CHAPTER IV

GROUP HYPERGRAPHS

4.1 Group Hypergraphs

Let $(V,\mathcal{E})$ be a hypergraph of rank $\gamma \geq 2$. Then $(V,\mathcal{E})$ is called a group hypergraph if there exists a binary operation o on V such that (V,o) is a group and there exists admissible set $\mathcal{A}$ of $(\gamma-1)$-subsets of V such that $\mathcal{E} = \mathcal{E}_{\mathcal{A}}$

4.1.1 Remark Let $(V,\mathcal{E}_{\mathcal{A}})$ be a group hypergraph. Then for each E in $\mathcal{E}_{\mathcal{A}}$ and each v in E, $v^{-1} o (E - \{v\})$ belongs to $\mathcal{A}$.

Proof Let $(V,\mathcal{E}_{\mathcal{A}})$ be a group hypergraph. Let E belong to $\mathcal{E}_{\mathcal{A}}$ and v belong to E. Hence, by lemma 3.2.1, there exists A in $\mathcal{A}$ such that $E - \{v\} = v \, o \, A$. Therefore $v^{-1} o (E - \{v\}) = A$. Hence $v^{-1} o (E - \{v\})$ belongs to $\mathcal{A}$. #

4.1.2 Remark Let $H = (V,\mathcal{E})$ and $H^* = (V^*,\mathcal{E}^*)$ be isomorphism hypergraphs. If H is a group hypergraph, then H^* is also a group hypergraph.

Proof Let $H = (V,\mathcal{E})$ and $H^* = (V^*,\mathcal{E}^*)$ be isomorphism hypergraphs. Let H be a group hypergraph. Let o be a group operation on V and $\mathcal{A}$ be an admissible set of subsets of V such that $\mathcal{E} = \mathcal{E}_{\mathcal{A}}$. Let α be an isomorphism from H^* to H. Define a binary operation * on V^* by putting

$$u^* * v^* = \alpha^{-1}(\alpha(u^*) \, o \, \alpha(v^*)) \text{ for every } u^*, v^* \text{ in } V^*$$

A straight forward verification shows that $(V^*,*)$ forms a group and α is an isomorphism from $(V^*,*)$ onto (V,o).

Let

$$\mathcal{A}^* = \{A^* \mid \alpha(A^*) \varepsilon \mathcal{A}\}.$$

Assume that A^* is any element of $\mathcal{A}^*$. Let u^* be any element in A^* and v^* be any element in V^*. Then $A = \alpha(A^*)$ belongs to $\mathcal{A}$, $u = \alpha(u^*)$ belongs to A and $v = \alpha(v^*)$ belongs to V. Since $\mathcal{A}$ is an admissible set, hence there exists $B_{u,v}$ in $\mathcal{A}$ such that $(\{v\} \cup v \circ A) - \{v \circ u\} = (v \circ u) \circ B_{u,v}$. Since α is onto, hence there exists $B \subseteq V$ such that $\alpha(B) = B_{u,v}$. Therefore, by definition of $\mathcal{A}^*$, B belongs to $\mathcal{A}^*$. Choose $B_{u^*,v^*} = B$. Hence B_{u^*,v^*} belongs to $\mathcal{A}^*$. For this choice of B_{u^*,v^*} we have

$$\begin{aligned}
&(\{v^*\} \cup (v^* * A^*)) - \{v^* * u^*\} \\
&= \alpha^{-1}(\alpha((\{v^*\} \cup (v^* * A^*)) - \{v^* * u^*\})) \\
&= \alpha^{-1}(\alpha(\{v^*\} \cup (v^* * A^*)) - \alpha(\{v^* * u^*\})) \\
&= \alpha^{-1}((\{\alpha(v^*)\} \cup \alpha(v^* * A^*)) - \alpha(\{v^* * u^*\})) \\
&= \alpha^{-1}((\{\alpha(v^*)\} \cup (\alpha(v^*) \circ \alpha(A^*))) - \{\alpha(v^*) \circ \alpha(u^*)\}) \\
&= \alpha^{-1}((\{v\} \cup (v \circ A)) - \{v \circ u\}) \\
&= \alpha^{-1}((v \circ u) \circ B_{u,v}) \\
&= \alpha^{-1}((\alpha(v^*) \circ \alpha(u^*)) \circ \alpha(B_{u^*,v^*})) \\
&= \alpha^{-1}(\alpha(v^* * u^*) \circ \alpha(B_{u^*,v^*})) \\
&= \alpha^{-1}(\alpha((v^* * u^*) * B_{u^*,v^*})) \\
&= (v^* * u^*) * B_{u^*,v^*}.
\end{aligned}$$

Therefore $\mathcal{A}^*$ is an admissible set.

Next we shall show that $\mathcal{E}^* = \mathcal{E}_{\mathcal{A}^*}$. Let E^* be any non-empty subset of V^* and v^* be any element of E^*.

Then

$$\begin{aligned} &\alpha(v^{*-1} * (E^* - \{v^*\})) \\ &= \alpha(v^{*-1}) \circ \alpha(E^* - \{v^*\}) \\ &= \alpha(v^{*-1}) \circ (\alpha(E^*) - \{\alpha(v^*)\}) \\ &= (\alpha(v^*))^{-1} \circ (\alpha(E^*) - \{\alpha(v^*)\}) \end{aligned}$$

If E^* belongs to $\mathcal{E}^*$, then $\alpha(E^*)$ belongs to $\mathcal{E}_{\mathcal{A}}$.
Hence, by remark 4.1.1, $\alpha(v^*)^{-1} \circ (\alpha(E^*) - \{\alpha(v^*)\})$ belongs to $\mathcal{A}$, i.e. $\alpha(v^{*-1} * (E^* - \{v^*\}))$ belongs to $\mathcal{A}$. Therefore $v^{*-1} * (E^* - \{v^*\})$ belongs to $\mathcal{A}^*$. Hence $E^* = \{v^*\} \cup v^* * (v^{*-1} * (E^* - \{v^*\}))$ belongs to $\mathcal{E}_{\mathcal{A}^*}$. Therefore $\mathcal{E}^* \subseteq \mathcal{E}_{\mathcal{A}^*}$. Conversely, if E^* belongs to $\mathcal{E}_{\mathcal{A}^*}$, then, by remark 4.1.1, $v^{*-1} * (E^* - \{v^*\})$ belongs to $\mathcal{A}^*$. Therefore $\alpha(v^{*-1} * (E^* - \{v^*\}))$ belongs to $\mathcal{A}$, i.e. $\alpha(v^*)^{-1} \circ (\alpha(E^*) - \{\alpha(v^*)\})$ belongs to $\mathcal{A}$. Hence $\alpha(E^*) = \{\alpha(v^*)\} \cup \alpha(v^*) \circ (\alpha(v^*)^{-1} \circ (\alpha(E^*) - \{\alpha(v^*)\}))$ belongs to $\mathcal{E}_{\mathcal{A}}$. Therefore E^* belongs to $\mathcal{E}^*$. Hence $\mathcal{E}_{\mathcal{A}^*} \subseteq \mathcal{E}^*$.
Therefore

$$\mathcal{E}^* = \mathcal{E}_{\mathcal{A}^*}$$

Hence $(V^*, \mathcal{E}^*)$ is a group hypergraph.

#

4.1.3 <u>Proposition</u> Let $(V, \mathcal{E}_{\mathcal{A}})$ be a group hypergraph. Then there exists a subgroup Δ of the automorphism group $\Gamma(V, \mathcal{E}_{\mathcal{A}})$ such that $|\Delta| = |V|$ and Δ acts transitively on V.

Proof Let $(V, \mathcal{E}_{\mathcal{A}})$ be a group hypergraph. For each element v of V, we define a mapping $\alpha_v : V \to V$ by putting

$$\alpha_v(u) = v \circ u \quad \text{for all } u \text{ in } V$$

Observe that for any elements u, u' in V if $\alpha_v(u) = \alpha_v(u')$, then we have $v \circ u = v \circ u'$, which implies that $u = u'$. Hence α_v is one-to-one. For any w in V, we see that $\alpha_v(v^{-1} \circ w) = v \circ (v^{-1} \circ w) = w$. Hence α_v is onto.

Next we shall show that α_v is an automorphism of $(V, \mathcal{E}_{\mathcal{A}})$. Let E be any non-empty subset of V. Then

$$\begin{aligned} \alpha_v(E) &= v \circ E \\ &= v \circ (\{u_o\} \cup (E - \{u_o\})), \end{aligned}$$

where u_o is an element in E. Hence

$$\begin{aligned} \alpha_v(E) &= \{v \circ u_o\} \cup (v \circ E - \{v \circ u_o\}) \\ &= \{v \circ u_o\} \cup (v \circ u_o) \circ ((v \circ u_o)^{-1} \circ (v \circ E - \{v \circ u_o\})) \\ &= \{v \circ u_o\} \cup (v \circ u_o) \circ A, \end{aligned}$$

where

$$\begin{aligned} A &= (v \circ u_o)^{-1} \circ (v \circ E - \{v \circ u_o\}) \\ &= u_o^{-1} \circ v^{-1} \circ (v \circ E - \{v \circ u_o\}) \\ &= u_o^{-1} \circ (E - \{u_o\}). \end{aligned}$$

If E belongs to $\mathcal{E}_{\mathcal{A}}$, hence, by remark 4.1.1, A belong to $\mathcal{A}$. Hence $\alpha_v(E)$ belongs to $\mathcal{E}_{\mathcal{A}}$. If $\alpha_v(E)$ belongs to $\mathcal{E}_{\mathcal{A}}$, i.e. $v \circ E$ belong to $\mathcal{E}_{\mathcal{A}}$, then, by remark 4.1.1, A belongs to $\mathcal{A}$. Hence E belongs to $\mathcal{E}_{\mathcal{A}}$. Therefore α_v is an automorphism of $(V, \mathcal{E}_{\mathcal{A}})$.

Let

$$\Delta = \{\alpha_v \mid v \in V\}.$$

Clearly Δ is not empty.

Let α_v, $\alpha_{v'}$ belong to Δ and u belong to V. Hence

$$\alpha_v \circ \alpha_{v'}(u) = \alpha_v(v' \circ u)$$

$$= v \circ (v' \circ u)$$

$$= (v \circ v') \circ u$$

$$= \alpha_{v \circ v'}(u),$$

which implies that $\alpha_v \circ \alpha_{v'}$ belongs to Δ. For any α_v in Δ and any u in V, $\alpha_v(\alpha_{v^{-1}}(u)) = v \circ (v^{-1} \circ u) = u$.

Hence

$$\alpha_v^{-1} = \alpha_{v^{-1}}$$

Therefore, for each α_v in Δ, α_v^{-1} belongs to Δ. Hence Δ is a subgroup of $\Gamma(V, \mathcal{E}_{\mathcal{A}})$.

Define a mapping $\Theta : V \to \Delta$ by

$$\Theta(v) = \alpha_v \quad \text{for all v in V.}$$

Clearly, Θ is onto. To see that it is one-to-one, assume that $\Theta(v) = \Theta(v')$, i.e. $\alpha_v = \alpha_{v'}$. Hence for each u in V, $\alpha_v(u) = \alpha_{v'}(u)$, i.e. $v \circ u = v' \circ u$ for all u in V. In particular when u = e, the identity, we have v = v'. Therefore Θ is one-to-one. Hence

$$|\Delta| = |V|$$

To show that Δ is transitive over V, let v, v' be any elements of V. Note that $\alpha_{v \circ v^{-1}}$ belongs to Δ and we have

$$\alpha_{v' \circ v^{-1}}(v) = (v' \circ v^{-1}) \circ v$$
$$= v'$$

Therefore Δ is transitive over V.
Hence $\Gamma(V, \mathcal{E}_{A})$ has a subgroup Δ of order $|V|$ such that Δ acts transitively on V.

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The converse of the above proposition is also true. To prove the converse, we need the following lemmas.

4.1.4 <u>Lemma</u> Let Δ be any transitive subgroup of the automorphism group $\Gamma(V, \mathcal{E})$ of $H = (V, \mathcal{E})$. Then for each w in V, $\Delta_w = \{\gamma \in \Delta \mid \gamma(w) = w\}$ is a subgroup of Δ of index $[\Delta : \Delta_w] = |V|$.

<u>Proof</u> Let $H = (V, \mathcal{E})$ be a hypergraph. Assume that Δ is a transitive subgroup of the automorphism group $\Gamma(V, \mathcal{E})$. Let w belong to V and

$$\Delta_w = \{\gamma \in \Delta \mid \gamma(w) = w\}.$$

Note that identity of Δ belongs to Δ_w, hence $\Delta_w \neq \emptyset$. Let γ_1, γ_2 belong to Δ_w, hence $\gamma_1(w) = w$ and $\gamma_2(w) = w$. Therefore $\gamma_1 \circ \gamma_2(w) = \gamma_1(w) = w$ and $\gamma_1^{-1}(w) = \gamma_1^{-1}(\gamma_1(w)) = w$. Thus $\gamma_1 \circ \gamma_2$ and γ_1^{-1} belongs to Δ_w. Hence Δ_w is a subgroup of Δ.
Since Δ_w is a subgroup of Δ, hence Δ has a left coset decomposition $\alpha_1 \circ \Delta_w \cup \alpha_2 \circ \Delta_w \cup \dots \cup \alpha_m \circ \Delta_w$, where α_i belongs to Δ for all $i = 1, 2, \dots, m$ and $\alpha_i \circ \Delta_w \cap \alpha_j \circ \Delta_w = \emptyset$ for $i \neq j$. To show $m = |V|$, let v belong to V. Then there exists β in Δ such that $\beta(w) = v$. Since β belongs to Δ, β belongs to $\alpha_i \circ \Delta_w$ for some i, $1 \leq i \leq m$. Then

$\beta = \alpha_i \circ \alpha$ for some α in Δ_w. Therefore $\beta(w) = \alpha_i \circ \alpha(w) = \alpha_i(w)$. Hence $v = \alpha_i(w)$. Then v belongs to $\{\alpha_i(w) \mid i = 1,2,\ldots, m\}$. Therefore $V \subseteq \{\alpha_i(w) \mid i = 1,2,\ldots,m\}$. Hence $|V|$ is less than or equal m. To show that opposite inequality. We show that the function $\psi : \alpha_i \circ \Delta_w \mapsto \alpha_i(w)$ is a one-to-one function from the cosets in to V. Suppose that $\psi(\alpha_i \circ \Delta_w) = \psi(\alpha_j \circ \Delta_w)$, i.e. $\alpha_i(w) = \alpha_j(w)$. Then $\alpha_j^{-1} \circ \alpha_i(w) = w$, which implies that $\alpha_j^{-1} \circ \alpha_i$ belongs to Δ_w, or equivalently $\alpha_i \circ \Delta_w = \alpha_j \circ \Delta_w$. Therefore m is less than or equal $|V|$. Hence $m = |V|$. Therefore

$$[\Delta : \Delta_w] = |V|.$$

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4.1.5 <u>Lemma</u> Let Δ be any transitive subgroup of the automorphism group $\Gamma(V,\mathcal{E})$ of $H = (V,\mathcal{E})$ of order $|V|$. If γ_o is an element of Δ such that $\gamma_o(u) = u$ for some u in V, then $\gamma_o = e$, the identity of Δ.

<u>Proof</u> Let $H = (V,\mathcal{E})$ be a hypergraph. Assume that Δ is a transitive subgroup of the automorphism group $\Gamma(V,\mathcal{E})$ of order $|V|$. Let γ_o in Δ and u in V be such that $\gamma_o(u) = u$. Let

$$\Delta_u = \{\gamma \in \Delta \mid \gamma(u) = u\}.$$

Hence γ_o belongs to Δ_u. By lemma 4.1.4, Δ_u is a subgroup of Δ and $[\Delta:\Delta_u] = |V|$. Since $|V| = |\Delta|$ and $[\Delta:\Delta_u] = \frac{|\Delta|}{|\Delta_u|}$, therefore $|\Delta_u| = 1$. Hence $\Delta_u = \{e\}$. Therefore $\gamma_o = e$.

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4.1.6 <u>Proposition</u> Let $H = (V,\mathcal{E})$ be a hypergraph of rank at least 2. If its automorphism group $\Gamma(V,\mathcal{E})$ contain a subgroup Δ of order $|V|$ such that Δ acts transitively on V, then H is a group hypergraph.

Proof Let $H = (V, \mathcal{E})$ be a hypergraph of rank at least 2. Assume that the automorphism group $\Gamma(H)$ contains a subgroup Δ of order $|V|$ such that Δ acts transitively on V.

Case I Suppose $\mathcal{E} = \emptyset$. Let $\circ$ be any group operation on V. Let $\mathcal{A} = \emptyset$. Then $\mathcal{E}_{\mathcal{A}} = \emptyset$, which implies that $\mathcal{E} = \mathcal{E}_{\mathcal{A}}$. Hence $(V, \mathcal{E})$ is group hypergraph.

Case II Suppose $\mathcal{E} \neq \emptyset$. Then there exists at least one element in $\mathcal{E}$. Let E belong to $\mathcal{E}$. Fix u_o in E, let

$$\mathcal{A} = \{A \mid A \subseteq \Delta - \{e\} \text{ and } (\{u_o\} \cup A(u_o)) \varepsilon \mathcal{E}\} ,$$

where $A(v) = \{\rho(v) \mid \rho \ \varepsilon \ A\}$.

For each v in $E - \{u_o\}$, there exists σ_v in $\Delta - \{e\}$ such that $\sigma_v(u_o) = v$. Observe that

$$\{\sigma_v(u_o) \mid v \ \varepsilon \ E - \{u_o\}\} = E - \{u_o\}.$$

Hence

$$E = \{u_o\} \cup \{\sigma_v(u_o) \mid v \ \varepsilon \ E - \{u_o\}\}$$

Therefore $\{u_o\} \cup \{\sigma_v(u_o) \mid v \ \varepsilon \ E - \{u_o\}\}$ belongs to $\mathcal{E}$. Hence $\{\sigma_v \mid v \, \varepsilon \, E - \{u_o\}\}$ belongs to $\mathcal{A}$. Therefore $\mathcal{A} \neq \emptyset$.

To see that $\mathcal{A}$ is admissible, let A be any element of $\mathcal{A}$, ρ be any element of A and σ be any element of Δ.

Choose

$$B_{\rho,\sigma} = \rho^{-1} \circ (\{e\} \cup A - \{\rho\}).$$

We shall show that $B_{\rho,\sigma}$ belongs to $\mathcal{A}$. Since $A \subseteq \Delta - \{e\}$, hence $A - \{\rho\} \subsetneq \Delta - \{e, \rho\}$. Therefore we have

$$\rho^{-1} \circ (A-\{\rho\}) \subseteq \rho^{-1} \circ (\Delta-\{e,\rho\})$$

$$= \rho^{-1} \circ \Delta - \rho^{-1} \circ (\{e,\rho\})$$

$$= \Delta - \{\rho^{-1}, e\}$$

$$\subseteq \Delta - \{e\}$$

Since $\rho \varepsilon A$ and $A \subseteq \Delta-\{e\}$, hence $\rho \neq e$. Therefore $\rho^{-1} \varepsilon \Delta-\{e\}$. Thus

$$\{\rho^{-1}\} \cup \rho^{-1} \circ (A-\{\rho\}) \subseteq \Delta - \{e\}$$

But

$$B_{\rho,\sigma} = \rho^{-1} \circ (\{e\} \cup A - \{\rho\})$$

$$= \{\rho^{-1}\} \cup \rho^{-1} \circ (A-\{\rho\}).$$

Hence we have

$$B_{\rho,\sigma} \subseteq \Delta - \{e\}.$$

Observe that

$$\{u_o\} \cup B_{\rho,\sigma}(u_o) = \{u_o\} \cup \rho^{-1} \circ (\{e\} \cup (A-\{\rho\}))(u_o)$$

$$= \{u_o\} \cup \rho^{-1}((\{e\} \cup (A-\{\rho\}))(u_o))$$

$$= \{u_o\} \cup \rho^{-1}(\{u_o\} \cup (A(u_o)-\{\rho(u_o)\}))$$

$$= \rho^{-1}(\{\rho(u_o)\} \cup \{u_o\} \cup (A(u_o)-\{\rho(u_o)\}))$$

$$= \rho^{-1}(\{u_o\} \cup A(u_o))$$

Since $\{u_o\} \cup A(u_o)$ belongs to $\mathcal{E}$, hence $\{u_o\} \cup B_{\rho,\sigma}(u_o)$ belongs to $\mathcal{E}$. Therefore $B_{\rho,\sigma}$ belongs to $\mathcal{A}$. From our choice of $B_{\rho,\sigma}$ we see that

$$(\sigma \circ \rho) \circ B_{\rho,\sigma} = (\sigma \circ \rho) \circ (\rho^{-1} \circ (\{e\} \cup A - \{\rho\}))$$

$$= \sigma \circ (\{e\} \cup (A - \{\rho\}))$$

$$= \{\sigma\} \cup (\sigma \circ A - \{\sigma \circ \rho\})$$

$$= (\{\sigma\} \cup \sigma \circ A) - \{\sigma \circ \rho\}$$

Hence $\mathcal{A}$ is an admissible set.

Next we shall show that $(\Delta, \mathcal{E}_{\mathcal{A}})$ is isomorphic to $(V, \mathcal{E})$. Define a mapping $\psi^* : \Delta \to V$ by

$$\psi^*(\sigma) = \sigma(u_o) \qquad \text{for all } \sigma \text{ in } \Delta$$

Let σ_1, σ_2 belong to Δ. Assume that $\psi^*(\sigma_1) = \psi^*(\sigma_2)$. Then $\sigma_1(u_o) = \sigma_2(u_o)$. Hence $\sigma_2^{-1} \circ \sigma_1(u_o) = u_o$. Therefore $\sigma_2^{-1} \circ \sigma_1$ is an element of Δ such that $(\sigma_2^{-1} \circ \sigma_1)(u_o) = u_o$. Hence, by lemma 4.1.5, we have $\sigma_2^{-1} \circ \sigma_1 = e$. Therefore $\sigma_1 = \sigma_2$. Hence ψ^* is one-to-one. Let u be any element of V. Hence there exists σ' in Δ such that $\sigma'(u_o) = u$. That is $u = \sigma'(u_o) = \psi^*(\sigma')$. Hence ψ^* is onto

Finally we shall show that ψ^* is an isomorphism from $(\Delta, \mathcal{E}_{\mathcal{A}})$ onto $(V, \mathcal{E})$.

Let F be any non-empty subset of Δ. Then

$$\psi^*(F) = F(u_o)$$

$$= (\{\gamma\} \cup (F - \{\gamma\}))(u_o),$$

for some γ in F. Hence

$$\psi^*(F) = \{\gamma(u_o)\} \cup (F - \{\gamma\})(u_o)$$

$$= \{\gamma(u_o)\} \cup \gamma \circ \gamma^{-1} \circ (F - \{\gamma\})(u_o)$$

$$= \{\gamma(u_o)\} \cup \gamma(\gamma^{-1} \circ (F - \{\gamma\})(u_o))$$

$$= \gamma(\{u_o\} \cup (\gamma^{-1} \circ (F - \{\gamma\})(u_o)).$$

If F belongs to $\mathcal{E}_{\mathcal{A}}$, then, by remark 4.1.1, $\gamma^{-1} \circ (F - \{\gamma\}) = A$ for some A in $\mathcal{A}$. Hence we have $\psi^*(F) = \gamma(\{u_o\} \cup A(u_o))$. Since $\{u_o\} \cup A(u_o)$ belongs to $\mathcal{E}$, and γ is an automorphism, hence $\gamma(\{u_o\} \cup A(u_o))$ belongs to $\mathcal{E}$, i.e. $\psi^*(F)$ belongs to $\mathcal{E}$. Conversely, if $\psi^*(F)$ belongs to $\mathcal{E}$, then $\gamma^{-1}(\psi^*(F))$ belongs to $\mathcal{E}$. But $\gamma^{-1}(\psi^*(F)) = \{u_o\} \cup A(u_o)$, where $A = \gamma^{-1} \circ (F - \{\gamma\})$. Therefore $\{u_o\} \cup A(u_o)$ belongs to $\mathcal{E}$. Hence A belongs to $\mathcal{A}$. Note that

$$F = \{\gamma\} \cup \gamma \circ (\gamma^{-1} \circ (F - \{\gamma\}))$$

$$= \{\gamma\} \cup \gamma \circ A,$$

which belongs to $\mathcal{E}_{\mathcal{A}}$. Hence ψ^* is an isomorphism from $(\Delta, \mathcal{E}_{\mathcal{A}})$ onto $(V,\mathcal{E})$.

Therefore, by remark 4.1.2, $(V,\mathcal{E})$ is a group hypergraph. #

We may now summarize proposition 4.1.3 and proposition 4.1.6 into the following.

4.1.7 Theorem A hypergraph $H = (V,\mathcal{E})$ of rank at least 2 is a group hypergraph if and only if its automorphism group contain a subgroup Δ of order $|V|$ such that Δ acts transitively on V.

APPENDIX

An Example of a Quasi-group Hypergraph. Let $Q = \{0,1,2,3,4,5\}$.

Let a binary operation $\circ$ on Q be given by the following table :

$\circ$	0	1	2	3	4	5
0	2	5	3	4	0	1
1	3	4	2	5	1	0
2	4	1	5	0	2	3
3	5	0	4	1	3	2
4	0	3	1	2	4	5
5	1	2	0	3	5	4

Clearly $(Q,\circ)$ forms a quasi-group. It can be verified that

$$\mathcal{A} = \{\{0,1\}, \{1,2\}, \{2,3\}\}$$

is an admissible set. For this $\mathcal{A}$ we have

$$\mathcal{E}_{\mathcal{A}} = \{\{0,2,5\},\{1,3,4\},\{2,4,1\},\{3,5,0\},\{4,0,3\},\{5,1,2\}\}.$$

Hence $(Q, \mathcal{E}_{\mathcal{A}})$ is a quasi-group hypergraph.