CHAPTER III

ENDOMORPHISM SEMIGROUP OF IDEMPOTENT ALGEBRAS

In this chapter, we will show that the number of idempotents in an idempotent algebra and the number of right annihilators in its endomorphism semigroup are equal. We also show that if an entire algebra has a zero to be its unique idempotent and also has three congruence relations then its endomorphism semigroup has an identity, zero and every element, which is not zero, satisfies a right cancellation law.

3.1 <u>Definition</u>. Let $< A; F >$ be an algebra. An element a in A is called an <u>idempotent</u> element of $< A; F >$ if for $0 < \gamma < O(\tau)$, $f_\gamma(a, \ldots, a) = a$.

$< A; F >$ is called an <u>idempotent algebra</u> if there is an element a in A such that a is an idempotent element of $< A; F >$.

Groups and rings are examples of algebras with unique idempotent, whereas lattices are algebras in which every element is an idempotent.

The following theorems show, among other things, that every idempotent of an algebra induces an endomorphism which is a right annihilator of the endomorphism semigroup.

3.2 Theorem. A semigroup $< S; \cdot >$ is isomorphic to the endomorphism semigroup of some unique idempotent algebra if and only if the semigroup $< S; \cdot >$ has an identity and zero.

Proof. Let $< S; \cdot >$ be a semigroup. Assume that $< A; F >$ is an algebra which has p as its unique idempotent and $E(A; F) \cong S$. Then by Theorem 2.1, S has an identity. We will show that E(A; F) has a zero. Define $\psi : A \longrightarrow A$ by

$$x\psi = p \qquad (x \in A).$$

Since

$$(f_\gamma(a_0, \ldots, a_{n_\gamma-1}))\psi = p = f_\gamma(p, \ldots, p) = f_\gamma(a_0\psi, \ldots, a_{n_\gamma-1}\psi)$$

for all $f_\gamma \in F$ and $a_0, \ldots, a_{n_\gamma-1} \in A$, we have $\psi \in E(A; F)$.

To show that ψ is the only constant map in E(A,F), let $\sigma \in E(A,F)$ and $A\sigma = \{a\}$ for some a in A. Suppose that $a \neq p$. Then there is an operation $f_{\gamma'}$ in F such that $f_{\gamma'}(a, \ldots, a) \neq a$ which implies that $< A\sigma; F >$ is not a subalgebra of $< A; F >$. Hence, $a = p$.

Next, we show that ψ is a zero of E(A; F). Let $\delta \in E(A; F)$ and $x \in A$. Then $x(\psi \circ \delta) = p\delta$, so that $p\delta = p$ since $A(\psi \circ \delta) = \{p\delta\}$. Hence, $x(\psi \circ \delta) = p\delta = p = x\psi$ implies $\psi \circ \delta = \psi$. And $x\delta \circ \psi = (x\delta)\psi = p = x\psi$ implies $\delta \circ \psi = \psi$.

Conversely, assume that $< S; \cdot >$ is a semigroup with identity and zero, namely, e and 0, respectively. Construct an algebra as follows : Let A = S and for each $a \in S$, define a unary operation f_a on A by

$$f_a(x) = ax \qquad (x \in A).$$

Set

$$F = \{f_a | a \in S\},$$

and consider the algebra < A; F >. Since a0 = 0 for all $a \in A$, we have $f_a(0) = a0 = 0$. Hence, 0 is the only idempotent of A.

For each $a \in S$, define a mapping $\psi_a : A \longrightarrow A$ by

$$x\psi_a = xa \qquad (x \in A).$$

This is a mapping of A into itself.

(i) $\psi_a = \psi_b$ if and only if $a = b$.

Indeed, $e\psi_a = a$, $e\psi_b = b$. Hence, $\psi_a = \psi_b$ is equivalent to $a = b$.

(ii) $\psi_a \in E(A; F)$ for all $a \in S$.

Since $(f_b(x))\psi_a = (bx)a = b(xa) = f_b(x\psi_a)$ for all a, b, x in A.

(iii) Let $\psi \in E(A; F)$. Set $a = e\psi$. Then $x\psi = (xe)\psi = (f_x(e))\psi = f_x(e\psi) = x(e\psi) = xa = x\psi_a$. Hence, $\psi = \psi_a$.

(iv) $\psi_a \circ \psi_b = \psi_{ab}$ for all a, b in A.

Indeed, $x(\psi_a \circ \psi_b) = (x\psi_a)\psi_b = (xa)b = x(ab) = x\psi_{ab}$ for all $x \in A$.

Consider the mapping $\theta : a \longrightarrow \psi_a$. θ is 1 - 1 by (i). θ is onto by (ii). It preserves multiplication by (iv). Therefore θ is an isomorphism. This completes the proof of the theorem. #

We now generalize the above theorem to the case where the idempotent algebra has an arbitrary number of idempotents.

3.3 <u>Theorem</u>. Let π be a nonzero cardinal number. A semigroup is isomorphic to the endomorphism semigroup of some idempotent algebra with π idempotents if and only if the semigroup has an identity and π right annihilators.

Proof. Let $< S; \cdot >$ be a semigroup and π be a nonzero cardinal number. To prove the 'only if' part, we assume that $< A; F >$ is an idempotent algebra which has π idempotent elements and $E(A; F) \cong S$. Then by Theorem 2.1, S has an identity element.

We will show that the cardinality of the set of right annihilators of E(A; F) is π. Let X be the set of idempotent of $< A; F >$.

(i) Let $p \in X$. If $A\psi = \{p\}$, then $\psi \in E(A; F)$, and we denote ψ by ψ^p.

Indeed, $(f_\gamma(a_0,\ldots, a_{n_\gamma -1}))\psi = p = f_\gamma(p,\ldots, p)$

$$= f_\gamma(a_0 \psi,\ldots, a_{n_\gamma -1}\psi)$$

for all $f_\gamma \in F$, $a_0, \ldots, a_{n_\gamma -1} \in A$. Hence, $\alpha o \psi = \psi$ for all $\alpha \in E(A;F)$.

(ii) If $\psi \in E(A; F)$ and $A\psi = \{a\}$ for some a in A, then $a = p$ for some p in X.

Suppose that $a \neq p$ for all $p \in X$. Then there is an operation $f_{\gamma'}$ in F such that $f_{\gamma'}(a,\ldots, a) \neq a$. Hence, a subalgebra $< A\psi ; F >$ is not closed under $f_{\gamma'}$. It is a contradiction.

(iii) For all $\alpha \in E(A; F)$ if $p \in X$, then $p\alpha \in X$.

Indeed, $x(\psi^p o \alpha) = p\alpha$ for all $p \in X$, $x \in A$, so $A(\psi^p o \alpha) = \{p\alpha\}$. Hence, by (ii), $p\alpha = q$ for some q in X.

(iv) If $\psi \in E(A; F)$ and $\alpha o \psi = \psi$ for all $\alpha \in E(A; F)$, then $A\psi = \{p\}$ for some p in X.

Let $q \in X$. Then $\psi^q \in E(A; F)$ and also $\psi^q o \psi = \psi$. Hence, $A(\psi^q o \psi) = \{q\psi\}$ and by (iii), $q\psi = p$ for some p in X. This implies $A\psi = \{p\}$.

(v) $p = q$ if and only if $\psi^p = \psi^q$.

Indeed, $A\psi^p = \{p\}$, $A\psi^q = \{q\}$. Hence, $\psi^p = \psi^q$ is equivalent to $p = q$.

Consider the mapping $\theta : p \to \psi^p$ for all $p \in X$. Then θ is $1 - 1$ by (v), θ is onto by (iv). Therefore, θ is a $1 - 1$ correspondence between X and $\{\psi^p | p \in X\}$ which is the set of all right annihilators of $E(A; F)$. Hence, S has π right annihilators.

To prove the converse, let S be a semigroup with an identity, say e, and it has π right annihilators. Construct an algebra as follows : Let $A = S$ and for $a \in S$, define a unary operation $f_a : A \longrightarrow A$ by

$$f_a(x) = ax \qquad (x \in A).$$

Set

$$F = \{f_a | a \in S\},$$

and consider the algebra $< A; F >$. Then by (i) - (iv) of the 'if' part of the Theorem 3.2, S is isomorphic to $E(A; F)$ by the isomorphism $\theta : a \longrightarrow \psi_a$ for all $a \in S$.

It remains to show that $< A; F >$ has π idempotents. Let T be the set of all right annihilators of S.

(i) If $p \in T$, then p is an idempotent of $< A; F >$.

Since $ap = p$ for all $a \in S$, $p \in T$, we have $f_a(p) = ap = p$.

(ii) Let a be an idempotent element of $< A; F >$. Then for every $x \in S$, $f_x(a) = xa = a$ implies that a is a right annihilator of S.

By (i) and (ii), T is the set of all idempotent elements of $< A; F >$, completing the proof of the theorem. #

Theorem 3.3 is a proper generalization of Theorem 3.2, since a semigroup S has a unique right annihilator if and only if it has a zero.

3.4 Remark. In general, Theorem 3.3 does not hold for $\pi = 0$. But it does hold for special algebras in this case, as will be shown in Corollary 3.9 and the fact that idempotent induces right annihilator.

Next, we will consider the endomorphism semigroup of an idempotent algebra with congruence relations.

3.5 Definition. Let $< A; F >$ be an algebra and θ be a binary relation defined on A. θ is called a congruence relation if it is an equivalence relation satisfying the substitution property (SP) :

(SP) If $\gamma < O(\tau)$, $a_i \equiv b_i$ (θ), $a_i, b_i \in A$, $0 \leq i < n_\gamma$,

then

$$f_\gamma(a_0, \ldots, a_{n_\gamma - 1}) \equiv f_\gamma(b_0, \ldots, b_{n_\gamma - 1}) \ (\theta).$$

The binary relations ω and ι on A are defined by the rules :

$$x \equiv y \ (\omega) \longleftrightarrow x = y \qquad (x, y \in A),$$

and

$$x \equiv y \ (\iota) \longleftrightarrow x, y \in A,$$

respectively.

An algebra $< A; F >$ is simple if and only if ω and ι are the only congruence relations on A.

Recall that G is a simple group if it is nontrivial and has no normal subgroups other than {1} and G itself, and S is a simple semigroup if S is its only ideal.

The following propositions show that if G is a simple group, then it is a simple algebra. But it is not true for a simple semigroup.

3.6 <u>Proposition</u>. A group $< G; \cdot >$ is a simple algebra if and only if $< G; \cdot >$ is a simple group.

<u>Proof</u>. Let $< G; \cdot >$ be a group with identity e. We first observe that θ is a congruence relation on G if and only if $e\theta$ is a normal subgroup of G ..... (*). Now, let θ be any congruence relation on G; i.e. $N = e\theta$ is a subgroup of G since for all $x \in N$ $x^{-1} = x^{-1}e \equiv x^{-1}x = e\ (\theta)$, hence for all $x \in N$, $x^{-1} \in e\theta$. To show that N is normal, let $a \in G$. Then $ana^{-1} \in aNa^{-1}$ for all $n \in N$. Since $n \equiv e\ (\theta)$, $ana^{-1} \equiv e\ (\theta)$ and hence $ana^{-1} \in N$. Conversely, let M be a normal subgroup of G. Define a binary relation θ_M on G by

$$x \equiv y\ (\theta_M) \longleftrightarrow xy^{-1} \in M \text{ or } y^{-1}x \in M$$

for all $x, y \in G$. Then θ_M is a congruence relation on G. Since $e \in M$, we obtain $M = e\theta_M$.

To show the 'if' part of the theorem, let $< G; \cdot >$ be a simple algebra. Then $e\omega = \{e\}$ and $e\iota = G$ are the only normal subgroups of G.

To prove the converse of the theorem, let $< G; \cdot >$ be a simple group. Suppose that $< G; \cdot >$ is not a simple algebra. Then there is a congruence relation θ on G such that $\theta \neq \omega$ and $\theta \neq \iota$.

Therefore, by (*), e is a normal subgroup of G, and hence we can conclude that $e\theta \neq \{e\}$ and $e\theta \neq G$. This contradicts to the assumption, completing the proof. #

3.7 Examples :

An example of a simple algebra which is not a simple semigroup.

Let $\langle S; \cdot \rangle$ be a semigroup such that $S = \{0, 1\}$ where 0 is a zero and 1 is the identity of $\langle S; \cdot \rangle$. Obviously, $\langle S; \cdot \rangle$ is a simple algebra. Since $\{0\}$ is an ideal of S and $S \neq \{0\}$, S is not a simple semigroup.

An example of a simple semigroup which is not a simple algebra.

Let S be a semigroup such that $|S| \geq 3$ and every element of S is right annihilator. Let be a binary relation on S such that

$$\theta = \omega \cup \{(a, b), (b, a)\}$$

for some a, b in S such that $a \neq b$. Then θ is a congruence relation on S and $\theta \neq \omega$. Since $|S| \geq 3$, there exists an element c in S such that $a \neq c\ (\theta)$ implies $\theta \neq \iota$. Hence, S is not a simple algebra.

The following result is Grätzer's theorem on simple algebras.

3.8 Theorem ([2]). A semigroup $\langle S; \cdot \rangle$ is isomorphic to the endomorphism semigroup of some simple algebra $\langle A; F \rangle$ if and only if $\langle S; \cdot \rangle$ has an identity and every element in S is either a right annihilator or a right cancellative element.

Proof. Let < S; $\cdot$ > be a semigroup. Assume that < A; F > is a simple algebra such that E(A; F) is isomorphic to a semigroup S. Then by theorem 2.1, S has an identity element.

Let $\psi \in E(A; F)$. Then ψ induces a congruence relation ε_ψ on A defined by

$$x \equiv y\ (\varepsilon_\psi) \longleftrightarrow x\psi = y\psi$$

for all x, $y \in A$. Therefore, $\varepsilon_\psi = \iota$ or $\varepsilon_\psi = \omega$.

case 1 $\varepsilon_\psi = \omega$. By the definition of ε_ψ, $x\psi = y\psi$ is equivalent to x = y, hence ψ is a 1 - 1 mapping. Thus, $\beta o \psi = \alpha o \psi$ implies $\beta = \alpha$ for all α, β in E(A; F). Hence, ψ satisfies the right cancellation law.

case 2 $\varepsilon_\psi = \iota$. Then $A\psi$ consists of a single element. This implies that $\alpha o \psi = \psi$ for all $\alpha \in E(A; F)$. Hence, ψ is a right annihilator of E(A; F). This proves the 'only if' part of the theorem.

To prove the converse, let $S = R \cup N$, where R consists of the right cancellative elements and N of the right annihilators of S. Let e denote the identity of S; then $e \in R$. If S = {e}, the statement is trivial. We may assume that $S \neq \{e\}$. Set $A = S \cup \{0, 1\}$, $0 \notin S$, $1 \notin S$, $0 \neq 1$. We define the following operations on A:-

$$P \quad : \quad P(x) = x\ ,$$
$$P(0) = 1\ ,$$
$$P(1) = 0\ ;$$

$$f_a, \text{ for } a \in S \quad : \quad f_a(x) = ax \qquad (x \in S),$$
$$= x \qquad (x \notin S);$$

$$*\quad : \quad x*x = x \qquad (x \in A),$$
$$x*y = 0 \qquad (x, y \in S,\ x \neq y),$$
$$1*x = x*1 = x \qquad (x \in A),$$
$$0*x = x*0 = 0 \qquad (x \in A).$$

Set

$$F = \{f_a \mid a \in S\} \cup \{*, P\},$$

and consider the algebra $< A; F >$.

To show that $< A; F >$ is simple, we first observe that if θ is a congruence relation and $0 \equiv 1\ (\theta)$ then for any $x, y \in A$, $x = x*1 \equiv x*0 = 0\ (\theta)$ and similarly $y \equiv 0\ (\theta)$, thus $x \equiv y\ (\theta)$, i.e. $\theta = \iota$. Now, let θ be any congruence relation such that $\theta \neq \omega$, i.e. $x \equiv y\ (\theta)$ for some $x \neq y$ in A. If $\{x, y\} = \{0, 1\}$, then $\theta = \iota$ was proved above. So let us assume that $x \notin \{0, 1\}$. If $y \notin \{0, 1\}$, then $x = x*x = x*y = 0\ (\theta)$ and so $x = P(x) \equiv P(0) = 1\ (\theta)$, thus $0 \equiv 1\ (\theta)$. Thus we may assume that $y \in \{0, 1\}$, say $y = 0$. Then $x \equiv 0\ (\theta)$ implies $x = P(x) \equiv P(0) = 1\ (\theta)$, thus $0 \equiv 1\ (\theta)$, completing the proof.

Now, we define mappings on A as follows :-

$$\psi_a, \text{ for } a \in R \quad : \quad x\psi_a = xa \qquad (x \in S),$$
$$= x \qquad (x \notin S);$$
$$\psi_a, \text{ for } a \in N \quad : \quad x\psi_a = a \qquad (x \in A).$$

(i) $\psi_a \in E(A; F)$ for all $a \in S$.

If $a \in R$, then for any x, y in A, $x \neq y$, $(x*x)\psi_a = x\psi_a = xa = xa*xa = x\psi_a*x\psi_a$; since $x \neq y$, we have $xa \neq ya$, so that $(x*y)\psi_a = 0\psi_a = 0 = xa*ya = x\psi_a*y\psi_a$; for $x \in S$, $(P(x))\psi_a = x\psi_a = xa = P(xa) = P(x\psi_a)$ and $(f_b(x))\psi_a = (bx)a = b(xa) = f_b(x\psi_a)$ for all $b \in S$; for $y \in \{0,1\}$, $(x*0)\psi_a = 0 = x\psi_a*0 = x\psi_a*0\psi_a$ and $(x*1)\psi_a = x\psi_a*1 = x\psi_a*1\psi_a$ for all $x \in A$;

for $x \notin S$, suppose $x = 0$, $(P(0))\psi_a = 1\psi_a = 1 = P(0) = P(0\psi_a)$ and $f_b(0\psi_a) = f_b(0) = 0 = 0\psi_a = f_b(0\psi_a)$ for all b in S, thus it is similarly for x = 1. If $a \in N$, then ψ_a is a constant map, hence ψ_a is an endomorphism.

(ii) If $\psi \in E(A; F)$ and ψ is 1 - 1, then $0\psi = 0$ and $1\psi = 1$. Indeed, $1\psi = a \neq 1$ would imply that for any $x \in S$, $x\psi = (x*1)\psi = x\psi*1\psi = x\psi*a$, thus $x\psi = a$ or $x\psi = 0$. It is a contradiction since $x\psi \neq 0$, so that $x\psi = a = 1\psi$ implies $1 = x \in S$. Hence $0\psi = (P(1))\psi = P(1\psi) = P(1) = 0$.

(iii) If $\psi \in E(A; F)$ and ψ is 1 - 1, then $\psi = \psi_a$ for some a in R.

Let $x \in S$; then $x\psi \in S$; otherwise $x\psi = 0\psi$ or $x\psi = 1\psi$. Thus ψ maps S into S, so $e\psi \in S$ and if $a(e\psi) = b(e\psi)$ for any a, b in S, then $a\psi = (ae)\psi = f_a(e\psi) = (f_a(e))\psi = a(e\psi) = b(e\psi) = f_b(e\psi) = (f_b(e))\psi = b\psi$ implies $a = b$, hence $e\psi \in R$. So, $x\psi = (xe)\psi = (f_x(e))\psi = x(e\psi) = x\psi_{e\psi}$. Hence, $\psi = \psi_{e\psi}$ and $e\psi \in R$.

(iv) If $\psi \in E(A; F)$ and ψ is not 1 - 1, then $A\psi$ consists of one element. This follows from the fact that $< A; F >$ is simple; otherwise ψ induces a congruence relation ε_ψ which is not ω and ι.

(v) If $A\psi = \{a\}$ for some a in N, then $a \in N$.

Obviously, $< A ; F >$ is a subalgebra. Since $\{a\}$ is closed under P only if $a \neq 0$, $a \neq 1$, we have $a \in S$. If $a \in R$, then for $b \in S$, $b \neq a$, we have $ba \neq a$ and thus $\{a\}$ is not closed under f_b. Hence, $a \in N$. Thus:

(vi) If $A\psi = \{a\}$, then $\psi = \psi_a$ for $a \in N$. By (i) - (vi),

$E(A;\ F) = \{\psi_a | a \in S\}$.

It follows now as in Theorem 2.1 that $a \rightarrow \psi_a$ is an isomorphism between $< S; \cdot >$ and $< E(A;\ F); \circ >$. This completes the proof of the theorem. #

The following corollary of the Theorem 3.8 shows that the theorem 3.3 is true for any cardinal number π on a simple algebra, even if it is not an idempotent algebra.

3.9 <u>Corollary</u>. Let $< A;\ F >$ be a simple algebra such that $|E(A;F)| > 1$ and which has no idempotent element, then $E(A;\ F)$ has no right annihilator.

<u>Proof</u>. Suppose that $E(A;\ F)$ has a right annihilator. Let ψ be a right annihilator of $E(A;\ F)$. Then $\varepsilon_\psi = \iota$. Let $x \in A$, $f_\gamma \in F$. Then $x\psi = b$ for some b in A, so that $x \equiv f_\gamma(x, \ldots, x)\ (\varepsilon_\psi)$ implies $b = x\psi = f_\gamma(x, \ldots, x)\psi = f_\gamma(x\psi, \ldots, x\psi) = f_\gamma(b, \ldots, b)$. Hence, b is an idempotent element of A, completing the proof. #

An example of an algebra with no idempotent, but its endomorphism semigroup has right annihilators.

Let $A = \{a, b, c\}$ and $f : A \rightarrow A$ be such that

$$f(x) = a \quad \text{if } x = b,$$
$$ = b \quad \text{if } x = a \text{ or } x = c.$$

Then $< A;\ F >$ is an algebra with no idempotent, where $F = \{f\}$.

Let $\psi_1, \psi_2 : A \rightarrow A$ be such that

$$\psi_1(x) = b \quad \text{if } x = b,$$
$$ = a \quad \text{if } x = a \text{ or } x = c;$$

and

$$\psi_2(x) = b \qquad \text{if } x = b,$$
$$\qquad = c \qquad \text{if } x = a \text{ or } x = c.$$

Then $\psi_1, \psi_2 \in E(A; F)$ and they are right annihilators of $E(A; F)$ since $E(A; F) = \{I_A, \psi_1, \psi_2, \sigma\}$ where $\sigma : A \longrightarrow A$ by $\sigma(a) = c$, $\sigma(b) = b$, $\sigma(c) = a$ and I_A is the identity mapping on A.

3.10 <u>Definition</u>. Let $< A; F >$ be an algebra. An element Z in A is called a <u>zero</u> of A if for $0 < \gamma < 0(\tau)$, $a_0, \ldots, a_{n_\gamma - 1} \in A$, $f_\gamma(a_0, \ldots, a_{i-1}, Z, a_{i+1}, \ldots, a_{n_\gamma - 1}) = Z$.

$< A; F >$ is called an <u>entire algebra</u> if A contains a zero element and $f_\gamma(a_0, \ldots, a_{n_\gamma - 1}) = Z$ implies $a_i = Z$ for some i, $0 \leq i < n_\gamma$, for all $f_\gamma \in F$, $a_0, \ldots, a_{n_\gamma - 1} \in A$.

$< \mathbb{Z}; \cdot >$, where $\mathbb{Z}$ is the set of integers and $\cdot$ is the natural multiplication, is an example of an entire algebra.

In the next theorems, let us consider the endomorphism semigroup of an entire algebra.

3.11 <u>Lemma</u>. Let $< A; F >$ be an entire algebra such that $|A| > 2$. Then $< A; F >$ has at least three congruence relations.

<u>Proof</u>. Let 0 denote the zero element of such an algebra $<A;F>$. Define a binary relation $\bar{\theta}$ on A by

$$x \equiv y\ (\bar{\theta}) \longleftrightarrow x, y \in A \setminus \{0\} \text{ or } x = y$$

for all $x, y \in A$. Then $\bar{\theta}$ is an equivalence relation on A. To show that $\bar{\theta}$ is a congruence relation, let $f_\gamma \in F$, $a_0, \ldots, a_{n_\gamma - 1}$,

$b_0, \ldots, b_{n_\gamma - 1} \in A$ be such that $a_i \equiv b_i(\overline{\theta})$ for $0 \leq i < n_\gamma$. If there is a $j \in \{0, \ldots, n_\gamma - 1\}$ such that $a_j = 0$, then $b_j = 0$ and so $f_\gamma(a_0, \ldots, a_{n_\gamma - 1}) = 0 = f(b_0, \ldots, b_{n_\gamma - 1})$. If $a_i \neq 0$ for all $i \in \{0, \ldots, n_\gamma - 1\}$, then $b_0, \ldots, b_{n_\gamma - 1}$ are all nonzero, thus $f(a_0, \ldots, a_{n_\gamma - 1}) \neq 0$ and $f_\gamma(b_0, \ldots, b_{n_\gamma - 1}) \neq 0$, hence $f(a_0, \ldots, a_{n_\gamma - 1}) \equiv f_\gamma(b_0, \ldots, b_{n_\gamma - 1})$ $(\overline{\theta})$. In either cases, $\overline{\theta}$ is a congruence relation on A. Therefore ω, ι and $\overline{\theta}$ are congruence relations on $< A; F >$. #

3.12 Theorem. A semigroup $< S; \cdot >$ is isomorphic to the endomorphism semigroup of some unique idempotent entire algebra which exactly three congruence relations if and only if $< S; \cdot >$ has an identity e, zero $\neq$ e and for each nonzero element a in S, a satisfies a right cancellation law, and $< S; \cdot >$ has no zero divisor.

Proof. Let $< A; F >$ be an entire algebra which has zero, denoted by 0, as its unique idempotent and has exactly three congruence relations such that $S \cong E(A; F)$. Then by Theorem 2.1, S has an identity element and by Theorem 3.2, S has zero. Since $< A; F >$ is an entire algebra, $\overline{\theta}$ is a congruence relation on A.

Let $\psi \in E(A; F)$ such that ψ is not right annihilator. Then ψ induces a congruence relation ε_ψ on A by

$$x \equiv y \ (\varepsilon_\psi) \longleftrightarrow x\psi = y\psi$$

for all x, y in A. Then $\varepsilon_\psi = \omega$ or $\varepsilon_\psi = \iota$ or $\varepsilon_\psi = \overline{\theta}$.

case 1. $\varepsilon_\psi = \omega$. Then $x\psi = y\psi$ if and only if $x = y$, hence ψ is a 1 - 1 mapping. Thus, $\beta o \psi = \alpha o \psi$ implies $\beta = \alpha$ for all α, β in E(A; F). Hence, ψ satisfies a right cancellation law.

case 2. $\varepsilon_\psi = \overline{\theta}$. Then $0\psi = a$ and $(A \setminus \{0\})\psi = \{b\}$ for some a,b in A, $a \neq b$; i.e. $A\psi = \{a, b\}$. Since $f_\gamma(a, \ldots, a) = f_\gamma(0\psi, \ldots, 0\psi) = (f_\gamma(0, \ldots, 0)\psi = 0\psi = a$ and $f_\gamma(b, \ldots, b) = f_\gamma(a_0\psi, \ldots, a_{n_\gamma - 1}\psi) = f_\gamma(a_0, \ldots, a_{n-1})\psi = a'\psi = b$ for all $f_\gamma \in F$, $a_0, \ldots, a_{n_\gamma - 1}$, $a' \in A \setminus \{0\}$, we obtain that a, b are idempotents of A. Therefore, $a = b = 0$ which contradicts to $a \neq b$. Hence, we have no such ψ to be endomorphism on A which induces $\bar{\theta}$.

<u>case 3</u> $\varepsilon_\psi = \iota$. Then $A\psi$ consists of a single element, say a. If $a \neq 0$, then there is an operation $f_{\gamma'}$ in F such that f $f_{\gamma'}(a, \ldots, a) \neq a$, and so $A\psi$ is not closed under $f_{\gamma'}$ which contradicts $\langle A\psi ; F \rangle$ being a subalgebra. Hence, $A\psi = \{0\}$. Let us denote ψ by ψ_0. Then by Theorem 3.3, ψ_0 is the only right annihilator of $E(A; F)$ which is a contradiction.

(i) If $\alpha \in E(A; F)$, then $0\alpha = 0$.

Since $f_\gamma(0\alpha, \ldots, 0\alpha) = (f_\gamma(0, \ldots, 0)\alpha = 0\alpha$ for all f_γ in F, 0α is an idempotent of $\langle A\psi; F \rangle$, hence $0\alpha = 0$.

(ii) ψ_0 is a left zero of $E(A; F)$.

Indeed, $x(\psi_0 o \alpha) = 0\alpha = 0 = x\psi_0$ for all $x \in A$, $\alpha \in E(A; F)$.

(iii) For any $\alpha, \beta \in E(A; F)$, if $\alpha o \beta = \psi_0$ then $\alpha = \psi_0$ or $\beta = \psi_0$. Suppose that α and β are not right annihilators of $E(A; F)$. Therefore, α and β satisfy the right cancellation law, so is $\alpha o \beta$ since $\gamma o (\alpha o \beta) = \gamma' o (\alpha o \beta)$ implies $\gamma = \gamma'$ for any $\gamma, \gamma' \in E(A; F)$. This contradicts that ψ_0 is a right annihilator. Hence, $\alpha = \psi_0$ or $\beta = \psi_0$, completing the proof.

To prove the converse, let $S = R \cup \{\bar{0}, e\}$, where R consists of the right cancellative elements, $\bar{0}$ and e denote the zero and an identity, respectively. Construct an algebra as follows : Set $A = S \cup \{0, 1\}$, $0 \notin S$, $1 \notin S$, $0 \neq 1$. We define the following operations on A :-

$$
\begin{array}{llll}
P & : P(x) & = x & (x \in S), \\
 & \;\; P(0) & = 1, & \\
 & \;\; P(1) & = 0; & \\
f_a, \text{ for } a \in S \setminus \{\bar{0}\} & : f_a(x) & = ax & (x \in S), \\
 & & = x & (x \notin S); \\
* & : x*y & = 0 & (x, y \in S \setminus \{\bar{0}\}, x \neq y), \\
 & & = \bar{0} & (x, y \in A, x \text{ or } y = \bar{0}), \\
 & \;\; x*x & = x & (x \in A), \\
 & \;\; 1*x & = x*1 = x & (x \in A), \\
 & \;\; 0*x & = x*0 = 0 & (x \in A \setminus \{\bar{0}\}).
\end{array}
$$

Set

$$F = \{f_a \mid a \in S\} \cup \{*, P\}.$$

Obviously, $\bar{0}$ is the zero and the only idempotent of the algebra $< A; F >$ which is entire. So by Lemma 3.11, ω, $\bar{\theta}$ and ι are congruence relations on $< A; F >$. Let θ be a congruence relation on A.

(i) If $0 \equiv 1$ (θ), then $x \equiv y$ (θ) for all $x, y \in A \setminus \{\bar{0}\}$. Indeed, for any $x, y \in A \setminus \{\bar{0}\}$, $x = x*1 \equiv x*0 = 0$ (θ) and similarly $y \equiv 0$ (θ), thus $x \equiv y$ (θ).

(ii) If there is an x in $A \setminus \{\bar{0}\}$ such that $x \equiv \bar{0}$ (θ), then $\theta = \iota$.

Since $0 = 0*x \equiv 0*\bar{0} = \bar{0}$ (θ) and so $1 = P(0) \equiv P(\bar{0}) = \bar{0}$ (θ), we have $0 \equiv 1$ (θ) and by (i), $\bar{0} \equiv x \equiv a$ (θ) for all $a \in A$. Hence, $\theta = \iota$.

To show that < A; F > has exactly three congruence relations, let θ be any congruence relation such that $\theta \neq \omega$ and $\theta \neq \iota$. Then there exists $x, y \in A \setminus \{\overline{0}\}$, $x \neq y$ such that $x \equiv y\ (\theta)$. Therefore, $x = x{*}x \equiv x{*}y = 0\ (\theta)$ and so $x = P(x) \equiv P(0) = 1(\theta)$, thus $0 \equiv 1\ (\theta)$. Hence, $\theta = \overline{\theta}$, completing the proof.

Now, we define mappings on A as follows :-

ψ_a, for $a \in R$: $x\psi_a = xa$ $(x \in S)$,

$= x$ $(x \notin S)$;

ψ_a, for $a = \overline{0}$: $x\psi_a = \overline{0}$ $(x \in A)$.

(i) $\psi_a \in E(A; F)$ for all $a \in S$.

If $a \in R$, then for any $x, y \in S$, $x \neq y$, we have $xa \neq ya$, thus $(x{*}x)\psi_a = x\psi_a = x\psi_a{*}x\psi_a$ and $(x{*}y)\psi_a = 0\psi_a = 0 = xa{*}ya = x\psi_a{*}y\psi_a$; for $y \in \{0,1\}$, $x \in A$, $(x{*}0)\psi_a = 0 = x\psi_a{*}0 = x\psi_a{*}0\psi_a$ and $(x{*}1)\psi_a = x\psi_a{*}1 = x\psi_a{*}1\psi_a$; and it is similarly for $x \in \{0, 1\}$, $y \in A$; for $x \in S$, $(P(x)\psi_a = x\psi_a = xa = P(xa) = P(x\psi_a)$ and $(f_b(x))\psi_a = (bx)a = b(xa) = f_b(x\psi_a)$ for all $b \in S$; for $x \notin S$, suppose $x = 0$, $(P(0)\psi_a = 1\psi_a = 1 = P(0) = P(0\psi_a)$ and $(f_b(0))\psi_a = 0\psi_a = 0 = f_b(0) = f_b(0\psi_a)$ for all $b \in S$, and it is similarly for $x = 1$. If $a = \overline{0}$, then ψ_a is a constant map, hence ψ_a is an endomorphism.

(ii) If $\psi \in E(A; F)$ and ψ is 1 - 1, then $0\psi = 0$ and $1\psi = 1$. Indeed, $1\psi = a \neq 1$ would imply that for any $x \in S \setminus \{\overline{0}\}$, $x\psi = (x{*}1)\psi = x\psi{*}1\psi = x\psi{*}a$, thus $x\psi = a$ or $x\psi = 0$. Since ψ is 1 - 1 and $(S \setminus \{\overline{0}\})\psi \neq \{0\}$, we have $x = 1$. It is a contradiction. Hence, $0\psi = (P(1))\psi = P(1\psi) = P(1) = 0$.

(iii) If $\psi \in E(A; F)$ and ψ is 1 - 1, then $\psi = \psi_a$ for some a in R.

Indeed, for any x, $y \in S$, $x(e\psi) = y(e\psi)$ implies $x = y$. Hence, $e\psi \in R$. Let $x \in S$. Then $x\psi \in S$; otherwise $x\psi = 0\psi$ or $x\psi = 1\psi$. Thus, $x\psi = (xe)\psi = (f_x(e))\psi = f_x(e\psi) = x(e\psi) = x\psi_{e\psi}$.

(iv) If $\psi \in E(A; F)$ and ψ is not 1 - 1, then $A\psi = \{\bar{0}\}$. Since < A; F > has three congruence relations and ψ is not 1 - 1, ψ induces a congruence relation ε_ψ such that $\varepsilon_\psi = \bar{\theta}$ or $\varepsilon_\psi = \iota$. If $\varepsilon_\psi = \bar{\theta}$, then $A\psi = \{a, b\}$ for some a, b in A, $a \neq b$,,hence a and b are idempotents of < A; F >, thus $a = \bar{0} = b$. Then $\bar{\theta} = \iota$ which is a contradiction. Hence, $\varepsilon_\psi = \iota$, i.e. $A\psi = \{a\}$ for some idempotent a in A so that $a = \bar{0}$.

Consider the mapping $\theta : a \to \psi_a$. It follows now as in Theorem 3.3 that θ is an isomorphism between < S, • > and < E(A; F), ∘ >. This completes the proof of the theorem. #

The following example is an example of an entire algebra which has two idempotents and three congruence relations.

<u>Example</u> : Let < G, • > be a nontrivial simple group with identity e. Set $A = G \cup \{0\}$, and $x \cdot 0 = 0 \cdot x = 0$ for all x in G. Then < A; • > is an entire algebra which has 0 and e to be its idempotents. By Lemma 3.11, $\bar{\theta}$ is a congruence relation of < A; • >.

To show that < A; • > has three congruence relations, let θ be any congruence relation on A such that $\theta \neq \omega$ and $\theta \neq \bar{\theta}$. Then there exists an $x \in G$ such that $x \equiv 0\ (\theta)$ and also $e = x^{-1} \cdot x \equiv x^{-1} \cdot 0 = 0\ (\theta)$, hence $a = a \cdot e \equiv a \cdot 0 = 0\ (\theta)$ for all $a \in G$, thus $\theta = \iota$.

Now, we consider a 2-idempotent entire algebra which has three congruence relations such as the above example. The following theorem shows that the endomorphism semigroup of such algebra contains an identity, two right annihilators, an identity of right annihilator, and the other satisfy a right cancellation law.

3.13 Theorem. A semigroup $< S; \cdot >$ is isomorphic to the endomorphism semigroup of some 2-idempotent entire algebra which has exactly three congruence relations if and only if S has an identity, two right annihilators, an identity of right annihilators which does not satisfy right cancellation law, say $\bar{e}$, and for each a in S, if a is not $\bar{e}$ and right annihilators then a satisfies a right cancellation law, and there is a right annihilator r_1 such that for all a, b $\in$ S, $ab = r_1$ implies $a = r_1$ or $b = r_1$.

Proof. Let $< S; \cdot >$ be a semigroup. Assume that $< A; F >$ is an 2-idempotent entire algebra which has exactly three congruence relations such that $E(A; F) \cong S$. Then by Theorem 2.1, S has an identity. Since $< A; F>$ is entire, $\bar{\theta}$ is a congruence relation on A.

Let $\psi \in E(A; F)$. Then ψ induces a congruence relation ε_ψ on A by

$$x \equiv y \ (\varepsilon_\psi) \longleftrightarrow x\psi = y\psi$$

for all x, y A. Therefore, $\varepsilon_\psi = \omega$ or $\varepsilon_\psi = \iota$ or $\varepsilon_\psi = \bar{\theta}$. If $\varepsilon_\psi = \omega$ or ι , then by Theorem 3.8, ψ is right cancellative or ψ is a right annihilator of E(A; F); and by Theorem 3.3 E(A; F) has two right annihilators, say ψ_0 and ψ_e, where $A\psi_0 = \{0\}$ and $A\psi_e = \{e\}$; e is denoted to be the other idempotent of A; also if $\alpha o \beta = \psi_0$ then $\alpha = \psi_0$ or $\beta = \psi_0$.

If $\varepsilon_\psi = \bar{\theta}$, then $A\psi = \{a, b\}$ for some $a, b \in A$. Let $0\psi = a$ and $(A \setminus \{0\})\psi = \{b\}$.

(i) $\{a, b\} = \{0, e\}$.

Indeed, $f_\gamma(b,\ldots, b) = f_\gamma(x_0\psi,\ldots, x_{n_\gamma-1})$

$= (f_\gamma(x_0,\ldots, x_{n_\gamma-1}))\psi = b,$

and $f_\gamma(a,\ldots, a) = f_\gamma(0\psi,\ldots, 0\psi)$

$= (f_\gamma(0,\ldots, 0))\psi = 0\psi = a$

for all $f_\gamma \in F$, $x_0,\ldots, x_{n_\gamma-1} \in A \setminus \{0\}$. Hence, a and b are idempotents of A.

(ii) $0\psi = 0$.

Suppose not. Then $0\psi = e$ and $(A \setminus \{0\})\psi = \{0\}$. Let $f_\gamma \in F$ and $x_0,\ldots, x_{n_\gamma-1}$ be all nonzero element of A. Then

$$\begin{aligned} 0 &= (f_\gamma(x_0,\ldots, x_{i-1}, 0, x_{i+1},\ldots, x_{n_\gamma-1}))\psi \\ &= f_\gamma(x_0\psi,\ldots, x_{i-1}\psi, 0\psi, x_{i+1}\psi,\ldots, x_{n_\gamma-1}\psi) \\ &= f_\gamma(0, \ldots, 0, e, 0,\ldots, 0) = 0. \end{aligned}$$

Therefore, $0\psi = x\psi$ for all $x \in A \setminus \{0\}$. Then by the definition of ε_ψ, $\varepsilon_\psi = \iota$ which is a contradiction.

(iii) $\psi_0 o \psi = \psi_0 = \psi o \psi_0$ and $\psi_e o \psi = \psi_e = \psi o \psi_e$.

Indeed, $x(\psi_0 o \psi) = 0\psi = 0 = x\psi_0$ and $x(\psi_e o \psi) = e\psi = e = x\psi_e$ for all x in A. Hence, $\psi_0 o \psi = \psi_0$ and $\psi_e o \psi = \psi_e$. Therefore ψ is not right concellative and $\psi = \bar{e}$.

To prove the converse, assume that $S = R \cup \{r_1, r_2, \bar{e}\}$, where r_1, r_2 are rights annihilators, $\bar{e}$ is the identity of r_1 and r_2, and R consists of the right concellative elements of S. Let e denote the identity of S. Then $e \in R$. Construct an algebra as follows : Set $A = S \cup \{0, 1\}$, $0 \notin S$, $1 \notin S$, $0 \neq 1$. We define the following operations on A :-

P :	$P(x) = x$	$(x \in S)$,
	$P(0) = 1$,	
	$P(1) = 0$;	
f_a, for $a \in S \setminus \{r_1\}$:	$f_a(x) = ax$	$(x \in S)$,
	$= x$	$(x \notin S)$;
$*$:	$x*y = 0$	$(x, y \in S \setminus \{r_1\}, x \neq y)$,
	$= r_1$	$(x, y \in A$, x or $y = r_1)$,
	$x*x = x$	$(x \in A)$,
	$1*x = x*1 = x$	$(x \in A)$,
	$0*x = x*0 = 0$	$(x \in A \setminus \{r_1\})$.

Set

$$F = \{f_a | a \in S\} \cup \{*, P\}.$$

Obviously, r_1 is the zero of $< A; F >$ which is an entire algebra, and r_1, r_2 are the only idempotents of $< A; F >$. So by Lemma 3.11, ω, ι and $\bar{\theta}$ are congruence relations on A. Let θ be a congruence relation on A.

(i) If $0 \equiv 1$ (θ), then $\theta = \bar{\theta}$ or ι.

Indeed, for any $x, y \in A \setminus \{r_1\}$, $x = x*1 \equiv x*0 = 0$ (θ) and similarly, $y \equiv 0$ (θ), thus $x \equiv y$ (θ). Hence, $\theta = \bar{\theta}$ or ι.

(ii) If there is an x in $A \setminus \{r_1\}$ such that $x \equiv r_1$ (θ), then $\theta = \iota$. Since $0 = 0*x \equiv 0*r_1 = r_1$ (θ) and so $1 = P(0) \equiv P(r_1) = r_1$ (θ), we have $0 \equiv 1$ (θ) and by (i), $\theta = \bar{\theta}$ or ι but $r_1 \equiv x$ (θ) where $x \neq r_1$. Hence, $\theta = \iota$.

To show that $< A; F >$ has exactly three congruence relations, let θ be any congruence relation such that $\theta \neq \omega$ and $\theta \neq \iota$. Then there exists $x, y \in A \setminus \{r_1\}$, $x \neq y$ such that $x \equiv y\ (\theta)$. Therefore, $x = x*x \equiv x*y = 0\ (\theta)$ and so $x = P(x) \equiv P(0) = 1\ (\theta)$, thus $0 \equiv 1\ (\theta)$. Hence, $\theta = \bar{\theta}$.

Now, we define mappings on A as follows :-

$$\psi_a, \text{for } a \in R, \quad : \quad x\psi_a = xa \quad (x \in S),$$
$$= x \quad (x \notin S);$$
$$\psi_a, \text{for } a \in \{r_1, r_2\} \quad : \quad x\psi_a = a \quad (x \in A),$$
$$\psi_a, \text{for } a = \bar{e} \quad : \quad x\psi_a = r_1 \quad (x = r_1),$$
$$= r_2 \quad (x \neq r_1).$$

(i) $\psi_a \in E(A; F)$ for all $a \in S$.

Obviously, by Theorem 3.8, if $a \in R \cup \{r_1, r_2\}$, then $\psi_a \in E(A; F)$. To show that if $a = \bar{e}$ then $\psi_a \in E(A; F)$, let $x, y \in A$, $x \neq y$. If x or y is r_1, say $x = r_1$, then $(r_1*y)\psi_a = r_1\psi_a = r_1 = r_1*y\psi_a = r_1\psi_a*y\psi_a$ and $(y*y)\psi_a = y\psi_a = r_2 = r_2*r_2 = y\psi_a*y\psi_a$; $(P(x))\psi_a = (P(r_1))\psi_a = r_1\psi_a = r_1 = P(r_1) = P(r_1\psi_a)$; $(f_b(x))\psi_a = (f_b(r_1))\psi_a = r_1 = r_1\psi_a = f_b(r_1\psi_a)$ and $(f_b(y))\psi_a = r_2\psi_a = r_2 = f_b(r_2) = f_b(y\psi_a)$ for all $b \in S$. Hence, $\psi_a \in E(A; F)$. If $x \neq r_1$ and $y \neq r_1$, then $(x*y)\psi_a = 0\psi_a = r_2 = r_2*r_2 = x\psi_a*y\psi_a$; $(P(x))\psi_a = r_2 = P(r_2) = P(x\psi_a)$; $(f_b(x))\psi_a = r_2 = f_b(r_2) = f_b(x\psi_a)$ for all $b \in S$. Hence, $\psi_a \in E(A; F)$.

(ii) If $\psi \in E(A; F)$, and ψ is 1 - 1, then $0\psi = 0$ and $1\psi = 1$. Obviously, by the same proof as Theorem 3.12.

(iii) If $\psi \in E(A; F)$ and ψ is 1 - 1, then $\psi = \psi_a$ for some $a \in R$.

Obviously, by the same proof as Theorem 3.12.

(iv) If $\psi \in E(A; F)$ and ψ is not 1 - 1, then $\psi = \psi_{r_1}$ or $\psi = \psi_{r_2}$ or $\psi = \psi_{\bar{e}}$. Since $< A; F >$ has three congruence relations and ψ is not 1 - 1, ψ induces a congruence relation ε_ψ such that $\varepsilon_\psi = \bar{\theta}$ or $\varepsilon_\psi = \iota$. If $\varepsilon_\psi = \bar{\theta}$, then $A\psi = \{a, b\}$ for some a, b in A, $a \neq b$ and so a, b are idempotents of $< A; F >$, thus $\{a, b\} = \{r_1, r_2\}$. Since $r_1\psi = (x*r_1)\psi = x\psi*r_1\psi$ for $x \in A \smallsetminus \{r_1\}$ implies $r_1\psi = r_1$ or $r_1\psi = 0$, we have $r_1\psi = r_1$. Hence, $\psi = \psi_{\bar{e}}$. If $\varepsilon_\psi = \iota$, then $A\psi = \{a\}$ for some a in A, hence a is an idempotent of A and $a = r_1$ or $a = r_2$. Hence, $\psi = \psi_{r_1}$ or $\psi = \psi_{r_2}$.

(v) $\psi_a \circ \psi_b = \psi_{ab}$ for all $a, b \in S$.

Obviously, if $\{a, b\} \subseteq R$ or $\{a, b\} = \{r_1, r_2\}$ or $\{a, b\} \subseteq R \cup \{r_1, r_2\}$ or $\{a, b\} \subsetneq \{r_1, r_2, \bar{e}\}$, then they are done. Let $\{a, b\} \subseteq R \cup \{\bar{e}\}$, say $a \in R$ and $b = \bar{e}$.

<u>case 1</u> $r_1 a = r_1$. Then $r_1\psi_a = r_1 a = r_1$. Since $a \in R$, $r_2 a = r_2$ so that $r_2\psi_a = r_2 a = r_2$. Since $r_1(a\bar{e}) = (r_1 a)\bar{e} = r_1\bar{e} = r_1$ and $r_2(a\bar{e}) = (r_2 a)\bar{e} = r_2\bar{e} = r_2$, we have $a\bar{e} = \bar{e}$. Therefore $x(\psi_a \circ \psi_b) = (x\psi_a)\psi_b = (xa)\psi_b = r_2 = x\psi_{\bar{e}} = x\psi_{a\bar{e}} = x\psi_{ab}$ for all $x \in A \smallsetminus \{r_1\}$. Hence, $\psi_a \circ \psi_b = \psi_{ab}$.

<u>case 2</u> $r_1 a = r_2$. Similarly, we can get $r_2 a = r_1$ and $a\bar{e} = \bar{e}$. Hence, $\psi_a \circ \psi_b = \psi_{ab}$.

(vi) Obviously, $\psi_a = \psi_b$ if and only if $a = b$.

Consider the mapping $\theta : a \to \psi_a$. θ is 1 - 1 by (vi). θ is onto by (iii) and (iv). It preserves multiplication by (v). Therefore, θ is an isomorphism. This completes the proof of the theorem. #