CHAPTER VI

THE FUNCTIONAL EQUATION $f(x \circ f(y)) = f(x) * f(y)$.

The materials of this chapter are drawn from references [2], [3], and [4].

In this chapter, we shall be concerned with the following functional equations :

(6.1) $$f(x \cdot f(y)) = f(x) \cdot f(y)$$

(6.2) $$f(x + f(y)) = f(x) + f(y)$$

(6.3) $$f(x + f(y)) = f(x) \cdot f(y)$$

(6.4) $$f(x \cdot f(y)) = f(x) + f(y) .$$

Of course, the domain and range of f need to be chosen in accordance with the specific equation. Moreover, these equations have been treated in references [3] and [4].

<u>Eq (6.1) and Eq (6.2)</u> :

Suppose f maps a group $(G_o, *)$ with zero into itself and satisfies the equation

(6.1)′ $$f(x * f(y)) = f(x) * f(y).$$

<u>Proposition 6.1</u>. If f is constant, then $f \equiv 0$ or 1.

<u>Proof</u>. Suppose $f \equiv c$. Then

$$c = c * c$$

Therefore, $c = 0$ or 1.

Hence the proposition is proved /

Proposition 6.2. $f(G_o)^* = f(G_o) \setminus \{0\}$ is a subgroup of $(G, *)$. providing $f \not\equiv 0$.

Proof. Since $f \not\equiv 0$ and since

$$f(x) * f(y)^{-1} = f(x * f(y)^{-1})$$

for $f(y) \neq 0$, $f(G_o) \setminus \{0\}$ is a subgroup of G /

Theorem 6.3. A function f mapping a group $(G_o, *)$ with zero into itself satisfies Eq (6.1)$'$ if and only if either $f \equiv 0$ or else there exists a subgroup M of G and a mapping

$$\tilde{f} : S \cup \{0\} \longrightarrow M \cup \{0\}$$

such that

$$\tilde{f}(0) = 0 \quad \text{if} \quad f \not\equiv 1 \quad \text{and}$$

$$f(x) = \tilde{f}(s_x) * m_x \tag{6.5}$$

where each $x \in G$ can be written uniquely as

$$x = s_x * m_x \quad (s_x \in S,\ m_x \in M)$$

and S is a set of representatives in G of the left cosets of G by M.

Proof. (Necessity). Suppose $f \not\equiv 0$ satisfies Eq (6.1)$'$. Then $M = f(G_o) \setminus \{0\}$ is a subgroup of G by Proposition 6.2. Let S be a set of representatives in G of the left cosets of G by M. For each $x \in G$,

$$x = s_x * m_x$$

for a unique $s_x \in S$ and a unique $m_x \in M$. Then

$$f(x) = f(s_x) * m_x$$

by Eq $(6.1)'$. Let

$$\tilde{f}(s) = \begin{cases} f(s) & \text{if } s \in S \\ 0 & \text{if } s = 0 \text{ (provided } f \not\equiv 1). \end{cases}$$

(Note if $f \not\equiv 0$ or 1, then $f(0) = f(0) * f(y)$ implies $f(0) = 0$ so that necessarily $\tilde{f}(0) = 0$).

Hence the necessity condition is proved.

(Sufficiency). Suppose we are given M and $\tilde{f}$ as stipulated by the theorem. Let f be defined by Eq (6.5) together with $f(0) = 0$. Suppose

$$x = s_x * m_x \quad \text{and} \quad y = s_y * m_y$$

with obvious notations. Then

$$f(y) = \tilde{f}(s_y) * m_y$$

so that

$$\begin{aligned} f(x * f(y)) &= f(s_x * m_x * \tilde{f}(s_y) * m_y) \\ &= \tilde{f}(s_x) * m_x * \tilde{f}(s_y) * m_y \\ &= f(x) * f(y) \end{aligned}$$

since m_x, $\tilde{f}(s_y)$, $m_y \in M$. Thus Eq $(6.1)'$ is satisfied by f /

Of course if the group G is commutative, we need not worry about the left or right cosets because they are identical.

<u>Corollary 6.4.</u> Let G be a commutative group. A function $f : G \longrightarrow G$ satisfies Eq $(6.1)'$ if and only if there exists a subgroup M of G and a mapping $\tilde{f} : S \longrightarrow M$ such that

$$f(x) = \tilde{f}(s_x) * m_x$$

where each $x \in G$ can be written uniquely as

$$x = s_x * m_x \qquad (s_x \in S,\ m_x \in M),$$

where S is a set of representatives in G of G/M /

In the important case where $G_0 = \mathbb{R}$, we can obtain quite strong results.

<u>Theorem 6.5</u>. If $f : (\mathbb{R}, .) \longrightarrow (\mathbb{R}, .)$ is a continuous function satisfying Eq (6.1), then f is of the form

$$f(x) \equiv 1 \ ; \ f(x) = cx \quad \text{for some } c \text{ in } \mathbb{R}$$

$$\text{or} \quad f(x) = \operatorname{Sup}\{ax, bx\} \quad \text{where } a \leqslant 0,\ b \geqslant 0,\ b > a.$$

<u>Proof</u>. We may assume that $f \not\equiv 0$ or 1. Then $M = f(\mathbb{R}) \setminus \{0\}$ is a subgroup of $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. Suppose $\{x_n\}$ is a sequence in M converging to a point x in $\mathbb{R}^*$.

Then

$$\lim_{n\to\infty} f(x_n) = f(x) .$$

But

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(1).\, x_n = f(1) . x$$

by Theorem 6.3 so that

$$f(1) . x = f(x).$$

Since $f(x) \neq 0$, $x^{-1} = f(1) . f(x)^{-1}$ is in M by Proposition 6.2 so that $x \in M$.

Hence M is a closed subgroup of $\mathbb{R}^*$. It then follows from Theorem A-6 that

$$M = \mathbb{R}^*,\ M = \mathbb{R}^*_+ = \{x \in \mathbb{R} \mid x > 0\}$$

or M is discrete. Since f is a non-constant continuous function, $f(\mathbb{R})$ is an interval which is not a singleton so that $M = f(\mathbb{R}) \smallsetminus \{0\}$ is not discrete.

If $M = \mathbb{R}^*$, then $\mathbb{R}^*/\mathbb{R}^*$ has exactly one coset and we may let $S = \{1\}$ in Theorem 6.3 so that

$$f(x) = f(1) . x = cx \quad (x \in \mathbb{R}).$$

If $M = \mathbb{R}^*_+$, then $\mathbb{R}^*/_M$ contains exactly two cosets and we may take $S = \{-1, 1\}$ in Theorem 6.3 so that

$$f(x) = \begin{cases} f(1) . x & \text{if } x \in M = \mathbb{R}^*_+ \\ \\ f(-1).(-1).x & \text{if } x \notin M. \end{cases}$$

$$= \begin{cases} ax & \text{if } x \in \mathbb{R}^*_+ \\ bx & \text{if } x \notin \mathbb{R}^*_+ \end{cases}$$

where $a \geqslant 0$, $b \leqslant 0$ since $f(\mathbb{R}) \subset M \cup \{0\} = \mathbb{R}_+$. In this case

$$f(x) = \operatorname{Sup}\{ax, bx\}, \quad a \geqslant 0 \text{ and } b \leqslant 0.$$

If $a = b$, then $a = b = 0$ and $f \equiv 0$ which has already exhibited. Hence we may assume $b > a$ /

<u>Theorem 6.6</u>. If $f : (\mathbb{R}, +) \longrightarrow (\mathbb{R}, +)$ is a continuous function satisfying Eq (6.2), then

$$f \equiv 0 \text{ or } f(x) = x + a \text{ for some } a \text{ in } \mathbb{R}.$$

<u>Proof</u>. As in the proof of Theorem 6.5, the set $M = f(\mathbb{R})$ is a closed subgroup of $(\mathbb{R}, +)$. But M is connected, being the continuous image of the connected set $\mathbb{R}$, so that M is a singleton or M is non-discrete. Since the only constant functiom $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying Eq (6.2) is $f \equiv 0$, $M = f(\mathbb{R})$ being a singleton implies $f \equiv 0$.

If M is non-discrete, then $M = \mathbb{R}$ by Corollary A-4 so that we may take $S = \{0\}$ in Theorem 6.3.

Hence

$$f(x) = f(0) + x = a + x$$

in this case /

As an application, we prove the following :

<u>Theorem 6.7</u>. A continuous function $f : \mathbb{R} \to \mathbb{R}$ is demi-multiplicative symmetric, i.e.,

$$f(x\, f(y)) = f(f(x)\, f(y)) \tag{6.6}$$

if and only if f is of one of the following forms :

$$f(x) = a \quad \text{for any constant } a \text{ in } \mathbb{R}$$

$$\text{or} \quad f(x) = x$$

$$\text{or} \quad f(x) = \operatorname{Sup}\{ax, x\}, \quad a < 0$$

$$\text{or} \quad f(x) = -|x| \quad \text{for all } x \text{ in } \mathbb{R}.$$

<u>Proof</u>. Assume f is a non-constant continuous DMS function. Then Theorem 5.2 implies that f or -f is SMS. Therefore Theorem 6.5 assures that f or -f is of the forms :

$$f(x) = ax$$

$$\text{or} \quad -f(x) = ax$$

$$\text{or} \quad f(x) = \operatorname{Sup}\{ax, bx\} \quad a \leqslant 0,\ b \geqslant 0,\ b > a,$$

$$\text{or} \quad -f(x) = \operatorname{Sup}\{ax, bx\} \quad a \leqslant 0,\ b \geqslant 0,\ b > a.$$

If follows from the first two cases that f is of the form

$$f(x) = cx \quad (x \in \mathbb{R})$$

for some c in $\mathbb{R}$. But f is DMS, Eq (6.6) and the last equation give

$$f(1.c.1) = f(c.1.c.1)$$

$$c(1c1) = c(c1.c1)$$

so that $c = 1$. Hence the only DMS function of the form $f(x) = cx$ is when $c = 1$. Thus

(6.7) $$f(x) = x \qquad (x \in \mathbb{R}).$$

Next consider function of the form $f(x) = \text{Sup}\{ax, bx\}$ with $a \leqslant 0$, $b \geqslant 0$ and $b > a$. Then Eq (6.6) gives

$$f(1.f(1)) = f(b) = b^2.$$

But $f(1.f(1)) = f(f(1).\ f(1)) = b^3$ so that $b = 1$. Thus

(6.8) $$f(x) = \text{Sup}\{ax, x\} \qquad (x \in \mathbb{R})$$

with $a \leqslant 0$.

Finally, the case $-f(x) = \text{Sup}\{ax, bx\}$ $a \leqslant 0$, $b \geqslant 0$ and $b > a$. As in the preceding case, we can show that

$$-f(x) = \text{Sup}\{ax, x\} \quad (x \in \mathbb{R}),\ a \leqslant 0.$$

That is

(6.9) $$f(x) = \text{Inf}\{cx, -x\} \quad (x \in \mathbb{R}),\ c \geqslant 0.$$

We claim that Eq (6.9) satisfies DMS property only if $c = 1$. By Eq (6.6) again, $f(-1.f(-1)) = -c$ and

$$f(-1 . f(-1)) = f(f(-1) . f(-1)) = -c^2$$

so that $c = 0$ or 1. Since $f(x) = -x$ does not satisfy Eq (6.6), $c = +1$.

Hence

(6.10) $$f(x) = \text{Inf}\{x, -x\}$$

$$= -|x| \qquad (x \in \mathbb{R}).$$

Obviously, functions given by Eq (6.7), (6.9) and (6.10) satisfy Eq (6.6).

Thus the theorem is now completely proved /

It is worth noting that the MS, DMS and SMS functions over $\mathbb{R}\ (>0)$ are much simpler than those over $(\mathbb{R},.)$.

<u>Theorem 6.8.</u> A continuous function $f : \mathbb{R}\ (>0) \longrightarrow \mathbb{R}\ (>0)$ is

i) MS if and only if f is of the forms :

$$f(x) \equiv a \text{ or } f(x) = ax \ (a > 0).$$

ii) SMS if and only if f is of the forms :

$$f(x) \equiv 1 \text{ or } f(x) = ax \ (a > 0).$$

iii) DMS if and only if f is of the forms :

$$f(x) \equiv a \ (a > 0) \text{ or } f(x) = x.$$

<u>Proof.</u> (i) follows at once from Theorem 4.6 when x and its values are in $\mathbb{R}\ (>0)$. Similarly, (ii) obtains from Theorem 6.5 and (iii) also obtains from Theorem 6.7 /

<u>Eq (6.3) and Eq (6.4).</u>

Here we are mainly concerned with functions $f : \mathbb{K} \longrightarrow \mathbb{K}$ where $\mathbb{K}$ is a field of characteristic zero, and satisfying one of the following equations :

(6.3) $$f(x + f(y)) = f(x)\, f(y)$$

(6.4) $$f(x\, f(y)) = f(x) + f(y).$$

As for Eq (6.4), we have

$$f(0) = f(0.\ f(x)) = f(0) + f(x)$$

for any $x \in \mathbb{K}$, so that the only function $f : \mathbb{K} \longrightarrow \mathbb{K}$ satisfying Eq (6.4) is the zero function.

From now on, we shall assume that the function $f : \mathbb{K} \longrightarrow \mathbb{K}$ satisfies Eq (6.3). Then we immediately get

$$f(x + z) = zf(x) \quad (x \in \mathbb{K}, \ z \in f(\mathbb{K}))$$

and

$$f(f(x)) = f(0)\, f(x) \quad (x \in \mathbb{K}).$$

Moreover, if f is identically $k \in \mathbb{K}$, then

$$k = f(0 + f(0)) = f(0) \, . \, f(0) = k^2$$

so that k = 0 or 1.

<u>Proposition 6.9</u>. If f is identically constant, then $f \equiv 0$ or $f \equiv 1$.

Suppose now that $f(x_o) = 0$ for some x_o in $\mathbb{K}$ then for any $x \in \mathbb{K}$,

$$f(x) = f(x + 0) = f(x + f(x_o))$$
$$= f(x)\, f(x_o) = 0$$

and we have proved :

<u>Proposition 6.10</u>. If f is 0 at one point of $\mathbb{K}$, then f is identically 0.

Henceforth we shall further assume that <u>f is non-vanishing</u>.

<u>Proposition 6.11</u>. For any $x \in \mathbb{K}$, $f(x - f(x)) = 1$.

<u>Proof</u>. For any $x \in \mathbb{K}$,

$$f(x) = f(x - f(x) + f(x)) = f(x - f(x)) \, . \, f(x)$$

so that the proposition is obtained /

Now

$$f(x + 1) = f(x + f(0 - f(0))) = f(x)\, f(0 - f(0))$$
$$= f(x) \,.\, 1$$

and we have just proved :

<u>Proposition 6.12</u>. For any $x \in \mathbb{K}$, $f(x + 1) = f(x)$.

<u>Proposition 6.13</u>. If $f : \mathbb{K} \longrightarrow \mathbb{K}$ satisfies Eq (6.3) and does not vanish identically, then $f(\mathbb{K})$ is a multiplicative subgroup of $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$.

<u>Proof</u>. Since $1 = f(0 - f(0))$ by Proposition 6.11, $1 \in f(\mathbb{K})$.

Let g_1 and g_2 be in $f(\mathbb{K})$. Then

$$f(0 - f(0) + g_1 + g_2) = f(0 - f(0)) \,.\, g_1 g_2$$
$$= 1 .\, g_1 g_2$$

so that $g_1 g_2 \in f(\mathbb{K})$.

If $g \in f(\mathbb{K})$, then

$$f(0) = f(-f(0) - g + g + f(0))$$
$$= f(-f(0) - g) \,.\, g f(0).$$

Since $f(0) \neq 0$, multiplying both side of the above equation by $f(0)^{-1}$ to obtain

$$1 = f(-f(0) - g) \,.\, g.$$

That is

g^{-1} is in $f(\mathbb{K})$.

Hence $f(\mathbb{K})$ is a subgroup of $\mathbb{K}^*$ /

Let Λ be the additive abelian group on the generators $f(\mathbb{K})$, i.e., Λ consists of all finite linear combinations of elements from $f(\mathbb{K})$ with coefficients in $\mathbb{Z}$. Then Λ is a subgroup of the additive group $\mathbb{K}$.

Theorem 6.14. If $f(\mathbb{K}) \neq \{0\}$, the mapping $\phi : \Lambda \longrightarrow f(\mathbb{K})$ defined by $\phi(\lambda) = f(-f(0) + \lambda)$ is a homomorphism from $(\Lambda, +)$ onto $(f(\mathbb{K}), .)$ satisfying :

$$\phi(g) = g \text{ for all } g \in f(\mathbb{K}).$$

Proof. Clearly ϕ is well-defined. By Eq (6.3) and $g \in f(\mathbb{K})$,

$$\phi(g) = f(-f(0) + g) = f(-f(0)).\, g\,.$$

But $f(-f(0)) = f(-f(0) + 0) = 1$ by Proposition 6.11. Hence $\phi(g) = g$ for all $g \in f(\mathbb{K})$.

Now let $\lambda = \sum_{i=1}^{n} a_i g_i$ with $g_i \in f(\mathbb{K})$ and $a_i \in \mathbb{Z}$, and assume, for instance, that $a_i > 0$ for $1 \leqslant i \leqslant k$, and $a_i = -b_i < 0$ for $k + 1 \leqslant i \leqslant n$. We claim that

$$\phi(\lambda) = \prod_{i=1}^{k} g_i^{a_i}\ \phi\Big(-\sum_{k+1}^{n} b_i g_i\Big).$$

For $n = 1$,

$$\phi(\lambda) = f(-f(0) + a_1 g_1) = f(-f(0))\, g_1^{a_1} = \phi(0)\, g_1^{a_1}\,.$$

Assume the claim true for lesser value of n. Then

$$\begin{aligned}\phi(\lambda) &= f\Big(-f(0) + \sum_{i=1}^{k} a_i g_i - \sum_{i=k+1}^{n} b_i g_i\Big)\\ &= f\Big(-f(0) - \sum_{i=k+1}^{n} b_i g_i + \sum_{i=1}^{k-1} a_i g_i + a_k g_k\Big)\\ &= f\Big(-f(0) - \sum_{i=k+1}^{n} b_i g_i + \sum_{i=1}^{k-1} a_i g_i\Big)\, g_k^{a_k}\end{aligned}$$

by Eq (6.3). It now follows from the induction hypothesis that

$$\phi(\lambda) = f(-f(0) - \sum_{i=k+1}^{n} b_i g_i + \sum_{i=1}^{k-1} a_i g_i)\, g_k^{a_k}$$

$$= \phi(- \sum_{i=k+1}^{n} b_i g_i) \prod_{i=1}^{k-1} g_i^{a_i} \; g_k^{a_k}$$

$$= \phi(- \sum_{i=k+1}^{n} b_i g_i) \prod_{i=1}^{k} g_i^{a_i} .$$

Hence the claim is proved.

Since $\phi(0) = f(-f(0) + 0) = 1$ by proposition 6.11 and the claim,

$$\phi(- \sum_{k+1}^{n} b_i g_i) = (\prod_{k+1}^{n} g_i^{b_i})^{-1}$$

$$= \prod_{k+1}^{n} g_i^{-b_i} = \prod_{k+1}^{n} g_i^{a_i}$$

so that

$$\phi(\lambda) = \prod_{i=1}^{k} g_i^{a_i} \cdot \prod_{i=k+1}^{n} g_i^{-b_i}$$

$$= \prod_{i=1}^{n} g_i^{a_i} .$$

Therefore for any α and λ in Λ, we can show that

$$\phi(\alpha + \lambda) = \phi(\alpha) . \phi(\lambda).$$

Thus ϕ is a homomorphism from $(\Lambda, +)$ onto $(f(\mathbb{K}), .)$.

Hence the theorem is proved /

Observe that if $f \not\equiv 0$ is a solution of Eq (6.3), then the associated homomorphism

$$\emptyset : \Lambda \xrightarrow{\text{onto}} f(\mathbb{K})$$

must be well-defined. This means that if

$$0 = \sum_{i=1}^{n} a_i g_i$$

is any representation of 0 in Λ, then

$$1 = \emptyset(0) = \emptyset\left(\sum_{i=1}^{n} a_i g_i\right) = \prod_{i=1}^{n} g_i^{a_i}$$

must hold. Thus we arrive at the consistency condition

$$(*) \qquad \sum_{i=1}^{n} a_i g_i = 0 \text{ implies } \prod_{i=1}^{n} g_i^{a_i} = 1$$

for all $g_i \in f(\mathbb{K})$ and $a_i \in \mathbb{Z}$.

<u>Theorem 6.15</u>. If f satisfies Eq (6.3) with $f(\mathbb{K}) \neq 0$, $x \in \mathbb{K}$ and $\lambda \in \Lambda$, then :

$$f(x + \lambda) = f(x)\,\emptyset(\lambda).$$

<u>Proof</u>. Suppose $\lambda = \sum_{i=1}^{n} a_i g_i$ with $a_i \in \mathbb{Z}$, $g_i \in f(\mathbb{K})$. As in the proof of Theorem 6.14, we obtain

$$f(x + \lambda) = f\left(x + \sum_{i=1}^{n} a_i g_i\right) = f(x) \prod_{i=1}^{n} g_i^{a_i} = f(x)\,\emptyset(\lambda).$$

Hence the theorem is proved /

This theorem enables us to give a lot of examples. Even more the previous necessary conditions for a subset G of $\mathbb{K}$ to be the range

$f(\mathbb{K})$ of a solution of Eq (6.3) appear to be sufficient.

<u>Theorem 6.16.</u> Given a multiplicative subgroup G of $\mathbb{K}^*$ such that the additive abelian group Λ generated by G satisfies the consistency ($*$), let S be a set of representatives of $\mathbb{K}/\Lambda$ in $\mathbb{K}$, so that any $x \in \mathbb{K}$ may be written as $x = s + \lambda$ where $s \in S$ and $\lambda \in \Lambda$, with a unique s and define

$$f(x) = f(s)\,\phi(\lambda)$$

where $f(s) = g_s \in G$ and ϕ is the natural homomorphism $\phi : (\Lambda, +) \longrightarrow (G, .)$, the existence of which is ensured by ($*$). (If $x = s + \lambda'$, then $\lambda - \lambda' = 0$ and $\phi(\lambda - \lambda') = 1$, because of ($*$), and f is unambiguously defined). Obviously f satisfies Eq (6.3)/

<u>Example 6.17.</u> Let $\mathbb{K} = \mathbb{R}$, α a transcendental number, $\mathcal{G}(\alpha)$ the cyclic group generated by α. Condition ($*$) is verified, since if $\sum_{i=m}^{n} a_i \alpha^i = 0$, then $a_i = 0$ for $i = m, \ldots, n$. Λ is the free abelian group $\mathbb{Z}\langle \mathcal{G}(x) \rangle$ and $x \in \mathbb{K}$ has a unique representation $x = s + \lambda$. We may define, without ambiguity

$$f(x) = f(s + \sum_{i=m}^{n} a_i \alpha^i) = \alpha^{1 + \sum i a_i}$$

The function f defined in this way satisfies Eq (6.3). The definition of the set of representatives S relies on the axiom of choice. In fact, if we assume that $S \subset [0,1[$, which is possible because of Proposition 6.12, it appears that S is basically the best known example of a non-measurable set in $\mathbb{R}$, with outer measure $\mu^*(S) > 0$ and inner measure $\mu_*(S) = 0$. (See for example Halmos [2]). The reciprocal image of α, $\Sigma = f^{-1}(\alpha)$ is the countable union of the $S + \lambda$, where λ runs through the countable set :

$$\left\{ \sum_{i=m}^{n} a_i \alpha^i \;\middle|\; a_i \in \mathbb{Z} , \quad \sum_{i=m}^{n} i\, a_i = 0 \right\} .$$

We have $\mu_*(\Sigma) = 0$, since the sum of any series with all terms equal to zero, is zero, but $\mu^*(\Sigma) \geqslant \mu^*(S) > 0$. We have just shown that the reciprocal image of the (closed) set $\{\alpha\}$ is not measurable so that the function f itself is not measurable.

Example 6.18. There exists non-measurable unbounded solutions for Eq (6.2) in the case $\mathbb{K} = \mathbb{R}$. Let α be a transcendental number, $\mathcal{G}(\alpha)$, $\mathcal{M}$ and S have the same meaning as in Example 6.17. For each $x \in \mathbb{R}$, x has the unique representation

$$x = s + \sum_{i=m}^{n} a_i \alpha^i , \; s \in S , \; a_i \in \mathbb{Z}$$

Define $$f(x) = \alpha + \sum_{i=m}^{n} a_i \alpha^i .$$

Obviously f satisfies Eq (6.2) and is non-measurable and unbounded.

Example 6.19. There exists discontinuous solutions for Eq (6.1) with $\mathbb{K} = \mathbb{R}$. α and $\mathcal{G}(\alpha)$ being as previously, let E be a set in $\mathbb{R}^*$ of representatives of $\mathbb{R}^*/\mathcal{G}(\alpha)$.

Now define :

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \alpha^{k+1} & \text{if } x = e.\alpha^k \text{ where } e \in E,\ k \in \mathbb{Z}. \end{cases}$$

Obviously, f satisfies Eq (6.1) and is not continuous.

From now on, take $\underline{\mathbb{K} = \mathbb{R}}$ and proved :

<u>Theorem 6.20.</u> If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a function satisfying

(6.3) $$f(x + f(y)) = f(x)\, f(y).$$

Then $f(\mathbb{R})$ contains no algebraic numbers except 0 or 1 (but not both).

<u>Proof.</u> Suppose that $f(x)$ is algebraic for some x in $\mathbb{R}$. Then there exist integers $a_o, a_1, \ldots, a_n$ such that

$$a_o + a_1\, f(x) + a_2\, f(x)^2 + \ldots + a_n\, f(x)^n = 0$$

so that

$$a_o\, f(x) + a_1\, f(x)^2 + \ldots + a_n f(x)^{n+1} = 0.$$

Thus

$$f(0) = f\left(\sum_{i=0}^{n} a_i \left[f(x)\right]^{i+1}\right)$$

$$= f(0) \,.\, \left[f(x)\right]^{\sum_{i=0}^{n}(i+1)a_i}$$

by Theorem 6.15. If $f(0) = 0$, then f is identically 0 by Proposition 6.10 and the theorem holds in this case. Assume that $f(0) \neq 0$. Then the obove equality gives

(6.11) $$1 = \left[f(x)\right]^N$$

where $N = \sum_{i=0}^{n}(i + 1)\, a_i$ is an integer. Then Eq (6.11) holds if $f(x) = 1$ or -1.

If there exists an x in $\mathbb{R}$ such that $f(x) = -1$, then by Proposition 6.11 and 6.12, we obtain

$$-1 = f(x) = f(x + 1) = f(x - f(x)) = 1$$

which is impossible so that $f(x) = 1$.

Hence the theorem is now completely proved /

And as an application of this theorem, we will prove the following :

<u>Theorem 6.21</u>. If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function satisfying

(6.3) $$f(x + f(y)) = f(x)\, f(y) \qquad (x, y \in \mathbb{R}),$$

then f is identically 0 or 1.

<u>Proof</u>. Suppose f is neither the 0 function nor the function 1. It follows from Proposition 6.9 that f cannot be a constant function so that there exist x and y in $\mathbb{R}$ such that $f(x) \neq f(y)$. Since f is continuous, f must assume some algebraic number, different from 1 between f(x) and f(y). This contradicts Theorem 6.20 so that the conclusion of the theorem must hold /

It seems that $f \equiv 0$ or 1 are the only possible regular solutions of Eq (6.3). Although we can not confirm this, we give more support to it.

<u>Lemma 6.22</u>. Let $x, x_1, x_2, \ldots, x_k$ be any element in $\mathbb{R}$, then

(6.12) $$f\left(x + \prod_{i=1}^{k} f(x_i)\right) = f(x) \prod_{i=1}^{k} f(x_i),$$

<u>Proof</u>. For k = 1, Eq (6.12) is just Eq (6.3). Assume that Eq (6.12) holds for lesser value of k. Then by induction hypothesis and Eq (6.3)

$$f\left(x + \prod_{i=1}^{k} f(x_i)\right) = f\left(x + \prod_{i=1}^{k-2} f(x_i) \cdot f(x_{k-1} + f(x_k))\right)$$

$$= f(x) \cdot \prod_{i=1}^{k-2} f(x_i) \cdot f(x_{k-1} + f(x_k))$$

$$= f(x) \cdot \prod_{i=1}^{k-2} f(x_i) \cdot f(x_{k-1}) \cdot f(x_k)$$

$$= f(x) \cdot \prod_{i=1}^{k} f(x_i).$$

The conclusion of the lemma follows /

<u>Proposition 6.23</u>. Let $x, x_1, x_2, \ldots, x_k$ be any element in $\mathbb{R}$ and n is any integer, then

$$(6.13) \qquad f(x + n \prod_{i=1}^{k} f(x_i)) = f(x) \cdot \left[\prod_{i=1}^{k} f(x_i) \right]^n ,$$

provided $\prod_{i=1}^{k} f(x_i) \neq 0.$

<u>Proof</u>. We will prove Eq (6.13) first for $n \geq 0$ by induction on n. Since Eq (6.13) is true for $n = 0$ and is Eq (6.12) for $n = 1$, assume Eq (6.13) holds for lesser values of n. Then by Lemma 6.22 and the induction hypothesis,

$$f(x + n \prod_{i=1}^{k} f(x_i)) = f(x + (n-1) \prod_{i=1}^{k} f(x_i) + \prod_{i=1}^{k} f(x_i))$$

$$= f(x + (n-1) \prod_{i=1}^{k} f(x_i)) \cdot \prod_{i=1}^{k} f(x_i)$$

$$= f(x) \left[\prod_{i=1}^{k} f(x_i) \right]^{n-1} \cdot \prod_{i=1}^{k} f(x_i)$$

$$= f(x) \left[\prod_{i=1}^{k} f(x_i) \right]^{n},$$

providing $\prod_{i=1}^{k} f(x_i) \neq 0$. Eq (6.13) holds for $n \geq 0$.

If n is positive integer, then from the first part of the proof

$$f(x) = f(x - n \prod_{i=1}^{k} f(x_i) + n \prod_{i=1}^{k} f(x_i))$$

$$= f(x - n \prod_{i=1}^{k} f(x_i)) \cdot \left[\prod_{i=1}^{k} f(x_i) \right]^{n}.$$

so that

$$f\left(x - n \prod_{i=1}^{k} f(x_i)\right) = f(x) \cdot \left[\prod_{i=1}^{k} f(x_i) \right]^{-n},$$

provided that $\prod_{i=1}^{k} f(x_i) \neq 0$. Hence Eq (6.13) holds for any integer n /

<u>Theorem 6.24</u>. If $f : \mathbb{R} \to \mathbb{R}$ is not a constant function satisfying

(6.3) $$f(x + f(y)) = f(x)\, f(y) ,$$

then f assumes arbitrarily large and arbitrarily small values.

<u>Proof</u>. Since the only constant functions satisfying Eq (6.3) are 0 and 1, the hypothesis on f together with Proposition 6.9 implies that there are x' and y' in $\mathbb{R}$ such that

$$f(x') \neq f(y')$$

and

$$f(x') \neq 0 \neq f(y').$$

Since $f(x) \cdot f(x) = f(x + f(x))$, we may assume that $f(x'), f(y') > 0$. Also assume that $f(x') < f(y')$.

It now follows from Proposition 6.23 that

$$f(x' - f(y')) = f(x')\, f(y')^{-1} < 1$$

and

$$f(y' - f(x')) = f(y') \cdot f(x')^{-1} > 1.$$

Hence there are always x and y in $\mathbb{R}$ such that

$$0 < f(x) < 1$$

and

$$f(y) > 1.$$

It now follows from Theorem 6.15 that

$$\lim_{n\to\infty} f\left(\sum_{i=0}^{n}\left[f(x)\right]^{(i+1)}\right) = \lim_{n\to\infty}\left(f(0).\left[f(x)\right]^{\sum_{i=0}^{n}(i+1)}\right)$$

$$= 0$$

and that

$$\lim_{n\to\infty} f\left(\sum_{i=0}^{n}\left[f(y)\right]^{i+1}\right) = \lim_{n\to\infty}\left(f(0).\left[f(y)\right]^{\sum_{i=0}^{n}(i+1)}\right)$$

$$= +\infty .$$

Hence the conclusion of the theorem now follows /

Corollary 6.25. If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a function satisfying Eq (6.3) and if there exists an $\varepsilon > 0$ such that

$$f(x) \geqslant \varepsilon \qquad (x \in \mathbb{R}),$$

then $f \equiv 1$.

Corollary 6.26. If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a function satisfying Eq (6. 3) and if f is uniformly bounded from above, then $f \equiv 0$ or 1.

APPENDIX

In this appendix, we will show that the closed subgroups of $(\mathbb{R}\setminus\{0\},\cdot)$ are :

i) $(\mathbb{R}\setminus\{0\},\cdot)$

ii) $(\mathbb{R}(>0),\cdot)$ and

iii) the discrete subgroups

where $\mathbb{R}$ is the set of real numbers. To show this, we will prove first that every closed proper subgroup of the additive group $(\mathbb{R},+)$ is of the form :

$$a\mathbb{Z} = \{ an \mid n \in \mathbb{Z} \} \qquad (a \in \mathbb{R}(\geq 0)).$$

<u>Lemma A-1</u>. Let A be any subgroup of $(\mathbb{R},+)$. If A is not discrete, then A is dense in $\mathbb{R}$.

<u>Proof</u>. Suppose A is not discrete. Then there exists $a \in A$ such that for every neighborhood (nbhd) N of a in $\mathbb{R}$, we have $N \cap A$ contains elements of $A \setminus \{a\}$: In particular, for any given $\varepsilon > 0$, there exists $x_\varepsilon \neq 0$ in A such that $x_\varepsilon \in (a-\varepsilon, a+\varepsilon)$.

We claim that for each $\varepsilon > 0$ there exists a $x \neq 0$ in A such that $x \in (-\varepsilon, \varepsilon)$. From above, there exists $x_\varepsilon \in (a-\varepsilon, a+\varepsilon)$ and since A is an additive subgroup of $\mathbb{R}$, $x_\varepsilon - a \in A$ and $|x_\varepsilon - a| < \varepsilon$. Therefore $x = x_\varepsilon - a \in (-\varepsilon, \varepsilon)$.

Next we claim that A is dense in $\mathbb{R}$. It is enough to show that every set of the form $(r-\varepsilon, r+\varepsilon)$ contains a point of A where $\varepsilon < 1$ and $r \in \mathbb{R}$. Without loss of generality, we may assume that $r > 0$. From the above claim, there exists $a (> 0)$ in A such that $a \in (-\varepsilon, \varepsilon)$ so that by Euclidean Algorithm, there exists $n_0 \in \mathbb{Z} (\geq 0)$ such that $\frac{r}{a} = n_0 + b$ where $|b| < a < \varepsilon$. Therefore $r = n_0 a + ba$, i.e, $|r - n_0 a| = |ba| = |b| a < \varepsilon^2$. But $\varepsilon < 1$; hence $\varepsilon^2 < \varepsilon$. Thus $n_0 a \in (r - \varepsilon, r + \varepsilon)$. Since $n_0 a \in A$, A is dense in $\mathbb{R}$/

Lemma A.2. Every closed proper subgroup of $(\mathbb{R}, +)$ is discrete.

Proof. Let A be any closed proper subgroup of $(\mathbb{R}, +)$. Suppose A is not discrete. Then by Lemma A.1, A is dense in $\mathbb{R}$. But A is closed; hence $A = \bar{A} = \mathbb{R}$, where $\bar{A}$ is the closure of A, so that A is not a proper subgroup of $\mathbb{R}$. Thus a proper closed subgroup of $(\mathbb{R}, +)$ must be discrete/

Theorem A-3. Discrete subgroups of $(\mathbb{R}, +)$ must be of the form :

$$a\mathbb{Z} = \{ an \mid n \in \mathbb{Z} \}.$$

for some $a \in \mathbb{R} (\geq 0)$.

Proof. It is clear that a set of the form $a\mathbb{Z}$ is a discrete subgroup of $(\mathbb{R}, +)$. To prove the converse, let D be a discrete subgroup of $(\mathbb{R}, +)$. If $D = \{0\}$, then $D = 0 \cdot \mathbb{Z}$.

If $D \neq \{0\}$, let $D_+ = \{ d \in D \mid d > 0 \}$. Then $D_+ \neq \phi$ and bounded below by 0. Therefore the infimum of D_+, say a, exists as a number in $\mathbb{R} (\geq 0)$. Since $a = \inf D_+$, for any given $\varepsilon > 0$, there exists $x \in D_+$ such that $|x - a| < \varepsilon$ which implies that the distance $d(a, D_+) = 0$. Hence $a \in \bar{D}_+$, the closure of D_+ in $\mathbb{R}$.

Since D_+ is discrete in $\mathbb{R}$, $\bar{D}_+ = D_+$ and $a \in D_+$.

We claim that $D = a\mathbb{Z}$. Since D is an additive subgroup of $(\mathbb{R},+)$, $a\mathbb{Z} \subset D$. To prove the converse, suppose there is a $d \in D$ such that $d \notin a\mathbb{Z}$. Since both d and -d are in D, we can assume that $d > 0$. By the property of real numbers and the choice of a, there exist $n \in \mathbb{Z}\ (> 0)$ such that

$$an < d < a(n+1).$$

Since $d < a(n+1)$, $d-an < a$. But $d-an > 0$ and $d-an \in D$ so that a is not the infimum of D_+, a contradiction. Therefore. we must have $D \subset a\mathbb{Z}$.

Now the theorem is completely proved/

Then follows from Lemma A-2 and Theorem A-3, we immediately obtain the following corollary :

<u>Corollary A-4</u>. Every closed proper subgroup of $(\mathbb{R},+)$ is of the form :

$$a\mathbb{Z} = \{ an \mid n \in \mathbb{Z} \}$$

for some a in $\mathbb{R}\ (\geq 0)$.

Before showing anything else, let us define some notations which will be used from now on.

<u>Notations</u>. Let $\mathbb{R}^*$ denote the set $\mathbb{R} \setminus \{0\}$ and $\mathbb{R}^*_+$ denote the set $\mathbb{R}^*\ (> 0)$.

Next we will prove that every closed proper subgroup of $(\mathbb{R}^*_+,\cdot)$ is discrete. In order to prove this, we consider the exponential mapping :

$$g : (\mathbb{R},+) \longrightarrow (\mathbb{R}^*_+,\cdot)$$
$$x \longmapsto e^x .$$

Since g is group isomorphism and homeomorphism, there is a one-to-one correspondence between closed subgroups of $(\mathbb{R},+)$ and those of $(\mathbb{R}^*_+,\cdot)$. By Corollary A-4, every closed proper subgroup of $(\mathbb{R},+)$ is of the form $a\mathbb{Z}$; hence all proper closed subgroups of $(\mathbb{R}^*_+,\cdot)$ are of the form

$g(a\mathbb{Z}) = \{g(an) \mid n \in \mathbb{Z}\} = \{(e^a)^n \mid n \in \mathbb{Z}\} = \{\lambda^n \mid n \in \mathbb{Z}\}$ for some $\lambda \in \mathbb{R}$. Hence we have proved :

<u>Theorem A-5</u>. Every closed proper subgroup of $(\mathbb{R}^*_+,\cdot)$ is discrete.

Before we get the final theorem of this appendix, let us construct some functions which are neccssary for the proof of the desired facts.

Let $\{-1,1\}$ be a set with binary operation " $\cdot$ " defined by the table :

$\cdot$	1	-1
1	1	-1
-1	-1	1

Then $(\{-1, 1\},\cdot)$ is a discrete topological group. Since the mappings :

$$\mathbb{R}^* \longrightarrow \mathbb{R}^*_+$$
$$x \longmapsto |x|$$

and

$$\mathbb{R}^* \longrightarrow \{-1, 1\}$$
$$x \longmapsto \frac{x}{|x|}$$

are continuous, onto and a group homomorphism, the map :

$$f : \mathbb{R}^* \longrightarrow \mathbb{R}^*_+ \times \{-1, 1\}$$
$$x \longmapsto (|x|, \frac{x}{|x|})$$

is continuous, onto and a group homomorphism. Actually f is one-to-one, for if $f(x_1) = f(x_2)$ then $(|x_1|, \frac{x_1}{|x_1|}) = (|x_2|, \frac{x_2}{|x_2|})$ which implies that $|x_1| = |x_2|$ and $\frac{x_1}{|x_1|} = \frac{x_2}{|x_2|}$ so that $x_1 = x_2$.

To show that f is open it suffices to prove that f((a,b)) is open for any open interval $(a,b) \subset \mathbb{R}^*$. This is clear, since f((a,b)) is the union of an open set in $\mathbb{R}^*_+ \times \{1\}$ and an open set in $\mathbb{R}^*_+ \times \{-1\}$. Therefore f is a homeomorphism and a group isomorphism.

Next consider the profection mapping :

$$\Pi : \mathbb{R}^*_+ \times \{-1, 1\} \longrightarrow (\mathbb{R}^*_+, \cdot)$$
$$(x, \alpha) \longmapsto x .$$

This mapping is a continuous homomorphism and since $\{-1, 1\}$ is compact, Π is closed.

Now let M be a closed subgroup of $(\mathbb{R}^*, \cdot)$. Then $A = \Pi$ of $[M]$ is a closed subgroup of $(\mathbb{R}^*_+, \cdot)$ which is just $M \cap \mathbb{R}^*_+$. Since Π of is closed, A being discrete will imply that M is discrete also.

Assume A is not discrete, then $A = M \cap \mathbb{R}^*_+ = \mathbb{R}^*_+$ by Theorem A-5.

If $-1 \in M$, then $-1 \cdot \mathbb{R}^*_+ \subset -1.M \subset M$ and $M = \mathbb{R}^*$.

If $-1 \notin M$, then $-1 \cdot \mathbb{R}^*_+ \cap M = \emptyset$ and $M = \mathbb{R}^*_+$.

We have thus proved the following theorem :

<u>Theorem A-6</u>. The only closed subgroups of $(\mathbb{R}^*, \cdot)$ are :

i) $(\mathbb{R}^*, \cdot)$

ii) $(\mathbb{R}^*_+, \cdot)$

and iii) the discrete subgroups.