CHAPTER V

THE FUNCTIONAL EQUATION $f(x * f(y)) = f(f(x) * f(y))$

Materials of this chapter are based on reference [3].

In this chapter, we study only the character of the DMS function. We will show that if f is a non-constant continuous DMS function over $(\mathbb{R}, .)$ then either f or -f is a SMS function.

Recall : A function f from a semi-group $(G, *)$ into itself is a DMS function if $f(x * f(y)) = f(f(x) * f(y))$.

Lemma 5.1. Let $(G, *)$ be a commutative semi-group, $f : (G, *) \longrightarrow (G, *)$ be a DMS function. Then

$$f(x * f(y) * f(z)) = f(x * f(y * f(z))) .$$

Proof.

$$\begin{aligned}
f(x * f(y * f(z))) &= f(f(x) * f(y * f(z))) && \text{(f is DMS)} \\
&= f(f(x) * f(f(y) * f(z))) && \text{(f is DMS)} \\
&= f(f(f(y) * f(z)) * f(x)) && \text{($*$ is commutative)} \\
&= f(f(y) * f(z) * f(x)) && \text{(f is DMS)} \\
&= f(f(x) * f(y) * f(z)) && \text{($*$ is associative)} \\
&= f(f(f(x) * f(y)) * f(z)) && \text{(f is DMS)} \\
&= f(f(x * f(y)) * f(z)) && \text{(f is DMS)} \\
&= f(x * f(y) * f(z)) && \text{(f is DMS)} .
\end{aligned}$$

Hence the lemma is proved /

Theorem 5.2. Let $f : (\mathbb{R}, .) \longrightarrow (\mathbb{R}, .)$ be a non-constant continuous DMS function. Then f or -f is SMS.

Proof. We put $\alpha = f(y)\,f(z)$ in Lemma 5.1 to get

$$(5.1) \qquad f(x\,\alpha) = f(x\,.\,f(yf(z))).$$

If $\alpha = 0$, then $f(x\,.\,f(yf(z))) = f(0)$. But by Lemma 3.16 and the fact that if f is DMS then it is MS, $f(0) = 0$. Therefore,

$$f(x\,.\,f(yf(z))) = 0$$

so that $f \equiv 0$ which is excluded or else $f(y\,.\,f(z)) = 0$. For the latter case, since f is DMS, we have

$$0 = f(y\,.\,f(z)) = f(f(y)\,.\,f(z)),$$

and $\qquad 0 = \alpha = f(y)\,f(z)$

so that $\qquad f(y.f(z)) = 0 = f(y)\,.\,f(z).$

If $\alpha \neq 0$, then we write $f(x)$ as $f\,(x.\frac{\alpha}{\alpha}) = f\,(\frac{x}{\alpha}\,.\,\alpha\,)$.

Therefore, by Eq (5.1)

$$(5.2) \qquad f(x) = f(\frac{x}{\alpha}\,.\,f(y\,f(z))) = f(x.\,\frac{f(y\,f(z))}{\alpha})\,.$$

We claim that

$$(5.3) \qquad f(x) = f(x\,(\,\frac{f(y\,f(z))}{\alpha})^n)\quad,\quad n \in \mathbb{Z}\,(>0).$$

To prove Eq (5.3), by induction on n. Since Eq (5.3) is Eq (5.2) when n = 1, assume that Eq (5.3) holds for $k < n$. Then by induction hypothesis and by Lemma 5.1,

$$\begin{aligned} f(x) &= f(x\,.\,(\frac{f(y\,f(z))}{\alpha})^k\,.\,\frac{\alpha}{\alpha}\,) \\ &= f(\,\frac{x}{\alpha}\,(\,\frac{f(y\,f(z))}{\alpha}\,)^k\,.\,\alpha) \\ &= f\left[\frac{x}{\alpha}\,(\,\frac{f(y\,f(z))}{\alpha}\,)^k\,.\,f(y)\,f(z)\right] \qquad \text{(by Eq (5.1))} \end{aligned}$$

$$= f\left[\frac{x}{\alpha}\left(\frac{f(y\ f(z))}{\alpha}\right)^k f(y\ f(z))\right]$$

$$= f\left[x \cdot \left(\frac{f(y\ f(z))}{\alpha}\right)^{k+1}\right].$$

Thus Eq (5.3) holds for all $n \in \mathbb{Z}(>0)$.

We now claim that $\left|\frac{f(y)\ f(z))}{\alpha}\right| = 1.$

Since f is non-constant, we can choose x such that $f(x) \neq 0$.

Since f is continuous,

$$f(x) = \lim_{n\to\infty} f(x(\frac{f(y\ f(z))}{\alpha})^n)$$

$$= f(\lim_{n\to\infty} x(\frac{f(y\ f(z))}{\alpha})^n)$$

(5.4) $$= f(x \cdot \lim_{n\to\infty}\left[\frac{f(y\ f(z))}{\alpha}\right]^n)$$

by Eq (5.3) so that $f(x) = f(0) = 0$ if $\left|\frac{f(y\ f(z))}{\alpha}\right| < 1$,

contradicting the choice of x. Thus

$$\left|\frac{f(y\ f(z))}{\alpha}\right| \geqslant 1.$$

But Eq (5.4) implies that the limit $\lim_{n\to\infty}(\frac{f(y\ f(z))}{\alpha})^n$ must exist

so that

$$\left|\frac{f(y\ f(z))}{\alpha}\right| = 1.$$

Thus we can write

$$f(y\ f(z)) = \operatorname{sgn}(y, z)\ f(y)\ f(z)$$

where sgn (y, z) is equal to 1 or -1 for (y, z) in

$$\Delta = \left\{(x, y) \in \mathbb{R}^2 \mid f(x)\cdot f(y) \neq 0\right\}.$$

If sgn $(y, z) = 1$ on Δ, then

$$f(y\, f(z)) = f(y)\, f(z).$$

Therefore, f is SMS.

If there exist points (x, y) in Δ such that sgn (y, z) = -1, then according to Eq (5.3), we have

$$f(x) = f\left(x \,.\, \left(\frac{f(y\, f(z))}{\alpha}\right)^n\right)$$

$$= f\left(x \left(\frac{\text{sgn}\,(y, z)\, f(y)\, f(z)}{f(y)\, f(z)}\right)^n\right)$$

$$= f(x \,.\, (-1)^n),$$

for all $n \in \mathbb{Z}\ (> 0)$. Letting n = 1, we get f(x) = f(-x) so that f is an even function.

First suppose that $K(f) = \{ x \mid f(x) = 0 \}$ is just $\{0\}$. Since

$$f(y\, f(z)) = \text{sgn}\,(y, z)\, f(y)\, f(z)$$

and f is continuous, sgn is continuous on Δ and constant on each component of Δ which are the four quadrants. But, since f is MS, we have

$$\text{sgn}\,(y, z) = \text{sgn}\,(z, y)$$

and since f is even,

$$\text{sgn}\,(-y, z) = \text{sgn}\,(y, z)$$

which implies that

$$\text{sgn}\,(y, z) \equiv -1$$

on Δ so that

$$f(y\, f(z)) = -f(y)\, f(z).$$

But f is even ; this gives at once that -f is SMS.

Suppose now that $K(f)$ is not equal to $\{0\}$. Since f is even,

$$|f|(y|f|z) = |f|(y f(z)) = |f|(y)\,|f|(z).$$

Therefore, $|f|$ is SMS.

Now we anticipate and assume the validity of Corollary 6.4 of Chapter VI. Since $|f|$ is continuous, positive and non-constant, this corollary gives that $K(f)$ is either $\mathbb{R}(>0)$ or $\mathbb{R}(<0)$.

Note that $(y, z) \in \mathbb{R}^2 \setminus \Delta \iff (y, z) \in \mathbb{R}^2 \wedge (y, z) \notin \Delta$

$$\iff (y, z) \in \mathbb{R}^2 \wedge f(y)\, f(z) = 0$$

$$\iff (y, z) \in \mathbb{R}^2 \wedge (\text{either } f(y) = 0 \text{ or } f(z) = 0)$$

$$\iff ((y, z) \in \mathbb{R}^2 \wedge f(y) = 0) \text{ or } ((y, z) \in \mathbb{R}^2 \wedge f(z) = 0)$$

$$\iff (y, z) \in \mathbb{R}^2 \wedge y \in K(f) \text{ or } (y, z) \in \mathbb{R}^2 \wedge z \in K(f)$$

$$\iff (y, z) \in K(f) \times \mathbb{R} \text{ or } (y, z) \in \mathbb{R} \times K(f).$$

so that $\mathbb{R}^2 \setminus \Delta = (K(f) \times \mathbb{R}) \cup (\mathbb{R} \times K(f))$.

If $K(f) = \mathbb{R}(>0)$, then $\mathbb{R}^2 \setminus \Delta = (\mathbb{R}(>0) \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{R}(>0))$ so that Δ is the third quadrant which is connected. Since sgn is continuous on Δ and since we suppose that there exists (y, z) such that $\text{sgn}(y, z) = -1$,

$$\text{sgn}(y, z) \equiv -1.$$

on Δ. Therefore $f(y f(z)) = -f(y)\, f(z)$ and $-f$ is SMS.

Similarly if K(f) $= \mathbb{R}\,(<0)$, we have that Δ is the first quadrant and -f is SMS /