CHAPTER VI

SOLUTION OF THE FUNCTIONAL EQUATION (1-f(x)f(y))f(xy) = f(x)+f(y)

6.1 Introduction

The purpose of this chapter is to determine all the solutions of the functional equation

$$(T) \qquad (1 - f(x)f(y))f(xy) = f(x) + f(y) \;,$$

where f is a function on a group G into $\mathbb{R}$, the set of real numbers.

6.2 Solution of (1 - f(x)f(y))f(xy)= f(x)+f(y) on a Group.

6.2.1 Theorem Let G be a group. Then $f : G \longrightarrow \mathbb{R}$ satisfies

$$(T) \qquad (1 - f(x)f(y))\, f(xy) = f(x) + f(y)$$

if and only if there exists a homomorphism $h : G \longrightarrow \Delta$ such that

$$f(x) = \frac{h(x) - 1}{i(h(x) + 1)} \;,$$

where $i^2 = -1$.

Proof Assume that $f : G \longrightarrow \mathbb{R}$ satisfies (T).
Let $h : G \longrightarrow \Delta$ be defined by

$$(6.2.1.1) \qquad h(x) = \frac{1 + i\, f(x)}{1 - i\, f(x)} \;.$$

Hence $$h(x)\,(1 - i\, f(x)) = 1 + i\, f(x) \;.$$

Thus $$h(x) - i\, h(x)f(x) = 1 + i\, f(x) \;,$$
$$h(x) - 1 = i\, f(x)\,(1 + h(x)).$$

Therefore $$f(x) = \frac{h(x) - 1}{i\,(1 + h(x))} .$$

It remains only to be proved that h defined by (6.2.1.1) is a homomorphism.

By (6.2.1.1), we have

$$\begin{aligned} h(xy) &= \frac{1 + i\, f(xy)}{1 - i\, f(xy)} , \\ &= \frac{1 + i\left(\dfrac{f(x) + f(y)}{1 - f(x)f(y)} \right)}{1 - i\left(\dfrac{f(x) + f(y)}{1 - f(x)f(y)} \right)} , \\ &= \frac{1 - f(x)f(y) + i(f(x) + f(y))}{1 - f(x)f(y) - i(f(x) + f(y))} , \\ &= \frac{1 + i\, f(x)}{1 - i\, f(x)} \cdot \frac{1 + i\, f(y)}{1 - i\, f(y)} , \\ &= h(x)h(y) . \end{aligned}$$

Then h is a homomorphism.

Conversely, we assume that

$$f(x) = \frac{h(x) - 1}{i\,(h(x) + 1)} ,$$

where h is a homomorphism from G into Δ .

Thus we have,

$$\begin{aligned} \left[1 - f(x)f(y)\right] f(xy) &= \left[1 - \left\{\frac{h(x)-1}{i(h(x)+1)}\right\}\left\{\frac{h(y)-1}{i(h(y)+1)}\right\}\right] \left[\frac{h(xy)-1}{i(h(xy)+1)}\right], \\ &= \left[1 + \frac{(h(x)-1)(h(y)-1)}{(h(x)+1)(h(y)+1)}\right] \left[\frac{h(x)h(y)-1}{i(h(x)h(y)+1)}\right], \end{aligned}$$

$$= \frac{2(h(x)h(y) + 1)}{(h(x) + 1)(h(y) + 1)} \cdot \frac{h(x)h(y) - 1}{i\,(h(x)h(y) + 1)},$$

$$= \frac{2(h(x)h(y) - 1)}{i(h(x) + 1)(h(y) + 1)},$$

and,

$$f(x) + f(y) = \frac{h(x) - 1}{i\,(h(x) + 1)} + \frac{h(y) - 1}{i\,(h(y) + 1)},$$

$$= \frac{2(h(x)h(y) - 1)}{i\,(h(x) + 1)(h(y) + 1)}.$$

Therefore, $f(x) + f(y) = \Big[1 - f(x)f(y)\Big]\, f(xy)$.

6.3 Continuous Solutions of $\big[1 - f(x)f(y)\big]\, f(xy) = f(x)+f(y)$ on a Topological Group.

6.3.1 Theorem Let G be a topological group. Then $f : G \longrightarrow \mathbb{R}$ is continuous and satisfies

$$\text{(T)} \qquad (1 - f(x)f(y))\; f(xy) = f(x) + f(y),$$

if and only if there exists a continuous homomorphism h from G into $\triangle$ such that

$$\text{(6.3.1.1)} \qquad f(x) = \frac{h(x) - 1}{i\,(h(x) + 1)}.$$

Proof By theorem 6.2.1, f satisfies (T) if and only if

$$f(x) = \frac{h(x) - 1}{i\,(h(x) + 1)},$$

where h is a homomorphism from G into $\triangle$.

It remains only to be proved that f is continuous if and only if h is continuous.

From (6.3.1.1), we see that if h is continuous, so is f. Again, from (6.3.1.1), we get

$$h(x) = \frac{1 + i\, f(x)}{1 - i\, f(x)} .$$

Hence, if f is continuous, so is h.

6.4 Continuous Solutions of $(1 - f(x)f(y))\, f(x+y)=f(x)+f(y)$ on $\mathbb{R}^n$.

6.4.1 Theorem Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuous. Then f satisfies

(T_1) $$(1 - f(x)f(y))\, f(x+ y) = f(x) + f(y)$$

if and only if there exist $k_1,\ldots, k_n \in \mathbb{R}$ such that

$$(6.4.1.1) \quad f(x) = \frac{e^{i(k_1x_1+\ldots+k_nx_n)} - 1}{i\,(e^{i(k_1x_1+\ldots+k_nx_n)} + 1)} .$$

Proof By theorem 6.3.1, any continuous function f satisfies (T_1) if and only if f is of the form

$$f(x) = \frac{h(x) - 1}{i\,(h(x) + 1)} ,$$

where h is a continuous homomorphism from $\mathbb{R}^n$ to Δ.

By theorem 3.2.6, we know that there exist homomorphisms $h_j : \mathbb{R} \longrightarrow \Delta$ such that

$$h(x) = \prod_{j=1}^{n} (h_j \circ p_j)(x) ,$$

where p_j's are given by $p_j(x_1,\ldots,x_n) = x_j$, $j = 1,\ldots,n$.

Such an h_j is given by

$$h_j = h \circ \pi_j ,$$

where π_j is defined as in the proof of theorem 3.2.6.

Since h and π_j are continuous, then so is h_j.

By theorem 3.3.3, $h_j(x_j) = e^{ik_j x_j}$, where $k_j \in \mathbb{R}$, $j = 1,\ldots,n$.

Thus every continuous solution of (T_1) on $\mathbb{R}^n$ must be of the form

$$f(x) = \frac{e^{i(k_1x_1+\ldots+k_nx_n)} - 1}{i(e^{i(k_1x_1+\ldots+k_nx_n)} + 1)} .$$

APPENDIX

In this appendix, we will show the existence of a Hamel basis by means of the following :

<u>Lemma</u> (Zorn's lemma)

A partially ordered system has a maximal element if every totally ordered subsystem has an upper bound.

For proof of this lemma we refer to [4] .

Let E be the family of all subsets of $\mathbb{R}$ which are linearly independent over Q. We partial order E by inclusion.

Let F be any totally ordered subset of E. Put $A' = \cup F$. We claim that A' is an upper bound of F. It is clear from the definition of A' that

(1) for every $A \in F$, then $A \subseteq A'$.

It remains to be shown that

(2) $A' \in E$.

Let X be any finite subset of A'. Since X is finite, we may assume $X = \{x_1, \dots, x_n\}$. Since $x_i \in X \subseteq A' = \cup F$, hence there exists $A_i \in F$ such that $x_i \in A_i$. Without loss of generality, we assume $A_1 \subseteq A_2 \subseteq \dots \subseteq A_n$. Hence $x_i \in A_n$ for every i. i.e. $X \subseteq A_n$, which is a linearly independent set. Thus X is independent. So $A' \in E$. By Zorn's lemma, E has a maximal element, say H.

We claim that H is a Hamel basis. To see that every $x \in \mathbb{R}$ can be written as a linear combination of elements in H, we suppose the contrary. Hence there exists $x_o \in \mathbb{R}$ such that x_o cannot be written as a linear combination of elements of H. Therefore $H \cup \{x_o\}$ is a linearly independent set. This contradicts the fact that H is maximal. Hence every real number x can be written as a linear combination of elements of H.

Next, suppose that some $x_o \in \mathbb{R}$ can be expressed as two linear combinations of elements of H :

$$x_o = \sum_{i=1}^{n} a_i V_{\alpha_i} \quad \text{and} \quad x_o = \sum_{i=1}^{n} b_i V_{\alpha_i} .$$

By substraction, we have

$$0 = x_o - x_o = \sum_{i=1}^{n} (a_i - b_i) V_{\alpha_i} .$$

Since $V_{\alpha_1}, \dots, V_{\alpha_n}$ are linearly independent, hence

$$a_i - b_i = 0 \quad \text{for } i = 1, \dots, n.$$

i.e. we have $a_i = b_i$ for $i = 1, \dots, n$.
Therefore, every $x \in \mathbb{R}$ can be uniquely represented as a linear combination of elements of H.