CHAPTER II

THE NILPOTENT ALGEBRAS OF DIMENSION 4.

In this chapter we present some partial results in classifying the nilpotent algebras of dimension 4 over an arbitrary field.

Let A be a nilpotent algebra of dimension 4 over an arbitrary field K. Then we have $A \supset A^2 \supset A^3 \supset \ldots\ldots \supset A^k = \{0\}$. The proof is the same as page 3. Thus we see that if dimension A = 4 then dimension $A^2 = 3$ or 2 or 1 or 0. Dimension $A^2 = 0$ is the trivial case. Now we consider the case where dimension $A^2 = 3$. If dimension $A^2 = 3$, then dimension A^3 is 2 or 1 or 0 and dimension $A^4 = 1$ or 0 and $A^5 = \{0\}$.

Now we investigate the case where the dimension A^2 is 3 and $A^3 = \{0\}$. Let $\{e_1, e_2, e_3, e_4\}$ be a basis in A such that $\{e_1, e_2, e_3\}$ is a basis of A^2. For each x, y in A we can write

$$x = \sum_{i=1}^{4} a_i e_i ,$$

$$y = \sum_{j=1}^{4} b_j e_j , \ \{a_i, b_j\} \subset K \quad i,j = 1,2,3,4$$

and thus we obtain

$$xy = \sum_{j=1}^{4} \sum_{i=1}^{4} a_i b_j e_i e_j .$$

Since $e_ie_4, e_4e_i \varepsilon A^3 = \{0\}$, $i = 1,2,3, e_1e_i, e_2e_i, e_3e_i \varepsilon A^4 = \{0\}$ $i=1,2,3,$ we have

$$xy = a_4b_4e_4^2$$

and consequently, dimension of A^2 is 1. This contradicts the hypothesis that dimension $A^2 = 3$, so this case is impossible.

Next, we shall consider the case where the dimension of A^2 is 3 and dimension of $A^3 = 2$, $A^4 = \{0\}$.

First, we shall show that the following linear maps of A to itself are 1-1 and onto by showing that their determinants are not zero. Let $\{e_1, e_2, e_3, e_4\}$ and $\{e_1', e_2', e_3', e_4'\}$ be two bases of A.

(I)
$$\begin{aligned} f(e_1') &= e_1, \\ f(e_2') &= k_1e_2, \\ f(e_3') &= k_1k_5e_4, \\ f(e_4') &= k_2e_3, \quad \{k_i \neq 0\} \subset K,\ i = 1,2,5. \end{aligned}$$

f is 1-1 and onto, since

$$\det[f] = \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & k_1 & 0 & 0 \\ 0 & 0 & 0 & k_1k_5 \\ 0 & 0 & k_2 & 0 \end{bmatrix} = -k_1^2k_2k_5 \neq 0.$$

(II)
$$\begin{aligned} f(e_1') &= e_1, \\ f(e_2') &= k_1e_2 + k_2e_3, \\ f(e_3') &= k_1k_4e_3, \\ f(e_4') &= k_3e_4, \quad \{k_i \neq 0\} \subset K,\ i = 1,2,3,4. \end{aligned}$$

f is 1-1 and onto, since

$$\det [f] = \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & k_1 & k_2 & 0 \\ 0 & 0 & k_1k_4 & 0 \\ 0 & 0 & 0 & k_3 \end{bmatrix} = k_1^2k_3k_4 \neq 0.$$

(III) $f(e_1') = e_1,$

$f(e_2') = k_1e_2 + k_3e_4,$

$f(e_3') = k_1k_5e_4,$

$f(e_4') = k_2e_3, \quad \{k_i \neq 0\} \subset K, \ i = 1,2,3,5.$

f is 1-1 and onto, since

$$\det [f] = \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & k_1 & 0 & k_3 \\ 0 & 0 & 0 & k_1k_5 \\ 0 & 0 & k_2 & 0 \end{bmatrix} = -k_1^2k_2k_5 \neq 0.$$

(IV) $f(e_1') = e_1,$

$f(e_2') = k_1e_2 - e_4,$

$f(e_3') = k_1k_4e_3 + k_1k_5e_4,$

$f(e_4') = e_4 , \quad \{k_i \neq 0\} \subset K, \ i = 1,4,5.$

f is 1-1 and onto, since

$$\det [f] = \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & k_1 & 0 & -1 \\ 0 & 0 & k_1k_4 & k_1k_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} = k_1^2k_4 \neq 0.$$

(V) $f(e_1') = e_1,$

$f(e_2') = k_1e_2,$

$f(e_3') = k_1k_4e_3 + k_1k_5e_4,$

$f(e_4') = k_2e_3, \quad \{k_i \neq 0\} \subset K, \; i = 1,2,4,5.$

f is 1-1 and onto, since

$$\det [f] = \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & k_1 & 0 & 0 \\ 0 & 0 & k_1k_4 & k_1k_5 \\ 0 & 0 & k_2 & 0 \end{bmatrix} = -k_1^2k_2k_5 \neq 0.$$

(VI) $f(e_1') = e_1,$

$f(e_2') = k_1e_2,$

$f(e_3') = k_1k_4e_3 + k_1k_5e_4,$

$f(e_4') = k_3e_4, \quad \{k_i \neq 0\} \subset K, \; i = 1,3,4,5.$

f is 1-1 and onto, since

$$\det [f] = \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & k_1 & 0 & 0 \\ 0 & 0 & k_1k_4 & k_1k_5 \\ 0 & 0 & 0 & k_3 \end{bmatrix} = k_1^2k_3k_4 \neq 0.$$

(VII) $f(e_1') = e_1$

$f(e_2') = k_1e_2 + k_2e_3,$

$f(e_3') = k_1k_4e_3 + k_1k_5e_4,$

$f(e_4') = k_3e_4, \quad \{k_i \neq 0\} \subset K, \; i = 1,2,3,4,5.$

f is 1-1 and onto, since

$$\det [f] = \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & k_1 & k_2 & 0 \\ 0 & 0 & k_1k_4 & k_1k_5 \\ 0 & 0 & 0 & k_3 \end{bmatrix} = k_1^2k_3k_4 \neq 0.$$

Theorem: Let A be a nilpotent algebra of dimension 4 over the field K. If dimension of A^2 is 3 and dimension of A^3 is 2 and $A^4 = \{0\}$, then the multiplication in A is uniquely determined up to isomorphism.

Proof: Since dimension A = 4, dimension $A^2 = 3$, dimension $A^3 = 2$ and $A^4 = \{0\}$, we let $\{e_1, e_2, e_3, e_4\}$ be a basis in A such that $\{e_2, e_3, e_4\}$ is a basis of A^2 and $\{e_3, e_4\}$ is a basis of A^3.

For each x, y in A we have that

$$x = \sum_{i=1}^{4} a_i e_i$$

$$y = \sum_{j=1}^{4} b_j e_j, \ \{a_i, b_j\} \subset K, \ i,j = 1,2,3,4,$$

and we obtain

$$xy = \sum_{j=1}^{4} \sum_{i=1}^{4} a_i b_j e_i e_j \ .$$

Since $e_2^2, e_1e_3, e_3e_1, e_1e_4, e_4e_1 \in A^4 = \{0\}$,

$e_2e_3, e_3e_2, e_2e_4, e_4e_2 \in A^5 = \{0\}$,

$e_3^2, e_4^2, e_3e_4, e_4e_3 \in A^6 = \{0\}$, we get

$$xy = a_1b_1e_1^2 + a_1b_2e_1e_2 + a_2b_1e_2e_1 .$$

Since $e_1^2 \varepsilon A^2$ and $e_1e_2, e_2e_1 \varepsilon A^3$, we can write

$$e_1^2 = k_1e_2 + k_2e_3 + k_3e_4 ,$$

$$e_1e_2 = k_4e_3 + k_5e_4,$$

$$e_2e_1 = k_6e_3 + k_7e_4 , \text{ for some } k_i \varepsilon K, \ i = 1,2,3,4,5,6,7.$$

Thus, the multiplication xy can be expressed in the form:

(*) $$xy = k_1a_1b_1e_2 + (k_2a_1b_1 + k_4a_1b_2 + k_6a_2b_1)e_3 + (k_3a_1b_1 + k_5a_1b_2 + k_7a_2b_1)e_4 .$$

Now we shall consider $k_i, i = 1,2,\ldots,7$. Since dimension of $A^2 = 3$, the case $k_1 = 0$ and $k_2 = k_4 = k_6 = 0$ and $k_3 = k_5 = k_7 = 0$ cannot occur. Therefore, we consider the following cases.

Case 1. If $k_1 \neq 0$, $k_2 \neq 0$, $k_3 \neq 0$ and $k_4 = k_5 = k_6 = k_7 = 0$, then the multiplication (*) becomes

$$xy = k_1a_1b_1e_2 + k_2a_1b_1e_3 + k_3a_1b_1e_4$$
$$= a_1b_1(k_1e_2 + k_2e_3 + k_3e_4).$$

This formula holds for all x, y in A and since $k_1e_2 + k_2e_3 + k_3e_4$ is a vector in A^2, we have dimension $A^2 = 1$ which contradicts the hypothesis. Therefore this case is impossible.

Case 2. Assume that $k_1 \neq 0$, $k_2 \neq 0$, $k_5 \neq 0$ and $k_3 = k_4 = k_6 = k_7 = 0$. Then the multiplication (*) is

(2.1) $$xy = k_1a_1b_1e_2 + k_2a_1b_1e_3 + k_5a_1b_2e_4 .$$

To check whether or not A is associative under the multiplication in this case, let $z = \sum_{=1}^{4} c_\ell e_\ell$, $\{c_\ell\}_{\ell=1,2,3,4} \subset K$.

Using (2.1) we have that

$$(xy)z = [(\sum_{i=1}^{4} a_i e_i)(\sum_{j=1}^{4} b_j e_j)](\sum_{\ell=1}^{4} c_\ell e_\ell)$$

$$= (k_1 a_1 b_1 e_2 + k_2 a_1 b_1 e_3 + k_5 a_1 b_2 e_4)(c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4)$$

$$= 0 ,$$

on the otherhand ,

$$x(yz) = (\sum_{i=1}^{4} a_i e_i)[(\sum_{j=1}^{4} b_j e_j)(\sum_{\ell=1}^{4} c_\ell e_\ell)]$$

$$= (a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4)(k_1 b_1 c_1 e_2 + k_2 b_1 c_1 e_3 + k_5 b_1 c_2 e_4)$$

$$= k_5 a_1 (k_1 b_1 c_1) e_4 .$$

Hence A is not associative under the multiplication in this case, i.e. the multiplication in this case is impossible.

Next, we shall consider the following cases.

<u>Case 3</u>. If $k_1 \neq 0$, $k_2 \neq 0$, $k_7 \neq 0$ and $k_3 = k_4 = k_5 = k_6 = 0$, then from (*) we have that

$$xy = k_1 a_1 b_1 e_2 + k_2 a_1 b_1 e_3 + k_7 a_2 b_1 e_4 .$$

<u>Case 4</u>. If $k_1 \neq 0$, $k_4 \neq 0$, $k_3 \neq 0$ and $k_2 = k_5 = k_6 = k_7 = 0$, then from (*) we have that

$$xy = k_1 a_1 b_1 e_2 + k_4 a_1 b_2 e_3 + k_3 a_1 b_1 e_4 .$$

Case 5. If $k_1 \neq 0$, $k_4 \neq 0$, $k_5 \neq 0$ and $k_2 = k_3 = k_6 = k_7 = 0$, then from (*) we have that

$$xy = k_1 a_1 b_1 e_2 + k_4 a_1 b_2 e_3 + k_5 a_1 b_2 e_4 .$$

Case 6. If $k_1 \neq 0$, $k_6 \neq 0$, $k_3 \neq 0$ and $k_2 = k_4 = k_5 = k_7 = 0$, then from (*) we have that

$$xy = k_1 a_1 b_1 e_2 + k_6 a_2 b_1 e_3 + k_3 a_1 b_1 e_4 .$$

Case 7. If $k_1 \neq 0$, $k_6 \neq 0$, $k_7 \neq 0$ and $k_2 = k_3 = k_4 = k_5 = 0$, then from (*) we have that

$$xy = k_1 a_1 b_1 e_2 + k_6 a_2 b_1 e_3 + k_7 a_2 b_1 e_4 .$$

Case 8. If $k_1 \neq 0$, $k_2 \neq 0$, $k_4 \neq 0$, $k_3 \neq 0$ and $k_5 = k_6 = k_7 = 0$, then from (*) we have that

$$xy = k_1 a_1 b_1 e_2 + (k_2 a_1 b_1 + k_4 a_1 b_2) e_3 + b_3 a_1 b_1 e_4 .$$

Case 9. If $k_1 \neq 0$, $k_2 \neq 0$, $k_4 \neq 0$, $k_5 \neq 0$ and $k_3 = k_6 = k_7 = 0$, then from (*) we have that

$$xy = k_1 a_1 b_1 e_2 + (k_2 a_1 b_1 + k_4 a_1 b_2) e_3 + k_3 a_1 b_1 e_4 .$$

Case 10. If $k_1 \neq 0$, $k_2 \neq 0$, $k_6 \neq 0$, $k_3 \neq 0$ and $k_4 = k_5 = k_7 = 0$, then from (*) we have that

$$xy = k_1 a_1 b_1 e_2 + (k_2 a_1 b_1 + k_6 a_2 b_1) e_3 + k_3 a_1 b_1 e_4 .$$

Case 11. If $k_1 \neq 0$, $k_2 \neq 0$, $k_6 \neq 0$, $k_7 \neq 0$ and $k_3 = k_4 = k_5 = 0$, then from (*) we have that

$$xy = k_1 a_1 b_1 e_2 + (k_2 a_1 b_1 + k_6 a_2 b_1) e_3 + k_7 a_2 b_1 e_4 .$$

Case 12. If $k_1 \neq 0$, $k_2 \neq 0$, $k_3 \neq 0$, $k_5 \neq 0$ and $k_4 = k_6 = k_7 = 0$, then from (*) we have that

$$xy = k_1 a_1 b_1 e_2 + k_2 a_1 b_1 e_3 + (k_3 a_1 b_1 + k_5 a_1 b_2) e_4.$$

Case 13. If $k_1 \neq 0$, $k_2 \neq 0$, $k_3 \neq 0$, $k_7 \neq 0$ and $k_4 = k_5 = k_6 = 0$, then from (*) we have that

$$xy = k_1 a_1 b_1 e_2 + k_2 a_1 b_1 e_3 + (k_3 a_1 b_1 + k_7 a_2 b_1) e_4.$$

Case 14. If $k_1 \neq 0$, $k_4 \neq 0$, $k_3 \neq 0$, $k_5 \neq 0$ and $k_2 = k_6 = k_7 = 0$, then from (*) we have that

$$xy = k_1 a_1 b_1 e_2 + k_4 a_1 b_2 e_3 + (k_3 a_1 b_1 + k_5 a_1 b_2) e_4.$$

Case 15. If $k_1 \neq 0$, $k_6 \neq 0$, $k_3 \neq 0$, $k_7 \neq 0$ and $k_2 = k_4 = k_5 = 0$, then from (*) we have that

$$xy = k_1 a_1 b_1 e_2 + k_6 a_2 b_1 e_3 + (k_3 a_1 b_1 + k_7 a_2 b_1) e_4.$$

Case 16. If $k_1 \neq 0$, $k_2 \neq 0$, $k_4 \neq 0$, $k_3 \neq 0$, $k_5 \neq 0$ and $k_6 = k_7 = 0$, then from (*) we have that

$$xy = k_1 a_1 b_1 e_2 + (k_2 a_1 b_1 + k_4 a_1 b_2) e_3 + (k_3 a_1 b_1 + k_5 a_1 b_2) e_4.$$

Case 17. If $k_1 \neq 0$, $k_2 \neq 0$, $k_6 \neq 0$, $k_3 \neq 0$, $k_7 \neq 0$ and $k_4 = k_5 = 0$, then from (*) we have that

$$xy = k_1 a_1 b_1 e_2 + (k_2 a_1 b_1 + k_6 a_2 b_1) e_3 + (k_3 a_1 b_1 + k_7 a_2 b_1) e_4.$$

We can prove that A is not associative under the multiplications of case 3 to case 17. The proof of these cases are similar to that of case 2. Therefore, the multiplications in these cases are impossible.

<u>Case 18</u>. Assume that $k_1 \neq 0$, $k_4 \neq 0$, $k_7 \neq 0$ and $k_2 = k_3 = k_5 = k_6 = 0$. Then the multiplication (*) is

$$(18.1) \qquad xy = k_1 a_1 b_1 e_2 + k_4 a_1 b_2 e_3 + k_7 a_2 b_1 e_4.$$

To check whether or not A is associative under the multiplication (18.1), let $z = \sum_{\ell=1}^{4} c_\ell e_\ell$, $\{c_\ell\}_{\ell=1,2,3,4} \subset K$.

By (18.1) we have that

$$\begin{aligned} (xy)z &= \left[\left(\sum_{i=1}^{4} a_i e_i\right)\left(\sum_{j=1}^{4} b_j e_j\right)\right]\left(\sum_{\ell=1}^{4} c_\ell e_\ell\right) \\ &= (k_1 a_1 b_1 e_2 + k_4 a_1 b_2 e_3 + k_7 a_2 b_1 e_4)\left(\sum_{\ell=1}^{4} c_\ell e_\ell\right) \\ &= k_7 (k_1 a_1 b_1) c_1 e_4, \end{aligned}$$

on the other hand,

$$\begin{aligned} x(yz) &= \left(\sum_{i=1}^{4} a_i e_i\right)\left[\left(\sum_{j=1}^{4} b_j e_j\right)\left(\sum_{\ell=1}^{4} c_\ell e_\ell\right)\right] \\ &= \left(\sum_{i=1}^{4} a_i e_i\right)(k_1 b_1 c_1 e_2 + k_4 b_1 c_2 e_3 + k_7 b_2 c_1 e_4) \\ &= k_4 a_1 (k_1 b_1 c_1) e_3. \end{aligned}$$

Therefore, A is not associative under the multiplication (18.1). (If not, it contradicts the independence of basis).

Similarly, A is not associative under the multiplications in the following cases.

Case 19. If $k_1 \neq 0, k_6 \neq 0, k_5 \neq 0$ and $k_2 = k_3 = k_4 = k_7 = 0$, then from (*) we have that

$$xy = k_1 a_1 b_1 e_2 + k_6 a_2 b_1 e_3 + k_5 a_1 b_2 e_4.$$

Case 20. If $k_1 \neq 0, k_2 \neq 0, k_4 \neq 0, k_7 \neq 0$ and $k_3 = k_5 = k_6 = 0$, then from (*) we have that

$$xy = k_1 a_1 b_1 e_2 + (k_2 a_1 b_1 + k_4 a_1 b_2) e_3 + k_7 a_2 b_1 e_4.$$

Case 21. If $k_1 \neq 0, k_2 \neq 0, k_6 \neq 0, k_5 \neq 0$ and $k_3 = k_4 = k_7 = 0$, then from (*) we have that

$$xy = k_1 a_1 b_1 e_2 + (k_2 a_1 b_1 + k_6 a_2 b_1) e_3 + k_5 a_1 b_2 e_4.$$

Case 22. If $k_1 \neq 0, k_4 \neq 0, k_6 \neq 0, k_5 \neq 0$ and $k_2 = k_3 = k_7 = 0$, then from (*) we have that

$$xy = k_1 a_1 b_1 e_2 + (k_4 a_1 b_2 + k_6 a_2 b_1) e_3 + k_5 a_1 b_2 e_4.$$

Case 23. If $k_1 \neq 0, k_4 \neq 0, k_6 \neq 0, k_7 \neq 0$ and $k_2 = k_3 = k_5 = 0$, then from (*) we have that

$$xy = k_1 a_1 b_1 e_2 + (k_4 a_1 b_2 + k_6 a_2 b_1) e_3 + k_7 a_2 b_1 e_4.$$

Case 24. If $k_1 \neq 0, k_4 \neq 0, k_3 \neq 0, k_7 \neq 0$ and $k_2 = k_5 = k_6 = 0$, then from (*) we have that

$$xy = k_1 a_1 b_1 e_2 + k_4 a_1 b_2 e_3 + (k_3 a_1 b_1 + k_7 a_2 b_1) e_4.$$

Case 25. If $k_1 \neq 0, k_4 \neq 0, k_5 \neq 0, k_7 \neq 0$ and $k_2 = k_3 = k_6 = 0$, then from (*) we have that

$$xy = k_1 a_1 b_1 e_2 + k_4 a_1 b_2 e_3 + (k_5 a_1 b_2 + k_7 a_2 b_1) e_4.$$

Case 26. If $k_1 \neq 0, k_6 \neq 0, k_3 \neq 0, k_5 \neq 0$ and $k_2 = k_4 = k_7 = 0$, then from (*) we have that

$$xy = k_1a_1b_1e_2 + k_6a_2b_1e_3 + (k_3a_1b_1 + k_5a_1b_2)e_4.$$

Case 27. If $k_1 \neq 0, k_6 \neq 0, k_5 \neq 0, k_7 \neq 0$ and $k_2 = k_3 = k_4 = 0$, then from (*) we have that

$$xy = k_1a_1b_1e_2 + k_6a_2b_1e_3 + (k_5a_1b_2 + k_7a_2b_1)e_4.$$

Case 28. If $k_1 \neq 0, k_2 \neq 0, k_4 \neq 0, k_6 \neq 0, k_5 \neq 0$ and $k_3 = k_7 = 0$, then from (*) we have that

$$xy = k_1a_1b_1e_2 + (k_2a_1b_1 + k_4a_1b_2 + k_6a_2b_1)e_3 + k_5a_1b_2e_4.$$

Case 29. If $k_1 \neq 0, k_2 \neq 0, k_4 \neq 0, k_6 \neq 0, k_7 \neq 0$ and $k_3 = k_5 = 0$, then from (*) we have that

$$xy = k_1a_1b_1e_2 + (k_2a_1b_1 + k_4a_1b_2 + k_6a_2b_1)e_3 + k_7a_2b_1e_4.$$

Case 30. If $k_1 \neq 0, k_4 \neq 0, k_3 \neq 0, k_5 \neq 0, k_7 \neq 0$ and $k_2 = k_6 = 0$, then from (*) we have that

$$xy = k_1a_1b_1e_2 + k_4a_1b_2e_3 + (k_3a_1b_1 + k_5a_1b_2 + k_7a_2b_1)e_4.$$

Case 31. If $k_1 \neq 0, k_6 \neq 0, k_3 \neq 0, k_5 \neq 0, k_7 \neq 0$ and $k_2 = k_4 = 0$, then from (*) we have that

$$xy = k_1a_1b_1e_2 + k_6a_2b_1e_3 + (k_3a_1b_1 + k_5a_1b_2 + k_7a_2b_1)e_4.$$

Case 32. If $k_1 \neq 0, k_2 \neq 0, k_4 \neq 0, k_3 \neq 0, k_7 \neq 0$ and $k_6 = k_5 = 0$, then from (*) we have that

$$xy = k_1a_1b_1e_2 + (k_2a_1b_2 + k_4a_1b_2)e_3 + (k_3a_1b_1 + k_7a_2b_1)e_4.$$

Case 33. If $k_1 \neq 0, k_2 \neq 0, k_4 \neq 0, k_5 \neq 0, k_7 \neq 0$ and $k_6 = k_3 = 0$, then from (*) we have that

$$xy = k_1a_1b_1e_2 + (k_2a_1b_1 + k_4a_1b_2)e_3 + (k_5a_1b_2 + k_7a_2b_1)e_4.$$

Case 34. If $k_1 \neq 0, k_2 \neq 0, k_6 \neq 0, k_3 \neq 0, k_5 \neq 0$ and $k_4 = k_7 = 0$, then from (*) we have that

$$xy = k_1a_1b_1e_2 + (k_2a_1b_1 + k_6a_2b_1)e_3 + (k_3a_1b_1 + k_5a_1b_2)e_4.$$

Case 35. If $k_1 \neq 0, k_2 \neq 0, k_6 \neq 0, k_5 \neq 0, k_7 \neq 0$ and $k_3 = k_4 = 0$, then from (*) we have that

$$xy = k_1a_1b_1e_2 + (k_2a_1b_1 + k_6a_2b_1)e_3 + (k_5a_1b_2 + k_7a_2b_1)e_4.$$

Case 36. If $k_1 \neq 0, k_4 \neq 0, k_6 \neq 0, k_3 \neq 0, k_5 \neq 0$ and $k_2 = k_7 = 0$, then from (*) we have that

$$xy = k_1a_1b_1e_2 + (k_4a_1b_2 + k_6a_2b_1)e_3 + (k_3a_1b_1 + k_5a_1b_2)e_4.$$

Case 37. If $k_1 \neq 0, k_4 \neq 0, k_6 \neq 0, k_3 \neq 0, k_7 \neq 0$ and $k_2 = k_5 = 0$, then from (*) we have that

$$xy = k_1a_1b_1e_2 + (k_4a_1b_2 + k_6a_2b_1)e_3 + (k_3a_1b_1 + k_7a_2b_1)e_4$$

Case 38. If $k_1 \neq 0, k_2 \neq 0, k_4 \neq 0, k_6 \neq 0, k_3 \neq 0, k_5 \neq 0$ and $k_7 = 0$, then from (*) we have that

$$xy = k_1a_1b_1e_2 + (k_2a_1b_1 + k_4a_1b_2 + k_6a_2b_1)e_3 + (k_3a_1b_1 + k_5a_1b_2)e_4.$$

Case 39. If $k_1 \neq 0, k_2 \neq 0, k_4 \neq 0, k_6 \neq 0, k_3 \neq 0, k_7 \neq 0$ and $k_5 = 0$, then from (*) we have that

$$xy = k_1a_1b_1e_2 + (k_2a_1b_1 + k_4a_1b_2 + k_6a_2b_1)e_3 + (k_3a_1b_1 + k_7a_2b_1)e_4.$$

<u>Case 40</u>. If $k_1 \neq 0, k_2 \neq 0, k_4 \neq 0, k_3 \neq 0, k_5 \neq 0, k_7 \neq 0$ and $k_6 = 0$, then from (*) we have that

$$xy = k_1a_1b_1e_2 + (k_2a_1b_1 + k_4a_1b_2)e_3 + (k_3a_1b_1 + k_5a_1k_2 + k_7a_2b_1)e_4.$$

<u>Case 41</u>. If $k_1 \neq 0, k_2 \neq 0, k_6 \neq 0, k_3 \neq 0, k_5 \neq 0, k_7 \neq 0$ and $k_4 = 0$, then from (*) we have that

$$xy = k_1a_1b_1e_2 + (k_2a_1b_1 + k_6a_2b_1)e_3 + (k_3a_1b_1 + k_5a_1b_2 + k_7a_2b_1)e_4.$$

Hence, the multiplications of case 19 to case 41 are impossible.

<u>Case 42</u>. Assume that $k_1 \neq 0, k_4 \neq 0, k_6 \neq 0, k_3 \neq 0$ and $k_2 = k_5 = k_7 = 0$. Then the multiplication (*) is

$$xy = k_1a_1b_1e_2 + (k_4a_1b_2 + k_6a_2b_1)e_3 + k_3a_1b_1e_4.$$

Choose a new basis e_1', e_2', e_3' such that $e_1' = e_1$, $e_2' = k_1e_2$, $e_3' = k_1k_4e_3$ and $e_4' = k_3e_4$. Therefore, we get that

$$xy = a_1'b_1'(e_1')^2 + a_1'b_2'e_1'e_2' + a_2'b_1'e_2'e_1',$$

for $x = \sum_{i=1}^{4} a_o'e_i'$, $y = \sum_{j=1}^{4} b_je_j$, $\{a_i', b_j'\} \subset K$, $i,j = 1,2,3,4$.

$$\text{Since} \quad (e_1')^2 = e_1^2 = k_1e_2 + k_2e_3 + k_3e_4 = k_1e_2 + k_3e_4 = e_2' + e_4' ,$$

$$e_1'e_2' = k_1e_1e_2 = k_1(k_4e_3 + k_5e_4) = k_1k_4e_3 = e_3' ,$$

$$\text{and} \quad e_2'e_1' = k_1e_2e_1 = k_1(k_6e_3 + k_7e_4) = k_1k_6e_3 = \frac{k_6}{k_4} e_3' ,$$

we get that $xy = a_1'b_1'e_2' + (a_1'b_2' + \frac{k_6}{k_4} a_2'b_1')e_3' + a_1'b_1'e_4'$.

To check associativity, let $z = \sum_{\ell=1}^{4} c'_\ell e'_\ell$, $\{c'_\ell\}_{\ell=1,2,3,4} \subset K$.

Then

$$(xy)z = \left[\left(\sum_{i=1}^{4} a_i e_i\right)\left(\sum_{j=1}^{4} b'_j e'_j\right)\right]\left(\sum_{\ell=1}^{4} c'_\ell e'_\ell\right)$$

$$= \left[a'_1 b'_1 e'_2 + (a'_1 b'_2 + \frac{k_6}{k_4} a'_2 b'_1)e'_3 + a'_1 b'_1 e'_4\right]\left(\sum_{\ell=1}^{4} c'_\ell e'_\ell\right)$$

$$= \frac{k_6}{k_4} (a'_1 b'_1) c'_1 e'_3,$$

on the other hand,

$$x(yz) = \left(\sum_{i=1}^{4} a'_i e'_i\right)\left[\left(\sum_{j=1}^{4} b'_j e'_j\right)\left(\sum_{\ell=1}^{4} c'_\ell e'_\ell\right)\right]$$

$$= \left(\sum_{i=1}^{4} a'_i e'_i\right)\left[b'_1 c'_1 e'_2 + (b'_1 c'_2 + \frac{k_6}{k_4} b'_2 c'_1)e'_3 + b'_1 c'_1 e'_4\right]$$

$$= a'_1 (b'_1 c'_1) e'_3.$$

Since A is an associative algebra, we must have that

$\frac{k_6}{k_4} a'_1 b'_1 c'_1 = a'_1 b'_1 c'_1$. That is $\frac{k_6}{k_4} = 1$. Hence the multiplication in this case can be written as

$$(42.1) \quad xy = a'_1 b'_1 e'_2 + (a'_1 b'_2 + a'_2 b'_1)e'_3 + a'_1 b'_1 e'_4.$$

Case 43. Let $k_1 \neq 0$, $k_2 \neq 0, k_5 \neq 0, k_7 \neq 0$ and $k_3 = k_4 = k_6 = 0$. Then from (*) we have that

$$x \circ y = k_1 a_1 b_1 e_2 + k_2 a_1 b_1 e_3 + (k_5 a_1 b_2 + k_7 a_2 b_1) e_4.$$

To check associativity, let $z = \sum_{\ell=1}^{4} c_\ell e_\ell$, $\{c_\ell\}_{\ell=1,2,3,4} \subset K$.

We have that

$$(xoy)oz = \left[\left(\sum_{i=1}^{4} a_i e_i\right) o \left(\sum_{j=1}^{4} b_j e_j\right)\right] o \left(\sum_{\ell=1}^{4} c_\ell e_\ell\right)$$

$$= \left[k_1 a_1 b_1 e_2 + k_2 a_1 b_1 e_3 + (k_2 a_1 b_2 + k_7 a_2 b_1) e_4\right] o \left(\sum_{\ell=1}^{4} c_\ell e_\ell\right)$$

$$= k_7 (k_1 a_1 b_1) c_1 e_4,$$

whereas,

$$xo(yoz) = \left(\sum_{i=1}^{4} a_i e_i\right) o \left[\left(\sum_{j=1}^{4} b_j e_j\right) o \left(\sum_{\ell=1}^{4} c_\ell e_\ell\right)\right]$$

$$= \left(\sum_{i=1}^{4} a_i e_i\right) o \left[k_1 b_1 c_1 e_2 + k_2 b_1 c_1 e_3 + (k_5 b_1 c_2 + k_7 b_2 c_1) e_4\right]$$

$$= k_5 a_1 (k_1 b_1 c_1) e_4.$$

Since A is an associative algebra, we must have that $k_1 k_7 a_1 b_1 c_1 = k_1 k_5 a_1 b_1 c_1$. Therefore, $k_7 = k_5$. Hence, the multiplication in this case is

$$(43.1) \qquad xoy = k_1 a_1 b_1 e_2 + k_2 a_1 b_1 e_3 + (k_5 a_1 b_2 + k_5 a_2 b_1) e_4.$$

We claim that this multiplication is isomorphic to the multiplication (42.1) in case 42. To prove this, let f: A → A be the linear map defined by

$$f(e_1') = e_1,$$

$$f(e_2') = k_1 e_2,$$

$$f(e_3') = k_1 k_5 e_4,$$

$$f(e_4') = k_2 e_3, \quad k_i \in K, \ i = 1,2,5.$$

Then f is 1-1 and onto by argument (I) page 24. (42.1) implies that

$$f(xy) = f[a_1'b_1'e_2' + (a_1'b_2' + a_2'b_1')e_3' + a_1'b_1'e_4']$$

$$= k_1a_1'b_1'e_2 + k_1k_5(a_1'b_2' + a_2'b_1')e_4 + k_2a_1'b_1'e_3,$$

whereas, (43.1) implies that

$$f(x)of(y) = f(\sum_{i=1}^{4} a_i'e_i')o\ f(\sum_{j=1}^{4} b_j'e_j')$$

$$= (a_1'e_1 + k_1a_2'e_2 + k_2a_4'e_3 + k_1k_5a_3'e_4)o(b_1'e_1 + k_1b_2'e_2 + k_2b_4'e_3 + k_1k_5b_3'e_4)$$

$$= k_1a_1'b_1'e_2 + k_2a_1'b_1'e_3 + [k_5a_1'(k_1b_2') + k_5(k_1a_2')b_1']e_4$$

$$= k_1a_1'b_1'e_2 + k_2a_1'b_1'e_3 + k_1k_5(a_1'b_2' + a_2'b_1')e_4.$$

That is f(xy) = f(x)of(y), and consequently these two multiplications are isomorphic

<u>Case 44.</u> Assume that $k_1 \neq 0, k_2 \neq 0, k_4 \neq 0, k_6 \neq 0, k_3 \neq 0$ and $k_5 = k_7 = 0$. Then the multiplication (*) is

$$xoy = k_1a_1b_1e_2 + (k_2a_1b_1 + k_4a_1b_2 + k_6a_2b_1)e_3 + k_3a_1b_1e_4.$$

To check associativity, let $z = \sum_{\ell=1}^{4} c_\ell e_\ell$, $\{c_\ell\}_{\ell=1,2,3,4} \subset K$.

Then

$$(xoy)oz = [(\sum_{i=1}^{4} a_ie_i)o(\sum_{j=1}^{4} b_je_j)]o(\sum_{\ell=1}^{4} c_\ell e_\ell)$$

$$= [k_1a_1b_1e_2 + (k_2a_1b_1 + k_4a_1b_2 + k_6a_2b_1)e_3 + k_3a_1b_1e_4]o(\sum_{\ell=1}^{4} c_\ell e_\ell)$$

$$= k_6(k_1a_1b_1)c_1e_3,$$

whereas,

$$xo(yoz) = (\sum_{i=1}^{4} a_i e_i)o\left[(\sum_{j=1}^{4} b_j e_j)o(\sum_{\ell=1}^{4} c_\ell e_\ell)\right]$$

$$= (\sum_{i=1}^{4} a_i e_i)o\left[k_1 b_1 c_1 e_2+(k_2 b_1 c_1+k_4 b_1 c_2+k_6 b_2 c_1)e_3 +k_3 b_1 c_1 e_4\right].$$

$$= k_4 a_1 (k_1 b_1 c_1)e_3.$$

Since A is an associative algebra, we must have that $k_1 k_6 a_1 b_1 c_1 = k_1 k_4 a_1 b_1 c_1$. That is $k_4 = k_6$. Hence, the multiplication in this case becomes

$$(44.1) \quad xoy = k_1 a_1 b_1 e_2+(k_2 a_1 b_1+k_4 a_1 b_2+k_4 a_2 b_1)e_3+k_3 a_1 b_1 e_4.$$

We claim that this multiplication is isomorphic to the multiplication (42.1) in case 42. To prove this, let $f: A \to A$ be the linear map defined by

$$f(e_1') = e_1$$

$$f(e_2') = k_1 e_2+k_2 e_3,$$

$$f(e_3') = k_1 k_4 e_3,$$

$$f(e_4') = k_3 e_4 \ , \ k_i \in K, \ i = 1,2,3,4.$$

Then by (II) page 24 f is 1-1 and onto. By (42.1) we have that

$$f(xy) = f\left[a_1' b_1' e_2'+(a_1' b_2'+a_2' b_1')e_3'+a_1' b_1' e_4'\right]$$

$$= k_1 a_1' b_1' e_2+(k_2 a_1' b_1'+k_1 k_4 a_1' b_2'+k_1 k_4 a_2' b_1')e_3+k_3 a_1' b_1' e_4,$$

whereas, (44.1) implies that

$$f(x)of(y) = f(\sum_{i=1}^{4} a_i'e_i') \circ f(\sum_{j=1}^{4} b_j'e_j')$$

$$= [a_1'e_1'+k_1a_2'e_2+(k_2a_2'+k_1k_4a_3')e_3+k_3a_4'e_4] \circ$$

$$[b_1'e_1+k_1b_2'e_2+(k_2b_2'+k_1k_4b_3')e_3+k_3b_4'e_4]$$

$$= k_1a_1'b_1'e_2+(k_2a_1'b_1'+k_1k_4a_1'b_2'+k_1k_4a_2'b_1')e_3$$

$$+k_3a_1'b_1'e_4.$$

Therefore, these two multiplications are isomorphic.

Case 45. If $k_1\neq 0, k_2\neq 0, k_3\neq 0, k_5\neq 0, k_7\neq 0$ and $k_4= k_6= 0$, then the multiplication (*) is

$$xoy = k_1a_1b_1e_2+k_2a_1b_1e_3+(k_3a_1b_1+k_5a_1b_2+k_7a_2b_1)e_4.$$

As in case 43, we can show that

$$(xoy)oz = k_1k_7a_1b_1c_1e_4 \quad \text{and}$$

$$xo(yoz) = k_1k_5a_1b_1c_1e_4 .$$

Therefore, $k_5= k_7$. Hence the multiplication in this case is

$$(45.1)\ xoy = k_1a_1b_1e_2+k_2a_1b_1e_3+(k_3a_1b_1+k_5a_1b_2+k_5a_2b_1)e_4.$$

Claim that this multiplication is isomorphic to the multiplication (42.1) in case 42. To prove this, let f: $A\rightarrow A$ be the linear map defined by

$$f(e'_1) = e_1,$$
$$f(e'_2) = k_1e_2+k_3e_4,$$
$$f(e'_3) = k_1k_5e_4,$$
$$f(e'_4) = k_2e_3, \quad k_i \in K,\ i = 1,2,3,5.$$

Then f is 1-1 and onto (III) page 19. (42.1) implies that

$$f(xy) = f[a'_1b'_1e'_2+(a'_1b'_2+a'_2b'_1)e'_3+a'_1b'_1e'_4]$$
$$= k_1a'_1b'_1e_2+k_2a'_1b'_1e_3+(k_3a'_1b'_1+k_1k_5a'_1b'_2+k_1k_5a'_2b'_1)e_4,$$

and (45.1) implies that

$$f(x)of(y) = f(\sum_{i=1}^{4} a'_ie'_i) \circ f(\sum_{j=1}^{4} b'_je'_j)$$
$$= [a'_1e_1+k_1a'_2e_2+k_2a'_4e_3+(k_3a'_2+k_1k_5a'_3)e_4] \circ$$
$$[b'_1e_1+k_1b'_2e_2+k_2b'_4e_3+(k_3b'_2+k_1k_5b'_3)e_4]$$
$$= k_1a'_1b'_1e_2+k_2a'_1b'_1e_3+(k_3a'_1b'_1+k_1k_5a'_1b'_2$$
$$+k_1k_5a'_2b'_1)e_4.$$

Therefore, (42.1) and (45.1) are isomorphic.

<u>Case 46.</u> Suppose that $k_1\neq 0, k_4\neq 0, k_6\neq 0, k_5\neq 0, k_7\neq 0$ and $k_2= k_3= 0$. Then the multiplication (*) is

$$xoy = k_1a_1b_1e_2+(k_4a_1b_2+k_6a_2b_1)e_3+(k_5a_1b_2+k_7a_2b_1)e_4.$$

To check associativity, let $z = \sum_{\ell=1}^{4} c_\ell e_\ell$, $\{c_\ell\}_{\ell=1,2,3,4} \subset K$. Then

$$xoy = [(\sum_{i=1}^{4} a_ie_i)o(\sum_{j=1}^{4} b_je_j)]o(\sum_{\ell=1}^{4} c_\ell e_\ell)$$
$$= [k_1a_1b_1e_2+(k_4a_1b_2+k_6a_2b_1)e_3+(k_5a_1b_2+k_7a_2b_1)e_4] \circ (\sum_{\ell=1}^{4} c_\ell e_\ell)$$
$$= k_6(k_1a_1b_1)c_1e_3+ k_7(k_1a_1b_1)c_1e_4,$$

whereas,

$$xo(yoz) = (\sum_{i=1}^{4} a_i e_i)o\left[(\sum_{j=1}^{4} b_j e_j)o(\sum_{\ell=1}^{4} c_\ell e_\ell)\right]$$

$$= (\sum_{i=1}^{4} a_i e_i)o\left[(k_1b_1c_1e_2+(k_4b_1c_2+k_6b_2c_1)e_3+ (k_5b_1c_2+k_7b_2c_1)e_4\right]$$

$$= k_4a_1(k_1b_1c_1)e_3+k_5a_1(k_1b_1c_1)e_4.$$

Since A is an associative algebra, we must have that $k_1k_4a_1b_1c_1 = k_1k_6a_1b_1c_1$ and $k_1k_5a_1b_1c_1 = k_1k_7a_1b_1c_1$. That is $k_4 = k_6$ and $k_5 = k_7$. Hence the multiplication in this case can be written as

$$(46.1) \quad xoy = k_1a_1b_1e_2+(k_4a_1b_2+k_4a_2b_1)e_3+(k_5a_1b_2+k_5a_2b_1)e_4.$$

We claim that (46.1) is isomorphic to (42.1). To prove this, let $f: A \to A$ be the linear map defined by

$$f(e_1') = e_1$$
$$f(e_2') = k_1e_2-e_4,$$
$$f(e_3') = k_1k_4e_3+k_1k_5e_4,$$
$$f(e_4') = e_4, \quad k_i \in K, \ i = 1,4,5.$$

Then f is 1-1 and onto by IV page 25. By (42.1) we have that

$$f(xy) = f\left[a_1'b_1'e_2'+(a_1'b_2'+a_2'b_1')e_3'+a_1'b_1'e_4'\right]$$

$$= k_1a_1'b_1'e_2-a_1'b_1'e_4+(k_1k_4a_1'b_2'+k_1k_4a_2'b_1')e_3 +(k_1k_5a_1'b_2'+k_1k_5a_2'b_1')e_4+a_1'b_1'e_4$$

$$= k_1a_1'b_1'e_2+(k_1k_4a_1'b_2'+k_1k_4a_2'b_1')e_3+ (k_1k_5a_1'b_2'+k_1k_5a_2'b_1')e_4,$$

and by (46.1) we have that

$$f(x)of(y) = f(\sum_{i=1}^{4} a_i'e_i')o\ f(\sum_{j=1}^{4} b_j'e_j')$$

$$= [a_1'e_1+k_1a_2'e_2+k_1k_4a_3'e_3+(-a_2'+k_1k_3a_3'+a_4')e_4]\ o$$

$$[b_1'e_1+k_1b_2'e_2+k_1k_4b_3'e_3+(-b_2'+k_1k_3b_3'+b_4')e_4]$$

$$= k_1a_1'b_1'e_2+(k_1k_4a_1'b_2'+k_1k_4a_2'b_1')e_3+$$

$$(k_1k_5a_1'b_2'+k_1k_5a_2'b_1')e_4.$$

This proves the claim.

<u>Case 47.</u> Assume that $k_1 \neq 0, k_2 \neq 0, k_4 \neq 0, k_6 \neq 0, k_5 \neq 0,$ $k_7 \neq 0$ and $k_3 = 0$. Then from (*) we have that

$$xoy = k_1a_1b_1e_2+(k_2a_1b_1+k_4a_1b_2+k_6a_2b_1)e_3+(k_5a_1b_2+k_7a_2b_1)e_4.$$

As in case 46, we can prove that

$$(xoy)oz = k_1k_6a_1b_1c_1e_3+ k_1k_7a_1b_1c_1e_4$$

$$xo(yoz) = k_1k_4a_1b_1c_1e_3+ k_1k_5a_1b_1c_1e_4.$$

Therefore, $k_4 = k_6$ and $k_5 = k_7$. Hence the multiplication in this case can be written as

$$(47.1)\quad xoy = k_1a_1b_1e_2+(k_2a_1b_1+k_4a_1b_2+k_4a_2b_1)e_3+(k_5a_1b_2+k_5a_2b_1)e_4.$$

Claim that (47.1) is isomorphic to (42.1). To prove this, let $f: A \to A$ be the linear map defined by

$$f(e_1') = e_1,$$
$$f(e_2') = k_1 e_2,$$
$$f(e_3') = k_1 k_4 e_3 + k_1 k_5 e_4,$$
$$f(e_4') = k_2 e_3, \quad k_i \varepsilon K, \ i = 1,2,4,5.$$

Then f is 1-1 and onto by V page 26. Thus (42.1) implies that

$$f(xy) = f\left[a_1'b_1'e_2' + (a_1'b_2' + a_2'b_1')e_3' + a_1'b_1'e_4'\right]$$
$$= k_1 a_1'b_1'e_2 + (k_2 a_1'b_1' + k_1 k_4 a_1'b_2' + k_1 k_4 a_2'b_1')e_3$$
$$+ (k_1 k_5 a_1'b_2' + k_1 k_5 a_2'b_1')e_4 ,$$

and from (47.1) we get that

$$f(x)of(y) = f(\sum_{i=1}^{4} a_i'e_i') \circ f(\sum_{j=1}^{4} b_j'e_j')$$
$$= \left[a_1'e_1 + k_1 a_2'e_2 + (k_1 k_4 a_3' + k_2 a_4')e_3 + k_1 k_5 a_3'e_4\right] \circ$$
$$\left[b_1'e_1 + k_1 b_2'e_2 + (k_1 k_4 b_3' + k_2 b_4')e_3 + k_1 k_5 b_3'e_4\right]$$
$$= k_1 a_1'b_1'e_2 + (k_2 a_1'b_1' + k_1 k_4 a_1'b_2' + k_1 k_4 a_2'b_1')e_3$$
$$+ (k_1 k_5 a_1'b_2' + k_1 k_5 a_2'b_1')e_4.$$

This proves the claim.

Case 48. Assume that $k_1 \neq 0, k_4 \neq 0, k_6 \neq 0, k_3 \neq 0, k_5 \neq 0$, $k_7 \neq 0$ and $k_2 = 0$. Then the multiplication (*) is

$$xoy = k_1 a_1 b_1 e_2 + (k_4 a_1 b_2 + k_6 a_2 b_1)e_3 + (k_3 a_1 b_1 + k_5 a_1 b_2 + k_7 a_2 b_1)e_4.$$

As in case 46, we have that $k_4 = k_6$ and $k_5 = k_7$. Therefore, the multiplication in this case can be written as

$$(48.1) \quad xoy = k_1 a_1 b_1 e_2 + (k_4 a_1 b_2 + k_4 a_2 b_1)e_3 + (k_3 a_1 b_1 + k_5 a_1 b_2 + k_5 a_2 b_1)e_4.$$

Claim that this multiplication is isomorphic to the multiplication (42.1). To prove this, let f: A → A be the linear map defined by

$$f(e_1') = e_1,$$
$$f(e_2') = k_1e_2,$$
$$f(e_3') = k_1k_4e_3+k_1k_5e_4,$$
$$f(e_4') = k_3e_4, \quad k_i \varepsilon K, \ i = 1,3,4,5.$$

By VI page 26, f is 1-1 and onto. Then (42.1) imples that

$$f(xy) = f[a_1'b_1'e_2'+(a_1'b_2'+a_2'b_1')e_3'+a_1'b_1'e_4']$$
$$= k_1a_1'b_1'e_2+(k_1k_4a_1'b_2'+k_1k_4a_2'b_1')e_3+ (k_3a_1'b_1'+k_1k_5a_1'b_2'+k_1k_5a_2'b_1')e_4$$

and (48.1) implies that

$$f(x)of(y) = f(\sum_{i=1}^{4} a_i'e_i') \circ f(\sum_{j=1}^{4} b_j'e_j')$$
$$= [a_1'e_1+k_1a_2'e_2+k_1k_4a_3'e_3+(k_1k_5a_3'+k_3a_4')e_4] \circ [b_1'e_1+k_1b_2'e_2+k_1k_4b_3'e_3+(k_1k_5b_3'+k_3b_4')e_4]$$
$$= k_1a_1'b_1'e_2+(k_1k_4a_1'b_2'+k_1k_4a_2'b_1')e_3+ (k_3a_1'b_1'+k_1k_5a_1'b_2'+k_1k_5a_2'b_1')e_4.$$

Hence (48.1) and (42.1) are isomorphic.

<u>Case 49</u>. In the final case, let all $k_i \neq 0$, $i = 1,\ldots,7$. Then from (*) we have that

$$xoy = k_1a_1b_1e_2+(k_2a_1b_1+k_4a_1b_2+k_6a_2b_1)e_3+(k_3a_1b_1+k_5a_1b_2+k_7a_2b_1)e_4.$$

As in case 46, we can prove that $k_4 = k_6$ and $k_5 = k_7$. Then the multiplication in this case can be written as

$$(49.1)\quad x o y = k_1a_1b_1e_2+(k_2a_1b_1+k_4a_1b_2+k_4a_2b_1)e_3+(k_3a_1b_1+k_5a_1b_2 +k_5a_2b_1)e_4.$$

Claim that (49.1) and (42.1) are isomorphic. To prove this, let $f: A \to A$ be the linear map defined by

$$f(e_1') = e_1,$$
$$f(e_2') = k_1e_2+k_2e_3,$$
$$f(e_3') = k_1k_4e_3+k_1k_5e_4,$$
$$f(e_4') = k_3e_4, \quad k_i \in K,\ i = 1,2,3,4,5.$$

Then f is 1-1 and onto by VII page 26. (42.1) implies that

$$f(xy) = f[a_1'b_1'e_2'+(a_1'b_2'+a_2'b_1')e_3'+a_1'b_1'e_4']$$
$$= k_1a_1'b_1'e_2+(k_2a_1'b_1'+k_1k_4a_1'b_2'+k_1k_4a_2'b_1')e_3 +(k_3a_1'b_1'+k_1k_5a_1'b_2'+k_1k_5a_2'b_1')e_4$$

and (49.1) implies that

$$f(x)of(y) = f(\sum_{i=1}^{4} a_i'e_i') \text{ o } f(\sum_{j=1}^{4} b_j'e_j')$$
$$= [a_1'e_1+k_1a_2'e_2+(k_2a_2'+k_1k_4a_3')e_3+(k_1k_5a_3'+k_3a_4')e_4] \text{ o } [b_1'e_1+k_1b_2'e_2+(k_2b_2'+k_1k_4b_3')e_3+(k_1k_5b_3'+k_3b_4')e_4]$$
$$= k_1a_1'b_1'e_2+(k_2a_1'b_1'+k_1k_4a_1'b_2'+k_1k_4a_2'b_1')e_3 +(k_3a_1'b_1'+k_1k_5a_1'b_2'+k_1k_5a_2'b_1')e_4.$$

This proves the claim.

Thus the multiplication in a nilpotent algebra A of dimension 4 over a field K with dimension $A^2 = 3$, dimension $A^3 = 2$ and $A^4 = \{0\}$ is uniquely determined up to isomorphism. #

Next, we begin to classify the multiplications in a 4-dimensional nilpotent algebra A over the field K with dimension $A^2 = 3$, dimension $A^3 = 1$ and $A^4 = \{0\}$.

Let $\{e_1, e_2, e_3, e_4\}$ be a basis in A such that $\{e_2, e_3, e_4\}$ is a basis of A^2 and e_4 is a basis of A^3.

For each x, y in A we can write

$$x = \sum_{i=1}^{4} a_i e_i,$$

$$y = \sum_{j=1}^{4} b_j e_j, \quad \{a_i, b_j\} \subset K,\ i,j = 1,2,3,4,$$

and we get that

$$xy = \sum_{j=1}^{4} \sum_{i=1}^{4} a_i b_j e_i e_j.$$

Since $e_2^2, e_1e_4, e_4e_1, e_2e_3, e_3e_2, e_3^2 \in A^4 = \{0\}$,

$e_2e_4, e_4e_2, e_3e_4, e_4e_3 \in A^5 = \{0\}$ and $e_4^2 \in A^6 = \{0\}$, we have that

$$xy = a_1b_1e_1^2 + a_1b_2e_1e_2 + a_1b_3e_1e_3 + a_2b_1e_2e_1 + a_3b_1e_3e_1.$$

Since $e_1^2 \in A^2$, we can write $e_1^2 = k_1e_2 + k_2e_3 + k_3e_4$ for some k_1, k_2, k_3 in K and since $e_1e_2, e_2e_1, e_1e_3, e_3e_1 \in A^3$, we can write $e_1e_2 = k_4e_4, e_2e_1 = k_5e_4$, $e_1e_3 = k_6e_4$ and $e_3e_1 = k_7e_4$ for some $k_i \in K$, $i = 4,5,6,7$. Therefore, by substituting e_1^2, e_1e_2, e_1e_3, e_2e_1, e_3e_1, we get

(**) $xy = k_1a_1b_1e_2+k_2a_1b_1e_3+(k_3a_1b_1+k_4a_1b_2+k_5a_2b_1+k_6a_1b_3+k_7a_3b_1)e_4.$

Since dimension of A^2 is 3, the case $k_1 = 0$ and $k_2 = 0$ and $k_3 = k_4 = k_5 = k_6 = k_7 = 0$ can not occur. Now we begin to consider the remaining cases.

<u>Case 1</u>. If $k_1 \neq 0$, $k_2 \neq 0$, $k_3 \neq 0$ and $k_4 = k_5 = k_6 = k_7 = 0$, then the multiplication (**) is

$$\begin{aligned} xy &= k_1a_1b_1e_2+k_2a_1b_1e_3+k_3a_1b_1e_4 \\ &= a_1b_1(k_1e_2+k_2e_3+k_3e_4). \end{aligned}$$

This multiplication holds for all x, y in A and since $k_1e_2+k_2e_3+k_3e_4$ is a vector in A^2, we have dimension $A^2 = 1$ which contradicts the hypothesis. Therefore, this case is impossible.

<u>Case 2</u>. Assume that $k_1 \neq 0$, $k_2 \neq 0$, $k_4 \neq 0$ and $k_3 = k_5 = k_6 = k_7 = 0$. Then from (**) we have

$$xy = k_1a_1b_1e_2+k_2a_1b_1e_3+k_4a_1b_2e_4.$$

To check whether or not A is associative under this multiplication, let $z = \sum_{\ell=1}^{4} c_\ell e_\ell$, $\{c_\ell\}_{\ell=1,2,3,4} \subset K$ then we have that

$$\begin{aligned} (xy)z &= \left[\left(\sum_{i=1}^{4} a_ie_i\right)\left(\sum_{j=1}^{4} b_je_j\right)\right]\left(\sum_{\ell=1}^{4} c_\ell e_\ell\right) \\ &= \left[k_1a_1b_1e_2+k_2a_1b_1e_3+k_4a_1b_2e_4\right]\left(\sum_{\ell=1}^{4} c_\ell e_\ell\right) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} x(yz) &= \left(\sum_{i=1}^{4} a_ie_i\right)\left[\left(\sum_{j=1}^{4} b_je_j\right)\left(\sum_{\ell=1}^{4} c_\ell e_\ell\right)\right] \\ &= \left(\sum_{i=1}^{4} a_ie_i\right)\left[k_1b_1c_1e_2+k_2b_1c_1e_3+k_4b_1c_2e_4\right] \\ &= k_4a_1(k_1b_1c_1)e_4. \end{aligned}$$

Therefore, A is not associative under this multiplication. So this case is impossible.

Now we consider the following cases.

Case 3. If $k_1 \neq 0, k_2 \neq 0, k_5 \neq 0$ and $k_3 = k_4 = k_6 = k_7 = 0$, then from (**) we have that

$$xy = k_1a_1b_1e_2+k_2a_1b_1e_3+k_5a_2b_1e_4.$$

Case 4. If $k_1 \neq 0, k_2 \neq 0, k_6 \neq 0$ and $k_3 = k_4 = k_5 = k_7 = 0$, then from (**) we have that

$$xy = k_1a_1b_1e_2+k_2a_1b_1e_3+k_6a_1b_3e_4.$$

Case 5. If $k_1 \neq 0, k_2 \neq 0, k_7 \neq 0$ and $k_3 = k_4 = k_5 = k_6 = 0$, then from (**) we have that

$$xy = k_1a_1b_1e_2+k_2a_1b_1e_3+k_7a_3b_1e_4$$

Case 6. If $k_1 \neq 0, k_2 \neq 0, k_3 \neq 0, k_4 \neq 0$ and $k_5 = k_6 = k_7 = 0$, then from (**) we have that

$$xy = k_1a_1b_1e_2+k_2a_1b_1e_3+(k_3a_1b_1+k_4a_1b_2)e_4.$$

Case 7. If $k_1 \neq 0, k_2 \neq 0, k_3 \neq 0, k_5 \neq 0$ and $k_4 = k_6 = k_7 = 0$, then from (**) we have that

$$xy = k_1a_1b_1e_2+k_2a_1b_1e_3+(k_3a_1b_1+k_5a_2b_1)e_4.$$

Case 8. If $k_1 \neq 0, k_2 \neq 0, k_3 \neq 0, k_6 \neq 0$ and $k_4 = k_5 = k_7 = 0$, then from (**) we have that

$$xy = k_1a_1b_1e_2+k_2a_1b_1e_3+(k_3a_1b_1+k_6a_1b_3)e_4.$$

Case 9. If $k_1 \neq 0, k_2 \neq 0, k_3 \neq 0, k_7 \neq 0$ and $k_4 = k_5 = k_6 = 0$, then from (**) we have that

$$xy = k_1a_1b_1e_2 + k_2a_1b_1e_3 + (k_3a_1b_1 + k_7a_3b_1)e_4.$$

As in case 2 we can prove that A is not associative under the above multiplications therefore, the multiplications in these cases are impossible.

Case 10. Assume that $k_1 \neq 0, k_2 \neq 0, k_4 \neq 0, k_5 \neq 0$ and $k_3 = k_6 = k_7 = 0$. Then from (**) we have that

$$xy = k_1a_1b_1e_2 + k_2a_1b_1e_3 + (k_4a_1b_2 + k_5a_2b_1)e_4.$$

Choose a new basis e_1', e_2', e_3', e_4' of A such that $e_1' = e_1$, $e_2' = k_1e_2$, $e_3' = k_2e_3$ and $e_4' = k_1k_4e_4$. Then we get that

$$xy = a_1'b_1'e_1'^2 + a_1'b_2'e_1'e_2' + a_1'b_3'e_1'e_3' + a_2'b_1'e_2'e_1' + a_3'b_1'e_3'e_1',$$

for $x = \sum_{i=1}^{4} a_i'e_i'$, $y = \sum_{j=1}^{4} b_j'e_j'$, $\{a_i', b_j'\} \subset K$, $i,j = 1,2,3,4$.

Since $e_1'^2 = e_1^2 = k_1e_2 + k_2e_3 + k_3e_4 = k_1e_2 + k_2e_3 = e_2' + e_3'$,

$$e_1'e_2' = k_1e_1e_2 = k_1k_4e_4 = e_4',$$

$$e_2'e_1' = k_1e_2e_1 = k_1k_5e_4 = \frac{k_5}{k_4}e_4',$$

$$e_1'e_3' = k_2e_1e_3 = k_2k_6e_4 = 0$$

and $e_3'e_1' = k_2e_3e_1 = k_2k_7e_4 = 0$, we have that

$$xy = a_1'b_1'e_2' + a_1'b_1'e_3' + (a_1'b_2' + \frac{k_5}{k_4}a_2'b_1')e_4'.$$

Let $z = \sum_{\ell=1}^{4} c'_\ell e'_\ell$, $\{c'_\ell\}_{\ell=1,2,3,4} \subset K$ then we have that

$$(xy)z = [(\sum_{i=1}^{4} a'_i e'_i)(\sum_{j=1}^{4} b'_j e'_j)](\sum_{\ell=1}^{4} c'_\ell e'_\ell)$$

$$= [a'_1 b'_1 e'_2 + a'_1 b'_1 e'_3 + (a'_1 b'_2 + \frac{k_5}{k_4} a'_2 b'_1) e'_4](\sum_{\ell=1}^{4} c'_\ell e'_\ell)$$

$$= \frac{k_5}{k_4}(a'_1 b'_1) c'_1 e'_4 ,$$

$$x(yz) = (\sum_{i=1}^{4} a'_i e'_i)[(\sum_{j=1}^{4} b'_j e'_j)(\sum_{\ell=1}^{4} c'_\ell e'_\ell)]$$

$$= (\sum_{i=1}^{4} a'_i e'_i)[b'_1 c'_1 e'_2 + b'_1 c'_1 e'_3 + (b'_1 c'_2 + \frac{k_5}{k_4} b'_2 c'_1) e'_4]$$

$$= a'_1 (b'_1 c'_1) e'_4 .$$

Since A is an associative algebra, we have that $\frac{k_5}{k_4} = 1$.

Therefore, the multiplication in this case can be written as

(10.1) $\quad xy = a'_1 b'_1 e'_2 + a'_1 b'_1 e'_3 + (a'_1 b'_2 + a'_2 b'_1) e'_4 .$

<u>Case 11.</u> Suppose that $k_1 \neq 0, k_2 \neq 0, k_6 \neq 0, k_7 \neq 0$ and $k_3 = k_4 = k_5 = 0$. Then the multiplication (**) is

$$x o y = k_1 a_1 b_1 e_2 + k_2 a_1 b_1 e_3 + (k_6 a_1 b_3 + k_7 a_3 b_1) e_4 .$$

Let $z = \sum_{\ell=1}^{4} c_\ell e_\ell$, $\{c_\ell\}_{\ell=1,2,3,4} \subset K$. Then we get that

$$(x o y) o z = [(\sum_{i=1}^{4} a_i e_i) o (\sum_{j=1}^{4} b_j e_j)] o (\sum_{\ell=1}^{4} c_\ell e_\ell)$$

$$= [k_1 a_1 b_1 e_2 + k_2 a_1 b_1 e_3 + (k_6 a_1 b_3 + k_7 a_3 b_1) e_4] o (\sum_{\ell=1}^{4} c_\ell e_\ell)$$

$$= k_7 (k_2 a_1 b_1) c_1 e_4 ,$$

on the other hand,

$$xo(yoz) = (\sum_{i=1}^{4} a_i e_i) o [(\sum_{j=1}^{4} b_j e_j) o (\sum_{\ell=1}^{4} c_\ell e_\ell)]$$

$$= (\sum_{i=1}^{4} a_i e_i) o [k_1 b_1 c_1 e_2 + k_2 b_1 c_1 e_3 + (k_6 b_1 c_3 + k_7 b_3 c_1) e_4]$$

$$= k_6 a_1 (k_2 b_1 c_1) e_4 .$$

Since A is an associative algebra, we have that $k_2 k_7 a_1 b_1 c_1 = k_2 k_6 a_1 b_1 c_1$ i.e. $k_6 = k_7$. Therefore, this multiplication can be written as

(11.1) $$xoy = k_1 a_1 b_1 e_2 + k_2 a_1 b_1 e_3 + (k_6 a_1 b_3 + k_6 a_3 b_1) e_4 .$$

We claim that (11.1) is isomorphic to (10.1). To prove this, let $f: A \to A$ be the linear map defined by

$$f(e_1') = e_1,$$
$$f(e_2') = k_2 e_3,$$
$$f(e_3') = k_1 e_2,$$
$$f(e_4') = k_2 k_6 e_4, \quad k_1, k_2, k_6 \in K.$$

Since det $[f] \neq 0$, f is 1-1 and onto. (10.1) implies that

$$f(xy) = f[a_1' b_1' e_2' + a_1' b_1' e_3' + (a_1' b_2' + a_2' b_1') e_4']$$

$$= k_2 a_1' b_1' e_3 + k_1 a_1' b_1' e_2 + (k_2 k_6 a_1' b_2' + k_2 k_6 a_2' b_1') e_4$$

and (11.1) implies that

$$f(x) o f(y) = f(\sum_{i=1}^{4} a_i' e_i') o f(\sum_{j=1}^{4} b_j' e_j')$$

$$= [a_1' e_1 + k_1 a_3' e_2 + k_2 a_2' e_3 + k_2 k_6 a_4' e_4] o [b_1' e_1 + k_1 b_3' e_2 + k_2 b_2' e_3 + k_2 k_6 b_4' e_4]$$

$$= k_1 a_1' b_1' e_2 + k_2 a_1' b_1' e_3 + (k_2 k_6 a_1' b_2' + k_2 k_6 a_2' b_1') e_4 .$$

Hence (11.1) and (10.1) are isomorphic.

Case 12. If $k_1 \neq 0, k_2 \neq 0, k_4 \neq 0, k_6 \neq 0$ and $k_3 = k_5 = k_7 = 0$, then the multiplication (**) is

$$xy = k_1a_1b_1e_2+k_2a_1b_1e_3+(k_4a_1b_2+k_6a_1b_3)e_4.$$

Let $z = \sum_{\ell=1}^{4} c_\ell e_\ell$, $\{c_\ell\} \subset K$, $\ell = 1,2,3,4$. It follows that

$$\begin{aligned}(xy)z &= \left[\left(\sum_{i=1}^{4} a_ie_i\right)\left(\sum_{j=1}^{4} b_je_j\right)\right]\left(\sum_{\ell=1}^{4} c_\ell e_\ell\right)\\ &= \left[k_1a_1b_1e_2+k_2a_1b_1e_3+(k_4a_1b_2+k_6a_1b_3)e_4\right]\left(\sum_{\ell=1}^{4} c_\ell e_\ell\right)\\ &= 0,\end{aligned}$$

on the other hand,

$$\begin{aligned}x(yz) &= \left(\sum_{i=1}^{4} a_ie_i\right)\left[\left(\sum_{j=1}^{4} b_je_j\right)\left(\sum_{\ell=1}^{4} c_\ell e_\ell\right)\right]\\ &= \left(\sum_{i=1}^{4} a_ie_i\right)\left[k_1b_1c_1e_2+k_2b_1c_1e_3+(k_4b_1c_2+k_6b_1c_3)e_4\right]\\ &= \left[k_4a_1(k_1b_1c_1)+k_6a_1(k_2b_1c_1)\right]e_4.\end{aligned}$$

Hence A is not associative. (If not, then for all x,y,z in A we have $(xy)z = x(yz) = 0$ which implies $A^3 = \{0\}$. This contradicts the hypothesis that dimension $A^3 = 1$.) The multiplication in this case is impossible.

Case 13. Assume that $k_1 \neq 0, k_2 \neq 0, k_4 \neq 0, k_7 \neq 0$ and $k_3 = k_5 = k_6 = 0$. Then the multiplication (**) is

$$x * y = k_1a_1b_1e_2+k_2a_1b_1e_3+(k_4a_1b_2+k_7a_3b_1)e_4.$$

We can choose a new basis $e_1'', e_2'', e_3'', e_4''$ such that $e_1'' = e_1$, $e_2'' = k_1e_2, e_3'' = k_2e_3, e_4'' = k_1k_4e_4$. Then

$$x * y = a_1''b_1''(e_1'')^2 + a_1''b_2''e_1''e_2'' + a_1''b_3''e_1''e_3'' + a_2''b_1''e_2''e_1'' + a_3''b_1''e_3''e_1'',$$

for $x = \sum_{i=1}^{4} a_i''e_i''$, $y = \sum_{j=1}^{4} b_j''e_j''$, $\{a_i'', b_j''\} \subset K$, $i,j = 1,2,3,4$.

Since $(e_1'')^2 = e_1^2 = k_1e_2 + k_2e_3 + k_3e_4 = k_1e_2 + k_2e_3 = e_2'' + e_3''$,

$$e_1''e_2'' = k_1e_1e_2 = k_1k_4e_4 = e_4'',$$

$$e_2''e_1'' = k_1e_2e_1 = k_1k_5e_4 = 0,$$

$$e_1''e_3'' = k_2e_1e_3 = k_2k_6e_4 = 0$$

and $e_3''e_1'' = k_2e_3e_1 = k_2k_7e_4 = \dfrac{k_2k_7}{k_1k_4} e_4$, we have that

$$x * y = a_1''b_1''e_2'' + a_1''b_1''e_3'' + (a_1''b_2'' + \frac{k_2k_7}{k_1k_4} a_3''b_1'')e_4''.$$

Let $z = \sum_{=1}^{4} c_\ell''e_\ell''$, $\{c_\ell''\}_{\ell=1,2,3,4} \subset K$. Then we have that

$$\begin{aligned}(x * y)*z &= \Big[(\sum_{i=1}^{4} a_i''e_i'')*(\sum_{j=1}^{4} b_j''e_j'')\Big]*(\sum_{\ell=1}^{4} c_\ell''e_\ell'') \\ &= \Big[a_1''b_1''e_2'' + a_1''b_1''e_3'' + (a_1''b_2'' + \frac{k_2k_7}{k_1k_4} a_3''b_1'')e_4''\Big]*(\sum_{\ell=1}^{4} c_\ell''e_\ell'') \\ &= \frac{k_2k_7}{k_1k_4}(a_1''b_1'')c_1''e_4'',\end{aligned}$$

whereas,

$$\begin{aligned}x*(y * z) &= (\sum_{i=1}^{4} a_i''e_i'')*\Big[(\sum_{j=1}^{4} b_j''e_j'')*(\sum_{\ell=1}^{4} c_\ell''e_\ell'')\Big] \\ &= (\sum_{i=1}^{4} a_i''e_i'')*\Big[b_1''c_1''e_2'' + b_1''c_1''e_3'' + (b_1''c_2'' + \frac{k_2k_7}{k_1k_4} b_3''c_1'')e_4''\Big] \\ &= a_1''(b_1''c_1'')e_4''.\end{aligned}$$

Since A is an associative algebra, we must have that

$\frac{k_2k_7}{k_1k_4}a_1''b_1''c_1'' = a_1''b_1''c_1''$ i.e. $\frac{k_2k_7}{k_1k_4} = 1$. Therefore, the multiplication in this case can be written as

(13.1) $\quad x * y = a_1''b_1''e_2''+a_1''b_1''e_3''+(a_1''b_2''+a_3''b_1'')e_4''$.

Observe that A is not commutative under this multiplication, but A is commutative under the multiplication (10.1) in case 10. Therefore, the multiplications (13.1) and (10.1) are not issomorphic

<u>Case 14.</u> Assume that $k_1 \neq 0, k_2 \neq 0, k_5 \neq 0, k_6 \neq 0$ and $k_3 = k_4 = k_7 = 0$. Then the multiplication (**) is

(14.1) $\quad xy = k_1a_1b_1e_2+k_2a_1b_1e_3+(k_5a_2b_1+k_6a_1b_3)e_4$.

Let $z = \sum_{\ell=1}^{4} c_\ell e_\ell$, $\{c_\ell\}_{\ell=1,2,3,4} \subset K$. We have that

$$(xy)z = \left[\left(\sum_{i=1}^{4} a_ie_i\right)\left(\sum_{j=1}^{4} b_je_j\right)\right]\left(\sum_{\ell=1}^{4} c_\ell e_\ell\right)$$

$$= \left[k_1a_1b_1e_2+k_2a_1b_1e_3+(k_5a_2b_1+k_6a_1b_3)e_4\right]\left(\sum_{\ell=1}^{4} c_\ell e_\ell\right)$$

$$= k_5(k_1a_1b_1)c_1e_4,$$

$$x(yz) = \left(\sum_{i=1}^{4} a_ie_i\right)\left[\left(\sum_{j=1}^{4} b_je_j\right)\left(\sum_{\ell=1}^{4} c_\ell e_\ell\right)\right]$$

$$= \left(\sum_{i=1}^{4} a_ie_i\right)\left[k_1b_1c_1e_2+k_2b_1c_1e_3+(k_5b_2c_1+k_6b_1c_3)e_4\right]$$

$$= k_6a_1(k_2b_1c_1)e_4.$$

Since A is an associative algebra, we have that $k_1k_5 = k_2k_6$. We claim that the multiplication (14.1) and (13.1) are isomorphic. To prove this, let $f\colon A \to A$ be the linear map defined by

$$\begin{aligned} f(e''_1) &= e_1, \\ f(e''_2) &= k_2e_3, \\ f(e''_3) &= k_1e_2, \\ f(e''_4) &= k_2k_6e_4, \quad k_1,k_2,k_6 \in K. \end{aligned}$$

Then f is 1-1 and onto. (13.1) implies that

$$\begin{aligned} f(x * y) &= f\left[a''_1b''_1e''_2+a''_1b''_1e''_3+(a''_1b''_2+a''_3b''_1)e''_4\right] \\ &= k_2a''_1b''_1e_3+k_1a''_1b''_1e_2+(k_2k_6a''_1b''_1+k_2k_6a''_3b''_1)e_4 \end{aligned}$$

and (14.1) implies that

$$\begin{aligned} f(x)f(y) &= f\left(\sum_{i=1}^{4} a''_ie''_i\right)f\left(\sum_{j=1}^{4} b''_je''_j\right) \\ &= \left[a''_1e_1+k_1a''_3e_2+k_2a''_2e_3+k_2k_6a''_4e_4\right]\left[b''_1e_1+k_1b''_3e_2 \right. \\ &\qquad \left. +k_2b''_2e_3+k_2k_6b''_4e_4\right] \\ &= k_1a''_1b''_1e_2+k_2a''_1b''_1e_3+(k_1k_5a''_3b''_1+k_2k_6a''_1b''_2)e_4 \\ &= k_1a''_1b''_1e_2+k_2a''_1b''_1e_3+(k_2k_6a''_1b''_2+k_2k_6a''_3b''_1)e_4, \end{aligned}$$

This proves the claim.

<u>Case 15</u>. If $k_1 \neq 0, k_2 \neq 0, k_5 \neq 0, k_7 \neq 0$ and $k_3 = k_4 = k_6 = 0$, then from (**) we have that

$$xy = k_1a_1b_1e_2+k_2a_1b_1e_3+(k_5a_2b_1+k_7a_3b_1)e_4.$$

As in case 12, we can prove that A is not associative under this multiplication. Therefore, this case is impossible

Case 16. Suppose that $k_1 \neq 0, k_2 \neq 0, k_3 \neq 0, k_4 \neq 0, k_5 \neq 0$ and $k_6 = k_7 = 0$. Then from (**) we have that

$$xoy = k_1a_1b_1e_2+k_2a_1b_1e_3+(k_3a_1b_1+k_4a_1b_2+k_5a_2b_1)e_4.$$

Let $z = \sum_{\ell=1}^{4} c_\ell e_\ell$, $\{c_\ell\}_{\ell=1,2,3,4} \subseteq K$. We consider (xoy)oz and xo(yoz).

$$\begin{aligned}(xoy)oz &= \left[\left(\sum_{i=1}^{4} a_ie_i\right)o\left(\sum_{j=1}^{4} b_je_j\right)\right]o\left(\sum_{\ell=1}^{4} c_\ell e_\ell\right)\\ &= \left[k_1a_1b_1e_2+k_2a_1b_1e_3+(k_3a_1b_1+k_4a_1b_2+k_5a_2b_1)e_4\right]o\left(\sum_{\ell=1}^{4} c_\ell e_\ell\right)\\ &= k_5(k_1a_1b_1)c_1e_4,\\ xo(yoz) &= \left(\sum_{i=1}^{4} a_ie_i\right)o\left[\left(\sum_{j=1}^{4} b_je_j\right)o\left(\sum_{\ell=1}^{4} c_\ell e_\ell\right)\right]\\ &= \left(\sum_{i=1}^{4} a_ie_i\right)o\left[k_1b_1c_1e_2+k_2b_1c_1e_3+(k_3b_1c_1+k_4b_1c_2+k_5b_2c_1)e_4\right]\\ &= k_4a_1(k_1b_1c_1)e_4.\end{aligned}$$

Since A is an associative algebra, we must have that $k_4 = k_5$. Therefore, this multiplication becomes

(16.1) $$xoy = k_1a_1b_1e_2+k_2a_1b_1e_3+(k_3a_1b_1+k_4a_1b_2+k_4a_2b_1)e_4.$$

Claim that this multiplication is isomorphic to the multiplication (10.1) in case 10. To prove this, let $f: A \to A$ be the linear map defined by

$$f(e_1') = e_1,$$
$$f(e_2') = k_1e_2+k_3e_4,$$
$$f(e_3') = k_2e_3,$$
$$f(e_4') = k_1k_4e_4, \quad k_i \in K, \ i = 1,2,3,4.$$

Then f is 1-1 and onto. (10.1) implies that

$$f(xy) = f[a_1'b_1'e_2'+a_1'b_1'e_3'+(a_1'b_2'+a_2'b_1')e_4']$$
$$= k_1a_1'b_1'e_2+k_2a_1'b_1'e_3+(k_3a_1'b_1'+k_1k_4a_1'b_2'+k_1k_4a_2'b_1')e_4,$$

where as, (16.1) implies that

$$f(x)of(y) = f(\sum_{i=1}^{4} a_1'e_1')of(\sum_{j=1}^{4} b_j'e_j')$$
$$= [a_1'e_1'+k_1a_2'e_2+k_2a_3'e_3+(k_3a_2'+k_1k_4a_4')e_4] \circ$$
$$[b_1'e_1+k_1b_2'e_2+k_2b_3'e_3+(k_3b_2'+k_1k_4b_4')e_4]$$
$$= k_1a_1'b_1'e_2+k_2a_1'b_1'e_3+(k_3a_1'b_1'+k_1k_4a_1'b_2'+k_1k_4a_2'b_1')e_4.$$

Therefore, (16.1) and (10.1) are isomorphic.

Case 17. Let $k_1\neq 0, k_2\neq 0, k_3\neq 0, k_6\neq 0, k_7\neq 0$ and $k_4=k_5=0$. Then from (**) we have that

$$xoy = k_1a_1b_1e_2+k_2a_1b_1e_3+(k_3a_1b_1+k_6a_1b_3+k_7a_3b_1)e_4.$$

As in case 11 (since A is an associative algebra), we get that $k_6= k_7$. Therefore, we can write xoy as

$$(17.1) \quad xoy = k_1a_1b_1e_2+k_2a_1b_1e_3+(k_3a_1b_1+k_6a_1b_3+k_6a_3b_1)e_4.$$

Claim that (17.1) and (10.1) are isomorphic. To prove this, let $f\colon A \to A$ be the linear map defined by

$$f(e_1') = e_1,$$
$$f(e_2') = k_2e_3+k_3e_4,$$
$$f(e_3') = k_1e_2,$$
$$f(e_4') = k_2k_6e_4,\ k_i \in K,\ i = 1,2,3,6.$$

Since det $[f] \neq 0$, f is 1-1 and onto. (10.1) implies that

$$f(xy) = f[a_1'b_1'e_2'+a_1'b_1'e_3'+(a_1'b_2'+a_2'b_1')e_4']$$
$$= k_2a_1'b_1'e_3+k_1a_1'b_1'e_2+(k_3a_1'b_1'+k_2k_6a_1'b_2'+k_2k_6a_2'b_1')e_4$$

and (17.1) implies that

$$f(x)of(y) = f(\sum_{i=1}^{4} a_i'e_1')of(\sum_{j=1}^{4} b_j'e_j')$$
$$= [a_1'e_1+k_1a_3'e_2+k_2a_2'e_3+(k_3a_2'+k_2k_6a_4')e_4]\ o$$
$$[b_1'e_1+k_1b_3'e_2+k_2b_2'e_3+(k_3b_2'+k_2k_6b_4')e_4]$$
$$= k_1a_1'b_1'e_2+k_2a_1'b_1'e_3+(k_3a_1'b_1'+k_2k_6a_1'b_2'+k_2k_6a_2'b_1')e_4.$$

Therefore, $f(xy) = f(x)of(y)$. This proves the claim.

Case 18. If $k_1\neq 0, k_2\neq 0, k_3\neq 0, k_4\neq 0, k_6\neq 0$ and $k_5= k_7= 0$, Then from (**) we have that

$$xy = k_1a_1b_1e_2+k_2a_1b_1e_3+(k_3a_1b_1+k_4a_1b_2+k_6a_1b_3)e_4.$$

As in case 12, we can prove that A is not associative under this multiplication. Therefore, this case is impossible.

Case 19. Suppose that $k_1\neq 0, k_2\neq 0, k_3\neq 0, k_4\neq 0, k_7\neq 0$ and $k_5= k_6= 0$. Then the multiplication (**) is

(19.1) $$xy = k_1a_1b_1e_2+k_2a_1b_1e_3+(k_3a_1b_1+k_4a_1b_2+k_7a_3b_1)e_4.$$

Let $z = \sum_{\ell=1}^{4} c_\ell e_\ell$, $\{c_\ell\}_{\ell=1,2,3,4} \subset K$. We consider $(xy)z$ and $x(yz)$.

$$(xy)z = \Big[\Big(\sum_{i=1}^{4} a_ie_i\Big)\Big(\sum_{j=1}^{4} b_je_j\Big)\Big]\Big(\sum_{\ell=1}^{4} c_\ell e_\ell\Big)$$

$$= \big[k_1a_1b_1e_2+k_2a_1b_1e_3+(k_3a_1b_1+k_4a_1b_2+k_7a_3b_1)e_4\big]\Big(\sum_{\ell=1}^{4} c_\ell e_\ell\Big)$$

$$= k_7(k_2a_1b_1)c_4e_4,$$

$$x(yz) = \Big(\sum_{i=1}^{4} a_ie_i\Big)\Big[\Big(\sum_{j=1}^{4} b_je_j\Big)\Big(\sum_{\ell=1}^{4} c_\ell e_\ell\Big)\Big]$$

$$= \Big(\sum_{i=1}^{4} a_ie_i\Big)\big[k_1b_1c_1e_2+k_2b_1c_1e_3+(k_3b_1c_1+k_4b_1c_2+k_7b_3c_1)e_4\big]$$

$$= k_4a_1(k_1b_1c_1)e_4.$$

Since A is an associative algebra, we must have that $k_2k_7 = k_1k_4$. We claim that (19.1) is isomorphic to (13.1). To prove this, let $f: A \to A$ be the linear map defined by

$$f(e_1'') = e_1,$$

$$f(e_2'') = k_1e_2+k_3e_4,$$

$$f(e_3'') = k_2e_3,$$

$$f(e_4'') = k_1k_4e_4, \quad k_i \in K,\ i = 1,2,3,4.$$

Then f is 1-1 and onto. By (13.1) we have that

$$f(x * y) = f\big[a_1''b_1''e_2''+a_1''b_1''e_3''+(a_1''b_2''+a_3''b_1'')e_4''\big]$$

$$= k_1a_1''b_1''e_2+k_2a_1''b_1''e_3+(k_3a_1''b_1''+k_1k_4a_1''b_2''+k_1k_4a_3''b_1'')e_4,$$

whereas, (19.1) implies that

$$f(x)f(y) = f(\sum_{i=1}^{4} a_i''e_i'')f(\sum_{j=1}^{4} b_j''e_j'')$$

$$= [a_1''e_1+k_1a_2''e_2+k_2a_3''e_3+(k_3a_2''+k_1k_4a_4'')e_4]$$

$$[b_1''e_1+k_1b_2''e_2+k_2b_3''e_3+(k_3b_2''+k_1k_4b_4'')e_4]$$

$$= k_1a_1''b_1''e_2+k_2a_1''b_1''e_3+(k_3a_1''b_1''+k_1k_4a_1''b_2''+k_2k_7a_3''b_1'')e_4.$$

Since $k_2k_7= k_1k_4$, we get that $f(x * y) = f(x)f(y)$. Therefore, this proves the claim.

<u>Case 20.</u> Assume that $k_1\neq 0, k_2\neq 0, k_3\neq 0, k_5\neq 0, k_6\neq 0$ and $k_4= k_7= 0$. Then from (**) we have that

(20.1) $$xy = k_1a_1b_1e_2+k_2a_1b_1e_3+(k_3a_1b_1+k_5a_2b_1+k_6a_1b_3)e_4.$$

Then $k_1k_5 = k_2k_6$ by the associativity of A (as in case 14). Claim that (20.1) and (13.1) are isomorphic. To prove this, let $f: A \to A$ be the linear map defined by

$$f(e_1'') = e_1,$$

$$f(e_2'') = k_2e_3+k_3e_4,$$

$$f(e_3'') = k_1e_2,$$

$$f(e_4'') = k_1k_5e_4, \quad k_i\epsilon K, i = 1,2,3,5.$$

Then f is 1-1 and onto. (13.1) implies that

$$f(x * y) = f[a_1''b_1''e_2''+a_1''b_1''e_3''+(a_1''b_2''+a_3''b_1'')e_4'']$$

$$= k_2a_1''b_1''e_3+k_1a_1''b_1''e_2+(k_3a_1''b_1''+k_1k_5a_1''b_2''+k_1k_5a_3''b_1'')e_4,$$

and (20.1) implies that

$$f(x)f(y) = f(\sum_{i=1}^{4} a_i'' e_i'') f(\sum_{j=1}^{4} b_j'' e_j'')$$

$$= [a_1'' e_1 + k_1 a_3'' e_2 + k_2 a_2'' e_3 + (k_3 a_2'' + k_1 k_5 a_4'') e_4][b_1'' e_1 + k_1 b_3'' e_2 + k_2 b_2'' e_3 + (k_3 b_2'' + k_1 k_5 b_4'') e_4]$$

$$= k_1 a_1'' b_1'' e_2 + k_2 a_1'' b_1'' e_3 + (k_3 a_1'' b_1'' + k_1 k_5 a_3'' b_1'' + k_2 k_6 a_1'' b_2'') e_4$$

$$= k_1 a_1'' b_1'' e_2 + b_2 a_1'' b_1'' e_3 + (k_3 a_1'' b_1'' + k_1 k_5 a_1'' b_2'' + k_1 k_5 a_3'' b_1'') e_4 .$$

Therefore, (20.1) and (13.1) are isomorphic

Case 21. If $k_1 \neq 0, k_2 \neq 0, k_3 \neq 0, k_5 \neq 0, k_7 \neq 0$ and $k_4 = k_6 = 0$, then from (**) we have that

$$xy = k_1 a_1 b_1 e_2 + k_2 a_1 b_1 e_3 + (k_3 a_1 b_1 + k_5 a_2 b_1 + k_7 a_3 b_1) e_4 .$$

As in case 12, we can prove that A is not associative under this multiplication. Therefore, this case is impossible.

Case 22. Assume that $k_1 \neq 0, k_2 \neq 0, k_4 \neq 0, k_5 \neq 0, k_6 \neq 0$ and $k_3 = k_7 = 0$. Then the multiplication (**) is

$$(22.1) \qquad xy = k_1 a_1 b_1 e_2 + k_2 a_1 b_1 e_3 + (k_4 a_1 b_2 + k_5 a_2 b_1 + k_6 a_1 b_3) e_4 .$$

Let $z = \sum_{\ell=1}^{4} c_\ell e_\ell$, $\{c_\ell\}_{\ell=1,2,3,4} \subset K$. It follows that

$$(xy)z = [(\sum_{i=1}^{4} a_i e_i)(\sum_{j=1}^{4} b_j e_j)](\sum_{\ell=1}^{4} c_\ell e_\ell)$$

$$= [k_1 a_1 b_1 e_2 + k_2 a_1 b_1 e_3 + (k_4 a_1 b_2 + k_5 a_2 b_1 + k_6 a_1 b_3) e_4](\sum_{\ell=1}^{4} c_\ell e_\ell)$$

$$= k_5 (k_1 a_1 b_1) c_1 e_4 ,$$

$$x(yz) = (\sum_{i=1}^{4} a_i e_i)[(\sum_{j=1}^{4} b_j e_j)(\sum_{\ell=1}^{4} c_\ell e_\ell)]$$

$$= (\sum_{i=1}^{4} a_i e_i)[k_1 b_1 c_1 e_2 + k_2 b_1 c_1 e_3 + (k_4 b_1 c_2 + k_5 b_2 c_1 + k_6 b_1 c_3) e_4]$$

$$= [k_4 a_1 (k_1 b_1 c_1) + k_5 a_1 (k_2 b_1 c_1)] e_4$$

Since A is an associative algebra, we must have that $k_1 k_5 = k_1 k_4 + k_2 k_6$. We claim that (22.1) is isomorphic to (13.1). To prove this, let $f\colon A \to A$ be the linear map defined by

$$f(e_1') = e_1'',$$

$$f(e_2) = \frac{k_4}{k_1 k_5} e_2'' + \frac{e_3''}{k_1},$$

$$f(e_3) = \frac{k_6}{k_1 k_5} e_2'',$$

$$f(e_4) = \frac{e_4''}{k_1 k_5}, \quad \{k_i \neq 0\}_{i=1,5,4,6} \subset K .$$

Since $\det [f] = \dfrac{-k_6}{k_1^3 k_5} \neq 0$, f is 1-1 and onto. (22.1) implies that

$$f(xy) = f[k_1 a_1 b_1 e_2 + k_2 a_1 b_1 e_3 + (k_4 a_1 b_2 + k_5 a_2 b_1 + k_6 a_1 b_3) e_4]$$

$$= \frac{k_1 k_4}{k_1 k_5} a_1 b_1 e_2'' + a_1 b_1 e_3'' + \frac{k_2 k_6}{k_1 k_5} a_1 b_1 e_2'' + (k_4 a_1 b_2 + k_5 a_2 b_1 + k_6 a_1 b_3) \frac{e_4''}{k_1 k_5},$$

$$= a_1 b_1 e_2'' + a_1 b_1 e_3'' + (k_4 a_1 b_2 + k_5 a_2 b_1 + k_6 a_1 b_3) \frac{e_4''}{k_1 k_5},$$

whereas, (13.1) implies that

$$f(x)*f(y) = f(\sum_{i=1}^{4} a_i e_i) * f(\sum_{j=1}^{4} b_j e_j)$$

$$= [a_1 e_1'' + (\frac{k_4}{k_1 k_5} a_2 + \frac{k_6}{k_1 k_5} a_3) e_2'' + \frac{a_2}{k_1} e_3'' + \frac{a_4}{k_1 k_5} e_4''] *$$

$$[b_1 e_1'' + (\frac{k_4}{k_1 k_5} b_2 + \frac{k_6}{k_1 k_5} b_3) e_2'' + \frac{b_2}{k_1} e_3'' + \frac{b_4}{k_1 k_5} e_4'']$$

$$= a_1 b_1 e_2'' + a_1 b_1 e_3'' + [a_1(\frac{k_4}{k_1 k_5} b_2 + \frac{k_6}{k_1 k_5} b_3) + \frac{a_2}{k_1} b_1] e_4''$$

$$= a_1 b_1 e_2'' + a_1 b_1 e_3'' + [k_4 a_1 b_2 + k_5 a_2 b_1 + k_6 a_1 b_3] \frac{e_4''}{k_1 k_5}.$$

This proves the claim

Case 23. Suppose that $k_1 \neq 0, k_2 \neq 0, k_4 \neq 0, k_5 \neq 0, k_7 \neq 0$ and $k_3 = k_6 = 0$. Then from (**) we have that

(23.1) $$xy = k_1 a_1 b_1 e_2 + k_2 a_1 b_1 e_3 + (k_4 a_1 b_2 + k_5 a_2 b_1 + k_7 a_3 b_1) e_4.$$

Let $z = \sum_{\ell=1}^{4} c_\ell e_\ell$, $\{c_\ell\}_{\ell=1,2,3,4} \subset K$. We see that

$$(xy)z = [k_1 a_1 b_1 e_2 + k_2 a_1 b_1 e_3 + (k_4 a_1 b_2 + k_5 a_2 b_1 + k_7 a_3 b_1) e_4](\sum_{\ell=1}^{4} c_\ell e_\ell)$$

$$= [k_5(k_1 a_1 b_1) c_1 + k_7 (k_2 a_1 b_1) c_1] e_4,$$

$$x(yz) = (\sum_{i=1}^{4} a_i e_i)[k_1 b_1 c_1 e_2 + k_2 b_1 c_1 e_3 + (k_4 b_1 c_2 + k_5 b_2 c_1 + k_7 b_3 c_1) e_4]$$

$$= k_4 a_1 (k_1 b_1 c_1) e_4.$$

Since A is an associative algebra, we get that $k_1 k_4 = k_1 k_5 + k_2 k_7$. We claim that this multiplication is isomorphic to the multiplication (13.1). To prove this, let f: A → A be the linear map defined by

$$f(e_1) = e_1'',$$

$$f(e_2) = \frac{e_2''}{k_1} + \frac{k_5 e_3''}{k_1 k_4},$$

$$f(e_3) = \frac{k_7 e_3''}{k_1 k_4}$$

$$f(e_4) = \frac{e_4''}{k_1 k_4}, \quad \{k_i \neq 0\}_{i=1,4,5,7} \subset K.$$

Since det $[f] \neq 0$, f is 1-1 and onto. (23.1) implies that

$$\begin{aligned} f(xy) &= f[k_1 a_1 b_1 e_2 + k_2 a_1 b_1 e_3 + (k_4 a_1 b_2 + k_5 a_2 b_1 + k_7 a_3 b_1) e_4] \\ &= a_1 b_1 e_2'' + \left(\frac{k_1 k_5 + k_2 k_7}{k_1 k_4}\right) a_1 b_1 e_3'' + (k_4 a_1 b_2 + k_5 a_2 b_1 + k_7 a_3 b_1) \frac{e_4''}{k_1 k_4} \\ &= a_1 b_1 e_2'' + a_1 b_1 e_3'' + (k_4 a_1 b_2 + k_5 a_2 b_1 + k_7 a_3 b_1) \frac{e_4''}{k_1 k_4}, \end{aligned}$$

whereas, (13.1) implies that

$$\begin{aligned} f(x) * f(y) &= f\left(\sum_{i=1}^{4} a_i e_i\right) * f\left(\sum_{j=1}^{4} b_j e_j\right) \\ &= \left[a_1 e_1'' + \frac{a_2}{k_1} e_2'' + \left(\frac{k_5}{k_1 k_4} a_2 + \frac{k_7}{k_1 k_4} a_3\right) e_3'' + \frac{a_4}{k_1 k_4} e_4''\right] * \left[b_1 e_1'' + \frac{b_2}{k_1} e_2'' \right. \\ &\qquad \left. + \left(\frac{k_5}{k_1 k_4} b_2 + \frac{k_7}{k_1 k_4} b_3\right) e_3'' + \frac{b_4}{k_1 k_4} e_4''\right] \\ &= a_1 b_1 e_2'' + a_1 b_1 e_3'' + \left[\frac{a_1 b_2}{k_1} + \left(\frac{k_5}{k_1 k_4} a_2 + \frac{k_7}{k_1 k_4} a_3\right) b_1\right] e_4'' \\ &= a_1 b_1 e_2'' + a_1 b_1 e_3'' + [k_4 a_1 b_2 + k_5 a_2 b_1 + k_7 a_3 b_1] \frac{e_4''}{k_1 k_4}. \end{aligned}$$

Therefore, (23.1) and (13.1) are isomorphic.

Case 24. Suppose that $k_1 \neq 0, k_2 \neq 0, k_4 \neq 0, k_6 \neq 0, k_7 \neq 0$ and $k_3 = k_5 = 0$. Then the multiplication (**) is

$$(24.1) \qquad xy = k_1a_1b_1e_2+k_2a_1b_1e_3+(k_4a_1b_2+k_6a_1b_3+k_7a_3b_1)e_4.$$

Let $z = \sum_{\ell=1}^{4} c_\ell e_\ell$, $\{c_\ell\}_{\ell=1,2,3,4} \subset K$. We have that

$$(xy)z = [(\sum_{i=1}^{4} a_ie_i)(\sum_{j=1}^{4} b_je_j)](\sum_{\ell=1}^{4} c_\ell e_\ell)$$

$$= [k_1a_1b_1e_2+k_2a_1b_1e_3+(k_4a_1b_2+k_6a_1b_3+k_7a_3b_1)e_4](\sum_{\ell=1}^{4} c_\ell e_\ell)$$

$$= k_7(k_2a_1b_1)c_1e_4,$$

$$\text{whereas, } x(yz) = (\sum_{i=1}^{4} a_ie_i)[(\sum_{j=1}^{4} b_je_j)(\sum_{\ell=1}^{4} c_\ell e_\ell)]$$

$$= (\sum_{i=1}^{4} a_ie_i)[k_1b_1c_1e_2+k_2b_1c_1e_3+(k_4b_1c_2+k_6b_1c_3+k_7b_3c_1)e_4]$$

$$= [k_4a_1(k_1b_1c_1)+k_6a_1(k_2b_1c_1)]\, e_4.$$

Since A is an associative algebra, we must have that $k_2k_7 = k_1k_4+k_2k_6$. Claim that this multiplication is isomorphic to the multiplication (13.1). Let $f: A \to A$ be the linear map defined by

$$f(e_1) = e_1'',$$

$$f(e_2) = \frac{k_4}{k_2k_7} e_2'' ,$$

$$f(e_3) = \frac{k_6}{k_2k_7} e_2'' + \frac{e_3''}{k_2} ,$$

$$f(e_4) = \frac{e_4''}{k_2k_7} , \quad \{k_i \neq 0\}_{i=2,4,6,7} \subset K.$$

Then f is 1-1 and onto. Using (24.1) we get that

$$f(xy) = f[k_1a_1b_1e_2+k_2a_1b_1e_3+(k_4a_1b_2+k_6a_1b_3+k_7a_3b_1)e_4]$$

$$= (\frac{k_1k_4+k_2k_6}{k_2k_7})a_1b_1e_2''+a_1b_1e_3''+(k_4a_1b_2+k_6a_1b_3+k_7a_3b_1)\frac{e_4''}{k_2k_7}$$

$$= a_1b_1e_2''+a_1b_1e_3''+(k_4a_1b_2+k_6a_1b_3+k_7a_3b_1)\frac{e_4''}{k_2k_7} ,$$

on the other hand, (13.1) implies that

$$f(x)*f(y) = f(\sum_{i=1}^{4} a_ie_i)*f(\sum_{j=1}^{4} b_je_j)$$

$$= [a_1e_1''+(\frac{k_4}{k_2k_7}a_2+\frac{k_6}{k_2k_7}a_3)e_2''+\frac{a_3}{k_2}e_3''+\frac{a_4}{k_2k_7}e_4'']*[b_1e_1''$$

$$+(\frac{k_4}{k_2k_7}b_2+\frac{k_6}{k_2k_7}b_3)e_2''+\frac{b_3}{k_2}e_3''+\frac{b_4}{k_2k_7}e_4'']$$

$$= a_1b_1e_2''+a_1b_1e_3''+[a_1(\frac{k_4}{k_2k_7}b_2+\frac{k_6}{k_2k_7}b_3)+\frac{a_3}{k_2}b_1]e_4''$$

$$= a_1b_1e_2''+a_1b_1e_3''+(k_4a_1b_2+k_6a_1b_3+k_7a_3b_1)\frac{e_4''}{k_2k_7}.$$

Therefore, (24.1) and (13.1) are isomorphic

<u>Case 25.</u> Assume that $k_1\neq 0, k_2\neq 0, k_5\neq 0, k_6\neq 0, k_7\neq 0$ and $k_3 = k_4 = 0$. Then the multiplication (**) is

(25.1) $$xy = k_1a_1b_1e_2+k_2a_1b_1e_3+(k_5a_2b_1+k_6a_1b_3+k_7a_3b_1)e_4.$$

Let $z = \sum_{\ell=1}^{4} c_\ell e_\ell$, $\{c_\ell\}_{\ell=1,2,3,4} \subset K$. We consider (xy)z and x(yz).

$$(xy)z = [(\sum_{i=1}^{4} a_ie_i)(\sum_{j=1}^{4} b_je_j)](\sum_{\ell=1}^{4} c_\ell e_\ell)$$

$$= [k_1a_1b_1e_2+k_2a_1b_1e_3+(k_5a_2b_1+k_6a_1b_3+k_7a_3b_1)e_4](\sum_{\ell=1}^{4} c_\ell e_\ell)$$

$$= [k_5(k_1a_1b_1)c_1+k_7(k_2a_1b_1)c_1]\, e_4,$$

$$x(yz) = (\sum_{i=1}^{4} a_ie_i)[(\sum_{j=1}^{4} b_je_j)(\sum_{\ell=1}^{4} c_\ell e_\ell)]$$

$$= (\sum_{i=1}^{4} a_ie_i)[k_1b_1c_1e_2+k_2b_1c_1e_3+(k_5b_2c_1+k_6b_1c_3+k_7b_3c_1)e_4]$$

$$= k_6a_1(k_2b_1c_1)e_4.$$

Since A is an associative algebra, we must have that $k_2k_6 = k_1k_5+k_2k_7$. We claim that (25.1) is isomorphic to (13.1). To prove this, let $f : A \to A$ be the linear map defined by

$$f(e_1) = e_1'',$$

$$f(e_2) = \frac{k_5}{k_2k_6} e_3'',$$

$$f(e_3) = \frac{e_2''}{k_2} + \frac{k_7}{k_2k_6} e_3'',$$

$$f(e_4) = \frac{e_4''}{k_2k_6}, \qquad \{k_i \neq 0\}_{i=2,5,6,7} \subset K.$$

Since $\det [f] = -\frac{k_5}{k_2^3k_6} \neq 0$, f is 1-1 and onto. (25.1) implies that

$$f(xy) = f[k_1a_1b_1e_2+k_2a_1b_1e_3+(k_5a_2b_1+k_6a_1b_3+k_7a_3b_1)e_4]$$

$$= (\frac{k_1k_5+k_2k_7}{k_2k_6})a_1b_1e_3''+a_1b_1e_2''+(k_5a_2b_1+k_6a_1b_3+k_7a_3b_1)\frac{e_4''}{k_2k_6}$$

$$= a_1b_1e_2''+a_1b_1e_3''+(k_5a_2b_1+k_6a_1b_3+k_7a_3b_1)\frac{e_4''}{k_2k_6},$$

whereas, (13.1) implies that

$$f(x)*f(y) = f(\sum_{i=1}^{4} a_ie_i)*f(\sum_{j=1}^{4} b_je_j)$$

$$= [a_1e_1''+\frac{a_3}{k_2}e_2''+(\frac{k_5}{k_2k_6}a_2+\frac{k_7}{k_2k_6}a_3)e_3''+\frac{a_4}{k_2k_6}e_4''] * [b_1e_1''$$

$$+\frac{b_3}{k_2}e_2''+(\frac{k_5}{k_2k_6}b_2+\frac{k_7}{k_2k_6}b_3)e_3''+\frac{b_4}{k_2k_6}e_4'']$$

$$= a_1b_1e_2''+a_1b_1e_3''+[a_1\frac{b_3}{k_2}+(\frac{k_5}{k_2k_6}a_2+\frac{k_7}{k_2k_6}a_3)b_1]e_4''$$

$$= a_1b_1e_2''+a_1b_1e_3''+[k_5a_2b_1+k_6a_1b_3+k_7a_3b_1]\frac{e_4''}{k_2k_6}.$$

Hence (25.1) and (13.1) are isomorphic.

<u>Case 26.</u> Suppose that $k_1\neq 0, k_2\neq 0, k_3\neq 0, k_4\neq 0, k_5\neq 0$, $k_6\neq 0$ and $k_7 = 0$. Then from (**) we have that

(26.1) $$xy = k_1a_1b_1e_2+k_2a_1b_1e_3+(k_3a_1b_1+k_4a_1b_2+k_5a_2b_1+k_6a_1b_3)e_4.$$

We can check the associativity of A as in case 22 and we get that $k_1k_5 = k_1k_4+k_2k_6$. Claim that (26.1) is isomorphic to (13.1). To prove this, let $f\colon A\to A$ be the linear map defined by

$$f(e_1) = e_1''+\frac{k_3}{k_1k_5}e_3'',$$

$$f(e_2) = \frac{k_4}{k_1k_5}e_2''+\frac{e_3''}{k_1},$$

$$f(e_3) = \frac{k_6}{k_1k_5}e_2'',$$

$$f(e_4) = \frac{e_4''}{k_1k_5}, \quad \{k_i\neq 0\}_{i=1,3,4,5,6} \subset K.$$

Then f is 1-1 and onto. (26.1) implies that

$$f(xy) = f[k_1a_1b_1e_2+k_2a_1b_1e_3+(k_3a_1b_1+k_4a_1b_2+k_5a_2b_1+k_6a_1b_3)e_4]$$

$$= (\frac{k_1k_4+k_2k_6}{k_1k_5})a_1b_1e''_2+a_1b_1e''_3+(k_3a_1b_1+k_4a_1b_2+k_5a_2b_1+k_6a_1b_3)\frac{e''_4}{k_1k_5}$$

$$= a_1b_1e''_2+a_1b_1e''_3+[k_3a_1b_1+k_4a_1b_2+k_5a_2b_1+k_6a_1b_3]\frac{e''_4}{k_1k_5}$$

and (13.1) implies that

$$f(x) * f(y) = f(\sum_{i=1}^{4} a_ie_i) * f(\sum_{j=1}^{4} b_je_j)$$

$$= [a_1e''_1+(\frac{k_4}{k_1k_5}a_2+\frac{k_6}{k_1k_5}a_3)e''_2+(\frac{k_3a}{k_1k_5}a_1+\frac{a_2}{k_1})e''_3+\frac{a_4}{k_1k_5}e''_4] *$$

$$[b_1e''_1+(\frac{k_4}{k_1k_5}b_2+\frac{k_6}{k_1k_5}b_3)e''_2+(\frac{k_3}{k_1k_5}b_1+\frac{b_2}{k_1})e''_3+\frac{b_4}{k_1k_5}e''_4]$$

$$= a_1b_1e''_2+a_1b_1e''_3+[a_1(\frac{k_4}{k_1k_5}b_2+\frac{k_6}{k_1k_5}b_3)+(\frac{k_3}{k_1k_5}a_1+\frac{a_2}{k_1})b_1]e''_4$$

$$= a_1b_1e''_2+a_1b_1e''_3+(k_3a_1b_1+k_4a_1b_2+k_5a_2b_1+k_6a_1b_3)\frac{e''_4}{k_1k_5} .$$

Thus we have that (26.1) and (13.1) are isomorphic.

<u>Case 27</u>. Suppose that $k_1 \neq 0, k_2 \neq 0, k_3 \neq 0, k_4 \neq 0, k_5 \neq 0$, $k_7 \neq 0$ and $k_6 = 0$. Then the multiplication (**) is

(27.1) $\quad xy = k_1a_1b_1e_2+k_2a_1b_1e_3+(k_3a_1b_1+k_4a_1b_2+k_5a_2b_1+k_7a_3b_1)e_4.$

As in case 23 we have that $k_1k_4 = k_1k_5+k_2k_7$. Claim that this multiplication is isomorphic to (13.1) case 13. To prove this, let $f\colon A \to A$ be the linear map defined by

$$f(e_1) = e_1'' + \frac{k_3}{k_1k_4} e_2'' ,$$

$$f(e_2) = \frac{e_2''}{k_1} + \frac{k_5}{k_1k_4} e_3'' ,$$

$$f(e_3) = \frac{k_7}{k_1k_4} e_3'' ,$$

$$f(e_4) = \frac{e_4''}{k_1k_4} , \ \{k_i \neq 0\} \subset K, \ i = 1,3,4,5,7.$$

Then f is 1-1 and onto. Using (27.1) we get that

$$f(xy) = f[k_1a_1b_1e_2 + k_2a_1b_1e_3 + (k_3a_1b_1 + k_4a_1b_2 + k_5a_2b_1 + k_7a_3b_1)e_4]$$

$$= a_1b_1e_2'' + a_1b_1e_3'' + (k_3a_1b_1 + k_4a_1b_2 + k_5a_2b_1 + k_7a_3b_1)\frac{e_4''}{k_1k_4} ,$$

whereas, from (13.1) we get that

$$f(x)*f(y) = f\left(\sum_{i=1}^{4} a_ie_i\right) * f\left(\sum_{j=1}^{4} b_je_j\right)$$

$$= \left[a_1e_1'' + \left(\frac{k_3}{k_1k_4}a_1 + \frac{a_2}{k_1}\right)e_2'' + \left(\frac{k_5}{k_1k_4}a_2 + \frac{k_7}{k_1k_4}a_3\right)e_3'' + \frac{a_4}{k_1k_4}e_4''\right]$$

$$* \left[b_1e_1'' + \left(\frac{k_3}{k_1k_4}b_1 + \frac{b_2}{k_1}\right)e_2'' + \left(\frac{k_5}{k_1k_4}b_2 + \frac{k_7}{k_1k_4}b_3\right)e_3'' + \frac{b_4}{k_1k_4}e_4''\right]$$

$$= a_1b_1e_2'' + a_1b_1e_3'' + \left[a_1\left(\frac{k_3}{k_1k_4}b_1 + \frac{b_2}{k_1}\right) + \left(\frac{k_5}{k_1k_4}a_2 + \frac{k_7}{k_1k_4}a_3\right)b_1\right]e_4''$$

$$= a_1b_1e_2'' + a_1b_1e_3'' + [k_3a_1b_1 + k_4a_1b_2 + k_5a_2b_1 + k_7a_3b_1]\frac{e_4''}{k_1k_4} .$$

Therefore, (27.1) and (13.1) are isomorphic.

<u>Case 28</u>. Assume that $k_1 \neq 0, k_2 \neq 0, k_3 \neq 0, k_4 \neq 0, k_6 \neq 0, k_7 \neq 0$ and $k_5 = 0$. Then from (**) we have that

(28.1) $$xy = k_1a_1b_1e_2+k_2a_1b_1e_3+(k_3a_1b_1+k_4a_1b_2+k_6a_1b_3+k_7a_3b_1)e_4.$$

As in case 24, we have that $k_2k_7= k_1k_4+k_2k_6$. This multiplication is isomorphic to (13.1) in case 13. To prove this, let $f\colon A \to A$ be the linear map defined by

$$f(e_1) = e_1''+\frac{k_3}{k_2k_7}e_3'' ,$$

$$f(e_2) = \frac{k_4}{k_2k_7}e_2'' ,$$

$$f(e_3) = \frac{k_6}{k_2k_7}e_2''+\frac{e_3''}{k_2} ,$$

$$f(e_4) = \frac{e_4''}{k_2k_7} , \quad \{k_i\neq 0\}\subset K, \ i = 2,3,4,6,7.$$

Then f is 1-1 and onto. (28.1) implies that

$$f(xy) = f[k_1a_1b_1e_2+k_2a_1b_1e_3+(k_3a_1b_1+k_4a_1b_2+k_6a_1b_3+k_7a_3b_1)e_4]$$

$$= a_1b_1e_2''+a_1b_1e_3''+(k_3a_1b_1+k_4a_1b_2+k_6a_1b_3+k_7a_3b_1)\frac{e_4''}{k_2k_7} ,$$

whereas, (13.1) implies that

$$f(x)*f(y) = f(\sum_{i=1}^{4} a_ie_i)*f(\sum_{j=1}^{4} b_je_j)$$

$$= [a_1e_1''+(\frac{k_4}{k_2k_7}a_2+\frac{k_6}{k_2k_7}a_3)e_2''+(\frac{k_3}{k_2k_7}a_1+\frac{a_3}{k_2})e_3''+\frac{a_4}{k_2k_7}e_4''] *$$

$$[b_1e_1''+(\frac{k_4}{k_2k_7}b_2+\frac{k_6}{k_2k_7}b_3)e_2''+(\frac{k_3}{k_2k_7}b_1+\frac{b_3}{k_2})e_3''+\frac{b_4}{k_2k_7}e_4'']$$

$$= a_1b_1e_2''+a_1b_1e_3''+[a_1(\frac{k_4}{k_2k_7}b_2+\frac{k_6}{k_2k_7}b_3)+(\frac{k_3}{k_2k_7}a_1+\frac{a_3}{k_2})b_1]e_4''$$

$$= a_1b_1e_2''+a_1b_1e_3''+[k_3a_1b_1+k_4a_1b_2+k_6a_1b_3+k_7a_3b_1]\frac{e_4''}{k_2k_7}.$$

Thus (28.1) and (13.1) are isomorphic.

Case 29. Assume that $k_1 \neq 0, k_2 \neq 0, k_3 \neq 0, k_5 \neq 0, k_6 \neq 0$, $k_7 \neq 0$ and $k_4 = 0$. Then the multiplication (**) is

$$(29.1) \qquad xy = k_1a_1b_1e_2+k_2a_1b_1e_3+(k_3a_1b_1+k_5a_2b_1+k_6a_1b_3+k_7a_3b_1)e_4.$$

As in case 25, we get that $k_2k_6 = k_1k_5+k_2k_7$. Claim that this multiplication is isomorphic to (13.1). To prove this, let $f\colon A \to A$ be the linear map defined by

$$f(e_1) = e_1''+\frac{k_3}{k_2k_6}e_2'' ,$$

$$f(e_2) = \frac{k_5}{k_2k_6}e_3'' ,$$

$$f(e_3) = \frac{e_2''}{k_2}+\frac{k_7}{k_2k_6}e_3'' ,$$

$$f(e_4) = \frac{e_4''}{k_2k_6} , \quad \{k_i \neq 0\} \subset K,\ i = 2,3,5,6,7.$$

Then f is 1-1 and onto. (29.1) implies that

$$f(xy) = f[k_1a_1b_1e_2+k_2a_1b_1e_3+(k_3a_1b_1+k_5a_2b_1+k_6a_1b_3+k_7a_3b_1)e_4]$$
$$= a_1b_1e_2''+a_1b_1e_3''+(k_3a_1b_1+k_5a_2b_1+k_6a_1b_3+k_7a_3b_1)\frac{e_4''}{k_2k_6}$$

and (13.1) implies that

$$f(x)*f(y) = f(\sum_{i=1}^{4} a_ie_i)*f(\sum_{j=1}^{4} b_je_j)$$
$$= [a_1e_1''+(\frac{k_3}{k_2k_6}a_1+\frac{a_3}{k_2})e_2''+(\frac{k_5}{k_2k_6}a_2+\frac{k_7}{k_2k_6}a_3)e_3''+\frac{a_4}{k_2k_6}e_4''] *$$
$$[b_1e_1''+(\frac{k_3}{k_2k_6}b_1+\frac{b_3}{k_2})e_2''+(\frac{k_5}{k_2k_6}+\frac{k_7}{k_2k_6}b_3)e_3''+\frac{b_4}{k_2k_6}e_4'']$$

$$= a_1b_1e_2''+a_1b_1e_3''+[a_1(\frac{k_3}{k_2k_6}b_1+\frac{b_3}{k_2})+(\frac{k_5}{k_2k_6}a_2+\frac{k_7}{k_2k_6}a_3)b_1]e_4''$$

$$= a_1b_1e_2''+a_1b_1e_3''+[k_3a_1b_1+k_5a_2b_1+k_6a_1b_3+k_7a_3b_1]\frac{e_4''}{k_2k_6}$$

Therefore, (29.1) and (13.1) are isomorphic.

<u>Case 30.</u> Suppose that $k_1\neq 0, k_2\neq 0, k_4\neq 0, k_5\neq 0, k_6\neq 0, k_7\neq 0$ and $k_3 = 0$. Then the multiplication (**) is

$$xy = k_1a_1b_1e_2+k_2a_1b_1e_3+(k_4a_1b_2+k_5a_2b_1+k_6a_1b_3+k_7a_3b_1)e_4.$$

Choose a new basis e_1', e_2', e_3', e_4' of A such that $e_1' = e_1, e_2' = k_1e_2$, $e_3' = k_2e_3$ and $e_4' = k_1k_4e_4$. Then we get that

$$xy = a_1'b_1'(e_1')^2+a_1'b_2'e_1'e_2'+a_1'b_3'e_1'e_3'+a_2'b_1'e_2'e_1'+a_3'b_1'e_3'e_1' ,$$

for $x = \sum_{i=1}^{4} a_i'e_i'$, $y = \sum_{j=1}^{4} b_j'e_j'$, $\{a_i', b_j'\} \subset K, i,j = 1,2,3,4.$

Since $(e_1')^2 = e_1^2 = k_1e_2+k_2e_3+k_3e_4 = e_2'+e_3'$,

$e_1'e_2' = k_1e_1e_2 = k_1k_4e_4 = e_4'$,

$e_2'e_1' = k_1e_2e_1 = k_1k_5e_4 = \frac{k_5}{k_4}e_4'$,

$e_1'e_3' = k_2e_1e_3 = k_2k_6e_4 = \frac{k_2k_6}{k_1k_4}e_4'$

and $e_3'e_1' = k_2e_3e_1 = k_2k_7e_4 = \frac{k_2k_7}{k_1k_4}e_4'$, we have that

$$xy = a_1'b_1'e_2'+a_1'b_1'e_3'+(a_1'b_2'+\frac{k_5}{k_4}a_2'b_1'+\frac{k_2k_6}{k_1k_4}a_1'b_3'+\frac{k_2k_7}{k_1k_4}a_3'b_1')e_4' .$$

Let $z = \sum_{\ell=1}^{4} c_\ell'e_\ell'$, $\{c_\ell'\}_{\ell=1,2,3,4} \subset K$. We consider (xy)z and x(yz).

$$(xy)z = [(\sum_{i=1}^{4} a_i'e_i')(\sum_{j=1}^{4} b_j'e_j')](\sum_{\ell=1}^{4} c_\ell'e_\ell')$$

$$= [a_1'b_1'e_2'+a_1'b_1'e_3'+(a_1'b_2'+\frac{k_5}{k_4}a_2'b_1'+\frac{k_2k_6}{k_1k_4}a_1'b_3'+\frac{k_2k_7}{k_1k_4}a_3'b_1')e_4'](\sum_{\ell=1}^{4} c_\ell'e_\ell')$$

$$= [\frac{k_5}{k_4}(a_1'b_1')c_1'+\frac{k_2k_7}{k_1k_4}(a_1'b_1')c_1']e_4' ,$$

$$x(yz) = (\sum_{i=1}^{4} a_i'e_i')[(\sum_{j=1}^{4} b_j'e_j')(\sum_{\ell=1}^{4} c_\ell'e_\ell')]$$

$$= (\sum_{i=1}^{4} a_1'e_1')[b_1'c_1'e_2'+b_1'c_1'e_3'+(b_1'c_2'+\frac{k_5}{k_4}b_2'c_1'+\frac{k_2k_6}{k_1k_4}b_1'c_3'+\frac{k_2k_7}{k_1k_4}b_3'c_1')e_4]$$

$$= (a_1'b_1'c_1'+\frac{k_2k_6}{k_1k_4}a_1'b_1'c_1')e_4'.$$

Since A is an associative algebra, we must have that $\frac{k_5}{k_4}+\frac{k_2k_7}{k_1k_4} = 1+\frac{k_2k_6}{k_1k_4}$. Let $k' = \frac{k_2k_6}{k_1k_4}$, $k'' = \frac{k_2k_7}{k_1k_4}$. Then $k' \neq 0$ and $k'' \neq 0$. Since $k_5 \neq 0$, we have that $1+k'-k'' \neq 0$. Therefore, the multiplication in this case can be written as

(30.1) $\qquad xy = a_1'b_1'e_2'+a_1'b_1'e_3'+[a_1'b_2'+(1+k'-k'')a_2'b_1'+k'a_1'b_3'+k''a_3'b_1']e_4'$

Notice that if $1+k' = 0$, then $(xy)z = x(yz) = 0$ for all x,y,z in A. This implies that $A^3 = \{0\}$ which contradicts the hypothesis of dimension $A^3 = 1$. Therefore $1+k' \neq 0$.

Subcase 1. If $k' = k''$, then we have that

(30.2) $\qquad xy = a_1'b_1'e_2'+a_1'b_1'e_3'+(a_1'b_2'+a_2'b_1'+k'a_1'b_3'+k'a_3'b_1')e_4'.$

We claim that this case is isomorphic to (10.1) in case 10. Recall that the multiplication in case 10 is

$$(10.1) \qquad x \circ y = a_1''b_1''e_2''+a_1''b_1''e_3''+(a_1''b_2''+a_2''b_1'')e_4'' ,$$

for $x = \sum_{i=1}^{4} a_i''e_i''$, $y = \sum_{j=1}^{4} b_j''e_j''$, $\{a_i'',b_j''\} \subset K$, $i,j = 1,2,3,4.$

To prove this, let $f: A \to A$ be the linear map defined by

$$f(e_1') = e_1'',$$

$$f(e_2') = \frac{e_2''}{1+k'} ,$$

$$f(e_3') = \frac{k'}{1+k'} e_2''+e_3'' ,$$

$$f(e_4') = \frac{e_4''}{1+k'} , \quad 1+k' \neq 0, \ k' \in K.$$

Then f is 1-1 and onto. We have from (30.2) that

$$f(xy) = f[a_1'b_1'e_2'+a_1'b_1'e_3'+(a_1'b_2'+a_2'b_1'+k'a_1'b_3'+k'a_3'b_1')e_4']$$

$$= \frac{a_1'b_1'}{1+k'}e_2''+\frac{k'}{1+k'}a_1'b_1'e_2''+a_1'b_1'e_3''+(a_1'b_2'+a_2'b_1'+k'a_1'b_3'+k'a_3'b_1')\frac{e_4''}{1+k'}$$

$$= a_1'b_1'e_2''+a_1'b_1'e_3''+(a_1'b_2'+a_2'b_1'+k'a_1'b_3'+k'a_3'b_1')\frac{e_4''}{1+k'}$$

and, by using (10.1), we get that

$$f(x)\circ f(y) = f(\sum_{i=1}^{4} a_i'e_i')\circ f(\sum_{j=1}^{4} b_j'e_j')$$

$$= [a_1'e_1''+(\frac{a_2'}{1+k'}+\frac{k'a_3}{1+k'})e_2''+a_3'e_3''+\frac{a_4'}{1+k'}e_4'']\circ[b_1'e_1''$$

$$+(\frac{b_2'}{1+k'}+\frac{k'b_3'}{1+k'})e_2''+b_3'e_3''+\frac{b_4'}{1+k'}e_4'']$$

$$= a_1'b_1'e_2''+a_1'b_1'e_3''+[a_1'(\frac{b_2'}{1+k'}+\frac{k'}{1+k'}b_3')+(\frac{a_2'}{1+k'}+\frac{k'a_3'}{1+k'})b_1']e_4''$$

$$= a_1'b_1'e_2''+a_1'b_1'e_3''+[a_1'b_2'+a_2'b_1'+k'a_1'b_3'+k'a_3'b_1']\frac{e_4''}{1+k'}$$

That is $f(xy) = f(x)\circ f(y)$. This proves the claim.

Subcase 2. Assume that $k' \neq k''$. In this case we claim that (30.1) is isomorphic to (13.1). To prove this, let $f:A \to A$ be the linear map defined by

$$f(e_1') = e_1'' ,$$

$$f(e_2') = \frac{e_2''}{1+k'} + \frac{(1+k'-k'')}{1+k'} e_3'' ,$$

$$f(e_3') = \frac{k'e_2''}{1+k'} + \frac{k''}{1+k'} e_3'' ,$$

$$f(e_4') = \frac{e_4''}{1+k'} , \quad k' \neq 0,\ k'' \neq 0 \text{ in } K.$$

Then $$\det[f] = \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{1+k'} & \frac{1+k'-k''}{1+k'} & 0 \\ 0 & \frac{k'}{1+k'} & \frac{k''}{1+k'} & 0 \\ 0 & 0 & 0 & \frac{1}{1+k'} \end{bmatrix} = \frac{k''-k'}{(1+k')^2} \neq 0.$$

So f is 1-1 and onto. By using (30.1), we get that

$$f(xy) = f[a_1'b_1'e_2'+a_1'b_1'e_3'+(a_1'b_2'+(1+k'-k'')a_2'b_1'+k'a_1'b_3'+k''a_3'b_1')e_4']$$

$$= a_1'b_1' \frac{e_2''}{1+k'} + \frac{(1+k'-k'')}{1+k'} a_1'b_1'e_3'' + \frac{k'}{1+k'} a_1'b_1'e_2'' + \frac{k''}{1+k'} a_1'b_1'e_3''$$

$$+ [a_1'b_2'+(1+k'-k'')a_2'b_1'+k'a_1'b_3'+k''a_3'b_1'] \frac{e_4''}{1+k'}$$

$$= a_1'b_1'e_2''+a_1'b_1'e_3''+[a_1'b_2'+(1+k'-k'')a_2'b_1'+k'a_1'b_3'+k''a_3'b_1'] \frac{e_4''}{1+k'}$$

and (13.1) implies that

$$f(x)*f(y) = f(\sum_{i=1}^{4} a_i'e_i')*f(\sum_{j=1}^{4} b_j'e_j')$$

$$= [a_1'e_1''+(\frac{a_2'}{1+k'}+\frac{k'}{1+k'}a_3')e_2''+(\frac{(1+k'-k'')}{1+k'} a_2'+\frac{k''}{1+k'}a_3')e_3''$$

$$+\frac{a_4'e_4''}{1+k'}]*[b_1'e_1''+(\frac{b_2'}{1+k'}+\frac{k'b_3'}{1+k'})e_2''+(\frac{(1+k'-k'')}{1+k'}b_2'+\frac{k''}{1+k'}b_3')e_3''+\frac{b_4'}{1+k'}e_4'']$$

$$= a_1'b_1'e_2''+a_1'b_1'e_3''+\left[a_1'\left(\frac{b_2'}{1+k'}+\frac{k'}{1+k'}b_3'\right)+\right.$$

$$\left.\left(\frac{(1+k'-k'')}{1+k'}a_2'+\frac{k''}{1+k'}a_3'\right)b_1'\right]e_4''$$

$$= a_1'b_1'e_2''+a_1'b_1'e_3''+\left[a_1'b_2'+(1+k'-k'')a_2'b_1+k'a_1'b_3'+k''a_3'b_1'\right]\frac{e_4''}{1+k'}.$$

Thus (30.1) and (13.1) are isomorphic.

Case 31. In the final case we assume that all the k_i, $i = 1,2,\ldots,7$ are not zero. Then the multiplication (**) is

(31.1) $\quad xoy = k_1a_1b_1e_2+k_2a_1b_1e_3+(k_3a_1b_1+k_4a_1b_2+k_5a_2b_1+k_6a_1b_3+k_7a_3b_1)e_4$

Let $z = \sum_{\ell=1}^{4} c_\ell e_\ell$, $\{c_\ell\}\subset K$, $= 1,2,3,4$. Then we have from (31.1) that

$$(xoy)oz = \left[\left(\sum_{i=1}^{4} a_ie_i\right)o\left(\sum_{j=1}^{4} b_je_j\right)\right]o\left(\sum_{\ell=1}^{4} c_\ell e_\ell\right)$$

$$= \left[k_1a_1b_1e_2+k_2a_1b_1e_3+(k_3a_1b_1+k_4a_1b_2+k_5a_2b_1+k_6a_1b_3+k_7a_3b_1)e_4\right]o\left(\sum_{\ell=1}^{4} c_\ell e_\ell\right)$$

$$= \left[k_5(k_1a_1b_1)c_1+k_7(k_2a_1b_1)c_1\right]e_4\ ,$$

and

$$xo(yoz) = \left(\sum_{i=1}^{4} a_ie_i\right)o\left[\left(\sum_{j=1}^{4} b_je_j\right)o\left(\sum_{\ell=1}^{4} c_\ell e_\ell\right)\right]$$

$$= \left(\sum_{i=1}^{4} a_ie_i\right)o\left[k_1b_1c_1e_2+k_2b_1c_1e_3+(k_3b_1c_1+k_4b_1c_2+k_5b_2c_1+k_6b_1c_3+k_7b_3c_1)e_4\right]$$

$$= \left[k_4a_1(k_1b_1c_1)+k_6a_1(k_2b_1c_1)\right]e_4.$$

Since A is an associative algebra, we get that $k_1k_5+k_2k_7= k_1k_4+k_2k_6$. We claim that this multiplication is isomorphic to (30.1) in case 30. To prove this, let f: A → A be the linear map defined by

$$f(e_1') = e_1,$$
$$f(e_2') = k_1e_2+k_3e_4,$$
$$f(e_3') = k_2e_3 ,$$
$$f(e_4') = k_1k_4e_4, \quad k_i \varepsilon K \quad i = 1,2,3,4.$$

Then f is 1-1 and onto. By using (30.1), we get that

$$f(xy) = f\Big[a_1'b_1'e_2'+a_1'b_1'e_3'+(a_1'b_2'+(1+\frac{k_2k_6}{k_1k_4}-\frac{k_2k_7}{k_1k_4})a_2'b_1'$$
$$+\frac{k_2k_6}{k_1k_4}a_1'b_3'+\frac{k_2k_7}{k_1k_4}a_3'b_1')e_4'\Big]$$
$$= k_1a_1'b_1'e_2+k_2a_1'b_1'e_3+\big[k_3a_1'b_1'+k_1k_4a_1'b_2'+$$
$$(k_1k_4+k_2k_6-k_2k_7)a_2'b_1'+k_2k_6a_1'b_3'+k_2k_7a_3'b_1'\big]e_4$$

and (31.1) implies that

$$f(x)of(y) = f(\sum_{i=1}^{4} a_i'e_i')of(\sum_{j=1}^{4} b_j'e_j')$$
$$= \big[a_1'e_1+k_1a_2'e_2+k_2a_3'e_3+(k_3a_2'+k_1k_4a_4')e_4\big]o\big[b_1'e_1+$$
$$k_1b_2'e_2+k_2b_3'e_3+(k_3b_2'+k_1k_4b_4')e_4\big]$$
$$= k_1a_1'b_1'e_2+k_2a_1'b_1'e_3+(k_3a_1'b_1'+k_1k_4a_1'b_2'+k_1k_5a_2'b_1'$$
$$k_2k_6a_1'b_3'+k_2k_7a_3'b_1')e_4.$$
$$= k_1a_1'b_1'e_2+k_2a_1'b_1'e_3+\big[k_3a_1'b_1'+k_1k_4a_1'b_2'+$$
$$(k_1k_4+k_2k_6-k_2k_7)a_2'b_1'+k_2k_6a_1'b_3'+k_2k_7a_3'b_1'\big]e_4.$$

That is $f(xy) = f(x)of(y)$. Therefore, (31.1) and (30.1) are isomorphic.

Hence, we have already proved that the nilpotent algebras A over the field K with dimension A = 4, dimension $A^2 = 3$, dimension $A^3 = 1$ and $A^4 = \{0\}$ have 2 non-isomorphic multiplications. That is, for each $x = \sum_{i=1}^{4} a_i e_i$, $y = \sum_{j=1}^{4} b_j e_j$, $\{a_i, b_j\} \subset K$, $i,j = 1,2,3,4$, we have that

1) $xy = a_1b_1e_2 + a_1b_1e_3 + (a_1b_2 + a_2b_1)e_4$.

2) $xy = a_1b_1e_2 + a_1b_1e_3 + (a_1b_2 + a_3b_1)e_4$.

Next, we begin to consider the multiplications in a 4-dimensional nilpotent algebra A over the field K with dimension $A^2 = 3$, dimension $A^3 = 2$, dimension $A^4 = 1$ and $A^5 = \{0\}$.

Let $\{e_1, e_2, e_3, e_4\}$ be a basis in A such that $\{e_2, e_3, e_4\}$ is a basis of A^2, $\{e_3, e_4\}$ is a basis of A^3 and e_4 is a basis of A^4. For each x,y in A we can write

$$x = \sum_{i=1}^{4} a_i e_i ,$$

$$y = \sum_{j=1}^{4} b_j e_j ; \quad \{a_i, b_j\} \subset K, \ i,j = 1,2,3,4.$$

Then

$$xy = \sum_{j=1}^{4} \sum_{i=1}^{4} a_i b_j e_i e_j .$$

Since $e_1e_4, e_4e_1, e_2e_3, e_3e_2 \in A^5 = \{0\}$, $e_2e_4, e_4e_2, e_3^2 \in A^6 = \{0\}$, $e_3e_4, e_4e_3 \in A^7 = \{0\}$ and $e_4^2 \in A^8 = \{0\}$, we have that

$$xy = a_1b_1e_1^2+a_1b_2e_1e_2+a_1b_3e_1e_3+a_2b_1e_2e_1+a_2b_2e_2^2+a_3b_1e_3e_1 .$$

Since $e_1^2 \varepsilon A^2$, $e_1e_2, e_2e_1 \varepsilon A^3$ and $e_1e_3, e_3e_1, e_2^2 \varepsilon A^4$, we can write

$$e_1^2 = k_1e_2+k_2e_3+k_3e_4,$$

$$e_1e_2 = k_4e_3+k_5e_4,$$

$$e_2e_1 = k_6e_3+k_7e_4,$$

$$e_1e_3 = k_8e_4 ,$$

$$e_2^2 = k_9e_4,$$

$$e_1e_1 = k_{10}e_4 , \text{ for some } k_i \varepsilon K, \ i = 1,2,\ldots,10.$$

Therefore, the multiplication xy can be written as

(***) $$xy = k_1a_1b_1e_2+(k_2a_1b_1+k_4a_1b_2+k_6a_2b_1)e_3+(k_3a_1b_1+k_5a_1b_2+k_7a_2b_1 +k_8a_1b_3+k_9a_2b_2+k_{10}a_3b_1)e_4 .$$

Since dimension of A^2 is 3, the case $k_1= 0, k_2= k_4= k_6= 0$ and $k_3= k_5= k_7= k_8= k_9= k_{10}= 0$ can not occur. We observe that there are many cases to consider, since there are 10 constants. However, by using the associativity of A, we can find relations between the k_i's and reduce the number of cases and give a complete classification.

We have that $(e_1e_2)e_1 = (k_4e_3+k_5e_4)e_1 = k_4k_{10}e_4$ and

$$e_1(e_2e_1) = e_1(k_6e_3+k_7e_4) = k_6k_8e_4 .$$

Since A is an associative algebra and $e_1, e_2 \varepsilon A$, we get that

(1) $$k_4k_{10} = k_6k_8 .$$

Similarly, we get the following results.

If $(e_1^2)e_2 = (k_1e_2+k_2e_3+k_3e_4)e_2 = k_1e_2^2 = k_1k_9e_4$ and

$e_1(e_1e_2) = e_1(k_4e_3+k_5e_4) = k_4e_1e_3 = k_4k_8e_4$, then we have that

(2) $$k_1k_9 = k_4k_8$$

If $(e_1^2)e_1 = (k_1e_2+k_2e_3+k_3e_4)e_1 = k_1e_2e_1+k_2e_3e_1$

$= k_1k_6e_3+(k_1k_7+k_2k_{10})e_4$

and $e_1(e_1^2) = e_1(k_1e_2+k_2e_3+k_3e_4) = k_1e_1e_2+k_2e_1e_3$

$= k_1k_4e_3+(k_1k_5+k_2k_8)e_4$,

then we have that

(3) $$k_1k_6 = k_1k_4$$

(4) $$k_1k_7+ k_2k_{10} = k_1k_5+ k_2k_8$$

If $e_2(e_1^2) = e_2(k_1e_2+k_2e_3+k_3e_4) = k_1e_2^2 = k_1k_9e_4$ and

$(e_2e_1)e_1 = (k_6e_3+k_7e_4)e_1 = k_6e_3e_1 = k_6k_{10}e_4$, then

we have that

(5) $$k_1k_9 = k_6k_{10} .$$

Notice that since $k_1 \neq 0$, we get from (3) that $k_4 = k_6$.

First, we consider the multiplication (***) when $k_2 \neq 0$, $k_4 = k_6 = 0$. Since $k_4 = 0$, from (2) we get that $k_9 = 0$. Therefore, the multiplication (***) becomes

$$xy = k_1a_1b_1e_2+k_2a_1b_1e_3+(k_3a_1b_1+k_5a_1b_2+k_7a_2b_1+k_8a_1b_3+k_{10}a_3b_1)e_4.$$

We see that this multiplication has a similar form to the multiplication (**) page 49. So we use the same method of proof as we use in the multiplication (**) and we get 2 non-isomorphic multiplications. That is, for each $x = \sum_{i=1}^{4} a_i'e_i'$, $y = \sum_{j=1}^{4} b_j'e_j'$, $\{a_i', b_j'\} \subset K$, $i,j = 1,2,3,4$, we have

(I) $\quad xy = a_1'b_1'e_2'+a_1'b_1'e_3'+(a_1'b_2'+a_2'b_1')e_4'$

(II) $\quad xy = a_1'b_1'e_2'+a_1'b_1'e_3'+(a_1'b_2'+a_3'b_1')e_4'$.

Next we consider the multiplication (***) when $k_2= 0, k_4\neq 0$. By equation (1), we get that $k_{10}= k_8$. This implies from (4) that $k_7= k_5$. Then the multiplication (***) becomes.

$$(0) \quad xy = k_1a_1b_1e_2+(k_4a_1b_2+k_4a_2b_1)e_3+(k_3a_1b_1+k_5a_1b_2+k_5a_2b_1 +k_8a_1b_3+k_9a_2b_2+k_8a_3b_1)e_4.$$

Notice that equation (2) implies that $k_8\neq 0$ iff $k_9\neq 0$. Therefore, we consider the following cases.

Case 1. Assume that $k_1\neq 0, k_4\neq 0, k_3\neq 0$ and $k_5= k_8= k_9 = 0$. Then from (0) we get that

$$(1.1) \quad xoy = k_1a_1b_1e_2+(k_4a_1b_2+k_4a_2b_1)e_4+k_3a_1b_1e_4.$$

We claim that (1.1) is isomorphic to (I). To prove this, let $f: A \to A$ be the linear map defined by

$$f(e_1') = e_1,$$
$$f(e_2') = k_1e_2,$$
$$f(e_3') = k_3e_4,$$
$$f(e_4') = k_1k_4e_3, \; k_1,k_3,k_4 \in K.$$

Then f is 1-1 and onto. We have from (I) that

$$f(xy) = f[a_1'b_1'e_2'+a_1'b_1'e_3'+(a_1'b_2'+a_2'b_1')e_4']$$
$$= k_1a_1'b_1'e_2+k_3a_1'b_1'e_4+(k_1k_4a_1'b_2'+k_1k_4a_2'b_1')e_3$$

and from (1.1) we have that

$$f(x)of(y) = f(\sum_{i=1}^{4} a_i'e_i')of(\sum_{i=1}^{4} b_j'e_j')$$

$$= [a_1'e_1+k_1a_2'e_2+k_4a_4'e_3+k_3a_3'e_4] \circ [b_1'e_1+k_1b_2'e_2 +k_4b_4'e_3+k_3b_3'e_4]$$

$$= k_1a_1'b_1'e_2+(k_1k_4a_1'b_2'+k_1k_4a_2'b_1')e_3+k_3a_1'b_1'e_4.$$

Thus (1.1) and (I) are isomorphic.

Case 2. Suppose that $k_1 \neq 0, k_4 \neq 0, k_5 \neq 0$ and $k_3 = k_8 = k_9 = 0$. Then from (0) we have that

(2.1) $$xoy = k_1a_1b_1e_2+(k_4a_1b_2+k_4a_2b_1)e_3+(k_5a_1b_2+k_5a_2b_1)e_4.$$

We claim that (2.1) is isomorphic to (I). To prove this, let $f: A \to A$ be the linear map defined by

$$f(e_1') = e_1,$$

$$f(e_2') = k_1e_2-e_3,$$

$$f(e_3') = e_3,$$

$$f(e_4') = k_1k_4e_3+k_1k_5e_4, \quad k_1,k_4,k_5 \in K.$$

Since $\det[f] = k_1^2k_5 \neq 0$, f is 1-1 and onto. We have from (I) that

$$f(xy) = f[a_1'b_1'e_2'+a_1'b_1'e_3'+(a_1'b_2'+a_2'b_1')e_4']$$

$$= k_1a_1'b_1'e_2-a_1'b_1'e_3+a_1'b_1'e_3+(a_1'b_2'+a_2'b_1')(k_1k_4e_3+k_1k_5e_4)$$

$$= k_1a_1'b_1'e_2+(k_1k_4a_1'b_2'+k_1k_4a_2'b_1')e_3+(k_1k_5a_1'b_2'+k_1k_5a_2'b_1')e_4$$

and from (2.1) we have that

$$f(x)of(y) = f(\sum_{i=1}^{4} a_i'e_i')of(\sum_{j=1}^{4} b_j'e_j')$$

$$= [a_1'e_1+k_1a_2'e_2+(a_3'-a_2'+k_1k_4a_4')e_3+k_1k_5a_4'e_4] \circ [b_1'e_1 +k_1b_2'e_2+(b_3'-b_2'-k_1k_4b_4')e_3+k_1k_5b_4'e_4]$$

$$= k_1a_1'b_1'e_2+(k_1k_4a_1'b_2'+k_1k_4a_2'b_1')e_3+(k_1k_5a_1'b_2'+k_1k_5a_2'b_1')e_4.$$

Therefore, (2.1) is isomorphic to (I).

Case 3. Assume that $k_1\neq 0, k_4\neq 0, k_3\neq 0, k_5\neq 0$ and $k_8 = k_9 = 0$. Then from (O) we have that

(3.1) $$xoy = k_1a_1b_1e_2+(k_4a_1b_2+k_4a_2b_1)e_3+(k_3a_1b_1+k_5a_1b_2+k_5a_2b_1)e_4.$$

We claim that this multiplication is isomorphic to (I). To prove this, let $f: A \to A$ be the linear map defined by

$$f(e_1') = e_1,$$
$$f(e_2') = k_1e_2-e_3+k_3e_4,$$
$$f(e_3') = e_3,$$
$$f(e_4') = k_1k_4e_3+k_1k_5e_4, \quad k_i \in K, i=1,3,4,5.$$

We see that $\det [f] = k_1^2k_5\neq 0$ so that f is 1-1 and onto. (I) implies that

$$f(xy) = f[a_1'b_1'e_2'+a_1'b_1'e_3'+(a_1'b_2'+a_2'b_1')e_4']$$
$$= k_1a_1'b_1'e_2-a_1'b_1'e_3+k_3a_1'b_1'e_4+a_1'b_1'e_3+(a_1'b_2'+a_2'b_1')(k_1k_4e_3+k_1k_5e_4)$$
$$= k_1a_1'b_1'e_2+(k_1k_4a_1'b_2'+k_1k_4a_2'b_1')e_3+(k_3a_1'b_1'+k_1k_5a_1'b_2'+k_1k_5a_2'b_1')e_4,$$

whereas, (3.1) implies that

$$f(x)of(y) = f(\sum_{i=1}^{4} a_i'e_i')of(\sum_{j=1}^{4} b_j'e_j')$$
$$=[a_1'e_1+k_1a_2'e_2+(a_3'-a_2'+k_1k_4a_4')e_3+(k_3a_2'+k_1k_5a_4')e_4]o[b_1'e_1$$
$$+k_1b_2'e_2+(b_3'-b_2'+k_1k_4b_4')e_3+(k_3b_2'+k_1k_5b_4')e_4]$$
$$= k_1a_1'b_1'e_2+(k_1k_4a_1'b_2'+k_1k_4a_2'b_1')e_3+$$
$$(k_3a_1'b_1'+k_1k_5a_1'b_2'+k_1k_5a_2'b_1')e_4.$$

This proves that (3.1) and (I) are isomorphic.

Case 4. Assume that $k_1 \neq 0, k_4 \neq 0, k_8 \neq 0, k_9 \neq 0$ and $k_3 = k_5 = 0$. Then the multiplication (0) is

$$xy = k_1a_1b_1e_2+(k_4a_1b_2+k_4a_2b_1)e_3+(k_8a_1b_3+k_9a_2b_2+k_8a_3b_1)e_4.$$

We can choose a new basis $e_1', e_2', e_3' e_4'$ of A such that $e_1' = e_1, e_2' = k_1e_2, e_3' = k_1k_4e_3, e_4' = k_1k_4k_8e_4$. Then we have that

$$xy = a_1'b_1'e_1'^2+a_1'b_2'e_1'e_2'+a_1'b_3'e_1'e_3'+a_2'b_1'e_2'e_1'+a_2'b_2'e_2'^2+a_3'b_1'e_3'e_1',$$

for $x = \sum_{i=1}^{4} a_i'e_i'$, $y = \sum_{j=1}^{4} b_j'e_j'$, $\{a_i', b_j'\} \subset K$, $i,j = 1,2,3,4$.

Since $e_1'^2 = e_1^2 = k_1e_2+k_2e_3+k_3e_4 = e_2'$,

$$e_1'e_2' = k_1e_1e_2 = k_1(k_4e_3+k_5e_4) = k_1k_4e_3 = e_3',$$

$$e_1'e_3' = k_1k_4e_1e_3 = k_1k_4(k_8e_4) = e_4',$$

$$e_2'e_1' = k_1e_2e_1 = k_1(k_6e_3+k_7e_4) = k_1k_4e_3 = e_3',$$

$$(e_2')^2 = (k_1e_2)^2 = k_1^2k_9e_4 = \frac{k_1k_9}{k_4k_8} e_4' = e_4' \quad \text{(by (2))}$$

and $e_3'e_1' = k_1k_4e_3e_1 = k_1k_4k_{10}e_4 = k_1k_4k_8e_4 = e_4'$, we have that

(4.1) $$xy = a_1'b_1'e_2'+(a_1'b_2'+a_2'b_1')e_3'+(a_1'b_3'+a_2'b_2'+a_3'b_1')e_4'.$$

This multiplication is not isomorphic to the multiplication (I) page 84 To see this note that the left center C_L of A under the multiplication (I) is $C_L = [e_3', e_4']$. Hence the dimension of C_L is 2. But the left center C_L' of A under the multiplication (4.1) is generated by e_4' and therefore the dimension of C_L' is 1. These imply that the left center C_L can not be isomorphic to the left center C_L' and hence, these two multiplications are not isomorphic.

Moreover, A is commutative under the multiplication (4.1) but A is not commutative under the multiplication (II). Therefore, these two multiplications are not isomorphic.

<u>Case 5.</u> Assume that $k_1 \neq 0, k_4 \neq 0, k_3 \neq 0, k_8 \neq 0, k_9 \neq 0$ and $k_5 = 0$. Then from (0) page 84 we have

(5.1) $$xoy = k_1a_1b_1e_2+(k_4a_1b_2+k_4a_2b_1)e_3+(k_3a_1b_1+k_8a_1b_3+k_9a_2b_2+k_8a_3b_1)e_4.$$

We have from the equation (2) that $k_1k_9 = k_4k_8$. Claim that this multiplication is isomorphic to (4.1) in case 4. To see this, let $f: A \to A$ be the linear map defined by

$$f(e_1') = e_1,$$
$$f(e_2') = k_1e_2+k_3e_4,$$
$$f(e_3') = k_1k_4e_3,$$
$$f(e_4') = k_1k_4k_8e_4,\ k_i \in K,\ i = 1,3,4,8.$$

Then f is 1-1 and onto. (4.1) implies that

$$f(xy) = f[a_1'b_1'e_2'+(a_1'b_2'+a_2'b_1')e_3'+(a_1'b_3'+a_2'b_2'+a_3'b_1')e_4']$$
$$= k_1a_1'b_1'e_2+k_1k_4(a_1'b_2'+a_2'b_1')e_3+(k_3a_1'b_1'+k_1k_4k_8a_1'b_3'$$
$$+k_1k_4k_8a_2'b_2'+k_1k_4k_8a_3'b_1')e_4,$$

whereas, (5.1) implies that

$$f(x)of(y) = f(\sum_{i=1}^{4} a_i'e_i')of(\sum_{j=1}^{4} b_j'e_j')$$
$$= [a_1'e_1+k_1a_2'e_2+k_1k_4a_3'e_3+(k_3a_2'+k_1k_4k_8a_4')e_4]o[b_1'e_1$$
$$+k_1b_2'e_2+k_1k_4b_3'e_3+(k_3b_2'+k_1k_4k_8b_4')e_4]$$
$$= k_1a_1'b_1'e_2+(k_1k_4a_1'b_2'+k_1k_4a_2'b_1')e_3+(k_3a_1'b_1'+k_1k_4k_8a_1'b_3'$$
$$+k_1^2k_9a_2'b_2'+k_1k_4k_8a_3'b_1')e_4$$

$$= k_1a_1'b_1'e_2+k_1k_4(a_1'b_2'+a_2'b_1')e_3+(k_3a_1'b_1'+k_1k_4k_8a_1'b_3'$$
$$+k_1k_4k_8a_2'b_2'+k_1k_4k_8a_3'b_1')e_4.$$

This proves that (5.1) and (4.1) are isomorphic.

Case 6. Suppose that $k_1\neq 0, k_4\neq 0, k_5\neq 0, k_8\neq 0, k_9\neq 0$ and $k_3=0$. Then the multiplication (0) is

$$(6.1)\qquad xoy = k_1a_1b_1e_2+(k_4a_1b_2+k_4a_2b_1)e_3+(k_5a_1b_2+k_5a_2b_1+k_8a_1b_3$$
$$+k_9a_2b_2+k_8a_3b_1)e_4.$$

We claim that (6.1) is isomorphic to (4.1). To prove this, let $f\colon A\to A$ be the linear map defined by

$$f(e_1') = e_1,$$
$$f(e_2') = k_1e_2,$$
$$f(e_3') = k_1k_4e_3+k_1k_5e_4,$$
$$f(e_4') = k_1k_4k_8e_4, \quad k_i\in K,\ i = 1,4,5,8.$$

We see that $\det[f]\neq 0$ so that f is 1-1 and onto. We have from (4.1) that

$$f(xy) = f[a_1'b_1'e_2'+(a_1'b_2'+a_2'b_1')e_3'+(a_1'b_3'+a_2'b_2'+a_3'b_1')e_4']$$
$$= k_1a_1'b_1'e_2+k_1k_4(a_1'b_2'+a_2'b_1')e_3+[k_1k_5a_1'b_2'+$$
$$k_1k_5a_2'b_1'+k_1k_4k_8(a_1'b_3'+a_2'b_2'+a_3'b_1')]e_4$$

and from (6.1) we get that

$$f(x)of(y) = f(\sum_{i=1}^{4} a_i'e_i')of(\sum_{j=1}^{4} b_j'e_j')$$
$$= [a_1'e_1+k_1a_2'e_2+k_1k_4a_3'e_3+(k_1k_5a_3'+k_1k_4k_8a_4')e_4]o[b_1'e_1$$
$$+k_1b_2'e_2+k_1k_4b_3'e_3+(k_1k_5b_3'+k_1k_4k_8b_4')e_4]$$

$$= k_1a_1'b_1'e_2+(k_1k_4a_1'b_2'+k_1k_4a_2'b_1')e_3+(k_1k_5a_1'b_2'+k_1k_5a_2'b_1'$$
$$+k_1k_4k_8a_1'b_3'+k_1^2k_9a_2'b_2'+k_1k_4k_8a_3'b_1')e_4.$$

We have from equation (2) that $k_1k_9 = k_4k_8$. Hence $f(xy) = f(x)of(y)$. This implies that (6.1) and (4.1) are isomorphic.

Case 7. Suppose that all k_i, $i = 1,4,3,5,8,9$ are not zero. Then we have that

(7.1) $$xoy = k_1a_1b_1e_2+(k_4a_1b_2+k_4a_2b_1)e_3+(k_3a_1b_1+k_5a_1b_2+k_5a_2b_1+ k_8a_1b_3+k_9a_2b_2+k_8a_3b_1)e_4.$$

Claim that this multiplication is isomorphic to the multiplication (4.1). To see this, let $f: A \to A$ be the linear map defined by

$$f(e_1') = e_1,$$
$$f(e_2') = k_1e_2+k_3e_4,$$
$$f(e_3') = k_1k_4e_3+k_1k_5e_4,$$
$$f(e_4') = k_1k_4k_8e_4, \quad k_i \in K,\ i = 1,3,4,5,8.$$

Since $\det [f] \neq 0$, f is 1-1 and onto. Then as in case 6 we can prove that $f(xy) = f(x)of(y)$. Hence the multiplications (7.1) and (4.1) are isomorphic.

Now we consider the multiplication (***) when $k_2 \neq 0$ and $k_4 \neq 0$. Since $k_4 \neq 0$, we get that $k_{10} = k_8$ and $k_7 = k_5$. Then the multiplication (***) becomes

(00) $$xy = k_1a_1b_1e_2+(k_2a_1b_1+k_4a_1b_2+k_4a_2b_1)e_3+(k_3a_1b_1+k_5a_1b_2+ k_5a_2b_1+k_8a_1b_3+k_9a_2b_2+k_8a_3b_1)e_4.$$

We have from (2) that $k_8 \neq 0$ iff $k_9 \neq 0$. Therefore, we consider the following cases.

Case 8. Assume that $k_1 \neq 0, k_2 \neq 0, k_4 \neq 0, k_3 \neq 0$ and $k_5 = k_8 = k_9 = 0$. Then from (00) we have that

(8.1) $xoy = k_1 a_1 b_1 e_2 + (k_2 a_1 b_1 + k_4 a_1 b_2 + k_4 a_2 b_1) e_3 + k_3 a_1 b_1 e_4.$

Then (8.1) is isomorphic to (I). To prove this, let $f\colon A \to A$ be the linear map defined by

$$f(e_1') = e_1,$$
$$f(e_2') = k_1 e_2 + k_2 e_3,$$
$$f(e_3') = k_3 e_4,$$
$$f(e_4') = k_1 k_4 e_3, \quad k_i \in K,\ i = 1,2,3,4.$$

Then f is 1-1 and onto. By (I) we have that

$$f(xy) = f[a_1'b_1'e_2' + a_1'b_1'e_3' + (a_1'b_2' + a_2'b_1')e_4']$$
$$= k_1 a_1'b_1'e_2 + (k_2 a_1'b_1' + k_1 k_4 a_1'b_2' + k_1 k_4 a_2'b_1')e_3 + k_3 a_1'b_1'e_4.$$

and by (8.1) we have that

$$f(x)of(y) = f\left(\sum_{i=1}^{4} a_i'e_i'\right)of\left(\sum_{j=1}^{4} b_j'e_j'\right)$$
$$= [a_1'e_1 + k_1 a_2'e_2 + (k_2 a_2' + k_1 k_4 a_4')e_3 + k_3 a_3'e_4]o[b_1'e_1 + k_1 b_2'e_2 + (k_2 b_2' + k_1 k_4 b_4')e_3 + k_3 b_3'e_4]$$
$$= k_1 a_1'b_1'e_2 + (k_2 a_1'b_1' + k_1 k_4 a_1'b_2' + k_1 k_4 a_2'b_1')e_3 + k_3 a_1'b_1'e_4.$$

That is $f(xy) = f(x)of(y)$.

Case 9. Suppose that $k_1 \neq 0, k_2 \neq 0, k_4 \neq 0, k_5 \neq 0$ and $k_3 = k_8 = k_9 = 0$. Then the multiplication (00) is

(9.1) $\quad x o y = k_1a_1b_1e_2+(k_2a_1b_1+k_4a_1b_2+k_4a_2b_1)e_3+(k_5a_1b_2+k_5a_2b_1)e_4.$

Claim that this multiplication is isomorphic to (I). To prove this, let $f: A \to A$ be the linear map defined by

$$f(e_1') = e_1,$$
$$f(e_2') = k_1e_2,$$
$$f(e_3') = k_2e_3,$$
$$f(e_4') = k_1k_4e_3+k_1k_5e_4, \ k_i \in K, \ i = 1,2,4,5.$$

Then f is 1-1 and onto. (I) implies that

$$f(xy) = f[a_1'b_1'e_2'+a_1'b_1'e_3'+(a_1'b_2'+a_2'b_1')e_4']$$
$$= k_1a_1'b_1'e_2+(k_2a_1'b_1'+k_1k_4a_1'b_2'+k_1k_4a_2'b_1')e_3+k_1k_5(a_1'b_2'+a_2'b_1')e_4$$

and (9.1) implies that

$$f(x)of(y) = f(\sum_{i=1}^{4} a_i'e_i')of(\sum_{j=1}^{4} b_j'e_j')$$
$$= [a_1'e_1+k_1a_2'e_2+(k_2a_3'+k_1k_4a_4')e_3+k_1k_5a_4'e_4]o[b_1'e_1 +k_1b_2'e_2+(k_2b_3'+k_1k_4b_4')e_3+k_1k_5b_4'e_4]$$
$$= k_1a_1'b_1'e_2+(k_2a_1'b_1'+k_1k_4a_1'b_2'+k_1k_4a_2'b_1')e_3+ (k_1k_5a_1'b_2'+k_1k_5a_2'b_1')e_4.$$

Thus (9.1) and (I) are isomorphic.

Case 10. Suppose that $k_1 \neq 0, k_2 \neq 0, k_4 \neq 0, k_3 \neq 0, k_5 \neq 0$ and $k_8 = k_9 = 0$. Then from (00) page 90 we have that

(10.1) $\quad x o y = k_1a_1b_1e_2+(k_2a_1b_1+k_4a_1b_2+k_4a_2b_1)e_3+(k_3a_1b_1+k_5a_1b_2 +k_5a_2b_1)e_4.$

We claim that this multiplication is isomorphic to (I). Let $f: A \to A$ be defined by

$$f(e_1') = e_1,$$
$$f(e_2') = k_1e_2+k_3e_4,$$
$$f(e_3') = k_2e_3,$$
$$f(e_4') = k_1k_4e_3+k_1k_5e_4, \quad k_i \in K,\ i = 1,2,3,4,5.$$

Since det $[f] \neq 0$, f is 1-1 and onto. We have from (I) that

$$\begin{aligned} f(xy) &= f[a_1'b_1'e_2'+a_1'b_1'e_3'+(a_1'b_2'+a_2'b_1')e_4'] \\ &= k_1a_1'b_1'e_2+(k_2a_1'b_1'+k_1k_4a_1'b_2'+k_1k_4a_2'b_1')e_3+(k_3a_1'b_1'+ \\ &\qquad k_1k_5a_1'b_2'+k_1k_5a_2'b_1')e_4 \end{aligned}$$

and from (10.1) we get that

$$\begin{aligned} f(x)of(y) &= f(\sum_{i=1}^{4} a_i'e_i')of(\sum_{j=1}^{4} b_j'e_j') \\ &= [a_1'e_1+k_1a_2'e_2+(k_2a_3'+k_1k_4a_4')e_3+(k_3a_2'+k_1k_4a_4')e_4] \\ &\quad o\ [b_1'e_1+k_1b_2'e_2+(k_2b_3'+k_1k_4b_4')e_3+(k_3b_2'+k_1k_4b_4')e_4] \\ &= k_1a_1'b_1'e_2+(k_2a_1'b_1'+k_1k_4a_1'b_2'+k_1k_4a_2'b_1')e_3+ \\ &\qquad (k_3a_1'b_1'+k_1k_5a_1'b_2'+k_1k_5a_2'b_1')e_4. \end{aligned}$$

This proves the claim.

Case 11. Let $k_1\neq 0, k_2\neq 0, k_4\neq 0, k_8\neq 0, k_9\neq 0$ and $k_3 = k_5 = 0$. Then from (00) we get that

(11.1) $xoy = k_1a_1b_1e_2+(k_2a_1b_1+k_4a_1b_2+k_4a_2b_1)e_3+(k_8a_1b_3+k_9a_2b_2+k_8a_3b_1)e_4.$

We have from (2) that $k_1k_9 = k_4k_8$. Claim that this multiplication is isomorphic to (4.1) in case 4. To prove this, let $f: A \to A$ be the linear map defined by

$$f(e_1') = e_1,$$
$$f(e_2') = k_1e_2+k_2e_3,$$
$$f(e_3') = k_1k_4e_3+k_2k_8e_4,$$

$$f(e_4') = k_1k_4k_8e_4, \quad k_i \varepsilon K, \ i = 1,2,4,8.$$

We see that det $[f] \neq 0$ so that f is 1-1 and onto. By using (4.1) we get that

$$f(xy) = f[a_1'b_1'e_2' + (a_1'b_2' + a_2'b_1')e_3' + (a_1'b_3' + a_2'b_2' + a_3'b_1')e_4']$$
$$= k_1a_1'b_1'e_2 + (k_2a_1'b_1' + k_1k_4a_1'b_2' + k_1k_4a_1'b_2')e_3 +$$
$$[k_2k_8(a_1'b_2' + a_2'b_1') + k_1k_4k_8(a_1'b_3' + a_2'b_2' + a_3'b_1')]e_4$$

and (11.1) implies that

$$f(x)of(y) = f(\sum_{i=1}^{4} a_i'e_i')of(\sum_{j=1}^{4} b_j'e_j')$$
$$= [a_1'e_1 + k_1a_2'e_2 + (k_2a_2' + k_1k_4a_3')e_3 + (k_2k_8a_3' + k_1k_4k_8a_4')e_4]$$
$$o\ [b_1'e_1 + k_1b_2'e_2 + (k_2b_2' + k_1k_4b_3')e_3 + (k_2k_8b_3' + k_1k_4k_8b_4')e_4]$$
$$= k_1a_1'b_1'e_2 + (k_2a_1'b_1' + k_1k_4a_1'b_2' + k_1k_4a_2'b_1')e_3 +$$
$$[k_8a_1'(k_2b_2' + k_1k_4b_3') + k_1^2k_9a_2'b_2' + k_8(k_2a_2' + k_1k_4a_3')b_1']e_4$$
$$= k_1a_1'b_1'e_2 + (k_2a_1'b_1' + k_1k_4a_1'b_2' + k_1k_4a_2'b_1')e_3 +$$
$$[k_2k_8(a_1'b_2' + a_2'b_1') + k_1k_4k_8(a_1'b_3' + a_2'b_2' + a_3'b_1')]e_4.$$

Thus (11.1) and (4.1) are isomorphic.

<u>Case 12.</u> Assume that $k_1 \neq 0, k_2 \neq 0, k_4 \neq 0, k_3 \neq 0, k_8 \neq 0, k_9 \neq 0$ and $k_5 = 0$. Then from (00) page 90 we have that

(12.1) $$xoy = k_1a_1b_1e_2 + (k_2a_1b_1 + k_4a_1b_2 + k_4a_2b_1)e_3 + (k_3a_1b_1 + k_8a_1b_3 + k_9a_2b_2 + k_8a_3b_1)e_4.$$

We claim that (12.1) is isomorphic to (4.1). We define f: A → A by

$$f(e_1') = e_1,$$
$$f(e_2') = k_1e_2 + k_2e_3 + k_3e_4,$$
$$f(e_3') = k_1k_4e_3 + k_2k_8e_4,$$
$$f(e_4') = k_1k_4k_8e_4, \quad k_i \varepsilon K, \ i = 1,2,3,4,8.$$

We have that det $[f] \neq 0$ implying that f is 1-1 and onto.

As in the other cases, we have from (4.1) that

$$f(xy) = f[a_1'b_1'e_2'+(a_1'b_2'+a_2'b_1')e_3'+(a_1'b_3'+a_2'b_2'+a_3'b_1')e_4']$$
$$= k_1a_1'b_1'e_2+(k_2a_1'b_1'+k_1k_4a_1'b_2'+k_1k_4a_2'b_1')e_3+[k_3a_1'b_1'$$
$$+k_2k_8(a_1'b_2'+a_2'b_1')+k_1k_4k_8(a_1'b_3'+a_2'b_2'+a_3'b_1')]e_4$$

and (12.1) implies that

$$f(x)of(y) = f(\sum_{i=1}^{4} a_i'e_i')of(\sum_{j=1}^{4} b_j'e_j')$$
$$= [a_1'e_1+k_1a_2'e_2+(k_2a_2'+k_1k_4a_3')e_3+(k_3a_2'+k_2k_8a_3'+k_1k_4k_8a_4')e_4]$$
$$o\ [b_1'e_1+k_1b_2'e_2+(k_2b_2'+k_1k_4b_3')e_3+(k_3b_2'+k_2k_8b_3'+k_1k_4k_8b_4')e_4]$$
$$= k_1a_1'b_1'e_2+(k_2a_1'b_1'+k_1k_4a_1'b_2'+k_1k_4a_2'b_1')e_3+[k_3a_1'b_1'$$
$$+k_2k_8(a_1'b_2'+a_2'b_1')+k_1k_4k_8(a_1'b_3'+a_2'b_2'+a_3'b_1')]e_4.$$

Hence (12.1) and (4.1) are isomorphic.

<u>Case 13.</u> Suppose that $k_1 \neq 0, k_2 \neq 0, k_4 \neq 0, k_5 \neq 0, k_8 \neq 0, k_9 \neq 0$ and $k_3 = 0$. Then from (00) page 90 we have that

(13.1) $$xoy = k_1a_1b_1e_2+(k_2a_1b_1+k_4a_1b_2+k_4a_2b_1)e_3+(k_5a_1b_2+k_5a_2b_1$$
$$+k_8a_1b_3+k_9a_2b_2+k_8a_3b_1)e_4.$$

Claim that this multiplication is isomorphic to (4.1) in case 4. To prove this, let $f: A \to A$ be the linear map defined by

$$f(e_1') = e_1,$$
$$f(e_2') = k_1e_2+k_2e_3,$$
$$f(e_3') = k_1k_4e_3+(k_1k_5+k_2k_8)e_4,$$
$$f(e_4') = k_1k_4k_8e_4, \quad k_i \in K,\ i = 1,2,4,5,8.$$

We see that det $[f] \neq 0$ hence f is 1-1 and onto. We have from (4.1) that

$$f(xy) = f\left[a_1'b_1'e_2'+(a_1'b_2'+a_2'b_1')e_3'+(a_1'b_3'+a_2'b_2'+a_3'b_1')e_4'\right]$$
$$= k_1a_1'b_1'e_2+(k_2a_1'b_1'+k_1k_4a_1'b_2'+k_1k_4a_2'b_1')e_3+\left[(k_1k_5+k_2k_8)(a_1'b_2'+a_2'b_1')+k_1k_4k_8(a_1'b_3'+a_2'b_2'+a_3'b_1')\right]e_4,$$

whereas, (13.1) implies that

$$f(x)of(y) = f(\sum_{i=1}^{4} a_i'e_i')of(\sum_{j=1}^{4} b_j'e_j'),$$
$$= \left[a_1'e_1+k_1a_2'e_2+(k_2a_2'+k_1k_4a_3')e_3+\{(k_1k_5+k_2k_8)a_3'+k_1k_4k_8a_4'\}e_4\right]$$
$$o\left[b_1'e_1+k_1b_2'e_2+(k_2b_2'+k_1k_4b_3')e_3+\{(k_1k_5+k_2k_8)b_3'+k_1k_4k_8b_4'\}e_4\right]$$
$$= k_1a_1'b_1'e_2+(k_2a_1'b_1'+k_1k_4a_1'b_2'+k_1k_4a_2'b_1')e_3+\left[k_1k_5(a_1'b_2'+a_2'b_1')+k_2k_8(a_1'b_2'+a_2'b_1')+k_1k_4k_8(a_1'b_3'+a_2'b_2'+a_3'b_1')\right]e_4$$
$$= k_1a_1'b_1'e_2+(k_2a_1'b_1'+k_1k_4a_1'b_2'+k_1k_4a_2'b_1')e_3+\left[(k_1k_5+k_2k_8)(a_1'b_2'+a_2'b_1')+k_1k_4k_8(a_1'b_3'+a_2'b_2'+a_3'b_1')\right]e_4.$$

Therefore, (13.1) and (4.1) are isomorphic.

Case 14. In the final case we assume that all $k_i, i=1,2,4,3,5,8,9$ are not zero. Then from (00) page 90 we get that

$$(14.1)\quad xoy = k_1a_1b_1e_2+(k_2a_1b_1+k_4a_1b_2+k_4a_2b_1)e_3+(k_3a_1b_1+k_5a_1b_2+k_5a_2b_1+k_8a_1b_3+k_9a_2b_2+k_8a_3b_1)e_4.$$

We see that this case is isomorphic to (4.1) in case 4. Let $f: A \to A$ be the linear map defined by

$$f(e_1') = e_1,$$
$$f(e_2') = k_1e_2+k_2e_3+k_4e_4,$$
$$f(e_3') = k_1k_4e_3+(k_1k_5+k_2k_8)e_4,$$
$$f(e_4') = k_1k_4k_8e_4,\quad k_i \in K,\ i = 1,2,3,4,5,8.$$

Since det $[f] \neq 0$, f is 1-1 and onto. Then, as in case 13, we can prove that $f(xy) = f(x) \circ f(y)$. Hence (14.1) and (4.1) are isomorphic.

Hence there are 3 non-isomorphism classes of nilpotent algebras A over a field K with dimension $A = 4$, dimension $A^3 = 2$, dimension $A^4 = 1$ and $A^5 = \{0\}$. That is for each $x = \sum_{i=1}^{4} a_i e_i$, $y = \sum_{j=1}^{4} b_j e_j$ $\{a_i, b_j\} \subset K$, $i,j = 1,2,3,4$ we have that

(I) $$xy = a_1b_1e_2 + a_1b_1e_3 + (a_1b_2 + a_2b_1)e_4,$$

(II) $$xy = a_1b_1e_2 + a_1b_1e_3 + (a_1b_2 + a_3b_1)e_4,$$

(III) $$xy = a_1b_1e_2 + (a_1b_2 + a_2b_1)e_3 + (a_1b_3 + a_2b_2 + a_3b_1)e_4.$$