)x^2(t)$ decreases to $\lambda P(t)x^2(t)$.

Since $P_n$ and $x$ are continuous in $[0,1]$, $P_n$ and $x$ are bounded so $\int_0^1 \lambda P_1(t)x^2(t)dt < \infty$. By a consequence of Monotone convergence Theorem,

$$\int_C \exp\left(\lambda\int_0^1 P(t)x^2(t)dt\right)dW(x) = \lim_{n\to\infty} \int_C \exp\left(\lambda\int_0^1 P_n(t)x^2(t)dt\right)d\,W(x)$$

and by (34) and (35), we have (32) and (33), respectively. #

<u>Example 2.4.1</u> Find $\int_C \exp(\lambda\int_0^1 t^\beta x^2(t)dt)dW(x)$ where $\beta$ is a positive real number and $0 \leq t \leq 1$. We consider the function $(t+\frac{1}{n})^\beta$ for $0 < n < \infty$. These are decreasing functions which converge to $t^\beta$ and they are positive and continuous on $[0,1]$. From example 2.1.5, the functions

$$f_1(t) = (t+\frac{1}{n})^{\frac{1}{2}} J_{\frac{1}{\beta+2}}\left[\frac{2(t+\frac{1}{n})^{\frac{\beta}{2}+1}\lambda^{\frac{1}{2}}}{\beta+2}\right]$$ and

$$f_2(t) = (t+\frac{1}{n})^{\frac{1}{2}} Y_{\frac{1}{\beta+2}}\left[\frac{2(t+\frac{1}{n})^{\frac{\beta}{2}+1}\lambda^{\frac{1}{2}}}{\beta+2}\right]$$ are two linearly

independent solutions of $f''(t) + \lambda(t+\frac{1}{n})^\beta f(t) = 0$.

Thus for $-\infty < \lambda < \lim_{n\to\infty} \lambda_{0,n}$, replacing (30) to the right side of (33) where $\alpha = \frac{1}{n}$, we obtain

$$\int_C \exp(\lambda\int_0^1 t^\beta x^2(t)dt)\, dW(x)$$

$$= \lim_{n\to\infty} \frac{(\beta+2)^{\frac{1}{2}}}{\pi^{\frac{1}{2}}\lambda^{\frac{1}{4}}(1+\frac{1}{n})^{\frac{\beta+1}{4}}(\frac{1}{n})^{\frac{1}{4}}}\left[Y_{\frac{1}{\beta+2}-1}\left(\frac{2(1+\frac{1}{n})^{\frac{\beta}{2}+1}\lambda^{\frac{1}{2}}}{\beta+2}\right)J_{\frac{1}{\beta+2}}\left(\frac{2(\frac{1}{n})^{\frac{\beta}{2}+1}\lambda^{\frac{1}{2}}}{\beta+2}\right)\right.$$

$$\left. - J_{\frac{1}{\beta+2}-1}\left(\frac{2(1+\frac{1}{n})^{\frac{\beta}{2}+1}\lambda^{\frac{1}{2}}}{\beta+2}\right)Y_{\frac{1}{\beta+2}}\left(\frac{2(\frac{1}{n})^{\frac{\beta}{2}+1}\lambda^{\frac{1}{2}}}{\beta+2}\right)\right]^{-\frac{1}{2}}$$

(36)

$$= \lim_{n\to\infty} \frac{(\beta+2)^{\frac{1}{2}}}{\pi^{\frac{1}{2}}\lambda^{\frac{1}{4}}(1+\frac{1}{n})^{\frac{\beta+1}{4}}} \left[ \left(\frac{1}{n}\right)^{\frac{1}{2}} Y_{\frac{1}{\beta+2}-1}\left(\frac{2(1+\frac{1}{n})^{\frac{\beta}{2}+1}\lambda^{\frac{1}{2}}}{\beta+2}\right) J_{\frac{1}{\beta+2}}\left(\frac{2(\frac{1}{n})^{\frac{\beta}{2}+1}\lambda^{\frac{1}{2}}}{\beta+2}\right)\right.$$

$$\left. - \left(\frac{1}{n}\right)^{\frac{1}{2}} J_{\frac{1}{\beta+2}-1}\left(\frac{2(1+\frac{1}{n})^{\frac{\beta}{2}+1}\lambda^{\frac{1}{2}}}{\beta+2}\right) Y_{\frac{1}{\beta+2}}\left(\frac{2(\frac{1}{n})^{\frac{\beta}{2}+1}\lambda^{\frac{1}{2}}}{\beta+2}\right)\right]^{-\frac{1}{2}} .$$

We consider that

$$\left(\frac{1}{n}\right)^{\frac{1}{2}} J_{\frac{1}{\beta+2}}\left(\frac{2(\frac{1}{n})^{\frac{\beta}{2}+1}\lambda^{\frac{1}{2}}}{\beta+2}\right)$$

$$= \left(\frac{1}{n}\right)^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\frac{1}{\beta+2}+1)} \left(\frac{(\frac{1}{n})^{\frac{\beta}{2}+1}\lambda^{\frac{1}{2}}}{\beta+2}\right)^{2k+\frac{1}{\beta+2}}$$

(37)

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{n}\right)^{k(\beta+2)+1}}{k!\ \Gamma(k+\frac{1}{\beta+2}+1)} \left(\frac{\lambda^{\frac{1}{2}}}{\beta+2}\right)^{2k+\frac{1}{\beta+2}} ,$$

and

$$\left(\frac{1}{n}\right)^{\frac{1}{2}} Y_{\frac{1}{\beta+2}}\left(\frac{2(\frac{1}{n})^{\frac{\beta}{2}+1}\lambda^{\frac{1}{2}}}{\beta+2}\right)$$

$$= \left(\frac{1}{n}\right)^{\frac{1}{2}} \frac{J_{\frac{1}{\beta+2}}\left(\frac{2(\frac{1}{n})^{\frac{\beta}{2}+1}\lambda^{\frac{1}{2}}}{\beta+2}\right)\cos(\frac{\pi}{\beta+2}) - J_{-\frac{1}{\beta+2}}\left(\frac{2(\frac{1}{n})^{\frac{\beta}{2}+1}\lambda^{\frac{1}{2}}}{\beta+2}\right)}{\sin\frac{\pi}{\beta+2}}$$

$$= \frac{1}{\sin\frac{\pi}{\beta+2}} \left[ \left(\frac{1}{n}\right)^{\frac{1}{2}} \cos \frac{\pi}{\beta+2} J_{\frac{1}{\beta+2}} \left( \frac{2\left(\frac{1}{n}\right)^{\frac{\beta}{2}+1} \lambda^{\frac{1}{2}}}{\beta+2} \right) - \left(\frac{1}{n}\right)^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k-\frac{1}{\beta+2}+1)} \left( \frac{\left(\frac{1}{n}\right)^{\frac{\beta}{2}+1} \lambda^{\frac{1}{2}}}{\beta+2} \right)^{2k-\frac{1}{\beta+2}} \right]$$

(38)

$$= \frac{1}{\sin\frac{\pi}{\beta+2}} \left[ \left(\frac{1}{n}\right)^{\frac{1}{2}} \cos \frac{\pi}{\beta+2} J_{\frac{1}{\beta+2}} \left( \frac{2\left(\frac{1}{n}\right)^{\frac{\beta}{2}+1} \lambda^{\frac{1}{2}}}{\beta+2} \right) - \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{1}{n}\right)^{(\beta+2)k}}{k!\,\Gamma(k-\frac{1}{\beta+2}+1)} \left( \frac{\lambda^{\frac{1}{2}}}{\beta+2} \right)^{2k-\frac{1}{\beta+2}} \right].$$

Hence, replacing (37) and (38) in (36), we obtain

$$\int_C \exp\left(\lambda \int_0^1 t^\beta x^2(t)\,dt\right) dW(x) = \frac{(\beta+2)^{\frac{1}{2}}}{\pi^{\frac{1}{2}} \lambda^{\frac{1}{4}}} \left[ 0 - \left(0 - \frac{(\beta+2)^{\frac{1}{\beta+2}}}{\sin\frac{\pi}{\beta+2} \Gamma(1-\frac{1}{\beta+2}) \lambda^{\frac{1}{2(\beta+2)}}} J_{\frac{1}{\beta+2}-1} \left( \frac{2\lambda^{\frac{1}{2}}}{\beta+2} \right) \right) \right]^{-\frac{1}{2}}$$

$$= \frac{(\beta+2)^{\frac{1}{2}-\frac{1}{2(\beta+2)}}}{\pi^{\frac{1}{2}} \lambda^{\frac{1}{4}-\frac{1}{4(\beta+2)}}} \left[ \frac{\sin \frac{\pi}{\beta+2} \Gamma\left(1-\frac{1}{\beta+2}\right)}{J_{\frac{1}{\beta+2}-1}\left( \frac{2\lambda^{\frac{1}{2}}}{\beta+2} \right)} \right]^{\frac{1}{2}} .$$