CHAPTER I

EIGENVALUE PROBLEM

In this chapter, we will show that there is a least positive characteristic value λ_0 of the system $f''(t) + \lambda P(t) f(t) = 0$, where P is a positive continuous function on $[0,1]$ and $f(0) = f'(1) = 0$. We will also show that the solution of the system with $\lambda = \lambda_0$ is nonvanishing in $0 < t \leq 1$. Moreover, we can prove that every non-trivial solution f_λ of the equation $f''(t) + \lambda P(t) f(t) = 0$ corresponding to any real value of $\lambda < \lambda_0$ and $f'_\lambda(1) = 0$ is nonvanishing in $0 \leq t \leq 1$. The wronskian of any two linearly independent solutions of the equation is constant. Finally, we will show that if P_n of $f''(t) + \lambda P_n(t) f(t) = 0$ decreases, then the first eigenvalue $\lambda_{0,n}$ increases.

A. Some Preliminary Results on the Sturm-Liouville System.

A normal first-order differential equation is the equation of the form $y' = F(x,y)$.

A function $F(x,y)$ satisfies a Lipschitz condition in a plane domain D if for some nonnegative constant L, the function satisfies the inequality $|F(x,y) - F(x,z)| \leq L|y-z|$ for all (x,y) and (x,z) in D.

<u>Lemma 1.1</u> If σ is a differentiable function satisfying the differential inequality

(1) $$\sigma'(x) \leq K\,\sigma(x) \qquad (a \leq x \leq b)$$

where K is a constant, then

$$\sigma(x) \leq \sigma(a)\,e^{K(x-a)} \qquad (a \leq x \leq b)$$

<u>Proof</u> : Multiplying both sides of (1) by e^{-Kx}, we obtain

$$e^{-Kx}\,\sigma'(x) \leq Ke^{-Kx}\,\sigma(x),$$

$$e^{-Kx}[\sigma'(x)-K\sigma(x)] \leq 0,$$

$$\frac{d}{dx}\{\sigma(x)e^{-Kx}\} \leq 0.$$

Thus, the function $\sigma(x)e^{-Kx}$ is nonincreasing for $a \leq x \leq b$.

Therefore $$\sigma(x)e^{-Kx} \leq \sigma(a)e^{-Ka},$$

$$\sigma(x) \leq \sigma(a)e^{K(x-a)}. \qquad \#$$

<u>Theorem 1.2</u> Let F satisfy a Lipschitz condition for $x \geq a$. If the function f satisfies the differential inequality $f'(x) \leq F(x, f(x))$ for $x \geq a$ and g is a solution of $y' = F(x,y)$ satisfying the initial condition $g(a) = f(a)$, then $f(x) \leq g(x)$ for $x \geq a$.

<u>Proof</u> : Suppose that $f(x_1) > g(x_1)$ for some $x_1 > a$. Let x_0 be the largest element in $[a, x_1]$ such that $f(x) \leq g(x)$. Then $f(x_0) = g(x_0)$.

Let $\sigma(x) = f(x) - g(x)$, we have

$$\sigma(x) \geq 0 \quad \text{for } x_0 \leq x \leq x_1 \quad \text{and}$$

$$\begin{aligned} \sigma'(x) &= f'(x) - g'(x) \\ &\leq F(x, f(x)) - F(x, g(x)) \\ &\leq L(f(x) - g(x)) \quad , L \text{ is the Lipschitz constant for } F \\ &= L\sigma(x) \quad , x_0 \leq x \leq x_1 . \end{aligned}$$

By Lemma 1.1, $\sigma(x) \leq \sigma(x_0)\, e^{L(x-x_0)} = 0$ for $x_0 \leq x \leq x_1$. So $\sigma(x) = 0$ for $x_0 \leq x \leq x_1$. This contradicts the hypothesis $f(x_1) > g(x_1)$. We conclude that $f(x) \leq g(x)$ for all $x \geq a$. #

<u>Theorem 1.3</u> (Comparison) Let f and g be solutions of the differential equations $y' = F(x,y)$ and $z' = G(x,z)$, respectively, where $F(x,y) \leq G(x,y)$ in the strip $a \leq x \leq b$ and F or G satisfies a Lipschitz condition. Let also $f(a) = g(a)$. Then $f(x) \leq g(x)$ for all $x \in [a,b]$.

<u>Proof</u> : We consider two cases.

<u>Case 1</u> : <u>G satisfies a Lipschitz condition</u>

Since $y' = F(x,y) \leq G(x,y)$, the functions f and g satisfy the condition of Theorem 1.2 with G in place of F. Therefore $f(x) \leq g(x)$ for $x \geq a$.

<u>Case 2</u> : <u>F satisfies a Lipschitz condition</u>

The function $u(x) = -f(x)$ and $v(x) = -g(x)$ satisfy the differential equations $u' = -F(x, -u)$ and $v' = -G(x, -v) \leq -F(x, -v)$, respectively. Thus, by Theorem 1.2, $v(x) \leq u(x)$ for $x \geq a$. This means that $f(x) \leq g(x)$ for $x \geq a$. #

<u>Corollary 1.4</u> Under the hypothesis of Theorem 1.3, if $f(a) < g(a)$ then $f(x) < g(x)$ for $x \geq a$.

<u>Proof</u> : Suppose that $f(x) \geq g(x)$ for some $x > a$, there would be a first point $x = x_1 > a$ such that $f(x_1) = g(x_1)$. The two functions $y = \emptyset(x) = f(-x)$ and $z = \psi(x) = g(-x)$ satisfy the differential equations $y' = -F(-x,y)$ and $z' = -G(-x,z)$, respectively, with the initial condition $\emptyset(-x_1) = \psi(-x_1)$. Since $-F(-x,y) \geq -G(-x,y)$ and $-F(-x,y)$ satisfies a Lipschitz condition, we can apply Theorem 1.3 in the interval $[-x_1, -a]$ and get $\emptyset(-a) \geq \psi(-a)$. This means that $f(a) \geq g(a)$ which is a contradiction. Thus $f(x) < g(x)$ for $x > a$. #

<u>Corollary 1.5</u> Under the hypothesis of Theorem 1.3 with $f \not\equiv g$, there is a point $x_0 \geq a$ such that $f(x) = g(x)$ for $a \leq x \leq x_0$ and $f(x) < g(x)$ for $x_0 < x \leq b$.

<u>Proof</u> : Let x_0 be the least upper bound of the set of points for which $f(x) = g(x)$, $a \leq x \leq b$. If x_1 is a point in $[a, x_0)$ such that $f(x_1) < g(x_1)$, then by Corollary 1.4, we have $f(x) < g(x)$ for $x > x_1$. Therefore $f(x_0) < g(x_0)$. This contradicts the properties of x_0. Hence $f(x) = g(x)$ for $a \leq x \leq x_0$. Since $f \not\equiv g$, by the properties of x_0 and Theorem 1.3, we have $f(x) < g(x)$ for $x_0 < x \leq b$. #

Corollary 1.6 Under the hypothesis of Theorem 1.3, if $F(x,y) < G(x,y)$ for all (x,y) in D, then $f(x) < g(x)$ for $x > a$.

Proof : Since $F(x,y) < G(x,y)$, $F(x,y) \leq G(x,y)$. By Corollary 1.5, we have a point $x_0 \geq a$ such that $f(x) = g(x)$, $a \leq x \leq x_0$ and $f(x) < g(x)$, $x_0 < x$. To show $x_0 = a$, suppose $x_0 > a$ and $f(x) = g(x)$, $a < x \leq x_0$. Then we have $f'(x) = F(x,f(x)) = G(x,g(x)) = g'(x)$, $a < x \leq x_0$. This contradicts the hypothesis $F(x,y) < G(x,y)$ for all (x,y) in D. Hence $x_0 = a$ and it follows that $f(x) < g(x)$, $x > a$.#

Let P and Q be continuous functions on an interval $[a,b]$. If x_0 is any point in this interval and α, β are any numbers, then by a well-known theorem of the initial value problem $\frac{d^2u}{dx^2} + P(x)\frac{du}{dx} + Q(x)\,u = 0$, $u(x_0) = \alpha$, $u'(x_0) = \beta$ has one and only one solution $u = u(x)$ on the interval $[a,b]$. Thus the following lemma is directly obtained:

Lemma 1.7 Any nontrivial solution of $\frac{d^2u}{dx^2} + P(x)\frac{du}{dx} + Q(x)\,u = 0$, $a \leq x \leq b$, which satisfies condition $u(x_0) = 0$ will have $u'(x_0) \neq 0$ for any $x_0 \in [a,b]$.

Next, we will prove Sturm Comparison Theorem. We consider the equation

$$(2) \qquad \frac{d}{dx}\left[P(x)\frac{du}{dx}\right] + Q(x)\,u = 0$$

where P and Q are positive continuous and P is differentiable in $[a,b]$. Let

$$(3) \qquad P(x)\, u'(x) = r(x) \cos \theta(x),$$

$$(4) \qquad u(x) = r(x) \sin \theta(x).$$

Differentiating (4) with respect to x, we have

$$u'(x) = r(x) \cos \theta(x) \frac{d\theta}{dx} + \sin \theta(x) \frac{dr}{dx}.$$

From (3), we obtain

$$(5) \qquad \frac{r(x) \cos \theta(x)}{P(x)} = r(x) \cos \theta(x) \frac{d\theta}{dx} + \sin \theta(x) \frac{dr}{dx}.$$

Differentiating $r(x) \cos \theta(x)$ with respect to x, we have

$$[r(x) \cos \theta(x)]' = -r(x) \sin \theta(x) \frac{d\theta}{dx} + \cos \theta(x) \frac{dr}{dx}.$$

From (2), (3) and (4), we obtain

$$(6) \qquad -Q(x) r(x) \sin \theta(x) = -r(x) \sin \theta(x) \frac{d\theta}{dx} + \cos \theta(x) \frac{dr}{dx}$$

Multiplying the equation (5) by $\cos \theta(x)$, the equation (6) by $\sin \theta(x)$ and subtracting, we get

$$(7) \qquad \frac{d\theta}{dx} = \frac{1}{P(x)} \cos^2 \theta(x) + Q(x) \sin^2 \theta(x).$$

Multiplying the equation (5) by $\sin \theta(x)$, the equation (6) by $\cos \theta(x)$ and adding, we get

$$(8) \qquad \frac{dr}{dx} = \left[\frac{1}{P(x)} - Q(x)\right] r(x) \sin \theta(x) \cos \theta(x).$$

To each solution u of (2), there correspond the solutions θ and r of (7) and (8) where $r^2 = u^2 + P^2 u'^2$, $\theta = \tan^{-1}(\frac{u}{Pu'})$. Since u and u' do not vanish simultaneously, it follows from Lemma 1.7 that $r^2(x) > 0$ on $[a,b]$ and thus it can be assumed that $r(x) > 0$. A consequence of this assumption is that

$u(x) = r(x) \sin \theta(x)$ can vanish only where $\theta(x) \equiv 0 \pmod{\pi}$.

<u>Theorem 1.8</u> (Sturm Comparison Theorem) Let $P_1(x) \geq P_2(x) > 0$ and $Q_2(x) \geq Q_1(x) > 0$ and let u_1, u_2 be nontrivial solutions of

$$(9) \qquad \frac{d}{dx}\left(P_1(x)\frac{du_1}{dx}\right) + Q_1(x)u_1 = 0 ,$$

$$(10) \qquad \frac{d}{dx}\left(P_2(x)\frac{du_2}{dx}\right) + Q_2(x)u_2 = 0 , \quad \text{respectively.}$$

Then, between any two zeros of u_1 there lies at least one zero of u_2.

<u>Proof</u> : First, we change (9) and (10) into the form

$$(11) \qquad \frac{d\theta}{dx} = \frac{1}{P_1(x)}\cos^2\theta(x) + Q_1(x)\sin^2\theta(x) = F_1(x,\theta)$$

and

$$(12) \qquad \frac{d\theta}{dx} = \frac{1}{P_2(x)}\cos^2\theta(x) + Q_2(x)\sin^2\theta(x) = F_2(x,\theta),$$

respectively.

Since $P_1(x) \geq P_2(x) > 0$ and $Q_2(x) \geq Q_1(x) > 0$ for all x in the interval $[a,b]$, $F_1(x,\theta) \leq F_2(x,\theta)$ in the strip $a \leq x \leq b$. Let $\theta_1(x)$ and $\theta_2(x)$ be solution of (11) and (12), respectively. Let x_1, x_2 be two consecutive zeros of u_1 where $x_1 < x_2$, then the curve $\theta_1(x)$ of $u_1(x)$ intersects the line $\theta = k\pi$ at $x = x_1$ and the line $\theta = (k+1)\pi$ at $x = x_2$ (because $\dfrac{d\theta_1}{dx} = \dfrac{1}{P_1(x)} > 0$ where $\theta_1(x) \equiv 0 \pmod{\pi}$, so θ_1 is an increasing function). We can assume without loss of generality that $k = 0$, that is, $\theta_1(x_1) = 0$, $\theta_1(x_2) = \pi$ and we can also assume that $0 \leq \theta_2(x_1) < \pi$ by a proper choice of n in $\theta_2(x) + n\pi$. By Corollary 1.4 or 1.5, we have $\theta_1(x) < \theta_2(x)$ for $x > x_1$. This implies that $\theta_2(\bar{x}) = \pi$ for some $\bar{x}$ in (x_1, x_2). Hence $u_2(\bar{x}) = 0$, and the theorem is proved. #

Now, we consider the Sturm-Liouville System.

A Sturm-Liouville equation is a second order homogeneous linear differential equation of the form

$$(13) \qquad \frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + \left[\lambda\rho(x) - q(x)\right] u = 0 ,$$

where λ is a real number ; p, ρ, q are real-valued continuous functions of x; the function p and ρ are positive and p is continuous differentiable, $a \leq x \leq b$.

A Sturm-Liouville system is a S-L equation together with two seperated endpoint conditions of the form

$$(14) \qquad \begin{cases} \alpha u(a) + \alpha' u'(a) = 0 \\ \beta u(b) + \beta' u'(b) = 0 \end{cases}$$

where α, α', β, β' are real numbers such that $\alpha^2 + \alpha'^2 \neq 0$, $\beta^2 + \beta'^2 \neq 0$.

A nontrivial solution of S-L equation is called an eigenfunction and the corresponding value λ is called its eigenvalue.

In (13) if we let $P(x) = p(x)$ and $Q(x) = \lambda\rho(x) - q(x)$, we obtain (2). Since $u = 0$ iff $\sin\theta = 0$ where θ is the solution of the equation

$$(15) \qquad \frac{d\theta}{dx} = \frac{1}{p(x)}\cos^2\theta + \left[\lambda\rho(x) - q(x)\right]\sin^2\theta \qquad (a \leq x \leq b),$$

the zerosof any solution of (13) are the points where $\theta \equiv 0 (\mathrm{mod}\ \pi)$. Given that $\theta(a,\lambda) = \gamma$ for all λ, where γ is defined by the condition $\tan \gamma = \dfrac{u(a)}{p(a)u'(a)} = \dfrac{-\alpha'}{p(a)\alpha}$, $0 \leqslant \gamma < \pi$. Thus for fixed value γ, $\theta(x, \lambda)$ is the solution of (15) which satisfies the initial condition $\theta(a, \lambda) = \gamma$ for $a \leqslant x \leqslant b$, $-\infty < \lambda < \infty$.

Theorem 1.9. Any S-L system has an infinite sequence of eigenvalues $\lambda_0 < \lambda_1 < \dots$ with $\lim_{n\to\infty} \lambda_n = \infty$. The eigenfunction u_n belonging to the eigenvalue λ_n has exactly n zeros in the interval $a < x < b$.

To prove this theorem we need the following lemmas :

Lemma 1.10 For a fixed point x such that $x > a$, $\theta(x, \lambda)$ is a strictly increasing function of the variable λ.

Proof : Let $\lambda_1 < \lambda_2$. Let $\theta(x, \lambda_1)$ be the solution of

$\dfrac{d\theta}{dx} = \dfrac{1}{p(x)} \cos^2\theta(x) + (\lambda_1\rho(x) - q(x)) \sin^2\theta(x)$ and $\theta(x, \lambda_2)$ be the solution of $\dfrac{d\theta}{dx} = \dfrac{1}{p(x)} \cos^2\theta(x) + (\lambda_2\rho(x) - q(x)) \sin^2\theta(x)$.

By Corollary 1.6, for a fixed point x such that $x > a$, $\theta(x, \lambda_1) < \theta(x, \lambda_2)$. #

Lemma 1.11 Suppose that for some $x_n > a$, $\theta(x_n, \lambda) = n\pi$ where n is nonnegative integer. Then $\theta(x, \lambda) > n\pi$ for all $x > x_n$.

Proof : If x_n is any point where $\theta(x_n,\lambda) = n\pi$, then by (15) we have $\frac{d}{dx_n}\theta(x_n,\lambda) = \frac{1}{p(x_n)} > 0$. Thus the function $\theta = \theta(x_n,\lambda)$, considered as a function of x_n, is increasing where it crosses the line $\theta = n\pi$. Hence $\theta(x,\lambda) > \theta(x_n,\lambda) = n\pi$ for $x > x_n$. #

Remark. Lemma 1.11 combined with the condition $0 \leq \theta(a,\lambda) < \pi$, makes the first zero of u in the open interval $a < x < b$ occur where $\theta = \pi$, and the n-th zero where $\theta = n\pi$.

Proof : We consider two cases :

Case 1 : $\theta(a,\lambda) = 0$. By Lemma 1.11, $\theta(x,\lambda) > \theta(a,\lambda) = 0$ for $x > a$. Since u has zero when $\theta \equiv 0 \pmod{\pi}$, the first zero of u in (a,b) occurs where $\theta = \pi$, and it follows that the n-th zero occurs where $\theta = n\pi$.

Case 2 : $0 < \theta(a,\lambda) < \pi$. Since u has a zero when $\theta \equiv 0 \pmod{\pi}$, the first zero of u in (a,b) occurs where $\theta = 0$ or $\theta = \pi$. If the first zero of u occurs where $\theta = 0$, then $\theta(x_1,\lambda) = 0$ for some $x_1 \in (a,b)$. Hence by following the same argument as in Lemma 1.11, we have $\theta(x,\lambda) < 0$ for all $x < x_1$ and in particular, $\theta(a,\lambda) < 0$. This contradicts the hypothesis $\theta(a,\lambda) > 0$. Therefore the first zero of u in (a,b) occurs where $\theta = \pi$ and by Lemma 1.11, it follows that the n-th zero occurs where $\theta = n\pi$. #

Let $x_n(\lambda)$ be the smallest x such that $\theta(x,\lambda) = n\pi$. Then the following lemma shows that $x_n(\lambda)$ exists for large λ.

Lemma 1.12 For a given fixed positive integer n and a sufficiently large λ, the function $\lambda \mapsto x_n(\lambda)$ is defined and continuous. It is a decreasing function of λ and $\lim_{\lambda\to\infty} x_n(\lambda) = a$.

Proof : Let $q_M = \max\{q(x) : a \leqslant x \leqslant b\}$, $p_M = \max\{p(x) : a \leqslant x \leqslant b\}$ and $\rho_m = \min\{\rho(x) : a \leqslant x \leqslant b\}$. A solution of the differential equation $p_M u'' + (\lambda\rho_m - q_M)\, u = 0$ where $\lambda > q_M/\rho_m$, is a function $u(x) = \sin kx$ where $k^2 = (\lambda\rho_m - q_M)/p_M$. This solution together with the condition $u(a)/p(a)u'(a) = \tan\gamma$ will give $a + (n\pi-\gamma)/k$, $n = 0,1,2,\ldots$ as the zeros. These zeros are spaced at a distance $\pi\sqrt{p_M/(\lambda\rho_m - q_M)}$ apart. Hence we can choose λ large enough so that u_1 has n+1 zeros in (a,b). By Theorem 1.8, any nontrivial solution u of the S-L equation must has at least one zero between any two zeros of u_1. It follows that u has at least n zeros in (a,b) and $\theta(x,\lambda)$ take the value $n\pi$ for some x. Hence $x_n(\lambda)$ is defined. Since $\theta(x,\lambda)$ is a continuous function of x and λ, we have $x_n(\lambda)$ is a continuous function of λ.

Next, we show that x_n is a decreasing function of λ. Since $\theta(x,\lambda)$ is an increasing function of λ, it follows that the zeros of u, if any, move to the left towards x = a as λ increases. So $x_n(\lambda)$ is a decreasing function of λ. But the number $x_n(\lambda)$ falls between the (n-1)th and the n-th zero of u_1 and both zeros tend to a as λ tends to ∞. Therefore $x_n(\lambda)$ tends to a as λ tend to ∞. #

Lemma 1.13 For any x such that $x > a$, $\lim_{\lambda\to\infty} \theta(x,\lambda) = \infty$ and $\lim_{\lambda\to-\infty} \theta(x,\lambda) = 0$.

Proof : Let x_1 be a fixed point in (a,b). Let $\varepsilon > 0$ and let n be a positive integer such that $n \geqslant \varepsilon$. By Lemma 1.12, there is a positive number N such that for $\lambda > N$, $\theta(x_n,\lambda) = n\pi$ for some $x_n \in (a,x_1)$. In addition, by Lemma 1.11, $\theta(x_1,\lambda) > \theta(x_n,\lambda) = n\pi \geqslant \varepsilon$. Since ε and x_1 are arbitrary, $\lim_{\lambda\to\infty} \theta(x,\lambda) = \infty$.

Next, we show that $\lim_{\lambda\to-\infty} \theta(x,\lambda) = 0$. Given $\varepsilon > 0$ and this ε is small enough so that $\gamma < \pi - \varepsilon$. If $\varepsilon \leqslant \theta \leqslant \pi-\varepsilon$ and $\lambda < 0$, then $\sin^2\varepsilon < \sin^2\theta$ and $\lambda = -|\lambda|$. Let $\rho_m = \min \{\rho(x) : a \leqslant x \leqslant b\}$, $Q_M = \max \{|q(x)| : a \leqslant x \leqslant b\}$, $p_m = \min \{p(x) : a \leqslant x \leqslant b\}$. Then ρ_m and p_m are positive and

$$\begin{aligned}
\frac{d}{dx}\theta(x,\lambda) &= \frac{1}{p(x)}\cos^2\theta + [\lambda\rho(x) - q(x)]\sin^2\theta \\
&< \frac{1}{p_m}\cos^2\theta - |\lambda|\,\rho(x)\sin^2\theta + |q(x)|\sin^2\theta \\
&< \frac{1}{p_m} - |\lambda|\,\rho_m \sin^2\varepsilon + Q_M .
\end{aligned}$$

We have that the slope of the segment in the $x\theta$ plane joining the points $(a, \pi-\varepsilon)$ and (x_1,ε) where $a < x_1 \leqslant b$ equals $\frac{2\varepsilon-\pi}{x_1-a}$. Then for a point (x,θ) on this segment, let

$$N = \max\left\{\left(\frac{1}{p_m} + Q_M - \frac{2\varepsilon-\pi}{x_1-a}\right)\frac{1}{\rho_m \sin^2\varepsilon}, \varepsilon\right\}.$$ Thus $N > 0$. Hence

for $\lambda < -N$, we have

$$\frac{d}{dx}\theta(x,\lambda) < \frac{1}{P_m} - \left(\frac{1}{P_m} + Q_M - \frac{2\varepsilon-\pi}{x_1-a}\right) + Q_M$$

$$= \frac{2\varepsilon-\pi}{x_1-a} \qquad \text{for } \varepsilon \leqslant \theta \leqslant \pi-\varepsilon \text{ , } a \leqslant x \leqslant x_1.$$

Claim $\theta(x,\lambda)$ lies below the segment for $a \leqslant x \leqslant x_1$. Suppose on the contrary that there is a first point x_0 in $(a,x_1]$ such that $\theta(x_0,\lambda)$ lies on this segment. Since we consider the curve $\theta(x,\lambda)$ on $[\varepsilon,\pi-\varepsilon]$, then, if the curve $\theta(x,\lambda)$ does not lie below the line $\theta = \varepsilon$ for all $x \in [a,x_0]$, we have $\dfrac{\theta(x_0,\lambda) - \theta(a,\lambda)}{x_0 - a} > \dfrac{2\varepsilon-\pi}{x_1-a}$.

By Mean Value Theorem, there exists a point $\bar{x} \in (a,x_0)$ such that $\theta'(\bar{x},\lambda) = \dfrac{\theta(x_0,\lambda) - \theta(a,\lambda)}{x_0 - a} > \dfrac{2\varepsilon-\pi}{x_1-a}$. This contradicts $\theta'(x,\lambda) < \dfrac{2\varepsilon-\pi}{x_1-a}$ for $x \in [a,x_1]$. If the curve $\theta(x,\lambda)$ lies below the line $\theta = \varepsilon$ for some $x \in [a, x_0]$, define x_2 be the last point in $[a,x_0]$ such that $\theta(x_2,\lambda) = \varepsilon$, then $\dfrac{\theta(x_0,\lambda) - \theta(x_2,\lambda)}{x_0 - x_2} > \dfrac{2\varepsilon-\pi}{x_1-a}$.

By Mean Value Theorem there is a point $\bar{x} \in (x_2,x_0)$ such that $\theta'(\bar{x},\lambda) > \dfrac{2\varepsilon-\pi}{x_1-a}$. This is a contradiction. Hence $\theta(x,\lambda)$ lies below the segment for $a \leqslant x \leqslant x_1$, in particular, $\theta(x_1,\lambda) < \varepsilon$ for $\lambda < -N$. Since $\theta(x_1,\lambda) > 0$ (by Remark), it follows that $|\theta(x,\lambda)| < \varepsilon$, and since ε and x_1 are arbitrary, $\lim_{\lambda\to-\infty} \theta(x,\lambda) = 0$. #

Proof of Theorem 1.9

First, we transform (14) into equivalent endpoint conditions for the function $\theta(x,\lambda)$ of the system

$$\frac{d\theta}{dx} = \frac{1}{p(x)} \cos^2\theta(x) + [\lambda\rho(x)-q(x)] \sin^2\theta(x),$$

$$\frac{dr}{dx} = [\frac{1}{p(x)} - \lambda\rho(x) + q(x)] \, r(x)\sin\theta(x)\cos\theta(x).$$

If $\alpha \neq 0$, then the function $\theta(x,\lambda)$ must satisfy the initial condition $\theta(a,\lambda) = \gamma$ where γ is the smallest number such that $0 \leq \gamma < \pi$, $\gamma \neq \frac{\pi}{2}$ and that $p(a) \tan\gamma = -\frac{\alpha'}{\alpha}$. When $\alpha = 0$, we choose $\gamma = \frac{\pi}{2}$. Similarly, we choose $0 < \delta \leq \pi$ so that $\tan\delta = \frac{-\beta'}{\beta p(b)}$.

A solution u of the equation (13) for $a \leq x \leq b$ is an eigen function of the S-L system iff for the corresponding phase function defined by $r^2 = u^2 + p^2u'^2$ and $\theta = \tan^{-1}(\frac{u}{pu'})$, $\theta(a,\lambda) = \gamma$ and $\theta(b,\lambda) = \delta+n\pi$ where $n = 0,1,2,\ldots$, $0 \leq \gamma < \pi$, $0 < \delta \leq \pi$. Any value of λ which satisfies these conditions is an eigenvalue of the S-L system.

Let $\theta(x,\lambda)$ be the solution of $\frac{d\theta}{dx} = \frac{1}{p(x)} \cos^2\theta(x)+[\lambda\rho(x) - q(x)] \sin^2\theta(x)$ for the initial condition $\theta(a,\lambda) = \gamma$. A solution $\theta(x,\lambda)$ is unique for each value of λ and by Lemma 1.10 and Lemma 1.13 with $x = b$, $\theta(x,\lambda)$ also satisfies the second condition $\theta(b,\lambda) = \delta+n\pi$ where n is nonnegative integer. Thus as λ increases from $-\infty$, there is an infinite sequence λ_n such that $\lambda_0 < \lambda_1 < \ldots$ and $\theta(b, \lambda_n) = \delta+n\pi$. Each of these values gives an eigenfunction $u_n(x) = r(x)\sin\theta(x, \lambda_n)$ of the S-L system. Since $\theta(a,\lambda_n) = \gamma$ and $\theta(b,\lambda_n) = \delta+n\pi$ and $\theta(x,\lambda_n)$

is a continuous function of x, it must take on each of the value π, $2\pi, \ldots, n\pi$ at least once on the interval (a,b). From Lemma 1.11, we note that none of these values can be taken on more than once. and since each corresponds to a zero of u_n, we have that the eigenfunction has exactly n zeros on (a,b).

To complete the proof we must show that $\lim_{k\to\infty} \lambda_k = \infty$. Suppose $\lim_{k\to\infty} \lambda_k = M$, $M < \infty$. Therefore, for given $\varepsilon > 0$ we can find a positive number N_ε such that for $n, m > N_\varepsilon$, $|\lambda_n - \lambda_m| < \varepsilon$. Since θ is a continuous function of λ, for any $\varepsilon_1 > 0$, there exists $\delta_{\varepsilon_1} > 0$ such that for $|\lambda_n - \lambda_m| < \delta_{\varepsilon_1}$, we have $|\theta(b, \lambda_n) - \theta(b, \lambda_m)| < \varepsilon_1$. Take $\varepsilon_1 < \pi$ and choose $\varepsilon = \delta_{\varepsilon_1}$, thus $|\theta(b, \lambda_n) - \theta(b, \lambda_m)| < \pi$. This contradicts $|\theta(b, \lambda_n) - \theta(b, \lambda_m)| \geq \pi$ since $\theta(b, \lambda_n) = \delta + n\pi$ and $\theta(b, \lambda_m) = \delta + m\pi$. Hence $\lim_{k\to\infty} \lambda_k = \infty$. #

<u>Theorem 1.14</u> If $q(x) < 0$ and u is a nontrivial solution of $u''(x) + q(x)u(x) = 0$, then u has at most one zero.

<u>Proof</u> : Let x_0 be a zero of u, so $u(x_0) = 0$. Since u is nontrivial, $u'(x_0) \not\equiv 0$ (Lemma 1.7)

<u>case 1 : $u'(x_0) > 0$</u>. Then u is a positive over some interval to the right of x_0. Since $u''(x) = -q(x)u(x)$ and $q(x)$ is negative, we have $u''(x) > 0$ on the same interval. It follows that u' is an increasing function to the right of x_0. To this end, we show that u has no zero to the right of x_0. Suppose on the contrary that

there is a point x_1 to the right of x_0 such that $u(x_1) = 0$, we may take x_1 to be the consecutive zero. Then by Rolle's Theorem, there is a point $\bar{x} \in (x_0, x_1)$ such that $u'(\bar{x}) = 0$. This contradicts the fact that u' is increasing on (x_0, x_1) which follows from the given equation that u'' is positive on (x_0, x_1). In similar way there is no zero to the left of x_0.

case 2 : $u'(x_0) < 0$. By similar argument as in the first case, we have that u has no zero to the right and to the left of x_0. Thus u has either no zero at all or only one. #

B. Properties of the Special Type of S-L System.

From Lemma 1.7, Theorem 1.9 and Theorem 1.14, we have Theorem 1.15 as follows.

Theorem 1.15. There is a least characteristic value λ_0 of the system

(16) $f''(t) + \lambda P(t)f(t) = 0$, P is positive on $[0, 1]$,

(17) $f(0) = f'(1) = 0$

and λ_0 is positive. The solution f of the system with $\lambda = \lambda_0$ is nonvanishing in $0 < t \leq 1$.

Theorem 1.16. Every nontrivial solution f_λ of (16) corresponding to any real value of $\lambda < \lambda_0$ and satisfying the single boundary condition $f'_\lambda(1) = 0$ is nonvanishing in $0 \leq t \leq 1$.

Proof : Let λ_0 be the least characteristic value of the system [(16),(17)] and let f be a solution of the system with $\lambda = \lambda_0$. For any $\lambda < \lambda_0$, $0 \leq t \leq 1$, f_λ is a nontrivial solution of (16) satisfying $f'_\lambda(1) = 0$. Thus

$$(18) \qquad f''(t) + \lambda_0 P(t)\, f(t) = 0 ,$$

$$(19) \qquad f''_\lambda(t) + \lambda P(t)\, f_\lambda(t) = 0 .$$

Multiplying the equation (18) by $f_\lambda(t)$, the equation (19) by $f(t)$ and subtracting, we get

$$f''(t)\, f_\lambda(t) - f''_\lambda(t) f(t) = (\lambda-\lambda_0) P(t) f(t) f_\lambda(t) ,$$

$$\frac{d}{dt}\left[f'(t) f_\lambda(t) - f'_\lambda(t) f(t)\right] = (\lambda-\lambda_0) P(t) f(t) f_\lambda(t) ,$$

$$\left[f'(s) f_\lambda(s) - f'_\lambda(s) f(s)\right]_t^1 = (\lambda-\lambda_0) \int_t^1 P(s) f(s) f_\lambda(s) ds,$$

where $0 \leq t \leq 1$. Since $f'(1) = f'_\lambda(1) = 0$,

$$(20) \qquad - f'(t) f_\lambda(t) + f'_\lambda(t) f(t) = (\lambda-\lambda_0) \int_t^1 P(s) f(s) f_\lambda(s) ds.$$

Because $f(0) = 0$, at $t = 0$ we have $f_\lambda(t) \neq 0$. If $f_\lambda(0) = 0$, then λ is the characteristic value of the system [(16,(17)] which is less than the least characteristic value λ_0, a contradiction. Also at $t = 1$ we have $f_\lambda(t) \neq 0$. If $f_\lambda(1) = 0$, then $f_\lambda(t)$ is trivial solution by Lemma 1.7. Now suppose there exists $t_0 \in (0,1)$ such that t_0 is the greastest zero, that is, $f_\lambda(t_0) = 0$. With this value of t_0, the relation (20) becomes

$$(21) \qquad f(t_0) f'_\lambda(t_0) = (\lambda-\lambda_0) \int_{t_0}^1 P(s) f(s)\, f_\lambda(s) ds .$$

Without loss nf of generality we assume

(22) $f_\lambda(t) > 0 \qquad (t_0 < t \leq 1)$

and $f(t) > 0 \qquad (0 < t \leq 1)$ (Theorem 1.15).

And since $\lambda < \lambda_0$, (21) yields that $f'_\lambda(t_0)$ is negative. But $f_\lambda(t_0) = 0$, so $f_\lambda(t) < 0$ $(t_0 < t \leq 1)$. This contradicts (22). Hence there is no greatest zero of f_λ in $(0,1)$. This yields Theorem 1.16. #

Let f_1 and f_2 be any two linearly independent solutions of the second order differential equation, the wronskian of f_1 and f_2 is $f_1(t)\, f'_2(t) - f_2(t) f'_1(t)$ and is denoted by $W(f_1,f_2)$.

Theorem 1.17 If $f''(t) + \lambda P(t)f(t) = 0$ and f_1, f_2 are linearly independent solutions, then its wronskian is constant.

Proof : We consider

(23) $f''_1(t) + \lambda P(t)\, f_1(t) = 0$ and

(24) $f''_2(t) + \lambda P(t)\, f_2(t) = 0$.

Multiplying (23) by $f_2(t)$ and (24) by $f_1(t)$ and subtracting, we have

$$f_1(t)f''_2(t) - f_2(t)f''_1(t) = 0$$

$$\frac{d}{dt}\left(f_1(t)f'_2(t) - f_2(t)f'_1(t)\right) = 0 .$$

Hence $f_1(t)f'_2(t) - f_2(t)f'_1(t) = k$, where k is constant. Therefore $W(f_1, f_2)$ is constant. #

C. Variational Behaviour of Eigenvalues

Theorem 1.18 Let

(25) $$f''(t) + \lambda P_1(t) f(t) = 0$$ and

(26) $$f''(t) + \lambda P_2(t) f(t) = 0$$

satisfy the conditions $f(0) = f'(1) = 0$ where P_1, P_2 are positive continuous functions on $[0,1]$. If $P_1(t) \geq P_2(t)$. Then the first eigenvalue $\lambda_{0,1}$ of (25) is less than or equal to the first eigenvalue $\lambda_{0,2}$ of (26).

Before proving this theorem, we will show the following theorems:

Theorem 1.19. The differential equation $f''(t) + \lambda P(t)f(t) = 0$ with $f(0) = f'(1) = 0$ must be satisfied by any extremizing function

(27) $$I = \int_0^1 f'^2(t)\, dt$$

with respect to continuously differentiable functions f which satisfy the normalization condition

(28) $$\int_0^1 P(t)\, f^2(t)\, dt = 1 .$$

Proof : We consider the problem of extremizing the quantity

(29) $$I = \int_0^1 \phi'^2(t)dt + a_1[\phi(0)]^2 + a_2[\phi(1)]^2$$

with respect to continuously differentiable functions ϕ which satisfy the normalization condition

(30) $$\int_0^1 P(t)\phi^2(t)\, dt = 1$$

where P is a positive continuous function and a_1, a_2 are nonnegative constants.

Let $a = a(t)$ be a continuous differentiable function with $a(0) = -a_1$, $a(1) = a_2$. Thus (29) can be written as

$$I = \int_0^1 [\emptyset'^2(t) + \frac{d}{dt}(a(t)\emptyset^2(t))]dt. \tag{31}$$

Using the method of isoperimetric problem ([8], Chapter 4), we form, from the integrand of (30) and (31),

$$\emptyset^* = \emptyset'^2 + \frac{d}{dt}(a\emptyset^2) - \lambda P\emptyset^2$$

$$= \emptyset'^2 + 2a\emptyset\emptyset' + \emptyset^2 \frac{da}{dt} - \lambda P\emptyset^2 ,$$

where $-\lambda$ is an undetermined multiplier, and it follows that

$$\frac{\partial\emptyset^*}{\partial\emptyset} - \frac{d}{dt}\left(\frac{\partial\emptyset^*}{\partial\emptyset'}\right) = 0 ,$$

$$2a\emptyset' + 2\emptyset\frac{da}{dt} - 2\lambda P\emptyset - \frac{d}{dt}(2\emptyset' + 2a\emptyset) = 0 ,$$

$$\emptyset'' + \lambda P\emptyset = 0 .$$

That is $\emptyset''(t) + \lambda P(t)\emptyset(t) = 0$ must be satisfied by any extremizing function for this problem.

In the free-end-point problem, we get $\frac{\partial\emptyset^*}{\partial\emptyset'} = 0$ at $t = 0$ and $t = 1$. This yields $\emptyset'(t) + a(t)\emptyset(t) = 0$ at $t = 0$ and $t = 1$. Since $a(0) = -a_1$, $a(1) = a_2$, we obtain

$$\begin{aligned} \emptyset'(0) - a_1\emptyset(0) &= 0 , \\ \emptyset'(1) + a_2\emptyset(1) &= 0 . \end{aligned} \tag{32}$$

In a fixed-end-point problem, we replace (32) by the condition

(33) $$\phi(0) = \phi(1) = 0$$

In a free-fixed problem, we must have one condition from each of (32) and (33). Hence the differential equation $f''(t) + \lambda P(t)f(t) = 0$ with the condition $f(0) = f'(1) = 0$ is a free-fixed problem with $a_2 = 0$, and this system is satisfied by an extremizing function $I = \int_0^1 f'^2(t)dt$ with respect to continuously differentiable functions f which satisfy $\int_0^1 P(t)f^2(t)dt = 1$. #

Note that this system is linear and homogeneous. Since $P(t) > 0$ for $0 \leq t \leq 1$, any $f(\not\equiv 0)$ may therefore be supposed, when necessary, to satisfy the normalization condition (28). If $\int_0^1 P(t)f^2(t)dt = c^2$ we replace f by f/c then we have (28).

Theorem 1.20. The first eigenvalue λ_0 is the minimum of the integral (27) with respect to those functions f which satisfy the normalization condition (28). Let f_m (m = 0,1,...) be the eigenfunction which satisfies $f_m''(t) + \lambda_m P(t)f_m(t) = 0$ in $[0,1]$ and $f_m(0) = f_m'(1) = 0$, then the minimum of I under $f(0) = f'(1) = 0$ is achieved when $f = f_0$.

Proof : We will show this theorem by using expansion Theorem. ([6],p.427) We expand the arbitrary function f eligible for the minimization of (27) in accordance with

(34)
$$\begin{cases} f(t) = \sum_{m=0}^{\infty} c_m f_m(t) \\ f'(t) = \sum_{m=0}^{\infty} c_m f'_m(t) \end{cases}$$

where $c_m = \int_0^1 P(t)\, f_m(t)\, f(t)\, dt.$

Subtracting the first equation of (34) for one factor of (28), we obtain

$$\int_0^1 P(t)f^2(t)dt = \sum_{m=0}^{\infty} c_m \int_0^1 P(t)\, f_m(t)f(t)dt = \sum_{m=0}^{\infty} c_m^2 = 1,$$

where the interchange of summation and integration is justified by the uniform convergence of the series expansions. Subtracting the second equation of (34) for one factor of (27), we obtain

$$I = \sum_{m=0}^{\infty} c_m \int_0^1 f'_m(t)f'(t)dt = \sum_{m=0}^{\infty} c_m \{[f'_m(t)f(t)]_0^1 - \int_0^1 f''_m(t)f(t)dt\}.$$

Since $f'_m(1) = 0$ and $f(0) = \sum_{m=0}^{\infty} c_m f_m(0) = 0$ where $m = 0,1,2,\ldots,$

$$I = -\sum_{m=0}^{\infty} c_m \int_0^1 f''_m(t)f(t)dt$$

$$= \sum_{m=0}^{\infty} c_m \lambda_m \int_0^1 P(t)f_m(t)f(t)dt$$

$$= \sum_{m=0}^{\infty} \lambda_m c_m^2$$

$$= \lambda_0 + \sum_{m=0}^{\infty} (\lambda_m - \lambda_0)c_m^2 .$$

Since $\lambda_m > \lambda_0$ if $m > 0$, it follows that $I \geq \lambda_0$. The equality sign holds if $c_0 = 1$ and $c_1 = c_2 = \ldots = 0$. This means that if $f = f_0$, then $I = \lambda_0$. #

<u>Proof of Theorem 1.18</u>

First, we transform (25) and (26) into an extremizing function $I^{(i)}$ for $P = P_i$, $i = 1,2$ (Use Theorem 1.19). Let $K^{(1)}$, $K^{(2)}$ be the classes of functions eligible for the minimization of $I^{(1)}$ and $I^{(2)}$, respectively, that is, members of $K^{(1)}$ and $K^{(2)}$ satisfy the same conditions. Since $P_1 \geq P_2$ in $[0,1]$, we have

$$(35) \qquad c^2 = \int_0^1 P_1(t)\, f^2_{(2)}(t)dt \geq \int_0^1 P_2(t) f^2_{(2)}(t)dt = 1,$$

where $f_{(2)}$ are any member of $K^{(2)}$ and c is a positive constant (ingeneral c is different for different member of $K^{(2)}$) defined by the left hand equation of (35). It is obvious that for any function $f_{(2)}$ in $K^{(2)}$, there is a corresponding member $f_{(2)}/c$ in $K^{(1)}$ with $c \geq 1$. Therefore, if any member $f_{(2)}$ in $K^{(2)}$ renders I of (27) equal to $I^{(2)}$, there exists a member $f_{(2)}/c$ in $K^{(1)}$ renders I equal to $I^{(1)}$ where $I^{(1)}(f_{(2)}/c) = \frac{1}{c^2} I^{(2)}(f_{(2)}) \leq I^{(2)}(f_{(2)})$. From this we have that the minimum of I with respect to $K^{(1)}$ is less than or equal to its minimum with respect to $K^{(2)}$. By Theorem 1.20, it follows that $\lambda_{0,1} \leq \lambda_{0,2}$. #