CHAPTER III

DISTRIBUTIONS

In this chapter, the definitions and some properties of distributions and Radon measures will be studied in so far as they can be applied in the following chapters. It will be shown that every Radon measure is a distribution but the converse is not true.

The materials of this chapter are drawn from references [1], [3], [5], [7], [10], [11] and [12].

3.1 Definition and Basic Properties

3.1.1 Definition. Let Ω be an open subset of $\mathbb{R}^n$. A mapping $T : \mathcal{D}(\Omega) \longrightarrow \mathbb{R}$ is a distribution (or generalized function) on Ω if

(i) T is linear, and

(ii) for every compact set K in Ω, the restriction of T to $\mathcal{D}(\Omega;K)$ is continuous, i.e., for all compact sets K in Ω, there exists an integer $m \geq 0$ and a real number $\beta > 0$ such that $|T(\varphi)| \leq \beta p_{m,K}(\varphi)$ for all $\varphi \in \mathcal{D}(\Omega;K)$.

The distributions on Ω form a vector space, denoted by $\mathcal{D}'(\Omega)$.

3.1.2 <u>Examples</u>. (i) The Dirac measure T_δ given by $T_\delta(\varphi) = \varphi(0)$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$ is a distribution on $\mathbb{R}^n$. The linearity is clear and since $|T_\delta(\varphi)| = |\varphi(0)| \leq \sup_{x \in K} |\varphi(x)| = p_{0,K}(\varphi)$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n;K)$, we have the result.

(ii) The Lebesgue measure T_λ given by $T_\lambda(\varphi) = \int_{\mathbb{R}^n} \varphi(x)dx$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$ is a distribution on $\mathbb{R}^n$. The linearity is clear and since $|T_\lambda(\varphi)| = \left|\int_K \varphi(x)dx\right| \leq \int_K |\varphi(x)|dx \leq \beta \sup_{x \in K} |\varphi(x)| = \beta\, p_{0,K}(\varphi)$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n;K)$, where $\beta = \int_K dx$ is a real number > 0, we have the result.

(iii) Let f be a locally integrable function (with respect to Lebesgue measure) on Ω, i.e., for every compact set K of Ω, $\int_K |f(x)|dx < +\infty$. The mapping T_f given by $T_f(\varphi) = \int_\Omega f(x)\varphi(x)dx$ for all $\varphi \in \mathcal{D}(\Omega)$ is a distribution on Ω and is called a <u>regular distribution</u>. The linearity is clear and since $|T_f(\varphi)| = \left|\int_K f(x)\varphi(x)dx\right| \leq \int_K |f(x)\varphi(x)|dx \leq \beta \sup_{x \in K} |\varphi(x)| = \beta\, p_{0,K}(\varphi)$ for all $\varphi \in \mathcal{D}(\Omega;K)$, where $\beta = \int_K |f(x)|\, dx$ is a real number > 0, we have the result

3.1.3 <u>Definition</u>. A distribution T on Ω is said to be <u>positive</u> if, for each function $\varphi \in \mathcal{D}(\Omega)$ such that $\varphi \geq 0$, we have $T(\varphi) \geq 0$.

3.1.4 Lemma. Let $\varphi_1, \varphi_2 \in \mathcal{D}(\Omega)$ and $\varphi_1 \geq \varphi_2$. A distribution T on Ω is positive if and only if $T(\varphi_1) \geq T(\varphi_2)$.

Proof : Apply (3.1.3) and the linearity of T to $\varphi_1 - \varphi_2$. The converse follows immediately by reversing.

3.1.5 Theorem. If T is a positive distribution on Ω, then T satisfies (ii) in (3.1.1) with m = 0, i.e., for every compact set K in Ω, there exists a real number $\beta > 0$ such that $|T(\varphi)| \leq \beta\, p_{0,K}(\varphi) = \beta \sup_{x \in K} |\varphi(x)|$ for all $\varphi \in \mathcal{D}(\Omega;K)$.

Proof : For every $\varphi \in \mathcal{D}(\Omega;K)$, we have

$$- \sup_{x \in K} |\varphi(x)| \leq \varphi(x) \leq \sup_{x \in K} |\varphi(x)| .$$

Since K is compact, there exists a $\psi \in \mathcal{D}(\Omega)$ such that $\psi(x) = 1$ on K and is between 0 and 1 on $\Omega - K$ (2.2.5), so we have

$$- \sup_{x \in K} |\varphi(x)|\, \psi(x) \leq \varphi(x) \leq \sup_{x \in K} |\varphi(x)|\, \psi(x).$$

By (3.1.4), we get

$$- \sup_{x \in K} |\varphi(x)|\; T(\psi) \leq T(\varphi) \leq \sup_{x \in K} |\varphi(x)|\; T(\psi)$$

or, equivalently $|T(\varphi)| \leq T(\psi) \sup_{x \in K} |\varphi(x)|$.

Let $T(\psi) = \beta$. Clearly β is a non-negative constant (3.1.3). Hence the result.

3.1.6 Theorem. Let T be a distribution on Ω. Then there exists a distribution T' on $\mathbb{R}^n$ such that $T'|_{\mathcal{D}(w)} = T|_{\mathcal{D}(w)}$ where w is an open set such that $w \subset \bar{w} \subset \Omega$.

Proof : Since $\bar{w}$ is compact, there exists a function $\psi \in \mathcal{D}(\Omega)$ such that $\psi(x) \leq 1$ for all $x \in \Omega$ and $\psi = 1$ on $\bar{w}$ (2.2.5). For any $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have $\psi\varphi \in \mathcal{D}(\Omega)$. We define

$$T'(\varphi) = T(\psi\varphi) \qquad (\varphi \in \mathcal{D}(\mathbb{R}^n)).$$

Clearly T' is linear and also for any compact set K in $\mathbb{R}^n$ and any $\varphi \in \mathcal{D}(\mathbb{R}^n;K)$

$$|T'(\varphi)| = |T(\psi\varphi)| \leq \beta \max_{|r| \leq m} \sup_{x \in K_0} |\partial^r(\psi(x)\varphi(x))| \quad (\psi\varphi \in \mathcal{D}(\Omega;K_0)),$$

where $\beta > 0$ and $K_0 = K \cap \bar{w}$. Since $\psi\varphi = \varphi$ on K_0 and equal to 0 otherwise, we see that

$$|T'(\varphi)| \leq \beta \max_{|r| \leq m} \sup_{x \in K_0} |\partial^r(\psi(x)\varphi(x))| \leq \beta \max_{|r| \leq m} \sup_{x \in K} |\partial^r\varphi(x)| \quad (\varphi \in \mathcal{D}(\mathbb{R}^n;K)).$$

Hence T' is a distribution on $\mathbb{R}^n$. For any $\eta \in \mathcal{D}(w)$, we have

$$T'(\eta) = T(\psi\eta) = T(\eta),$$

since $\psi(x) = 1$ on $\bar{w}$. Therefore $T'|_{\mathcal{D}(w)} = T|_{\mathcal{D}(w)}$.

3.2 The Radon Measure

3.2.1 <u>Notations</u>. Let Ω be an open subset of $\mathbb{R}^n$. Let $\mathcal{K}(\Omega)$ denote the set of all real-valued continuous functions on Ω with compact support. Clearly $\mathcal{K}(\Omega)$ is a real vector space. For every compact subset K of Ω, let $\mathcal{K}(\Omega;K)$ denote the real vector subspace of $\mathcal{K}(\Omega)$ consisting of the functions whose support is contained in K (and is therefore compact).

3.2.2 <u>Definition</u>. μ is a <u>Radon measure</u> on Ω if μ is a linear form on $\mathcal{K}(\Omega)$ with the following property : for each compact subset K of Ω, there exists a real number $\beta > 0$ (in general depending on K) such that $|\mu(f)| \leq \beta \sup_{x \in K} |f(x)|$ for all $f \in \mathcal{K}(\Omega;K)$.

3.2.3 <u>Examples</u>. (i) Let Ω be an open subset of $\mathbb{R}^n$, and let $x_o \in \Omega$. Let δ be the mapping of $\mathcal{K}(\Omega)$ into $\mathbb{R}$ given by $\delta(f) = f(x_o)$ for all $f \in \mathcal{K}(\Omega)$. Clearly δ is linear and we have $|f(x_o)| \leq \sup_{x \in K} |f(x)|$ for all $f \in \mathcal{K}(\Omega;K)$. Then δ is a Radon measure on Ω. This measure is called the <u>Dirac measure</u> at the point x_o.

(ii) Let λ be the mapping of $\mathcal{K}(\mathbb{R}^n)$ into $\mathbb{R}$ given by $\lambda(f) = \mathrm{R}\int_{\mathbb{R}^n} f(x)dx$ for all $f \in \mathcal{K}(\mathbb{R}^n)$. Clearly λ is linear and we have $|\lambda(f)| = |\int_K f(x)dx| \leq \int_K |f(x)|\,dx \leq \beta \sup_{x \in K} |f(x)|$ for all $f \in \mathcal{K}(\mathbb{R}^n;K)$, where $\beta = \int_K dx$ is a real number > 0. Then λ is a Radon measure on $\mathbb{R}^n$. This measure is called the <u>Lebesgue measure</u> on $\mathbb{R}^n$.

(iii) Let $S_r = \left\{ x : x \in \mathbb{R}^n \text{ and } \|x\| = r \right\}$. The Riemann integral of a function f defined on $\mathcal{K}(S_r)$ relative to the surface area element ds on S_r is denoted by $\eta(f) = R \int f(t) d\, s(t)$. Clearly η is linear and we have $|\eta(f)| = \left| \int_K f(t) d\, s(t) \right| \leq \int_K |f(t)| d\, s(t) \leq \beta \sup_{t \in K} |f(t)|$ for all $f \in \mathcal{K}(S_r;K)$, where $\beta = \int_K d\, s(t)$ is a real number > 0. Then η is a Radon measure on S_r. This measure is called the <u>surface area measure</u> on S_r.

3.2.4 <u>Definition</u>. A Radon measure μ on an open subset Ω of $\mathbb{R}^n$ is said to be <u>positive</u> if, for each function $f \in \mathcal{K}(\Omega)$ such that $f \geq 0$, we have $\mu(f) \geq 0$.

3.2.5 <u>Theorem</u>. If f and g are two functions belonging to $\mathcal{K}(\Omega)$ such that $f \leq g$, and μ is a Radon measure on Ω, then μ is positive if and only if $\mu(f) \leq \mu(g)$.

<u>Proof</u> : Applying (3.2.4) to $g-f \geq 0$, and by the linearity of μ, we can get the results.

3.2.6 <u>Theorem</u>. Every Radon measure μ on Ω is a distribution T_μ on Ω, where T_μ denotes the restriction of μ on $\mathcal{D}(\Omega)$.

<u>Proof</u> : Since $\mathcal{D}(\Omega)$ is the subspace of $\mathcal{K}(\Omega)$, we have

$$|T_\mu(\varphi)| = |\mu(\varphi)| \leq \beta \sup_{x \in K} |\varphi(x)| \qquad (\varphi \in \mathcal{D}(\Omega;K).$$

Therefore T_μ is a distribution on Ω.

3.2.7 <u>Example</u>. A mapping $T : \mathcal{D}(\mathbb{R}) \longrightarrow \mathbb{R}$ given by $T(\varphi) = \varphi'(0)$ is a distribution on $\mathbb{R}$ but not a Radon measure on $\mathbb{R}$. The linearity is clear, and also we have $|T(\varphi)| = |\varphi'(0)| \leq \sup_{x \in K} |\varphi'(x)| \leq p_{m,K}(\varphi)$ for all $\varphi \in \mathcal{D}(\mathbb{R};K)$ and $m \leq 1$. Hence T is a distribution on $\mathbb{R}$. Next, we will prove that T is not a Radon measure. We first produce a sequence (φ_m) of functions in $\mathcal{D}(\mathbb{R})$ such that $\varphi_m \longrightarrow 0$ uniformly but $\varphi_m'(0) \not\rightarrow 0$. To do this we define the functions f_m by

$$f_m(x) = \frac{\sin m x}{\sqrt{m}} \qquad (x \in \mathbb{R}).$$

Then f_m is infinitely differentiable. By (2.2.5), there exists a function $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\varphi = 1$ on $[-1, 1]$, and $0 \leq \varphi \leq 1$ on $\mathbb{R}$. Next we define the functions φ_m by

$$\varphi_m(x) = \varphi(x) f_m(x) \qquad (x \in \mathbb{R}).$$

Then we can see that for all m, Supp $(\varphi_m) \subset K$ for some compact subset in $\mathbb{R}$ and $\varphi_m \in \mathcal{D}(\mathbb{R})$. We observe that $\sup_{x \in K} |\varphi_m(x)| = \frac{1}{\sqrt{m}}$, and $\varphi_m'(0) = \sqrt{m}$ for all m. From this we can see that

$$|T(\varphi_m)| = |\varphi_m'(0)| = \sqrt{m} = m \cdot \frac{1}{\sqrt{m}}$$

which implies that

$$|T(\varphi_m)| = m \sup_{x \in K} |\varphi_m(x)| \qquad (\varphi_m \in \mathcal{D}(\mathbb{R};K)).$$

This shows that we cannot find any real number β such that

$$|T(f)| \leq \beta \sup_{x \in K} |f(x)| \qquad (f \in \mathcal{K}(\mathbb{R};K)).$$

Therefore T is not a Radon measure (3.2.2).

3.2.8 Theorem. If T is a positive distribution on Ω, then there exists a positive Radon measure μ on Ω such that $T = T_\mu$, i.e.,

$$T(\varphi) = \int_{\mathbb{R}^n} \varphi(x)\, d\mu(x) \qquad (\varphi \in \mathcal{D}(\Omega)).$$

Proof : Let $f \in \mathcal{K}(\Omega)$. By (2.2.4), there exists a sequence (φ_m) in $\mathcal{D}(\Omega)$ with supports contained in a compact neighborhood of Supp (f), say K such that $\varphi_m \to f$ uniformly as $m \to +\infty$. Define

$$\mu(f) = \lim_{m \to +\infty} T(\varphi_m).$$

We first show that the limit exists. Since K is compact, there exists a $\beta > 0$ such that, for all m, $|T(\varphi_m)| \leq \beta \sup_{x \in K} |\varphi_m(x)|$ (3.1.1(ii)). And since $\varphi_m \to f$ uniformly as $m \to +\infty$, for any $\varepsilon > 0$, there exists a $N(\varepsilon)$ such that, for all m, n $\geq$ N,

$$|\varphi_m(x)-f(x)| < \varepsilon/4\beta \quad \text{and} \quad |f(x)-\varphi_n(x)| < \varepsilon/4\beta.$$

Then we can conclude that

$$|\varphi_m(x)-\varphi_n(x)| < \varepsilon/2\beta$$

for all m, n $\geq$ N and for all $x \in \Omega$, which implies that for all m, n $\geq$ N,

$$|T(\varphi_m)-T(\varphi_n)| = |T(\varphi_m-\varphi_n)| \leq \beta \sup_{x \in K} |(\varphi_m-\varphi_n)(x)| \leq \frac{\varepsilon}{2} < \varepsilon.$$

Hence $(T(\varphi_m))$ is a Cauchy sequence in $\mathbb{R}$ which implies that $(T(\varphi_m))$ converges to c, say, in $\mathbb{R}$.

Let (ψ_n) be another sequence in $\mathcal{D}(\Omega)$ such that $\psi_n \longrightarrow f$ uniformly as $n \to +\infty$, and $\mathrm{Supp}(\psi_n) \subseteq K$ for all n. Let $\lim_{n \to +\infty} T(\psi_n) = c'$.

Since $\varphi_m \longrightarrow f$ uniformly as $m \longrightarrow +\infty$, for any $\varepsilon > 0$, there exists a $N_1(\varepsilon)$ such that for all $m \geq N_1$,

$$|\varphi_m(x) - f(x)| < \varepsilon/6\beta \quad .$$

Similarly, there exists a $N_2(\varepsilon)$ such that for all $n \geq N_2$,

$$|f(x) - \psi_n(x)| < \varepsilon/6\beta \quad .$$

Then for all $m, n \geq N_3 = \max\{N_1, N_2\}$, we have

$$|\varphi_m(x) - \psi_n(x)| < \varepsilon/3\beta \quad ,$$

which implies that for all $m, n \geq N_3$,

$$|T(\varphi_m) - T(\psi_n)| = |T(\varphi_m - \psi_n)| \leq \beta \sup_{x \in K} |(\varphi_m - \psi_n)(x)| \leq \varepsilon/3 \; .$$

Since the limits exist, for $\varepsilon > 0$, there exist N_4 and N_5 such that for all $m \geq N_4$, $n \geq N_5$,

$$|c - T(\varphi_m)| < \varepsilon/3 \quad \text{and} \quad |T(\psi_n) - c'| < \varepsilon/3 \; .$$

Then for all $m, n \geq N_6 = \max\{N_3, N_4, N_5\}$,

$$\begin{aligned} |c - c'| &= |c - T(\varphi_m) + T(\varphi_m) - T(\psi_n) + T(\psi_n) - c'| \\ &\leq |c - T(\varphi_m)| + |T(\varphi_m) - T(\psi_n)| + |T(\psi_n) - c'| \\ &< \varepsilon \, . \end{aligned}$$

Since ε is arbitrary, we can conclude that $c = c'$.

To prove the linearity of μ, we need the following facts :

(i) $\varphi_m \to f$ and $\psi_m \to g$ implies that $(\varphi_m + \psi_m) \longrightarrow (f+g)$, and

(ii) $\varphi_m \to f$ implies that $(\alpha\varphi_m) \longrightarrow \alpha f$ for any real number α ([8]).

By the linearity of T and the property of limit, we have

$$\mu(f+g) = \lim_{m\to+\infty} T(\varphi_m+\psi_m) = \lim_{m\to+\infty} (T(\varphi_m)+T(\psi_m))$$

$$= \lim_{m\to+\infty} T(\varphi_m) + \lim_{m\to+\infty} T(\psi_m) = \mu(f)+\mu(g),$$

and
$$\mu(\alpha f) = \lim_{m\to+\infty} T(\alpha\varphi_m) = \lim_{m\to+\infty} \alpha T(\varphi_m)$$

$$= \alpha \lim_{m\to+\infty} T(\varphi_m) = \alpha \mu(f).$$

Since $\varphi_m \longrightarrow f$ uniformly as $m \to +\infty$, for any $\varepsilon > 0$, there exists m_o such that for all $m \geq m_o$

$$\sup_{x\in K} \left|\varphi_m(x)-f(x)\right| \leq \varepsilon .$$

By using the facts that if $g(x) \leq h(x)$ on K, then $\sup_{x\in K} g(x) \leq \sup_{x\in K} h(x)$ and $\sup_{x\in K} g(x) + \inf_{x\in K} h(x) \leq \sup_{x\in K} (g(x)+h(x))$, we see that

$$\sup_{x\in K} \left|\varphi_m(x)\right| - \sup_{x\in K} \left|f(x)\right| \leq \varepsilon$$

or

$$\sup_{x\in K} \left|\varphi_m(x)\right| \leq \sup_{x\in K} \left|f(x)\right| + \varepsilon \qquad \text{for all } m \geq m_o.$$

Since ε is arbitrary and the inequality is true for all $m \geq m_o$, we conclude that

$$\lim_{m \to +\infty} \sup_{x \in K} \left| \varphi_m(x) \right| \leq \sup_{x \in K} \left| f(x) \right| .$$

From this we have

$$\left| \mu(f) \right| = \left| \lim_{m \to +\infty} T(\varphi_m) \right| = \lim_{m \to +\infty} \left| T(\varphi_m) \right|$$

$$\leq \beta \lim_{m \to +\infty} \sup_{x \in K} \left| \varphi_m(x) \right|$$

$$\leq \beta \sup_{x \in K} \left| f(x) \right| \qquad (f \in \mathcal{K}(\Omega;K)).$$

Hence μ is a Radon measure on Ω. That is, there exists a positive Radon measure μ such that $T = T_\mu$, i.e.,

$$T(\varphi) = \int_{\mathbb{R}^n} \varphi(x) d\mu(x) \qquad (\varphi \in \mathcal{D}(\Omega)).$$

3.3 Support

3.3.1 <u>Definition</u>. A distribution $T \in \mathcal{D}'(\Omega)$ is said to be <u>zero</u> on an open set U of Ω if $T(\varphi) = 0$ for every $\varphi \in \mathcal{D}(\Omega)$ with support contained in U.

3.3.2 <u>Theorem</u>. If $T \in \mathcal{D}'(\Omega)$ is zero on each of the open subsets U_λ of Ω, then T is zero on their union $U = \bigcup_{\lambda \in \Lambda} U_\lambda$.

<u>Proof</u> : Let $\varphi \in \mathcal{D}(\Omega)$ be a function with Supp $(\varphi) = K \subset U$. Since K is compact, a finite number of the U_λ (say $U_1, U_2, \ldots, U_m$) cover K. By (2.2.6) there are non-negative functions $\varphi_j \in \mathcal{D}(U_j)$ such that

$$\varphi(x) = \sum_{j=1}^{m} \varphi(x)\,\varphi_j(x) \qquad \left\{ \begin{array}{ll} \leq 1 & \text{on } \Omega, \\ = 1 & \text{on } K, \end{array} \right.$$

So we have

$$T(\varphi) = T\Big(\sum_{j=1}^{m} \varphi\cdot\varphi_j\Big) = \sum_{j=1}^{m} T(\varphi\cdot\varphi_j) = 0,$$

since $\varphi\cdot\varphi_j \in \mathcal{D}(U_j)$ and (3.3.1).

3.3.3 <u>Definition</u>. Let T be a distribution in Ω. The complement of the largest open subset of Ω in which T vanishes is called the <u>support</u> of T and will be denoted by Supp (T). Thus $T(\varphi) = 0$ whenever Supp $(T) \cap$ Supp $(\varphi) = \emptyset$.

3.3.4 Theorem. If S and T belong to $\mathcal{D}'(\Omega)$, then

(i) $\operatorname{Supp}(S+T) \subseteq \operatorname{Supp}(S) \cup \operatorname{Supp}(T)$, and

(ii) $\operatorname{Supp}(\alpha T) = \operatorname{Supp}(T)$ for all constants $\alpha \neq 0$.

Proof: (i) If $\varphi \in \mathcal{D}(\Omega)$ such that $\operatorname{Supp}(\varphi) \subset (\operatorname{Supp}(S) \cup \operatorname{Supp}(T))^c$ $= (\operatorname{Supp}(S))^c \cap (\operatorname{Supp}(T))^c$, then $(S+T)(\varphi) = S(\varphi)+T(\varphi) = 0$, and therefore $(\operatorname{Supp}(S) \cup \operatorname{Supp}(T))^c \subseteq (\operatorname{Supp}(S+T))^c$ or, equivalently

$$\operatorname{Supp}(S+T) \subseteq \operatorname{Supp}(S) \cup \operatorname{Supp}(T).$$

(ii) Since $\alpha T(\varphi) = T(\alpha\varphi)$ and $\operatorname{Supp}(\alpha\varphi) = \operatorname{Supp}(\varphi)$ for all constants $\alpha \neq 0$, we get the result immediately.

3.4 Differentiation

3.4.1 Definition. Let $T \in \mathcal{D}'(\Omega)$. For any multi-index r, $\partial^r T$ is given by

$$\partial^r T(\varphi) = (-1)^{|r|} T(\partial^r \varphi) \qquad (\varphi \in \mathcal{D}(\Omega)).$$

3.4.2 Theorem. Let $T \in \mathcal{D}'(\Omega)$. For any multi-index r, $\partial^r T$ also belongs to $\mathcal{D}'(\Omega)$.

Proof : By (3.4.1) and the linearity of T,

$$\partial^r T(\varphi_1+\varphi_2) = (-1)^{|r|} T(\partial^r(\varphi_1+\varphi_2)) = (-1)^{|r|} T(\partial^r\varphi_1+\partial^r\varphi_2)$$

$$= (-1)^{|r|}(T(\partial^r\varphi_1)+T(\partial^r\varphi_2)) = \partial^r T(\varphi_1)+\partial^r T(\varphi_2),$$

and $$\partial^r T(\alpha\varphi) = (-1)^{|r|} T(\partial^r(\alpha\varphi)) = \alpha(-1)^{|r|} T(\partial^r\varphi)$$

$$= \alpha\partial^r T(\varphi),$$

for all $\varphi_1, \varphi_2, \varphi \in \mathcal{D}(\Omega)$ and for all constants α. Hence $\partial^r T$ is linear. Since

$$\left|\partial^r T(\varphi)\right| = \left|(-1)^{|r|} T(\partial^r\varphi)\right| = \left|T(\partial^r\varphi)\right|$$

and Supp $(\partial^r\varphi) \subseteq$ Supp (φ) for all $\varphi \in \mathcal{D}(\Omega)$ and for every multi-index r, we have

$$\left|\partial^r T(\varphi)\right| = \left|T(\partial^r\varphi)\right| \leq \beta \max_{|s|\leq m} \sup_{x\in K} \left|\partial^s(\partial^r\varphi(x))\right|$$

$$= \beta \max_{|t|\leq m+|r|} \sup_{x\in K} \left|\partial^t\varphi(x)\right|$$

for all $\varphi \in \mathcal{D}(\Omega;K)$, where $\beta > 0$ and $m \geq 0$. That is, $\partial^r T$ is continuous. Therefore $\partial^r T \in \mathcal{D}'(\Omega)$ for every multi-index r.

3.4.3 <u>Theorem.</u> If $T \in \mathcal{D}'(\Omega)$, then for every multi-index r,

$$\text{Supp } (\partial^r T) \subseteq \text{Supp } (T).$$

<u>Proof</u> : Since for every multi-index r, and for all $\varphi \in \mathcal{D}(\Omega)$, $\text{Supp } (\partial^r \varphi) \subseteq \text{Supp } (\varphi)$, and by (3.4.1), the result follows immediately.

4.3.4 <u>Definition.</u> Let T be a distribution on Ω and ψ an infinitely differentiable function on Ω. The product ψT is given by

$$(\psi T)(\varphi) = T(\psi\varphi) \qquad (\varphi \in \mathcal{D}(\Omega)).$$

4.3.5 <u>Theorem.</u> Let T be a distribution on Ω and ψ an infinitely differentiable function on Ω. Then

$$\partial_j(\psi T)(\varphi) = \psi(\partial_j T)(\varphi) + (\partial_j \psi)T(\varphi) \qquad (\varphi \in \mathcal{D}(\Omega)).$$

This is called Leibniz's formula.

<u>Proof</u> : $\partial_j(\psi T)(\varphi) = -(\psi T)(\partial_j \varphi) = -T(\psi \partial_j \varphi)$

$$= -T(\partial_j(\psi\varphi) - (\partial_j\psi)\varphi) = -T(\partial_j(\psi\varphi)) + T((\partial_j\psi)\varphi)$$

$$= (\partial_j T)(\psi\varphi) + (\partial_j\psi)T(\varphi) = \psi(\partial_j T)(\varphi) + (\partial_j\psi)T(\varphi).$$

3.5 Distributions with Compact Support

3.5.1 Definition. Let Ω be an open subset of $\mathbb{R}^n$. The space of infinitely differentiable functions defined on Ω will be denoted by $\mathcal{E}(\Omega)$.

If T is any distribution on Ω, the subset of $\mathcal{E}(\Omega)$ consisting of those function φ for which Supp (T) $\cap$ Supp (φ) is compact is a vector subspace of $\mathcal{E}(\Omega)$ which we denote temporarily by F. We also note that $\mathcal{D}(\Omega) \subset$ F. The linear form T can be extended to F by taking for each $\varphi \in$ F, a function $\psi \in \mathcal{D}(\Omega)$ taking the value 1 on a neighbourhood of Supp (T) $\cap$ Supp (φ), and then putting $T(\varphi) = T(\psi\varphi)$. This value does not depend on the particular function ψ chosen, for, if ψ_1 and ψ_2 are any two such functions, $(\psi_1 - \psi_2)\varphi$ has support disjoint from Supp (T), and so $T((\psi_1-\psi_2)\varphi) = 0$ (3.3.3). Further, this extension of T to F is linear : if $\varphi_1, \varphi_2 \in$ F we can take ψ so as to be 1 on a neighbourhood of Supp (T) $\cap$ (Supp(φ_1) $\cup$ Supp(φ_2), and then

$$T(\varphi_1+\varphi_2) = T(\psi(\varphi_1+\varphi_2)) = T(\psi\varphi_1)+T(\psi\varphi_2) = T(\varphi_1)+T(\varphi_2).$$

Similarly $T(\alpha\varphi) = \alpha\, T(\varphi)$ for all constants α.

In particular, if T has compact support, F = $\mathcal{E}(\Omega)$ and T can be extended linearly to the whole of $\mathcal{E}(\Omega)$; the extension just decribed is the only one that does not increase Supp (T).

The functions $p_{m,K}$ defined in (2.1.5) are seminorms on $\mathcal{E}(\Omega)$; as m runs through the non-negative integers and K runs through an increasing sequence of compact sets whose union is Ω, and each of which is contained in the interior of its sucessor, and so define a topology there. A distribution T with compact support K satisfies

$$(*) \qquad |T(\varphi)| \leq \beta \, p_{m,K'}(\varphi) \qquad (\varphi \in \mathcal{E}(\Omega))$$

for some contant β and some compact neighborhood K' of K (containing the support of the function φ above). Hence T is continuous for this topology on $\mathcal{E}(\Omega)$. Conversely, any continuous linear form T satisfies an inequality of the type (*) and so defines a distribution with support contained in K' (since if Supp (φ) does not meet K', (*) implies that $T(\varphi) = 0$).

Thus the distribution of compact support form a vector space, denoted by $\mathcal{E}'(\Omega)$.