CHAPTER 2

DERIVATION OF FORMULAE

2.1 The S-beam

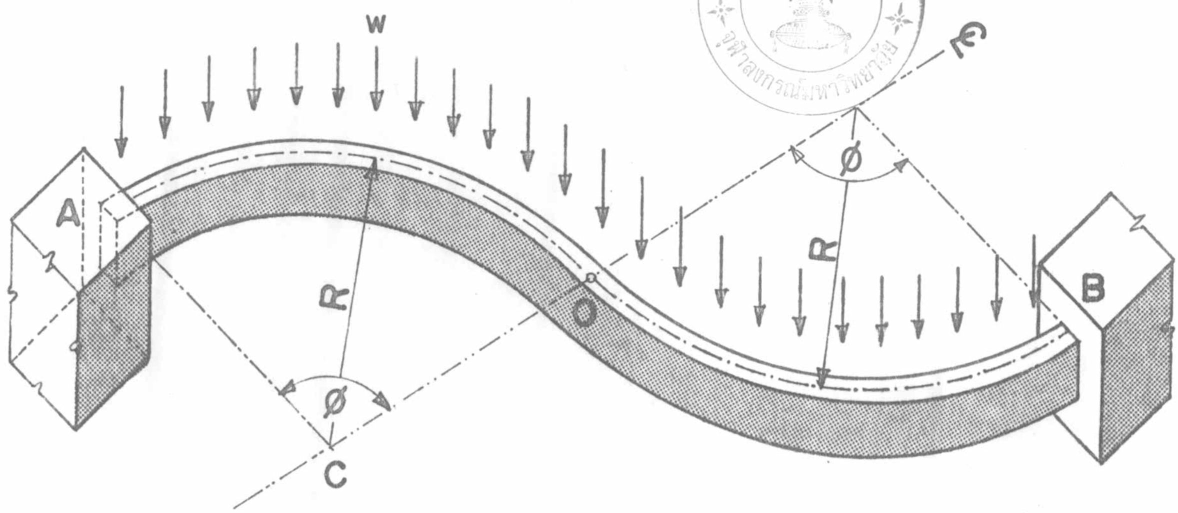

FIGURE 2.1 The Uniformly Loaded S-beam with Fixed Supports

Figure 2.1 portrays the S-beam under investigation. The horizontal alignment of the beam encompasses a reverse curve, each arc constituting the curve being circular with a radius R and a subtending angle ø. The constant cross-sectional dimension of the beam, above other features, is essential to analytical imposition of symmetry. Placement of a line load of uniform intensity w on the beam retains the symmetry requirement. Severance of the continuity at centre-span O calls for two redundants to restore structural integrity of each severed arc. These two redundants comprise the bending moment M_o and the torsional moment T_o.

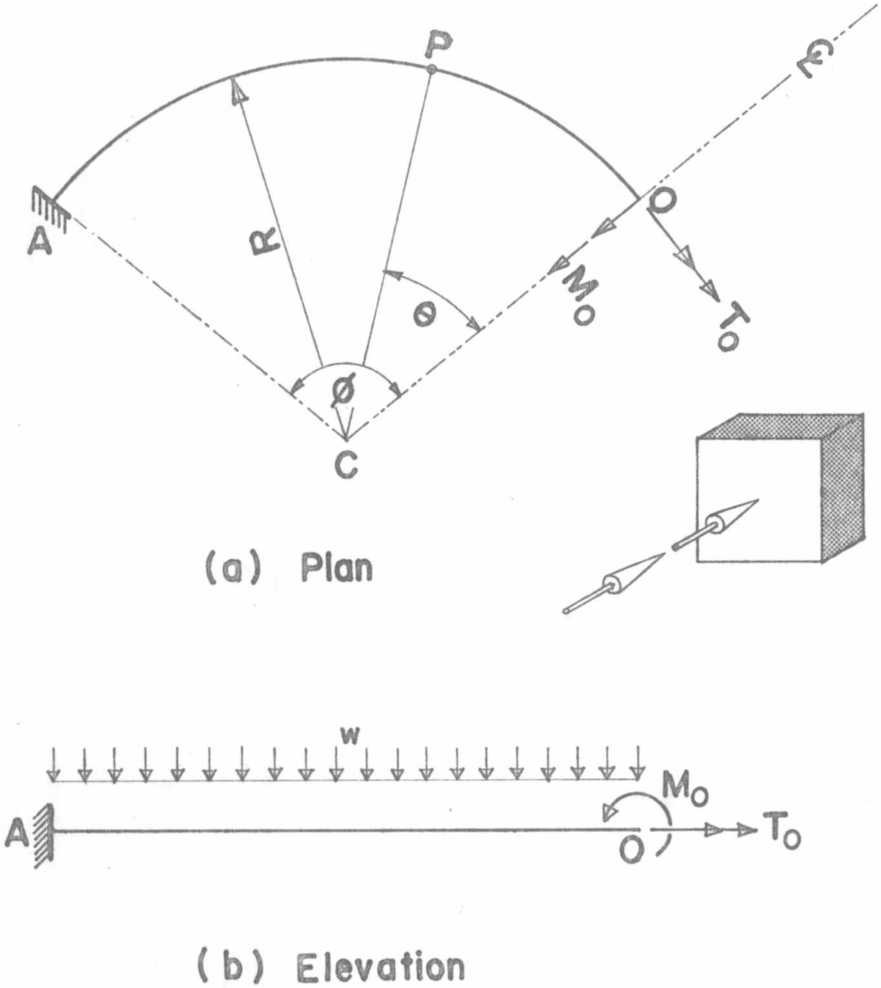

FIGURE 2.2 Half-beam under Action of Uniform Load and Redundants

Figure 2.2 shows one of the severed arcs under the action of the uniform load w and the redundants M_o and T_o. With the centre-span 0 as origin the bending and torsional moments at any point defined by an angle θ can be expressed as

$$M = M_o \cos\theta + T_o \sin\theta - wR^2(1-\cos\theta) \tag{2.1}$$

$$T = M_o \sin\theta - T_o \cos\theta - wR^2(\theta-\sin\theta) \tag{2.2}$$

Details pertinent to the formulation of these expressions are given in Appendix A.

According to Castigliano's second theorem connected with the strain energy approach the rotation and the angle of twist at centre-span respectively satisfy the relations.

$$\psi_o = \frac{R}{EI}\int M \frac{\partial M}{\partial M_o}\, d\theta + \frac{R}{GJ}\int T \frac{\partial T}{\partial M_o}\, d\theta \tag{2.3}$$

$$\tau_o = \frac{R}{EI}\int M \frac{\partial M}{\partial T_o}\, d\theta + \frac{R}{GJ}\int T \frac{\partial T}{\partial T_o}\, d\theta \tag{2.4}$$

wherein EI denotes the flexural rigidity and GJ the torsional rigidity of the beam section. It is inferred from symmetry that both the rotation and the angle of twist at centre-span equal zero. The imposition of these boundary conditions on relations (2.1), (2.2), (2.3), and (2.4) brings forth the following simultaneous equations in two unknowns:

$$\int_o^{\phi}\left[M_o \cos\theta + T_o \sin\theta - wR^2(1-\cos\theta)\right]\cos\theta d\theta$$
$$+ m\int_o^{\phi}\left[M_o \sin\theta - T_o \cos\theta - wR^2(\theta-\sin\theta)\right]\sin\theta d\theta = 0 \tag{2.5}$$

$$\int_o^{\phi}\left[M_o \cos\theta + T_o \sin\theta - wR^2(1-\cos\theta)\right]\sin\theta d\theta$$
$$- m\int_o^{\phi}\left[M_o \sin\theta - T_o \cos\theta - wR^2(\theta-\sin\theta)\right]\cos\theta d\theta = 0 \tag{2.6}$$

in which $m = \frac{EI}{GJ}$. Intergration reduces this pair of equations to:

$$\left[\frac{1}{2}(1+m)\phi + \frac{1}{4}(1-m)\sin 2\phi\right]M_o$$
$$+ \frac{1}{4}(1-m)(1-\cos 2\phi)\ T_o$$

$$+\left[\frac{1}{2}(1+m)\ \phi - (1+m)\ \sin\phi + \frac{1}{4}(1-m)\ \sin 2\phi + m\phi\ \cos\phi\right] wR^2 = 0 \quad (2.7)$$

$$\frac{1}{4}(1-m)(1-\cos 2\phi)\ M_o$$

$$+\left[\frac{1}{2}(1+m)\ \phi - \frac{1}{4}(1-m)\ \sin 2\phi\right] T_o$$

$$-\left[\frac{1}{4}(3+5m) - m\ \phi\ \sin\phi - (1+m)\ \cos\phi + \frac{1}{4}(1-m)\ \cos 2\phi\right] wR^2 = 0 \quad (2.8)$$

It is chosen to present the solution to these simultaneous equations in the following manner:

$$M_o = \frac{1}{N_5}(N_1 N_2 - N_3 N_4)\ wR^2 \quad (2.9)$$

$$T_o = \frac{1}{N_5}(N_4 N_6 - N_2 N_3)\ wR^2 \quad (2.10)$$

wherein

$$N_1 = 2(1+m)\ \phi - (1-m)\ \sin 2\phi \quad (2.11)$$

$$N_2 = (1+m)(\sin\phi - \frac{1}{2}\ \phi) - \frac{1}{4}(1-m)\ \sin 2\phi - m\phi\ \cos\phi \quad (2.12)$$

$$N_3 = (1-m)(1-\cos 2\phi) \quad (2.13)$$

$$N_4 = \frac{1}{4}(3+5m) - m\phi\ \sin\phi - (1+m)\ \cos\phi + \frac{1}{4}(1-m)\ \cos 2\phi \quad (2.14)$$

$$N_5 = (1+m)^2\phi^2 - \frac{1}{2}(1-m)^2(1-\cos 2\phi) \quad (2.15)$$

$$N_6 = 2(1+m)\ \phi + (1-m)\ \sin 2\phi \quad (2.16)$$

With the redundants M_o and T_o rendered explicit as such the bending and torsional moments at any location between the centre-span and the support can be enumerated with ease using relations (2.1) and (2.2). Reference is made to the graphical presentation in Chapter 3 for rapid quantification of the redundants M_o and T_o in a beam with known values of parameters R, Ø, and m.

2.2 The Z-beam

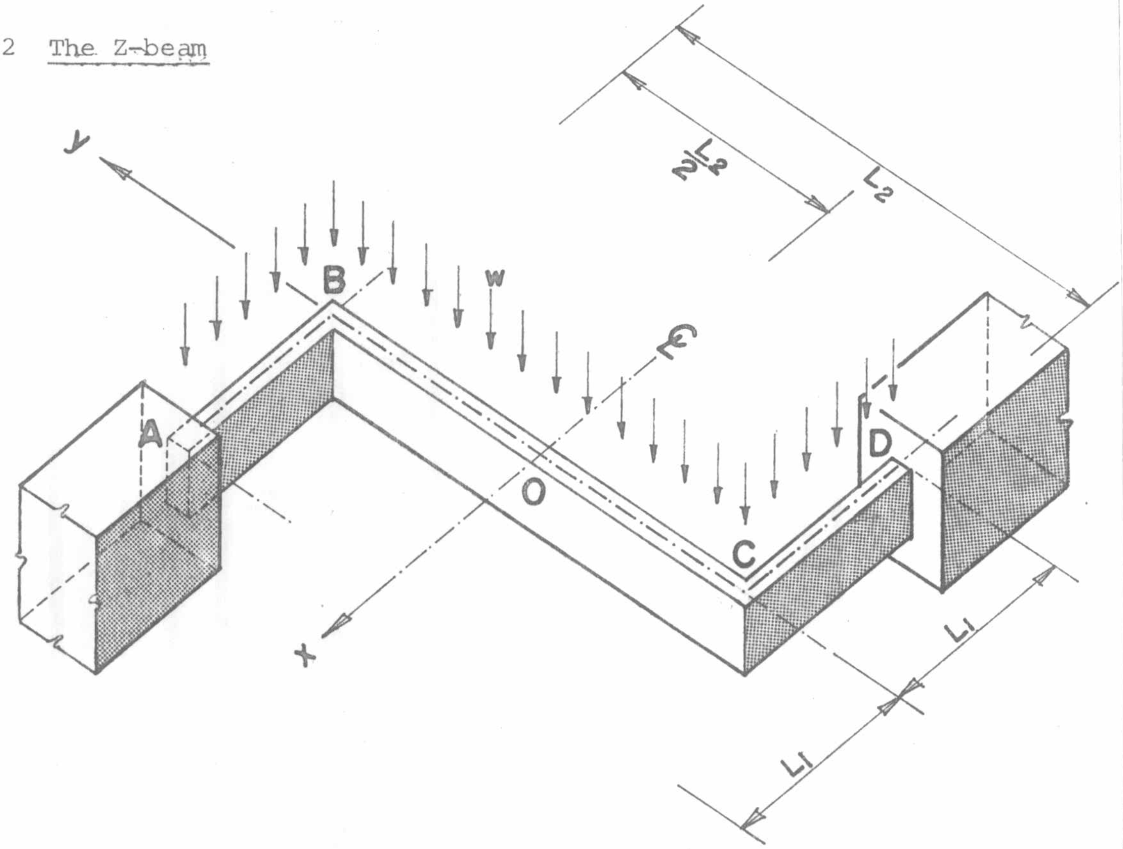

FIGURE 2.3 The Uniformly Loaded Z-beam with Fixed Supports

The perused Z-beam possesses such geometrical symmetry as depicted in Figure 2.3. Each approach part identifies with length L_1, the central transverse part with length L_2. Each of the longitudinal members makes an angle of 90 degrees with the transverse member. The entire beam with constant rigidities carries a uniform load of intensity w. The diagram of Figure 2.4 represents half of the whole system, under the action of redundants M_o and T_o.

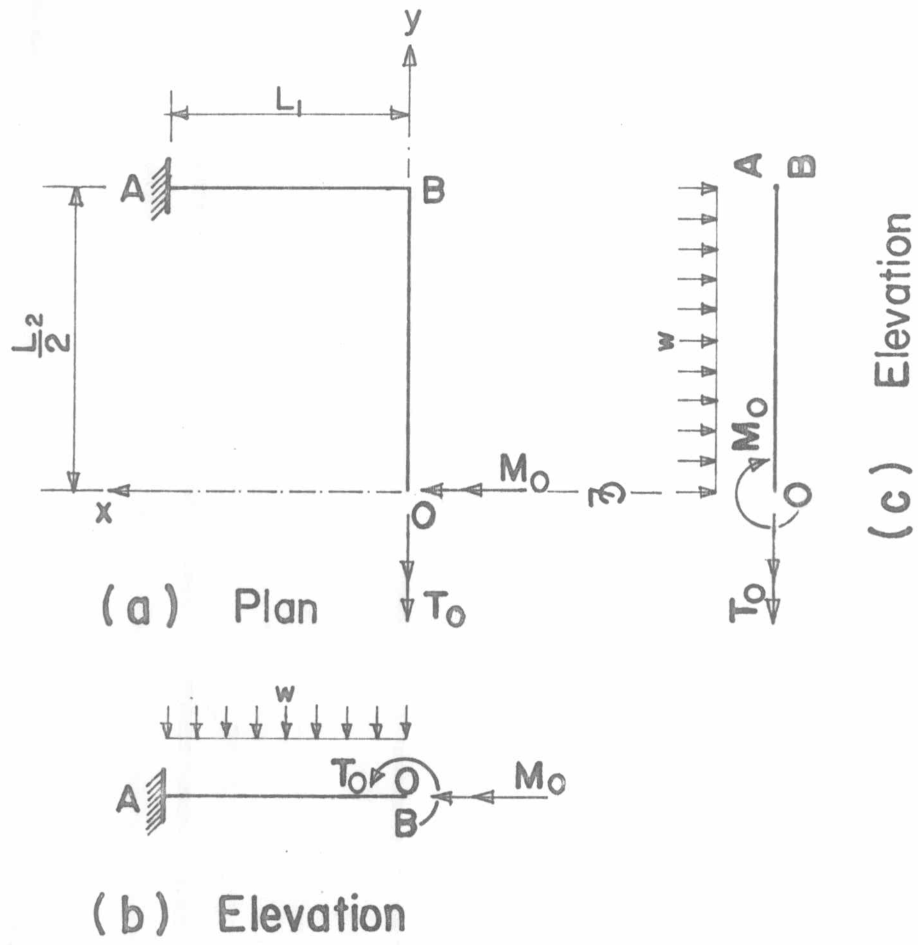

FIGURE 2.4 Half-beam under Action of Uniform Load and Redundants

It is analytically essential to divide the half-beam into two portions: the component between the centre-span and the junction of the transverse member with the longitudinal member; and the component between the junction and the support. With x and y denoting lineal variables such as indicated the bending and torsional moments can be written in terms of the redundants as follows.

For the transverse component,

$$M = M_o - \frac{1}{2} wy^2 \tag{2.17}$$

$$T = T_o \tag{2.18}$$

For the longitudinal component,

$$M = T_o - \frac{1}{2} wL_2 x - \frac{1}{2} wx^2 \tag{2.19}$$

$$T = \frac{1}{8} wL_2^2 - M_o \tag{2.20}$$

On account of symmetry the rotation and the angle of twist at centre-span equal zero. Employment of these conditions in conjunction with the strain energy approach results in the following equations:

$$\int_0^{\frac{L_2}{2}} (M_o - \frac{1}{2} wy^2)\, dy - m \int_0^{L_1} (\frac{1}{8} wL_2^2 - M_o)\, dx = 0 \tag{2.21}$$

$$\int_0^{L_1} (T_o - \frac{1}{2} wL_2 x - \frac{1}{2} wx^2)\, dx + mT_o \int_0^{\frac{L_2}{2}} dy = 0 \tag{2.22}$$

from which

$$M_o = \frac{1}{24} \cdot \frac{(1+6km)}{(1+2km)} \cdot wL_2^2 \tag{2.23}$$

$$T_o = \frac{k^2}{6} \cdot \frac{(2k+3)}{(2k+m)} \cdot wL_2^2 \tag{2.24}$$

where $k = \frac{L_1}{L_2}$. The bending and torsional moments at any point between the centre-span and the support can be readily determined via relations (2.17), (2.18), (2.19), and (2.20). Graphical aids empowering rapid quantification of M_o and T_o are contained in Chapter 3.