CHAPTER I

FACTORIZABLE TRANSFORMATION SEMIGROUPS

It has been proved by Tirasupa in [4] that the partial transformation semigroup on a set X is factorizable if and only if X is finite. The purpose of this chapter is to give an application of this work.

If X is a set, then for $\alpha, \beta \in T_X$, we have that $\Delta\alpha\beta = (\nabla\alpha \cap \Delta\beta)\alpha^{-1} \subseteq \Delta\alpha$ and $\nabla\alpha\beta = (\nabla\alpha \cap \Delta\beta)\beta \subseteq \nabla\beta$.

Let X be a set. For $\alpha \in T_X$, the <u>shift</u> of α is the set of all elements x in the domain of α such that $x\alpha \neq x$, and the shift of α is denoted by $S(\alpha)$, that is,

$$S(\alpha) = \{x \in \Delta\alpha \mid x\alpha \neq x\}.$$

Then, for $\alpha \in T_X$, α is almost identical if and only if $S(\alpha)$ is finite. Observe that if $\alpha \in T_X$, $\Delta\alpha \setminus S(\alpha) = \{x \in \Delta\alpha \mid x\alpha = x\}$. If $\alpha, \beta \in T_X$, then $S(\alpha\beta) \subseteq S(\alpha) \cup S(\beta)$. To prove this, let $\alpha, \beta \in T_X$. Let $x \in S(\alpha\beta)$. Then $x \in \Delta\alpha\beta$ and $x\alpha\beta \neq x$, so $x \in \Delta\alpha$ and $x\alpha \in \Delta\beta$. If $x\alpha = x$, then $x \in \Delta\beta$ and $x \neq x\alpha\beta = x\beta$ which implies $x \in S(\beta)$. If $x\alpha \neq x$, then $x \in S(\alpha)$.

Let X be a set and let U_X be the set of all almost identical partial transformations, that is,

$$U_X = \{\alpha \in T_X \mid |S(\alpha)| < \infty\}.$$

Then $U_X \subseteq T_X$. If $\alpha, \beta \in U_X$, then $|S(\alpha)| < \infty$ and $|S(\beta)| < \infty$ and so

$|S(\alpha\beta)| \leq |S(\alpha) \cup S(\beta)| \leq |S(\alpha)| + |S(\beta)| < \infty$, hence $\alpha\beta \in U_X$. Therefore U_X is a subsemigroup of T_X which is called the <u>semigroup of almost identical partial transformations</u> on X. Clearly, $0, 1 \in U_X$. Then U_X is a semigroup with zero and identity. Therefore $E(U_X) = E(T_X) \cap U_X$, that is, for $\alpha \in T_X$, α is an idempotent of U_X if and only if $\nabla\alpha \subseteq \Delta\alpha$, $x\alpha = x$ for all $x \in \nabla\alpha$ and $|S(\alpha)| < \infty$. We show in the next proposition that for any set X, U_X is regular.

1.1 <u>Proposition</u>. For any set X, the semigroup U_X is a regular semigroup.

<u>Proof</u> : Let X be a set and let $\alpha \in U_X$. Then $|S(\alpha)| < \infty$. For each element $x \in \nabla\alpha$, choose $d_x \in \Delta\alpha$ such that $d_x \in x\alpha^{-1}$. Note that $d_x\alpha = x$ for all $x \in \nabla\alpha$, and $d_x \neq d_y$ if $x \neq y$ in $\nabla\alpha$. Define $\alpha' : \nabla\alpha \to X$ by $x\alpha' = d_x$ for all $x \in \nabla\alpha$. Then $\Delta\alpha' = \nabla\alpha$ and $\nabla\alpha' = \{d_x \mid x \in \nabla\alpha\} \subseteq \Delta\alpha$. Therefore $\Delta\alpha'\alpha = (\nabla\alpha' \cap \Delta\alpha)\alpha'^{-1} = (\nabla\alpha')\alpha'^{-1} = \Delta\alpha' = \nabla\alpha$ and hence $\Delta\alpha\alpha'\alpha = (\nabla\alpha \cap \Delta\alpha'\alpha)\alpha^{-1} = (\nabla\alpha)\alpha^{-1} = \Delta\alpha$. If $x \in \Delta\alpha$, then $x\alpha\alpha'\alpha = (x\alpha)\alpha'\alpha = d_{x\alpha}\alpha = x\alpha$. Hence $\alpha = \alpha\alpha'\alpha$. Claim that $\alpha' \in U_X$. For any $x \in \Delta\alpha' = \nabla\alpha$ such that $x \in S(\alpha')$ we have that $x\alpha' \neq x$ and $x\alpha' = d_x$ which implies $d_x \neq x$, thus $d_x\alpha = x \neq d_x$, so $d_x \in S(\alpha)$. Hence if $x \in S(\alpha')$ then $d_x \in S(\alpha)$. Define $\psi : S(\alpha') \to S(\alpha)$ by $y\psi = d_y$ for all $y \in S(\alpha')$. Since for any $x, y \in \Delta\alpha' = \nabla\alpha$ such that $x \neq y$ implies $d_x \neq d_y$, it follows that ψ is one-to-one. Then $|S(\alpha')| \leq |S(\alpha)| < \infty$, so $\alpha' \in U_X$. This proves that U_X is regular. #

The group of units of the semigroup of almost identical partial transformations on a set is given in the next proposition.

1.2 Lemma. Let X be any set. If $\alpha \in G_X$, then $S(\alpha) = S(\alpha^{-1})$.

Proof : Assume $\alpha \in G_X$. Then $\Delta\alpha = \Delta\alpha^{-1} = X$. Let $x \in S(\alpha)$. Then $x\alpha \neq x$. To show that $x \in S(\alpha^{-1})$, suppose not. Then $x \notin S(\alpha^{-1})$, so $x\alpha^{-1} = x$ which implies $x\alpha = x$, a contradiction. Therefore $x \in S(\alpha^{-1})$. Thus $S(\alpha) \subseteq S(\alpha^{-1})$. From the above proof, we have that $S(\alpha^{-1}) \subseteq S((\alpha^{-1})^{-1}) = S(\alpha)$. Hence $S(\alpha) = S(\alpha^{-1})$. #

1.3 Proposition. For any set X, the set $G_X \cap U_X$ is the group of units of the semigroup U_X.

Proof : Let X be a set. Let $\alpha \in G_X \cap U_X$. Then $\alpha \in G_X$. By Lemma 1.2, $S(\alpha) = S(\alpha^{-1})$, so $|S(\alpha^{-1})| = |S(\alpha)| < \infty$ since $\alpha \in U_X$. Then $\alpha^{-1} \in U_X$. Because $\alpha\alpha^{-1} = 1 = \alpha^{-1}\alpha$ and $\alpha^{-1} \in U_X$, it follows that α belongs to the group of units of U_X. This proves that $G_X \cap U_X$ is a subset of the group of units of U_X.

Conversely, let α be a unit in U_X. Then there exists $\alpha' \in U_X$ such that $\alpha\alpha' = \alpha'\alpha = 1$. Since $G_X = \{\beta \in T_X \mid \beta\beta' = \beta'\beta = 1 \text{ for some } \beta' \in T_X\}$, we have that $\alpha \in G_X$. Therefore $\alpha \in G_X \cap U_X$.

Hence, $G_X \cap U_X$ is the group of units of U_X, as desired. #

We proceed to prove a main theorem of this chapter. First, we recall that the following have been proved in [4]. If S is a semigroup with identity and is factorizable as S = GE, then G is the group of units of S. For any set X, the partial transformation semigroup (T_X) is factorizable if and only if X is finite.

1.4 Theorem. For any set X, the semigroup of almost identical partial transformations on X is factorizable.

Proof : Let X be a set. Let $\alpha \in U_X$. Then $|S(\alpha)| < \infty$. Since α is a map, $|S(\alpha)\alpha| = |\{x\alpha | x \in S(\alpha)\}| \leq |S(\alpha)| < \infty$. Then $|S(\alpha) \cup S(\alpha)\alpha| \leq |S(\alpha)| + |S(\alpha)\alpha| < \infty$. Let $Y = S(\alpha) \cup S(\alpha)\alpha$. Then $Y \subseteq X$ and $|Y| < \infty$. Let $\beta = \alpha|_{\Delta\alpha \cap Y}$, the restriction of α to $\Delta\alpha \cap Y$. Then $\Delta\beta = \Delta\alpha \cap Y \subseteq Y$ and $\nabla\beta = S(\alpha)\alpha \subseteq Y$, so $\beta \in T_Y$. Since $|Y| < \infty$, $\beta = \lambda\gamma$ for some $\lambda \in G_Y$ and $\gamma \in E(T_Y)$ [4, Theorem 3.1]. Define the map $\mu : X \to X$ by

$$x\mu = \begin{cases} x\lambda & \text{if } x \in Y, \\ x & \text{if } x \notin Y. \end{cases}$$

Since $\lambda \in G_Y$, $\mu \in G_X$. From the definition of μ, it is clear that $S(\mu) = S(\lambda) \subseteq Y$. Then $|S(\mu)| \leq |Y| < \infty$, so $\mu \in U_X$. Thus $\mu \in G_X \cap U_X$ which is the unit group of U_X by Proposition 1.3. Claim that $\Delta\gamma \subseteq (\Delta\alpha)\mu$. Let $x \in \Delta\gamma$. Since $\Delta\gamma \subseteq Y$, $x \in \Delta\gamma \cap Y = \Delta\gamma \cap \nabla\lambda$. Then $x\lambda^{-1} \in (\Delta\gamma \cap \nabla\lambda)\lambda^{-1} = \Delta\lambda\gamma = \Delta\beta = \Delta\alpha \cap Y \subseteq Y$, so there exists $y \in Y$ such that $y = x\lambda^{-1}$. Since $y \in Y$, $y\mu = y\lambda = x$. Then $x\mu^{-1} = y = x\lambda^{-1}$, so $x\mu^{-1} = x\lambda^{-1} \in \Delta\beta = \Delta\alpha \cap Y \subseteq \Delta\alpha$ which implies $x \in (\Delta\alpha)\mu$. Thus $\Delta\gamma \subseteq (\Delta\alpha)\mu$. Let $\delta : (\Delta\alpha)\mu \to X$ be defined by

$$x\delta = \begin{cases} x\gamma & \text{if } x \in \Delta\gamma , \\ x & \text{if } x \in (\Delta\alpha)\mu \smallsetminus \Delta\gamma. \end{cases}$$

Then $\nabla\delta = \nabla\gamma \cup ((\Delta\alpha)\mu \setminus \Delta\gamma)$. Since $\gamma \in E(T_Y)$, we have that $\nabla\gamma \subseteq \Delta\gamma$ and $x\gamma = x$ for all $x \in \nabla\gamma$. It then follows that $\nabla\delta = \nabla\gamma \cup ((\Delta\alpha)\mu \setminus \Delta\gamma) \subseteq \Delta\gamma \cup ((\Delta\alpha)\mu \setminus \Delta\gamma) = (\Delta\alpha)\mu = \Delta\delta$ and $x\delta = x$ for all $x \in \nabla\delta$. Hence $\delta \in E(T_X)$. From the definition of δ, it is immediate that $S(\delta) = S(\gamma) \subseteq Y$ which is finite. Thus $\delta \in E(U_X)$. Claim that $\alpha = \mu\delta$. Since $\mu \in G_X$, $((\Delta\alpha)\mu)\mu^{-1} = \Delta\alpha$, so $\Delta\mu\delta = (\nabla\mu \cap \Delta\delta)\mu^{-1} = (\Delta\delta)\mu^{-1} = ((\Delta\alpha)\mu)\mu^{-1} = \Delta\alpha$. Let $x \in \Delta\alpha$.

<u>Case $x \in Y$.</u> Then $x\mu = x\lambda$, and $x\beta = x\alpha$ since $x \in \Delta\alpha \cap Y$. But since $x \in \Delta\alpha \cap Y = \Delta\beta = \Delta\lambda\gamma = (\nabla\lambda \cap \Delta\gamma)\lambda^{-1}$, it follows that $x\lambda \in \nabla\lambda \cap \Delta\gamma \subseteq \Delta\gamma$. Therefore $(x\lambda)\delta = (x\lambda)\gamma$. Hence $x\mu\delta = (x\lambda)\delta = x\lambda\gamma = x\beta = x\alpha$.

<u>Case $x \notin Y$.</u> Then $x \notin S(\alpha)$, so $x\alpha = x$. By the definition of μ, $x\mu = x$. Then $x \in (\Delta\alpha)\mu$. Because $x \notin Y$ and $\Delta\gamma \subseteq Y$, $x \notin \Delta\gamma$. Then $x\delta = x$. Thus $x\mu\delta = x\delta = x = x\alpha$.

Hence $\alpha = \mu\delta$.

This proves that the semigroup U_X is factorizable,as required. #

Let X be a set and set

$$V_X = \{\alpha \in \mathcal{T}_X \mid |S(\alpha)| < \infty\}$$

and

$$W_X = \{\alpha \in I_X \mid |S(\alpha)| < \infty\}.$$

Then V_X is a subsemigroup of $\mathcal{T}_X$ and of U_X with identity and W_X is a subsemigroup of I_X and of U_X with zero and identity. The semigroups V_X and W_X are called the <u>semigroup of almost identical full transformations</u> on X and the <u>semigroup of almost identical 1-1 partial transforamtions</u> on X, respectively. Observe that

$$V_X = \{\alpha \in U_X \mid \Delta\alpha = X\} = U_X \cap \mathcal{I}_X,$$

$$W_X = \{\alpha \in U_X \mid \alpha \text{ is one-to-one}\} = U_X \cap I_X,$$

$$E(V_X) = \{\alpha \in E(U_X) \mid \Delta\alpha = X\} = E(U_X) \cap \mathcal{I}_X,$$

and $$E(W_X) = \{\alpha \in E(U_X) \mid \alpha \text{ is one-to-one}\} = E(U_X) \cap I_X.$$

Let X be a set. Since $V_X \subseteq U_X$ and $W_X \subseteq U_X$, $G_X \cap V_X \subseteq G_X \cap U_X$ and $G_X \cap W_X \subseteq G_X \cap U_X$. If $\alpha \in G_X \cap U_X$, then $\alpha \in G_X$ which implies $\Delta\alpha = X$ and α is one-to-one, hence $\alpha \in G_X \cap V_X$ and $\alpha \in G_X \cap W_X$. Therefore, $G_X \cap V_X = G_X \cap U_X = G_X \cap W_X$. Thus $G_X \cap U_X \subseteq V_X \subseteq U_X$ and $G_X \cap U_X \subseteq W_X \subseteq U_X$. But since $G_X \cap U_X$ is the greatest subgroup of U_X having 1 as its identity, it follows that $G_X \cap U_X$ is the greatest subgroup of V_X and of W_X having 1 as its identity. Hence $G_X \cap U_X$ is the group of units of V_X and W_X.

We show in the following proposition that for any set X, the semigroup V_X is regular and the semigroup W_X is inverse.

1.5 <u>Proposition</u>. For any set X, the semigroup V_X is a regular semigroup and the semigroup W_X is an inverse semigroup.

<u>Proof</u> : Let X be a set. Let $\alpha \in W_X$. Then $\alpha \in U_X$. Define α' from α as in the proof of Proposition 1.1. Then $\alpha\alpha'\alpha = \alpha$. Since $d_x \neq d_y$ if $x \neq y$ in $\Delta\alpha' = \nabla\alpha$, it follows that α' is one-to-one, hence $\alpha' \in W_X$. This proof that W_X is regular. But W_X is a subsemigroup of the inverse semigroup I_X, it then follows that W_X is an inverse semigroup.

Let $\alpha \in V_X$. Then $\alpha \in U_X$. Define α' from α as in the proof of Proposition 1.1. Define the map $\beta : X \to X$ by

$$x\beta = \begin{cases} x\alpha' & \text{if } x \in \Delta\alpha' = \nabla\alpha, \\ x & \text{if } x \in X \setminus \Delta\alpha'. \end{cases}$$

Then $\beta \in \mathcal{I}_X$ and $S(\beta) = S(\alpha')$, so $\beta \in V_X$. Let $x \in X$. Then $x\alpha \in \nabla\alpha = \Delta\alpha'$ which implies $(x\alpha)\beta = (x\alpha)\alpha'$. Thus $x\alpha\beta\alpha = ((x\alpha)\beta)\alpha = ((x\alpha)\alpha')\alpha = x\alpha\alpha'\alpha = x\alpha$. Hence $\alpha\beta\alpha = \alpha$. This proves that V_X is regular. #

Using Theorem 1.4, we show in the next two theorems that the semigroups V_X and W_X are factorizable for any set X.

1.6 <u>Theorem.</u> For any set X, the semigroup of almost identical full transformations on X is factorizable.

<u>Proof</u> : Let X be a set. Let $\alpha \in V_X$. Then $\alpha \in U_X$ and $\Delta\alpha = X$. By Theorem 1.4, $\alpha = \beta\gamma$ for some $\beta \in G_X \cap U_X$ and $\gamma \in E(U_X)$. Then $\gamma = \beta^{-1}\alpha$. Since $\Delta\beta^{-1} = \Delta\alpha = X$, $\Delta\beta^{-1}\alpha = X$, so $\Delta\gamma = X$. Then $\gamma \in V_X$. Thus $\gamma \in E(U_X) \cap V_X = E(V_X)$. Because $G_X \cap U_X$ is the group of units of V_X, it is proved that V_X is factorizable. #

1.7 <u>Theorem.</u> For any set X, the semigroup of almost identical 1-1 partial transformations on X is factorizable.

<u>Proof</u> : Let X be a set. Let $\alpha \in W_X$. Then $\alpha \in U_X$ and α is one-to-one. By Theorem 1.4, $\alpha = \beta\gamma$ for some $\beta \in G_X \cap U_X$ and $\gamma \in E(U_X)$. Then $\beta^{-1}\alpha = \gamma$. Since β^{-1} and α are one-to-one, γ is one-to-one. Then $\gamma \in W_X$, so $\gamma \in E(U_X) \cap W_X = E(W_X)$. But $G_X \cap U_X$ is the group of units of W_X, so W_X is factorizable. #